If $\sigma \in L_{\rm s}(\mathcal{X}, \mathcal{X}^{\#})$, we denote by $O(\mathcal{X}, \sigma)$ the linear (pseudo-)orthogonal group on $\mathcal{X}$. Similarly if $\sigma \in L_{\rm a}(\mathcal{X}, \mathcal{X}^{\#})$ is non-degenerate, i.e. $(\mathcal{X}, \sigma)$ is a symplectic space, we denote by $Sp(\mathcal{X}, \sigma)$ the linear symplectic group on $\mathcal{X}$.

If $\mathcal{X}$ is a complex vector space, we denote by $\mathcal{X}_{\mathbb{R}}$ its *real form*, i.e. $\mathcal{X}$ considered as a real vector space. We denote by $\overline{\mathcal{X}}$ a *conjugate vector space* to $\mathcal{X}$, i.e. a complex vector space $\overline{\mathcal{X}}$ with an anti-linear isomorphism $\mathcal{X} \ni x \mapsto \overline{x} \in \overline{\mathcal{X}}$. The *canonical conjugate vector space* to $\mathcal{X}$ is simply the real vector space $\mathcal{X}_{\mathbb{R}}$ equipped with the complex structure $-{\rm i}$, if ${\rm i}$ is the complex structure of $\mathcal{X}$. In this case the map $x \to \overline{x}$ is the identity. If $a \in L(\mathcal{X}_1, \mathcal{X}_2)$, we denote by $\overline{a} \in L(\overline{\mathcal{X}}_1, \overline{\mathcal{X}}_2)$ the linear map defined by:

$$\overline{a}\overline{x}_1 := \overline{a x_1}, \ \overline{x}_1 \in \overline{\mathcal{X}}_1. \tag{3.1}$$

We denote by $\mathcal{X}^*$ the *anti-dual* of $\mathcal{X}$, i.e. the space of anti-linear forms on $\mathcal{X}$. Clearly $\mathcal{X}^*$ can be identified with $\overline{\mathcal{X}^{\#}} \sim \overline{\mathcal{X}}^{\#}$.

Sesquilinear forms on $\mathcal{X}$ are identified with elements of $L(\mathcal{X}, \mathcal{X}^*)$, and we use the notation $(x_1|bx_2)$ or sometimes $\overline{x_1}\cdot bx_2$ for $b \in L(\mathcal{X}, \mathcal{X}^*)$, $x_1, x_2 \in \mathcal{X}$.

The space of hermitian (resp. anti-hermitian) sesquilinear forms on $\mathcal{X}$ is denoted by $L_{\rm s}(\mathcal{X}, \mathcal{X}^*)$ (resp. $L_{\rm a}(\mathcal{X}, \mathcal{X}^*)$).

If $q \in L_{\rm s}(\mathcal{X}, \mathcal{X}^*)$ is non-degenerate, i.e. $(\mathcal{X}, q)$ is a pseudo-unitary space, we denote by $U(\mathcal{X}, q)$ the linear pseudo-unitary group on $\mathcal{X}$.

If b is a bilinear form on the real vector space $\mathcal{X}$, its canonical sesquilinear extension to $\mathbb{C}\mathcal{X}$ is by definition the sesquilinear form $b_{\mathbb{C}}$ on $\mathbb{C}\mathcal{X}$ given by

$$(w_1|b_{\mathbb{C}}w_2) := x_1\cdot bx_2 + y_1\cdot by_2 + {\rm i}x_1\cdot by_2 - {\rm i}y_1\cdot bx_2, \quad w_i = x_i + {\rm i}y_i$$

for $x_i, y_i \in \mathcal{X}$, $i = 1, 2$. This extension maps (anti-)symmetric forms on $\mathcal{X}$ onto (anti-)hermitian forms on $\mathbb{C}\mathcal{X}$.

Conversely if $\mathcal{X}$ is a complex vector space and $\mathcal{X}_{\mathbb{R}}$ is its real form, i.e. $\mathcal{X}$ considered as a real vector space, then for $b \in L_{\rm s/a}(\mathcal{X}, \mathcal{X}^*)$ the form ${\rm Re}b$ belongs to $L_{\rm s/a}(\mathcal{X}_{\mathbb{R}}, \mathcal{X}_{\mathbb{R}}^{\#})$.

4. Tensor algebras and Fock spaces

4.1. **Introduction.** Quantum field theory in most physics textbooks starts with the introdution of *Fock spaces*. These are Hilbert spaces describing systems with a finite, but not fixed number of particles. The possibility of creation and annihilation of particles is one of the essential new features of QFT, compared to non-relativistic quantum mechanics.

The particles can be either *bosons* or *fermions*, which leads to the bosonic/fermionic Fock spaces. Fock spaces are very useful to introduce some basic notions, however they are not really central objects in QFT. Relying too much on them can lead to some misconceptions.

4.2. **Tensor algebras.** We start by defining the tensor algebra over a vector space, which could also be called the Fock space without statistics. Let $\mathcal{Y}$ be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Vectors in $\mathcal{Y}$ correspond to *one-particle states*.

Definition 4.1. *Let* $\overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}$ *(or* $\mathcal{Y}^{\overset{\mathrm{al}}{\otimes} n}$*) denote the* n-th algebraic tensor power of $\mathcal{Y}$*. We will write* $\overset{\mathrm{al}}{\otimes}{}^0\mathcal{Y} := \mathbb{K}$*. The* algebraic tensor algebra over $\mathcal{Y}$ *is defined as*

$$\overset{\mathrm{al}}{\otimes}\mathcal{Y} := \overset{\mathrm{al}}{\underset{0\leq n<\infty}{\oplus}} \overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}.$$

The element $1 \in \overset{\mathrm{al}}{\otimes}{}^0\mathcal{Y}$ *is called the* vacuum *and denoted by* Ω*.*

If $\mathcal{Y}$ is a Hilbert space, we will write $\otimes^n\mathcal{Y}$ (or $\mathcal{Y}^{\otimes n}$) for the *n-th tensor power of $\mathcal{Y}$ in the sense of Hilbert spaces.* We set

$$\otimes\mathcal{Y} := \underset{0\leq n<\infty}{\oplus} \otimes^n\mathcal{Y} = \left(\overset{\mathrm{al}}{\otimes}\mathcal{Y}\right)^{\mathrm{cpl}}.$$

$\otimes\mathcal{Y}$ is called the *complete tensor algebra* or the *full Fock space.*

Elements of $\overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}$ or of $\otimes^n\mathcal{Y}$ are called *n particle vectors.* We set also:

$$\otimes^{\mathrm{fin}}\mathcal{Y} := \overset{\mathrm{al}}{\underset{0\leq n<\infty}{\oplus}} \otimes^n\mathcal{Y},$$

which is the space of *finite particle vectors.*

The vector spaces $\overset{\mathrm{al}}{\otimes}\mathcal{Y}$ and $\otimes\mathcal{Y}$ are associative algebras, when equipped with the tensor product $\otimes$ and identity Ω.

4.3. Some operators on $\otimes\mathcal{Y}$.

Definition 4.2. *Let p be a linear operator from $\mathcal{Y}_1$ to $\mathcal{Y}_2$. Then we define $\Gamma^n(p) := p^{\otimes n}$ with domain $\overset{\mathrm{al}}{\otimes}{}^n\mathrm{Dom}\, p$, and the operator $\Gamma(p)$ from $\otimes\mathcal{Y}_1$ to $\otimes\mathcal{Y}_2$*

$$\Gamma(p) := \overset{\infty}{\underset{n=0}{\oplus}} \Gamma^n(p)$$

with domain $\overset{\mathrm{al}}{\otimes}\mathrm{Dom}\, p$*.*

It is easy to see that $\Gamma(p)$ is closable, (resp. essentially selfadjoint, unitary) iff p is so. $\Gamma(p)$ is bounded iff $\|p\| \leq 1$.

Definition 4.3. *If h is a linear operator on $\mathcal{Y}$, we set*

$$\mathrm{d}\Gamma^n(h) := \sum_{j=1}^{n} 1\!\!1_{\mathcal{Y}}^{\otimes j-1} \otimes h \otimes 1\!\!1_{\mathcal{Y}}^{\otimes(n-j)}$$

with domain $\overset{\mathrm{al}}{\otimes}{}^n\mathrm{Dom}\, h$*, and*

$$\mathrm{d}\Gamma(h) := \overset{\infty}{\underset{n=0}{\oplus}} \mathrm{d}\Gamma^n(h)$$

with domain $\overset{\mathrm{al}}{\otimes}\mathrm{Dom}\, h$*.*

The operator

$$N := \mathrm{d}\Gamma(1\!\!1)$$

is called the *number operator*. Note that $N = n1\!\!1$ on $\otimes^n\mathcal{Y}$, which explains its name.

The following identity is often useful:

$$\Gamma(\mathrm{e}^a) = \mathrm{e}^{\mathrm{d}\Gamma(a)},$$

for a a linear operator on $\mathcal{Y}$.

4.4. **Bosonic and fermionic Fock spaces.** Let S_n the permutation group. Clearly S_n acts on $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$ by

$$\Theta(\sigma) y_1 \otimes \cdots \otimes y_n := y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)},\ \sigma \in S_n.$$

We define the following operators on $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$:

$$\begin{aligned}\Theta_{\mathrm{s}}^n &:= \frac{1}{n!} \sum_{\sigma \in S_n} \Theta(\sigma),\\ \Theta_{\mathrm{a}}^n &:= \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}\sigma \Theta(\sigma).\end{aligned}$$

and

$$\Theta_{\mathrm{s/a}} := \underset{0 \leq n < \infty}{\oplus} \Theta_{\mathrm{s/a}}^n.$$

The operators $\Theta_{\mathrm{s/a}}$ are projections on the subspaces of *symmetric/anti-symmetric tensors*. If $\mathcal{Y}$ is a Hilbert space, then $\Theta_{\mathrm{s/a}}$ are orthogonal projections.

Definition 4.4. *The space*

$$\Gamma_{\mathrm{s/a}}(\mathcal{Y}) := \Theta_{\mathrm{s/a}} \otimes \mathcal{Y}$$

is called the bosonic resp. fermionic Fock space over $\mathcal{Y}$.

The operators $\Gamma(p)$, $\mathrm{d}\Gamma(h)$ commute with $\Theta_{\mathrm{s/a}}$, hence act also on $\Gamma_{\mathrm{s/a}}(\mathcal{Y})$.

Definition 4.5. *Let* $\Psi, \Phi \in \overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y})$. *We define the* symmetric, *resp.* anti-symmetric tensor product of Φ and Ψ*:*

$$\Psi \otimes_{\mathrm{s/a}} \Phi := \Theta_{\mathrm{s/a}} \Psi \otimes \Phi.$$

Note that Θ_{a} is slightly different from the wedge product $\wedge$ often used for anti-symmetric tensors. In fact one has: The *wedge product of vectors* Φ *and* Ψ is defined as

$$\Psi \wedge \Phi := \frac{(p+q)!}{p!q!} \Psi \otimes_{\mathrm{a}} \Phi, \ \text{for } \Psi \in \overset{\mathrm{al}}{\Gamma}{}_{\mathrm{a}}^{p}(\mathcal{Y}),\ \Phi \in \overset{\mathrm{al}}{\Gamma}{}_{a}^{q}(\mathcal{Y}). \tag{4.1}$$

4.5. **Creation-annihilation operators.** The creation-annihilation operators are operators acting on $\Gamma_{\mathrm{s/a}}(\mathcal{Y})$ describing the operation of creating or annihilating one particle in $\mathcal{Y}$.

Definition 4.6. *Let* $y \in \mathcal{Y}$. *The* creation operator of y*, resp. the* annihilation operator of y*, are defined as operators on* $\Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Y})$ *by*

$$\begin{aligned} a^*(y)\Psi &:= \sqrt{n+1} y \otimes_{\mathrm{s/a}} \Psi,\\ a(y)\Psi &:= \sqrt{n} (y| \otimes 1\!\!1_{\mathcal{Y}}^{\otimes(n-1)} \Psi, \quad \Psi \in \Gamma_{\mathrm{s/a}}^n(\mathcal{Y}).\end{aligned}$$

We use here the standard practice of denoting creation operators by the symbol $a^*(y)$, which suggests that it is the adjoint of the annihilation operator $a(y)$. This is always true in the fermionic case, and becomes true in the bosonic case when appropriate domains are introduced.

Note that the map

$$\mathcal{Y} \ni \mapsto a^*(y) \text{ resp. } a(y) \in L(\Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Y}))$$

is $\mathbb{C}-$linear, resp. $\mathbb{C}-$anti-linear.

Note the following identity, valid in both the bosonic and fermionic case:

$$\Gamma(u)a^{(*)}(y)\Gamma(u^{-1}) = a^{(*)}(uy),\ u \in U(\mathcal{Y}).$$

In particular if ϵ is a selfadjoint operator on $\mathcal{Y}$ we have:

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\epsilon)}a^{(*)}(y)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\epsilon)} = a^{(*)}(\mathrm{e}^{\mathrm{i}t\epsilon}y),\ y \in \mathcal{Y}. \tag{4.2}$$

4.5.1. *Bosonic case.* In the bosonic case, the creation-annihilation operators are unbounded operators. In fact one has:

$$\|a(y)\Psi\| \leq \|y\|\|N^{\frac{1}{2}}\Psi\|,\quad \|a^*(y)\Psi\| \leq \|y\|\|(N+\mathbb{1})^{\frac{1}{2}}\Psi\|,\ \Psi \in \Gamma_{\mathrm{s}}(\mathcal{Y}),\ y \in \mathcal{Y}. \tag{4.3}$$

This follows easily from:

$$a^*(y)a(y) = \mathrm{d}\Gamma(|y)(y|) \leq \mathrm{d}\Gamma(\|y\|^2\mathbb{1}) = \|y\|^2 N.$$

The most important property are the CCR or *canonical commutation relations.* The following relations are valid on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Y})$:

$$\begin{aligned}
&(\Psi_1|a(y)\Psi_2) = (a^*(y)\Psi_1|\Psi_2)\\
&(\Psi_1|a^*(y)\Psi_2) = (a(y)\Psi_1|\Psi_2),\\
&[a^*(y_1), a^*(y_2)] = [a(y_1), a(y_2)] = 0,\\
&[a(y_1), a^*(y_2)] = (y_1|y_2)\mathbb{1},
\end{aligned} \tag{4.4}$$

where $[A, B] = AB - BA$ denotes the commutator of A and B.

4.5.2. *Fermionic case.* In the fermionic case, the creation-annihilation operators are bounded. In fact one has:

$$\|a(y)\| = \|a^*(y)\| = \|y\|,$$

which follows easily from the identity:

$$a^*(y)a(y)a^*(y)a(y) = \|y\|^2 a^*(y)a(y),$$

which in turns follows from the CAR just below:

$$\begin{aligned}
&(\Psi_1|a(y)\Psi_2) = (a^*(y)\Psi_1|\Psi_2)\\
&(\Psi_1|a^*(y)\Psi_2) = (a(y)\Psi_1|\Psi_2),\\
&[a^*(y_1), a^*(y_2)]_+ = [a(y_1), a(y_2)]_+ = 0,\\
&[a(y_1), a^*(y_2)]_+ = (y_1|y_2)\mathbb{1},
\end{aligned}$$

where $[A, B]_+ = AB + BA$ denotes the anti-commutator of A and B.

4.6. **Fields operators.** It is convenient to introduce selfadjoint operators, called the *fields*, which generate the same algebra as the creation-annihilation operators. We use the same notation in the bosonic or fermionic case.

Definition 4.7. *We set:*

$$\begin{aligned} \phi(y) :=& \ \tfrac{1}{\sqrt{2}}(a^*(y)+a(y)), \ \textit{acting on } \Gamma_{\rm s}^{\rm fin}(\mathcal{Y}) \textit{ in the bosonic case,} \\ \phi(y) :=& \ a^*(y)+a(y), \ \textit{acting on } \Gamma_{\rm a}(\mathcal{Y}) \textit{ in the fermionic case.} \end{aligned}$$

In the bosonic case the operators $\phi(y)$ are often called *Segal field operators.* Note that the map

$$\mathcal{Y} \ni y \mapsto \phi(y) \in L(\Gamma_{\rm s/a}^{\rm fin}(\mathcal{Y}))$$

is now only $\mathbb{R}-$linear on $\mathcal{Y}$. This means that when considering field operators the complex structure of $\mathcal{Y}$ is lost.

Recall that the space $\mathcal{Y}$ considered as a real vector space is denoted by $\mathcal{Y}_{\mathbb{R}}$.The space $\mathcal{Y}_{\mathbb{R}}$ inherits two natural structures from the Hilbertian scalar product on $\mathcal{Y}$:

(1) if we set:
$$y_1 \cdot \sigma y_2 := {\rm Im}(y_1|y_2),$$
then $\sigma \in L_{\rm a}(\mathcal{Y}_{\mathbb{R}}, \mathcal{Y}_{\mathbb{R}}^{\#})$ is non-degenerate, i.e. $(\mathcal{Y}_{\mathbb{R}}, \sigma)$ is a *symplectic space.*

(2) if we set:
$$y_1 \cdot \nu y_2 := {\rm Re}(y_1|y_2),$$
then $\nu \in L_{\rm s}(\mathcal{Y}_{\mathbb{R}}, \mathcal{Y}_{\mathbb{R}}^{\#})$ is non-degenerate and positive, i.e. $(\mathcal{Y}_{\mathbb{R}}, \nu)$ is an *Euclidean space.*

We will come back to this later when we will discuss the notion of a *Kähler structure.*

4.6.1. *Bosonic case.* In the bosonic case the operators $\phi(y)$ are unbounded, hence need a domain to be properly defined. From (11.1) we see that we can take ${\rm Dom}\phi(y) = {\rm Dom}N^{\frac{1}{2}}$. However with this domain $\phi(y)$ are not selfadjoint , but only hermitian, i.e. $\phi(y) \subset \phi^*(y)$.

However it is easy to prove, using Nelson's commutator theorem that $\phi(y)$ are *essentially selfadjoint*, i.e. their closure is selfadjoint. By abuse of notation we will still denote the closure of $\phi(y)$ by $\phi(y)$.

Proposition 4.8. *We have:*

$$\begin{aligned} &\phi(y) = \phi^*(y), \ y \in \mathcal{Y} \\ &\phi(\lambda y) = \lambda\phi(y), \ \phi(y_1)+\phi(y_2) \subset \phi(y_1+y_2), \ \lambda \in \mathbb{R}, \ y_i \in \mathcal{Y}, \\ &[\phi(y_1), \phi(y_2)] = {\rm i} y_1 \cdot \sigma y_2 \mathbb{1}, \ \textit{as quadratic form on } {\rm Dom}\phi(y_1) \cap {\rm Dom}\phi(y_2). \end{aligned}$$

The second line expresses the $\mathbb{R}-$linearity of $y \mapsto \phi(y)$ (with domain problems taken into account). The third line is also called the *Heisenberg form* of the CCR.

Note that we can recover the creation-annihilation operators from the field operators. In fact if we set:

$$a(y) := \phi(y) + {\rm i}\phi({\rm i}y), \ a^*(y) := \phi(y) - {\rm i}\phi({\rm i}y), \ y \in \mathcal{Y},$$

with domain ${\rm Dom}\phi(y) \cap {\rm Dom}\phi({\rm i}y)$, then $a(y)$, $a^*(y)$ are closed, densely defined and adjoint from one another. Moreover they satisfy the CCR (4.4) in quadratic form sense.

4.6.2. *Fermionic case.* The situation is much simpler in the fermionic case, since $\phi(y)$ is clearly bounded and selfadjoint.

Proposition 4.9. *We have:*

$$\begin{aligned}&\phi(y) = \phi^*(y),\ y \in \mathcal{Y}\\&\phi(\lambda y) = \lambda\phi(y),\ \phi(y_1) + \phi(y_2) = \phi(y_1 + y_2),\ \lambda \in \mathbb{R},\ y_i \in \mathcal{Y},\\&[\phi(y_1), \phi(y_2)]_+ = 2y_1 \cdot \nu y_2 \mathbb{1}.\end{aligned}$$

The last line is called the (euclidean) *Clifford relations.*

Again one can recover the creation-annihilation operators from the fields. One has:

$$a(y) = \frac{1}{2}(\phi(y) + \mathrm{i}\phi(\mathrm{i}y)),\ a^*(y) = \frac{1}{2}(\phi(y) - \mathrm{i}\phi(\mathrm{i}y)),\ y \in \mathcal{Y}.$$

4.7. **Weyl operators.** In the bosonic case ,it is inconvenient to work with the unbounded field operators. To avoid this problem one can introduce the *Weyl operators.*

Definition 4.10. *We set*

$$W(y) := \mathrm{e}^{\mathrm{i}\phi(y)} \in U(\Gamma_{\mathrm{s}}(\mathcal{Y})),\ y \in \mathcal{Y},$$

which are called the Weyl operators*. They satisfy:*

$$\begin{aligned}&W(0) = \mathbb{1},\ W(y)^* = W(-y),\ y \in \mathcal{Y},\\&W(y_1)W(y_2) = \mathrm{e}^{-\mathrm{i}y_1\cdot\sigma y_2}W(y_1 + y_2),\ y_i \in \mathcal{Y}.\end{aligned}$$

The second line above are called the *Weyl form* of the CCR. Again one can recover the fields from the Weyl operators. In fact the map

$$\mathbb{R} \ni t \mapsto W(ty) = \mathrm{e}^{\mathrm{i}t\phi(y)}$$

is a strongly continuous unitary group and:

$$\phi(y) := \mathrm{i}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}W(ty)_{|t=0}.$$

5. Quantization of field equations in Minkowski space

5.1. **Introduction.** In this section we will explain the quantization of the two main field equations on Minkowski space, the *Klein-Gordon equation*, describing scalar neutral bosons, and the *Dirac equation* describing charged fermions.

5.2. **The Minkowski space.** We recall that the Minkowski space $\mathbb{R}^{1,d}$ is the space $\mathbb{R}^{1+d}$ equipped with the pseudo-euclidean quadratic form

$$\langle x|x\rangle = -(x^0)^2 + \sum_{i=1}^{d}(x^i)^2.$$

Definition 5.1. (1) *a point* $x \in \mathbb{R}^{1,d}$ *is called* time-like *resp.* light-like*,* causal *,* space-like *if* $\langle x|x\rangle < 0$ *resp.* $\langle x|x\rangle = 0$*,* $\langle x|x\rangle \leq 0$*,* $\langle x|x\rangle > 0$*.*