Exercice 1. Write the details of the proof of Prop. 7.7 from the notes.

Exercice 2. Write the details of the proof of Thm. 7.9 from the notes.

Exercice 3. Formulate the conditions for a pair $\lambda_{\pm} \in L_{\rm h}(\mathcal{Y}, \mathcal{Y}^*)$ to be the complex covariances of a pure gauge invariant state in the fermionic case.

Exercice 4. Let $\phi_1, \dots, \phi_n \in B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Abusing the language one says that $\{\phi_1, \dots, \phi_n\}$ is a CAR representation over $\mathbb{R}^n$ if $\phi_i = \phi_i^*$ and $[\phi_i, \phi_j]_+ = 2\delta_{ij}\mathbb{1}$.

1) Check that then $\mathbb{R}^n \ni x \mapsto \sum_{i=1}^n x_i\phi_i$ is a representation of $\mathrm{CAR}(\mathbb{R}^n)$, where we equip $\mathbb{R}^n$ with its canonical euclidean structure.

2) We recall the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Check that $\{\sigma_1, \sigma_2\}$ is a CAR representation over $\mathbb{R}^2$, and $\{\sigma_1, \sigma_2, \sigma_3\}$ is a CAR representation over $\mathbb{R}^3$.

3) Let $\{\phi_1, \phi_2\}$ a CAR representation over $\mathbb{R}^2$ and $I := \mathrm{i}\phi_1\phi_2$. Show that $I^2 = \mathbb{1}$, $I = I^*$ and $I \neq \mathbb{1}$. Let $\mathcal{K} = \mathrm{Ker}(I - \mathbb{1})$. Show that the map

$$\mathcal{H} \ni \Psi \mapsto U\Psi := (\frac{1}{2}(\phi_1 - \phi_1 I)\Psi, \frac{1}{2}(\mathbb{1} + I)\Psi) \in \mathcal{K} \oplus \mathcal{K} \sim \mathbb{C}^2 \otimes \mathcal{K}$$

is unitary.

4) Show that $U\phi_1 U^{-1} = \sigma_1 \otimes \mathbb{1}_{\mathcal{K}}$ and $U\phi_2 U^{-1} = \sigma_2 \otimes \mathbb{1}_{\mathcal{K}}$.

Exercice 5. (Jordan-Wigner representation)

1) On $B(\otimes^m \mathbb{C}^2)$ we set:

$$\sigma_i^{(j)} := \mathbb{1}^{\otimes(j-1)} \otimes \sigma_i \otimes \mathbb{1}^{\otimes(m-j)}, \quad i = 1, 2, 3, \quad j = 1, \dots, m.$$

We set also $I_0 := \mathbb{1}$, $I_j := \sigma_3^{(1)} \cdots \sigma_3^{(j)}$ for $j = 1, \dots, m$. Let now:

$$\phi_{2j-1}^{\rm JW} := I_{j-1}\sigma_1^{(j)}, \quad \phi_{2j}^{\rm JW} := I_{j-1}\sigma_2^{(j)}, \ j = 1, \dots, m, \tag{0.1}$$

Prove that $\{\phi_1^{\rm JW}, \dots, \phi_{2m}^{\rm JW}\}$ is a CAR representation over $\mathbb{R}^{2m}$, called the Jordan-Wigner representation.

2) Prove that $\{\phi_1^{\rm JW}, \dots, \phi_{2m}^{\rm JW}, \pm I_m\}$ are two CAR representations over $\mathbb{R}^{2m+1}$.

3) Prove that these two representations are not equivalent.