Exercice 1. (Bases in Fock spaces) We denote by $c_{\mathrm{c}}(\mathbb{N},\mathbb{N})$ the set of sequences of integers equal to 0 except for a finite number of terms. For $\vec{k}\in c_{\mathrm{c}}(\mathbb{N},\mathbb{N})$, $k=(k_1,\dots,k_d)$ we set

$$|\vec{k}| = k_1+\cdots+k_d,\ \vec{k}! = k_1!\dots k_d!.$$

Let $\mathcal{Y}$ a separable Hilbert space with an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. We set

$$e_{\vec{k}} := e_1^{\otimes k_1}\otimes\cdots e_d^{\otimes k_d}\in\Gamma_{\mathrm{s}}(\mathcal{Y}).$$

1) Show that

$$\left\{\frac{\sqrt{|\vec{k}|!}}{\sqrt{\vec{k}!}}e_{\vec{k}}\ : |\vec{k}|=n\right\}$$

is an o.n. basis of $\Gamma_{\mathrm{s}}^n(\mathcal{Y})$.

2) Formulate the appropriate extension if $\mathcal{Y}$ is not separable.

3) For $J=\{i_1,\cdots,i_n\}\subset\{1,\dots,d\}$ with $1\leq i_1<\cdots<i_n\leq d$, set

$$e_J := e_{i_1}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_{i_n}.$$

Prove that

$$\left\{\sqrt{\#J!}e_J\ :\ J\subset I,\ ,\#J=n\right\}$$

is an o.n. basis of $\Gamma_{\mathrm{a}}^n(\mathcal{Y})$.

4) Same question as 2).

Exercice 2. (Exponential law for Fock spaces)

1) Find the formula giving

$$\|\prod_{i=1}^n a^*(y_i)\Omega\|^2,\ y_i\in\mathcal{Y},$$

both in the bosonic and fermionic case.

Hint: use the CCR /CAR and properties of the vacuum vector.

2) Let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be two Hilbert spaces. Show that there exists a unique linear map:

$$U:\Gamma_{\mathrm{s}}(\mathcal{Y}_1)\otimes\Gamma_{\mathrm{s}}(\mathcal{Y}_2)\to\Gamma_{\mathrm{s}}(\mathcal{Y}_1\oplus\mathcal{Y}_2)$$

such that:

$$a^{(*)}(y_1\oplus y_2)U = U(a^{(*)}(y_1)\otimes\mathbb{1}+\mathbb{1}\otimes a^{(*)}(y_2)),\ \forall\ y_i\in\mathcal{Y}_i,$$
$$U\Omega\otimes\Omega=\Omega.$$

3) Show without lengthy computation that U is unitary.

Hint use 1).

Exercice 3.

1) Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D}\subset\mathcal{H}$ a dense subspace, and c, a two linear operators on $\mathcal{D}$ such that

(1) $c,a:\mathcal{D}\to\mathcal{D}$;

(2) $c\subset a^*$, $a\subset c^*$;

(3) $a^2=c^2=0$, $ac+ca=\mathbb{1}$ as operator identities on $\mathcal{D}$.

Show that c, a extend as bounded operators on $\mathcal{H}$, $c = a^*$ and $\|a\| = \|c\| = 1$.

2) Deduce from 1) that the fermionic creation/annihilation operators are bounded with $\|a^{(*)}(y)\| = \|y\|$.

Exercice 4. (Essential selfadjointness of field operators)
Use Nelson's commutator theorem to prove that the bosonic field operators $\phi(y)$ $y \in \mathcal{Y}$ are essentially selfadjoint on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Y})$.

Exercice 5. (Creation/annihilation operators from field operators)
Let $\phi(y)$, $y \in \mathcal{Y}$ the bosonic field operators, acting on the Fock space. We admit that $\mathrm{Dom}\phi(y) \cap \mathrm{Dom}\phi(\mathrm{i}y)$ is dense in $\Gamma_{\mathrm{s}}(\mathcal{Y})$.

1) Prove that the operators

$$a(y) := \frac{1}{\sqrt{2}}(\phi(y) + \mathrm{i}\phi(\mathrm{i}y)),\ a^*(y) := \frac{1}{\sqrt{2}}(\phi(y) - \mathrm{i}\phi(\mathrm{i}y)),$$

with domain $\mathrm{Dom}\phi(y) \cap \mathrm{Dom}\phi(\mathrm{i}y)$ are closed and adjoint from one another.