Exercice 1. Let α^i, $\beta \in L(\mathcal{V})$ the operators defined in Sect. 5.4 of the notes.

1)Show that α^i and β are selfadjoint for $(\cdot|\cdot)$.

2) Show that for $k \in \mathbb{R}^d$ $\sigma(\alpha^i k_i + m\beta) = \{\epsilon(k), -\epsilon(k)\}$ for $\epsilon(k) = (k^2 + m^2)^{1/2}$.

Exercice 2. Check the properties listed in Subsect. 5.5.1 of the map:

$$U : \left\{ \begin{array}{l} \mathrm{Sol}_{\mathrm{sc}}(KG) \sim C_0^\infty(\mathbb{R}^d) \oplus C_0^\infty(\mathbb{R}^d) \to (2\epsilon)^{\frac{1}{2}}\mathfrak{h} \\ (\pi, \varphi) \mapsto \pi + \mathrm{i}\epsilon\varphi. \end{array} \right.$$

Exercice 3. Check that the Klein-Gordon field $\Phi(x)$ defined in Subsect. 5.5.2 satisfies the Klein-Gordon equation:

$$(-\Box + m^2)\Phi(x) = 0,$$

in distribution sense.

Same question for the Dirac field $\Psi(x)$ defined in Subsect. 5.6.2.

Exercice 4. We use the notation in Subsect. 5.6.1. On $\Gamma_{\mathrm{a}}(\mathcal{Z})$ we define the *total charge operator* by:

$$Q := \mathrm{d}\Gamma(\mathbb{1}_{\mathfrak{h}^+} \oplus -\mathbb{1}_{\overline{\mathfrak{h}^-}}).$$

Show that

$$\mathrm{spec}(Q) = \mathbb{Z}.$$

Check that

$$\mathrm{e}^{\mathrm{i}\theta Q}\psi^{(*)}(h)\mathrm{e}^{-\mathrm{i}\theta Q} = \psi^{(*)}(\mathrm{e}^{\mathrm{i}\theta}h),\ \theta \in \mathbb{R},\ h \in \mathfrak{h}.$$

Deduce from this fact that:

$$Q\psi(h) = \psi(h)(Q-1),\ Q\psi^*(h) = \psi^*(h)(Q+1),$$

i.e. that $\psi(h)$ resp. $\psi^*(h)$ decreases , resp. increases the total charge by 1.