Thermal Quantum Fields with Spatially Cut-off Interactions in 1+1 Space-time Dimensions

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Abstract

We construct interacting quantum fields in 1+1 space-time dimensions, representing char-ged or neutral scalar bosons at positive temperature and zero chemical potential. Our work is based on prior work by Klein and Landau and Høegh-Krohn. Generalized path space methods are used to add a spatially cut-off interaction to the free system, which is described in the Araki-Woods representation. It is shown that the interacting KMS state is normal w.r.t. the Araki-Woods representation. The observable algebra and the modular conjugation of the interacting system are shown to be identical to the ones of the free system and the interacting Liouvillean is described in terms of the free Liouvillean and the interaction.

Key words: Constructive field theory, thermal field theory, KMS states.

1 Introduction

Thermal quantum field theory is supposed to unify both quantum statistical mechanics and elementary particle physics. The formulation of the general framework should be wide enough to allow a QED description of ordinary matter. It should also provide the necessary tools for the QCD description of several experiments currently envisaged with the new Large Hadron Collider (LHC) at CERN. While the general theory of thermal quantum fields has made

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substantial progress in recent years, the actual construction of interacting models, which fit into the axiomatic setting, has not yet started (with the exception of the very early contributions by Høegh-Krohn [H-K1] and Fröhlich [Fr2]).

Let us briefly recall the formal description of charged scalar fields in physics. Examples of scalar particle-antiparticle pairs are the mesons π^+, π^-, K^+, K^-, or K^0, $\overline{K^0}$. (In the last case the 'charge' is strangeness). One starts with the classical Lagrangian density

$$\mathcal{L} = (\partial_\nu \varphi)(\partial^\nu \varphi^*) - m^2 \varphi\varphi^* - \frac{\lambda}{4}(\varphi\varphi^*)^2.$$

Here $\varphi(t,x)$ is a complex scalar field over space–time. The Lagrangian density $\mathcal{L}(t,x)$ is invariant under the global gauge transformations $\varphi \mapsto \mathrm{e}^{\mathrm{i}\alpha}\varphi$, $\alpha \in \mathbb{R}$. By Noether's theorem this invariance leads to a conserved current

$$j_\nu = i(\varphi^* \partial_\nu \varphi - \varphi \partial_\nu \varphi^*), \qquad \nu = 0, \ldots, 3,$$

and to a conserved charge

$$q = \int \mathrm{d}^3 x \, j_0(t,x).$$

The next step, according to the physics literature, is to setup real or imaginary time perturbation theory.

The state of art of perturbative thermal field theory is covered in three recent books by Kapusta [K], Le Bellac [L-B] and Umezawa [U]. The authors concentrate on theoretical efforts to understand various hot quantum systems (e.g., ultra-relativistic heavy-ion collisions or the phase transitions in the very early universe) and various physical implications (e.g., spontaneous symmetry breaking and restoration, deconfinement phase transition).

Constructive thermal field theory allows one to circumvent (at least in lower space-time dimensions) the severe problems (see, e.g., Steinmann [St]) of thermal perturbation theory, which can otherwise only be removed partially by applying certain "resummation schemes".

A class of models representing scalar neutral bosons with polynomial interactions in 1+1 space–time dimensions was constructed by Høegh-Krohn [H-K1] more than twenty years ago. As he could show, thermal equilibrium states for these models exist at all positive temperatures. For neutral particles, the particle density (and the energy density) adjust themselves to the given temperature; contrary to the non-relativistic case, a chemical potential adjusting the particle density can not be introduced, since the mass is no longer a conserved quantity. Shortly afterwards, several related results on the construction and properties of self-interacting thermal fields in 1+1 space–time dimensions were announced by Fröhlich [Fr2].

Our goal in this and a subsequent paper [GeJ] was twofold: first we wanted to fully understand the neutral scalar thermal field with polynomial interaction as constructed by Høegh-Krohn [H-K1], with the aim to study thermal scattering theory, using the framework introduced by Bros and Buchholz in [BB1], [BB2]. Secondly we wanted to generalize this construction to charged fields. This would allow us to study the system at different temperatures and chemical potentials, i.e., different charge densities. A possibility to change the charge density would put this model closer to non-relativistic models, where the mass is a conserved quantity, giving rise to the existence of a chemical potential.

The construction of the full interacting thermal quantum field without cutoffs in [GeJ] includes several of the original ideas of Høegh-Krohn [H-K1], but instead of starting from the interacting system in a box we start from the Araki-Woods representation for the free system in infinite volume. Using a general method developed by Klein and Landau [KL1] to treat spatially cutoff perturbations of the free system in infinite volume, we can eliminate some cumbersome limiting procedures due to the introduction of boxes, when we remove the spatial cutoff.

The present paper is devoted to the construction of neutral and charged thermal fields with *spatially cutoff* interactions in 1+1 space–time dimensions, using the method of Klein and Landau [KL1]. Although the excellent paper [KL1] is rather self contained, it did not include the discussion of examples. Twenty years ago it might have been evident for the experts in the field how to apply their method to thermal quantum fields, but we find it worthwhile to present this application in some detail.

A difference between this paper and [KL1] is the use of generalized path spaces as in [K], instead of stochastic processes. This compact formulation is convenient for our applications. In addition we prove several new results concerning the interacting KMS systems obtained by perturbations of path spaces.

1.1 Content of this paper

Our paper can be divided into several parts. The first part, presented in Section 2, discusses the description of neutral and charged scalar fields in terms of operator algebras. Its application to Klein-Gordon fields is discussed in Section

8. As usual the starting point is a real symplectic space (X, σ), which allows the construction of the Weyl algebra $\mathfrak{W}(X, \sigma)$. The next step is to introduce on (X, σ) a Kähler structure, i.e., a compatible Hermitian structure. For charged scalar fields, the symplectic space (X, σ) possesses also a canonical 'charge' complex structure j and a 'charge' sesquilinear form $\mathfrak{q}$, such that $\sigma = \mathrm{Im}\mathfrak{q}$. The maps $X \ni x \mapsto \mathrm{e}^{\mathrm{j}\alpha} x$ for $\alpha \in \mathbb{R}$ generate the *gauge transformations*. Given a regular CCR representation, complex quantum fields are defined.

This leads to the notion of a *charged Kähler structure*, corresponding to the introduction of another complex structure i and of the *charge operator* q, relating the two complex structures. Finally the notion of *charge conjugation* is discussed in this abstract framework.

For Klein-Gordon fields, a conjugation inducing charge-time reflections is used to distinguish an appropriate abelian sub-algebra of the Weyl algebra to which the interaction terms considered later on will be affiliated.

Section 3 recalls the characterization of a thermal equilibrium state by the KMS property. The GNS representation associated to a KMS state has a number of interesting properties which are briefly recalled. For instance, the GNS vector is cyclic and separating for the field algebra $\mathcal{F}$ (in our case the weak closure of the Weyl algebra in the GNS representation), and therefore one can always go over to the weak closure of the relevant operator algebras, and we will do so in the sequel. Since a KMS state is invariant under time translations, a Liouvillean implementing the time evolution is always available. As has been shown by Araki, the KMS condition allows us to introduce Euclidean Green's functions. The notion of *stochastically positive KMS systems* due to Klein and Landau is presented. This notion rests on the introduction of a distinguished abelian subalgebra $\mathcal{U}$ of the field algebra $\mathcal{F}$. In physics, this algebra is the algebra generated by the time-zero fields. It is also shown that stochastically positive KMS systems are invariant under a *time reversal* transformation.

In Section 4 we recall the notion of a quasi-free KMS system associated to a positive selfadjoint operator acting on the one-particle space. The GNS representation for a quasi-free KMS system has been analyzed by Araki and Woods. We briefly recall this framework and its connection to the Fock representation in a modern notation. It is shown that the field algebra $\mathcal{F}$ is generated by the time-translates of the abelian algebra $\mathcal{U}$. The observable algebra, consisting of elements of the field algebra which are invariant under gauge transformations, is introduced. In Subsection 4.5 it is shown that the KMS system for the (quasi-)free charged thermal field is indeed stochastically positive, if the chemical potential vanishes. However, if the chemical potential is non-zero, then the charge distinguishes a time direction, and consequently, the system is no longer invariant under time reversal. Thus it fails to be stochastically positive too, as we show in Subsection 8.3.

Following Klein and Landau, a cyclicity property of the Araki-Woods representation, which will imply the so-called *Markov property* for the free system later on, is shown. The Markov property has the consequence that the physical Hilbert space can naturally be considered as an L^2-space.

Section 5 recalls the notion of a *generalized path space*, both for the 0-temperature case and the case of positive temperature. We follow here [K], [KL1]. Although the 0-temperature case is not needed in this paper, it will be useful later on in [GeJ]. A generalized path space consists of a probability space (Q, Σ, μ), a distinguished σ-algebra Σ_0, a one-parameter group $t \mapsto U(t)$ and a reflection R. We recall the definition of *OS-positivity* and the *Markov property* for both cases.

Section 6 is devoted to a discussion of the Osterwalder-Schrader *reconstruction theorem* in the framework of generalized path spaces. This reconstruction theorem associates to a β-periodic, OS-positive path space a stochastically positive β-KMS system.

In Section 7 we recall from [KL1] how to deal with of perturbations, which are given in terms of *Feynman-Kac-Nelson kernels*. The main examples of FKN kernels are those obtained from a selfadjoint operator V on the physical Hilbert space $\mathcal{H}$, where V is affiliated to $\mathcal{U}$.

We show that for a class of perturbations V considered in [KL1], the perturbed Hilbert space can be canonically identified with the free Hilbert space in such a way that the interacting algebras $\mathfrak{F}, \mathcal{U}$ and the modular conjugation J coincide with the free ones. Moreover, we prove that the perturbed Liouvillean L_V is equal to $\overline{L + V - JVJ}$, if L is the free Liouvillean. Here $\overline{H}$ denotes the closure of a linear operator H.

Finally we show that the Markov property of a generalized path space is preserved by the perturbations associated to FKN kernels.

In Section 8 we apply the framework of Sections 2 and 4 to charged and neutral Klein-Gordon fields at positive temperature. The case of the neutral Klein-Gordon field is well known and reviewed only for completeness. We give more details on the charged Klein-Gordon field which provides an example of a charge symmetric Kähler structure. We also compare our setup with the one used in physics textbooks. Using the results of Section 4, we present the quasi-free KMS system describing a free charged or neutral Klein-Gordon field at positive temperature. Note that the conjugation used in the definition of the abelian algebra $\mathcal{U}$ corresponds to time reversal in the neutral case and to the composition of time-reversal and charge conjugation in the charged case. We show that the KMS system for the charged Klein-Gordon field is not stochastically positive, if the chemical potential is unequal to zero. The physical reason is that the dynamics of charged particles is only invariant under the combination of time reversal and charge conjugation. A non-zero chemical potential introduces a disymmetry between particles of positive and negative charge and hence breaks time reversal invariance, which itself is a property shared by all stochastically positive KMS systems.

In Section 9 we consider Klein-Gordon fields at positive temperature with spatially cutoff interactions in $1+1$ space-time dimensions. In the neutral case we will treat the $P(\phi)_2$ and the ${\rm e}^{\alpha\phi}{}_2$ interactions (the later being also known as the *Høegh-Krohn model*). In the charged case we treat the (gauge invariant) $P(\overline{\varphi}\varphi)_2$ interaction.

The UV divergences of the interactions are eliminated by Wick ordering, which is discussed in some details in Subsections 9.1 and 9.2. As it turns out, the leading order in the UV divergences is independent of the temperature. Thus it is a matter of convenience whether one uses thermal Wick ordering or Wick ordering w.r.t. the vacuum state.

The L^p-properties of the interactions needed to apply the abstract results of Section 7 are shown in Subsections 9.3, 9.4 and 9.5.

Finally, the main results of this paper, namely the construction and description of a KMS system representing a Klein-Gordon field at positive temperature with spatially cutoff interactions, is given in Subsection 9.6.

In a forthcoming paper we will consider the translation invariant $P(\phi)_2$ model at positive temperature. Following again ideas of Høegh-Krohn [H-K1], Nelson symmetry will be used to establish the existence of the model in the thermodynamic limit.

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2 Real and complex quantum fields

In this section we present real and complex quantum fields in an abstract framework. Usually in the physics literature complex quantum fields are described in the case of Klein-Gordon fields. Although the results of this section are probably known, we have not found them in the literature.

2.1 Notation

Let X be a real vector space. If X is equipped with a complex structure i, then we will denote by (X, i) the complex vector space X. If (X, i) is equipped with a hermitian form $(\,.\,,\,.\,)$, then we will denote by $(X, \mathrm{i}, (\,.\,,\,.\,))$ the Hermitian space X. If it is clear from the context which complex or Hermitian structure is used, (X, i) or $(X, \mathrm{i}, (\,.\,,\,.\,))$ will simply be denoted by X. As a rule the complex structure of a Hermitian space X will be denoted by the letter i. Sometimes another 'charge' complex structure appears; it will be denoted by the letter j.

2.2 Real fields

We start by recalling the formalism of real quantum fields.

CCR Algebra

Let (X, σ) be a real symplectic space. Let $\mathfrak{W}(X, \sigma)$ be the (uniquely determined) C^*-algebra generated by nonzero elements $W(x)$, $x \in X$, satisfying

$$W(x_1)W(x_2) = \mathrm{e}^{-\mathrm{i}\sigma(x_1,x_2)/2} W(x_1 + x_2),$$

$$W^*(x) = W(-x), \qquad W(0) = \mathbb{1}.$$

$\mathfrak{W}(X, \sigma)$ is called the *Weyl algebra* associated to (X, σ).

Regular representations

Let $\mathcal{H}$ be a Hilbert space. We recall that a representation

$$\pi\colon \mathfrak{W}(X, \sigma) \ni W(x) \mapsto W_\pi(x) \in \mathcal{U}(\mathcal{H})$$

is called a *regular CCR representation* if

$$t \mapsto W_\pi(tx) \text{ is strongly continuous for any } x \in X.$$

One can then define *field operators*

$$\phi_\pi(x) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W_\pi(tx)\Big|_{t=0},\ x \in X,$$

which satisfy in the sense of quadratic forms on $\mathcal{D}(\phi_\pi(x_1)) \cap \mathcal{D}(\phi_\pi(x_2))$ the commutation relations

$$[\phi_\pi(x_1), \phi_\pi(x_2)] = \mathrm{i}\sigma(x_1, x_2),\ x_1,\ x_2 \in X. \tag{2.1}$$

Kähler structures

Let (X, σ) be a real symplectic space and i a complex structure on X. The space (X, i, σ) is called a *Kähler space* if

$$\sigma(\mathrm{i}x_1, x_2) = -\sigma(x_1, \mathrm{i}x_2) \text{ and } \sigma(x, \mathrm{i}x) \text{ is positive definite.}$$

If (X, i, σ) is a Kähler space, then $(X, \mathrm{i}, (\,.\,,\,.\,))$ is a Hermitian space for

$$(x_1, x_2) := \sigma(x_1, \mathrm{i}x_2) + \mathrm{i}\sigma(x_1, x_2).$$

The typical example of a Kähler space is a Hermitian space $(X, \mathrm{i}, (\,.\,,\,.\,))$ with its natural complex structure and symplectic form $\sigma = \mathrm{Im}(\,.\,,\,.\,)$.

Creation and annihilation operators

If π is a regular CCR representation of the Weyl algebra $\mathfrak{W}(X, \sigma)$, and (X, σ) is equipped with a Kähler structure, then the *creation* and *annihilation operators* are defined as follows:

$$a_\pi^*(x) := \frac{1}{\sqrt{2}}(\phi_\pi(x) - \mathrm{i}\phi_\pi(\mathrm{i}x)),\ a_\pi(x) := \frac{1}{\sqrt{2}}(\phi_\pi(x) + \mathrm{i}\phi_\pi(\mathrm{i}x)).$$

Clearly,

$$\phi_\pi(x) = \frac{1}{\sqrt{2}}(a_\pi^*(x) + a_\pi(x)),\ x \in X.$$

The operators $a_\pi^*(x)$ and $a_\pi(x)$ with domain $\mathcal{D}(\phi_\pi(x)) \cap \mathcal{D}(\phi_\pi(\mathrm{i}x))$ are closed and satisfy canonical commutation relations in the sense of quadratic forms:

$$[a_\pi(x_1), a_\pi^*(x_2)] = (x_1, x_2)\mathbb{1},\ [a_\pi(x_2), a_\pi(x_1)] = [a_\pi^*(x_2), a^*(x_1)] = 0.$$

Let (X, j) be a complex vector space. Let us assume that X is equipped with a sesquilinear, symmetric non-degenerate form $\mathfrak{q}$. If $a \in L(X)$, we say that a is isometric (resp. symmetric, skew-symmetric) if $[a, \mathrm{j}] = 0$ and $\mathfrak{q}(ax_1, ax_2) = \mathfrak{q}(x_1, x_2)$ (resp. $\mathfrak{q}(ax_1, x_2) = \mathfrak{q}(x_1, ax_2)$, $\mathfrak{q}(ax_1, x_2) = -\mathfrak{q}(x_1, ax_2)$). Clearly $(X, \mathrm{Im}\mathfrak{q})$ is a real symplectic space. The quadratic form $\mathfrak{q}$ is called the *charge quadratic form.*

Gauge transformations

The maps $X \ni x \mapsto \mathrm{e}^{\mathrm{j}\alpha} x \in X$ for $\alpha \in \mathbb{R}$ are called *gauge transformations.* They are symplectic on $(X, \mathrm{Im}\mathfrak{q})$ and isometric on $(X, \mathfrak{q})$. We have

$$\mathfrak{q}(x_1, x_2) = \mathrm{Im}\,\mathfrak{q}(x_1, \mathrm{j}x_2) + \mathrm{i}\mathrm{Im}\mathfrak{q}(x_1, x_2). \tag{2.2}$$

Complex fields

Let now π be a regular CCR representation of $\mathfrak{W}(X, \mathrm{Im}\mathfrak{q})$ on a Hilbert space $\mathcal{H}$ and let $\phi_\pi(x)$ be the associated field.

Using the complex structure j, we can define the *complex fields*

$$\varphi_\pi^*(x) := \tfrac{1}{\sqrt{2}}(\phi_\pi(x) - \mathrm{i}\phi_\pi(\mathrm{j}x)),$$

$$\varphi_\pi(x) := \tfrac{1}{\sqrt{2}}(\phi_\pi(x) + \mathrm{i}\phi_\pi(\mathrm{j}x)),$$

with domains $\mathcal{D}(\phi_\pi(x)) \cap \mathcal{D}(\phi_\pi(\mathrm{j}x))$. The maps $X \ni x \mapsto \varphi_\pi^*(x)$ (resp. $x \mapsto \varphi_\pi(x)$) are j-linear (resp. j-antilinear).

Lemma 2.1 *The operators $\varphi_\pi^\sharp(x)$ are closed. In the sense of quadratic forms on $\mathcal{D}(\phi_\pi(x)) \cap \mathcal{D}(\phi_\pi(\mathrm{j}x))$ they satisfy the commutation relations*

$$[\varphi_\pi(x_1), \varphi_\pi^*(x_2)] = \mathfrak{q}(x_1, x_2)\mathbb{1},\ [\varphi_\pi(x_1), \varphi_\pi(x_2)] = [\varphi_\pi^*(x_1), \varphi^*(x_2)] = 0.$$

Proof. The commutation relations are easily deduced from (2.1). Let $u \in \mathcal{D}(\phi_\pi(x)) \cap \mathcal{D}(\phi_\pi(\mathrm{j}x))$. To prove that $\varphi_\pi^\sharp(x)$ is closed, we write

$$2\|\varphi_\pi(x)u\|^2 = \|\phi_\pi(x)u\|^2 + \|\phi_\pi(\mathrm{j}x)u\|^2 - \mathfrak{q}(x, \mathrm{j}x)\|u\|^2.$$

This easily implies that $\varphi_\pi(x)$ is closed. The case of $\varphi^*_\pi(x)$ is treated similarly. $\Box$

2.4 Charge operator

Definition 2.2 *Let $(X, \mathrm{j}, \mathfrak{q})$ be as in Subsection 2.3 and* i *another complex structure on X. Then $(X, \mathrm{j}, \mathrm{i}, \mathfrak{q})$ is called a* charged Kähler space *if* $[\mathrm{i}, \mathrm{j}] = 0$ *and $(X, \mathrm{i}, \mathrm{Im}\mathfrak{q})$ is a Kähler space.*

Let $(X, \mathrm{j}, \mathrm{i}, \mathfrak{q})$ be a charged Kähler space. Then i is antisymmetric for $\mathfrak{q}$, i.e., $\mathfrak{q}(x_1, \mathrm{i}x_2) = -\mathfrak{q}(\mathrm{i}x_1, x_2)$, and j is antisymmetric for $(\,.\,,\,.\,)$.

We can introduce the *charge operator*:

$$\mathrm{q} := -\mathrm{ij}.$$

Note that $[\mathrm{q}, \mathrm{i}] = [\mathrm{q}, \mathrm{j}] = 0$, $\mathrm{q}^2 = 1$ and that q is symmetric and isometric both for $\mathfrak{q}$ and $(\,.\,,\,.\,)$. Since $\mathrm{i} = \mathrm{jq}$ we have $\mathrm{e}^{\mathrm{j}\alpha} = \mathrm{e}^{\mathrm{i}\alpha\mathrm{q}}$ and the gauge transformations $x \mapsto \mathrm{e}^{\mathrm{j}\alpha}x$, $\alpha \in \mathbb{R}$, form a unitary group on $(X, \mathrm{i}, (\,.\,,\,.\,))$ with infinitesimal generator q.

The typical example of a charged Kähler space is a Hermitian space $(X, \mathrm{i}, (\,.\,,\,.\,))$ with a distinguished symmetric operator q such that $\mathrm{q}^2 = 1$. Let us denote by $X^\pm := \mathrm{Ker}(\mathrm{q} \mp 1\!\!1)$ the spaces of positive (resp. negative) charge and by $x^\pm$ the orthogonal projection of $x \in X$ onto $X^\pm$. If we set $\mathfrak{q}(x_1, x_2) = (x_1^+, x_2^+) - (x_2^-, x_1^-)$, then $(X, \mathrm{iq}, \mathrm{i}, \mathfrak{q})$ is a charged Kähler space. Note that X^+ or X^- may be equal to $\{0\}$.

Using the fact that q is symmetric for $(\,.\,,\,.\,)$ and $\mathfrak{q}$, we see that the spaces $X^\pm$ are orthogonal both for $(\,.\,,\,.\,)$ and $\mathfrak{q}$. If we set $x^\pm = \frac{1}{2}(x \pm \mathrm{q}x)$, then the map

$$\begin{array}{rccc} U\colon & X & \to & X^+ \oplus X^- \\ & x & \mapsto & x^+ \oplus x^- \end{array}$$

is unitary from $(X, \mathrm{i}, (\,.\,,\,.\,))$ to $(X^+, \mathrm{i}, (\,.\,,\,.\,)) \oplus (X^-, \mathrm{i}, (\,.\,,\,.\,))$ and isometric from $(X, \mathrm{j}, \mathfrak{q})$ to $(X^+, \mathrm{i}, (\,.\,,\,.\,)) \oplus (X^-, -\mathrm{i}, -\overline{(\,.\,,\,.\,)})$.

If $\pi\colon \mathfrak{W}(X, \mathrm{Im}\mathfrak{q}) \to \mathcal{U}(\mathcal{H})$ is a regular CCR representation on a Hilbert space $\mathcal{H}$, then we can introduce, just as in Subsection 2.2, *creation* and *annihilation operators*

$$a^*_\pi(x) := \frac{1}{\sqrt{2}}(\phi_\pi(x) - \mathrm{i}\phi_\pi(\mathrm{i}x)),\ a_\pi(x) := \frac{1}{\sqrt{2}}(\phi_\pi(x) + \mathrm{i}\phi_\pi(\mathrm{i}x)),$$

with domains $\mathcal{D}(\phi_\pi(x)) \cap \mathcal{D}(\phi_\pi(\mathrm{i}x))$. The maps $X \ni x \mapsto a_\pi^*(x)$ (resp. $a_\pi(x)$) are i-linear (resp. i-antilinear). If $x = x^+ + x^-$, with $x^\pm \in X^\pm$, then

$$\varphi_\pi(x) = a_\pi(x^+) + a_\pi^*(x^-) \text{ and } \varphi_\pi^*(x) = a_\pi^*(x^+) + a_\pi(x^-).$$

Note that this is consistent with fact that the maps $X \ni x \mapsto \varphi_\pi^*(x)$(resp. $x \mapsto \varphi_\pi(x)$) are j-linear (resp. j-antilinear).

2.5 Charge conjugation

Let $(X, \mathrm{j}, \mathrm{i}, \mathfrak{q})$ be a charged Kähler space. Assume that there exists some $\mathrm{c} \in L(X)$ such that

$$\mathrm{c}^2 = 1\!\!1,\ \mathrm{ci} = \mathrm{ic},\ \mathrm{cq} = -\mathrm{qc},\ (x_1, \mathrm{c}x_2) = (\mathrm{c}x_1, x_2),\ x_1, x_2 \in X. \tag{2.3}$$

I.e., c is a symmetric involution for $(. , .)$, which anticommutes with the charge operator q. An operator c satisfying (2.3) is called a *charge conjugation*. Charge conjugations exist in charge-symmetric quantum field theories. A charged Kähler space $(X, \mathrm{j}, \mathrm{i}, \mathfrak{q}, \mathrm{c})$ equipped with a charge conjugation c will be called a *charge-symmetric Kähler space*.

It follows from (2.3) that $\mathfrak{q}(x_1, \mathrm{c}x_2) = -\mathfrak{q}(\mathrm{c}x_1, x_2)$, i.e., c is antisymmetric for $\mathfrak{q}$. Since $\mathrm{cq} = -\mathrm{qc}$, we see that c is a unitary map from $(X^-, \mathrm{i}, (. , .))$ to $(X^+, \mathrm{i}, (. , .))$.

3 Stochastically positive KMS systems

In this section we recall the notion of a *stochastically positive KMS system* due to Klein and Landau [KL1]. We prove that stochastically positive KMS systems are invariant under time-reversal.

3.1 KMS systems

Let $\mathfrak{F}$ be a C^*-algebra and $\{\tau_t\}_{t \in \mathbb{R}}$ a group of *-automorphisms of $\mathfrak{F}$. Let ω be a (τ, β)-*KMS state* on $\mathfrak{F}$, i.e., a state such that for each $A, B \in \mathfrak{F}$ there exists a function $F_{A,B}(z)$ holomorphic in the strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Im} z < \beta\}$ and continuous on its closure such that

$$F_{A,B}(t) = \omega(A\tau_t(B)),\ F_{A,B}(t + \mathrm{i}\beta) = \omega(\tau_t(B)A),\ t \in \mathbb{R}.$$

A triple $(\mathfrak{F}, \tau, \omega)$ such that ω is a (τ, β)-KMS state is called a β-*KMS system*.

Let us now recall some standard facts about KMS systems. By the GNS construction, one associates to $(\mathfrak{F}, \tau, \omega)$ a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathfrak{F}$ on $\mathcal{H}_\omega$, a unit vector Ω_ω, cyclic for π_ω, and a strongly continuous unitary group $\{\mathrm{e}^{-\mathrm{i}tL}\}_{t\in\mathbb{R}}$ such that

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega),\ \pi_\omega(\tau_t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL},\ L\Omega_\omega = 0.$$

The KMS condition implies that Ω_ω is separating for the von Neumann algebra $\pi_\omega(\mathfrak{F})''$, i.e., $A\Omega_\omega = 0 \Rightarrow A = 0$ for $A \in \pi_\omega(\mathfrak{F})''$. Consequently, the image of $\mathfrak{F}$ under π_ω is isomorphic to $\mathfrak{F}$; it will therefore not be distinguished from $\mathfrak{F}$. Moreover, we will identify an element A of $\mathfrak{F}$ with its image $\pi_\omega(A)$.

The selfadjoint operator L is called the *Liouvillean* associated to the KMS system $(\mathfrak{F}, \tau, \omega)$. It is the unique selfadjoint operator whose associated unitary group generates the dynamics τ and such that $L\Omega_\omega = 0$ (see e.g. [DJP, Prop. 2.14]).

Proposition 3.1 *Let $\mathfrak{F}_1 \subset \mathfrak{F}$ be the set of $A \in \mathfrak{F}$ such that $\tau\colon t \mapsto \tau_t(A)$ is C^1 for the strong topology on $\mathcal{B}(\mathcal{H}_\omega)$. Then $\mathfrak{F}_1\Omega_\omega \subset \mathcal{D}(L)$ is a core for L.*

Proof. Note first that $A \in \mathfrak{F}_1$ iff A is of class $C^1(L)$ (see [ABG, Def. 6.2.2]). Clearly $\mathfrak{F}_1$ is dense in $\mathfrak{F}$ for the strong operator topology. In fact, if $A \in \mathfrak{F}$, then the strong integral $A_\epsilon = \epsilon^{-1}\int_0^\epsilon \tau_t(A)\mathrm{d}t$ belongs to $\mathfrak{F}_1$ and converges strongly to A when $\epsilon \to 0$.

Since Ω_ω is cyclic for $\mathfrak{F}$, this implies that $\mathfrak{F}_1\Omega_\omega$ is dense in $\mathcal{H}_\omega$. Moreover, since $L\Omega_\omega = 0$, we have $\mathrm{e}^{\mathrm{i}tL}\mathfrak{F}_1\Omega_\omega = \mathfrak{F}_1\Omega_\omega$ and $\mathfrak{F}_1\Omega_\omega \subset \mathcal{D}(L)$. Thus Nelson's theorem implies that $\mathfrak{F}_1\Omega_\omega$ is a core for L.

Euclidean Green's functions

Let

$$I_\beta^{n+} := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \mathrm{Im}z_j < \mathrm{Im}z_{j+1},\ \mathrm{Im}z_n - \mathrm{Im}z_1 < \beta\}. \tag{3.4}$$

It follows from a result of Araki [Ar1,?] that, for $A_1, \dots, A_n \in \mathfrak{F}$, the Green's function

$$G(t_1, \dots, t_n; A_1, \dots, A_n) := \omega(\prod_1^n \tau_{t_i}(A_i))$$

extends to an holomorphic function in I_β^{n+}, continuous on $\overline{I_\beta^{n+}}$. In particular, one can uniquely define the *Euclidean Green's functions*

$$^{E}G(s_1, \dots, s_n; A_1, \dots, A_n) := G(\mathrm{i}s_1, \dots, \mathrm{i}s_n; A_1, \dots, A_n)$$

for all $(s_1, \dots, s_n)$ such that $s_1 \leq \cdots \leq s_n$ and $s_n - s_1 \leq \beta$. The correct way to view such an n-tuple $(s_1, \dots, s_n)$ is as an n-tuple of points on the circle of length β, *ordered counter-clockwise.*

3.2 Stochastically positive KMS systems

In [KL1] Klein and Landau introduced a class of KMS systems which they called *stochastically positive KMS systems.* To a stochastically positive KMS system one can associate a (unique up to equivalence) *generalized path space* $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ (see Section 5) which has some special properties, the most important being the β-periodicity in t and the *Osterwalder-Schrader* (OS)*-positivity.*

Conversely Klein and Landau have shown in [KL1] that to a generalized path space satisfying the properties in Definition 5.1 one can associate a (unique up to unitary equivalence) stochastically positive KMS system. This is an example of a *reconstruction theorem*; similar results are well-known in Euclidean QFT. A reconstruction theorem allowing to go from Euclidean Green's functions to a KMS system has recently been proved in a general context by Birke and Fröhlich in [BF].

The advantage of the Klein and Landau formalism is that it is relatively easy to perturb the stochastic process associated to a KMS system, using functional integral methods.

Definition 3.2 *Let $(\mathfrak{F}, \tau, \omega)$ be a KMS system and $\mathfrak{U} \subset \mathfrak{F}$ an abelian *-subalgebra. The KMS system $(\mathfrak{F}, \mathfrak{U}, \tau, \omega)$ is called* stochastically positive *if*

(i) *the C^*-algebra generated by $\bigcup_{t\in\mathbb{R}} \tau_t(\mathfrak{U})$ is equal to $\mathfrak{F}$;*
(ii) *the Euclidean Green's functions $^{E}G(s_1, \dots, s_n; A_1, \dots, A_n)$ are positive for all $A_1, \dots, A_n \in \mathfrak{U}^+ = \{A \in \mathfrak{U} \mid A \geq 0\}$ and for all $(s_1, \dots, s_n)$ such that $s_1 \leq \cdots \leq s_n$ and $s_n - s_1 \leq \beta$.*

It is often more convenient to consider instead of the C^*-algebras $\mathfrak{F}$ and $\mathfrak{U}$ their weak closures in the GNS representation, which we denote by $\overline{\mathfrak{F}}$ and $\overline{\mathfrak{U}}$. We denote by $\overline{\tau}$ the group $\{\overline{\tau}_t\}_{t\in\mathbb{R}}$ of *-automorphisms of $\overline{\mathfrak{F}}$ defined by $\overline{\tau}_t(A) := \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}$. The state ω extends to $\overline{\mathfrak{F}}$ by setting $\overline{\omega}(A) := (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$. The following fact has been shown in [KL1, Prop. 3.4].

Proposition 3.3 *Let $(\mathfrak{F}, \mathfrak{U}, \tau, \omega)$ be a stochastically positive KMS system. Then $(\overline{\mathfrak{F}}, \overline{\mathfrak{U}}, \overline{\tau}, \overline{\omega})$ is also a stochastically positive KMS system (in the W^*-sense). I.e.,*

(i) *the W^*-algebra generated by $\bigcup_{t\in\mathbb{R}} \tau_t(\overline{\mathfrak{U}})$ is equal to $\overline{\mathfrak{F}}$;*
(ii) *the Euclidean Green's functions ${}^{E}G(s_1, \ldots, s_n; A_1, \ldots, A_n)$ are positive for all $A_1, \ldots, A_n \in \overline{\mathfrak{U}}^+$ and for all n-tuples $(s_1, \ldots, s_n)$ such that $s_1 \le \cdots \le s_n$ and $s_n - s_1 \le \beta$.*

Now we show that stochastically positive KMS systems are invariant under *time reversal*, a fact that is well known for 0-temperature field theories (see for example [Si1]).

Proposition 3.4 *Let $(\mathfrak{F}, \mathfrak{U}, \tau, \omega)$ be a* stochastically positive *KMS system. Then there exists an anti-unitary involution T on $\mathcal{H}_\omega$ such that*

(i) *$T\overline{\mathfrak{F}}T^{-1} = \overline{\mathfrak{F}}$, $TAT^{-1} = A^*$ for $A \in \overline{\mathfrak{U}}$;*
(ii) *$T\Omega_\omega = \Omega_\omega$, $T\,\overline{\tau}_t(A) = \overline{\tau}_{-t}(A)T$ for $A \in \overline{\mathfrak{F}}$, $t \in \mathbb{R}$.*

From the properties of T we see that T implements the *time reversal transformation.*

Proof. Let $A_1, A_2 \in \mathfrak{U}$. The map $z \mapsto \omega(A_1\tau_t(A_2))_{|t=iz}$ is holomorphic in $\{0 < \mathrm{Re} z < \beta\}$. By stochastic positivity it is real on $\{\mathrm{Im} z = 0\}$ if $A_i = A_i^*$. The Schwarz's reflection principle implies

$$\omega(A_1\tau_t(A_2))_{|t=iz} = \overline{\omega(A_1\tau_t(A_2))_{|t=i\bar{z}}} \quad \text{for} \quad A_i \in \mathfrak{U},\ A_i = A_i^*.$$

For $z = -it$ this yields

$$\omega(A_1\tau_t(A_2)) = \overline{\omega(A_1\tau_{-t}(A_2))} = \omega(\tau_{-t}(A_2)A_1) \quad \text{for} \quad A_i \in \mathfrak{U},\ A_i = A_i^*. \tag{3.5}$$

By $\mathbb{C}$-linearity this identity extends to all $A_i \in \mathfrak{U}$. We can now define the antilinear operator

$$T\colon \sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}t_j L} A_j \Omega_\omega \mapsto \sum_{j=1}^{n} \mathrm{e}^{-\mathrm{i}t_j L} A_j^* \Omega_\omega. \tag{3.6}$$

For $u = \sum_{j=1}^n \mathrm{e}^{\mathrm{i}t_j L} A_j \Omega_\omega$ identity (3.5) implies

$$\begin{aligned}
\|u\|^2 &= (\textstyle\sum_{j=1}^n \mathrm{e}^{\mathrm{i}t_j L} A_j \Omega_\omega, \sum_{k=1}^n \mathrm{e}^{\mathrm{i}t_k L} A_k \Omega_\omega) \\
&= \textstyle\sum_{j,k} (\Omega_\omega, A_j^* \mathrm{e}^{\mathrm{i}(t_k - t_j)L} A_k \Omega_\omega) = \sum_{j,k} \omega(A_j^* \tau_{t_k - t_j}(A_k)) \\
&= \textstyle\sum_{j,k} \omega(\tau_{t_j - t_k}(A_k) A_j^*) = \sum_{j,k} (\Omega_\omega, A_k \mathrm{e}^{\mathrm{i}(t_k - t_j)L} A_j^* \Omega_\omega) \\
&= \textstyle\sum_{j,k} (\mathrm{e}^{-\mathrm{i}t_k L} A_k^* \Omega_\omega, \mathrm{e}^{-\mathrm{i}t_j L} A_j^* \Omega_\omega) = \|Tu\|^2.
\end{aligned}$$

Thus T is a well defined antilinear operator. Moreover, using property (i) of Definition 3.2 and the fact that Ω_ω is cyclic for $\mathfrak{F}$, we conclude that T has a dense domain and a dense range. Hence T extends uniquely to an anti-unitary operator. Clearly T is an involution. The other properties of T follow directly from (3.6). □

4 Quasi-free KMS states

In this section we recall some well-known facts about quasi-free KMS states and describe a class of quasi-free KMS states which generate stochastically positive KMS systems (see [KL2], [GO]).

4.1 Quasi-free KMS states

Let X_0 be a pre-Hilbert space, X the completion of X_0. Then (X_0, σ) is a real symplectic space for $\sigma = \mathrm{Im}(\,.\,,\,.\,)$, and we denote by $\mathfrak{W}(X_0)$ the Weyl algebra $\mathfrak{W}(X_0, \sigma)$. Let $\mathrm{a} \geq 0$ be a selfadjoint operator on X such that $X_0 \subset \mathcal{D}(\mathrm{a}^{-\frac{1}{2}})$ and $\mathrm{e}^{-\mathrm{i}t\mathrm{a}}$ preserves X_0. Given $\mathrm{a} \geq 0$ the canonical choice for X_0 is $\mathcal{D}(\mathrm{a}^{-\frac{1}{2}})$.

For $\beta > 0$ one defines a state ω_β on $\mathfrak{W}(X_0)$ by the functional

$$\omega_\beta(W(x)) := \mathrm{e}^{-\frac{1}{4}(x,(1+2\rho)x)},\ x \in X_0, \tag{4.7}$$

where $\rho := (\mathrm{e}^{\beta \mathrm{a}} - 1)^{-1}$. Since $1 + 2\rho = \frac{1+\mathrm{e}^{-\beta\mathrm{a}}}{1-\mathrm{e}^{-\beta\mathrm{a}}}$ and $\mathrm{a} \geq 0$ the form domain of $1 + 2\rho$ is equal to $D(\mathrm{a}^{-\frac{1}{2}}) \supset X_0$.

The state ω_β is a (τ°, β)-KMS state for the dynamics $\tau^\circ\colon t \mapsto \tau_t^\circ$ defined by

$$\begin{array}{rccc}
\tau_t^\circ\colon & \mathfrak{W}(X_0) & \rightarrow & \mathfrak{W}(X_0) \\
& W(x) & \mapsto & W(\mathrm{e}^{\mathrm{i}t\mathrm{a}}x).
\end{array}$$

The state ω_β is *quasi-free* (see [BR]) and the KMS system $(\mathfrak{W}(X_0), \tau^\circ, \omega_\beta)$ defined above is called the quasi-free KMS system *associated to* a.

The standard example is the following one: let $\mathrm{h} \geq 0$ be a selfadjoint operator representing the *one particle energy*. Assume that there exists a selfadjoint operator q on X representing the *one particle charge* such that $q^2 = \mathbb{1}$, $[\mathrm{h}, q] = 0$. Then we can associate a group of *gauge transformations* $\{\alpha_t\}_{t\in[0,2\pi[}$,

$$\begin{array}{rccc} \alpha_t\colon & \mathfrak{W}(X_0) & \to & \mathfrak{W}(X_0) \\ & W(x) & \mapsto & W(\mathrm{e}^{\mathrm{i}tq}x), \end{array}$$

to the charge operator q. Let $\mu \in \mathbb{R}$ such that $\mathrm{h} - \mu q \geq \lambda > 0$. Thus the range for the value of the chemical potential μ, which we consider, excludes Bose-Einstein condensation. It follows that $\mathrm{a} := \mathrm{h} - \mu q > 0$ and hence $X_0 = \mathcal{D}(\mathrm{a}^{-\frac{1}{2}}) = X$. Therefore the unique quasi-free KMS state on $\mathfrak{W}(X)$ at inverse temperature β and chemical potential μ is the state ω_β defined by (4.7).

4.2 Araki-Woods representation

Let us consider a quasi-free KMS system associated to a selfadjoint operator a as in Subsection 4.1. Let $\overline{X}$ be the conjugate Hilbert space to X. Elements of $\overline{X}$ will be denoted by $\overline{x}$. Equivalently, we denote by $X \ni x \mapsto \overline{x} \in \overline{X}$ the identity operator, which is antilinear. If a is a linear operator on X, we denote by $\overline{\mathrm{a}}$ the linear operator on $\overline{X}$ defined by $\overline{\mathrm{a}}\,\overline{x} := \overline{\mathrm{a}x}$. If $\mathfrak{h}$ is a Hilbert space, then

$$\Gamma(\mathfrak{h}) = \bigoplus_{n=0}^{+\infty} \otimes_{\mathrm{s}}^n \mathfrak{h}$$

denotes the bosonic Fock space over $\mathfrak{h}$.

We set:

$$\mathcal{H}_\omega := \Gamma(X \oplus \overline{X}),$$

$$\Omega_\omega := \Omega,$$

$$W_{\omega,\mathrm{l}}(x) := W_{\mathrm{F}}((1+\rho)^{\frac{1}{2}} x \oplus \overline{\rho}^{\frac{1}{2}} \overline{x}),\ x \in X_0,$$

$$W_{\omega,\mathrm{r}}(x) := W_{\mathrm{F}}(\rho^{\frac{1}{2}} x \oplus (1+\overline{\rho})^{\frac{1}{2}} \overline{x}),\ x \in X_0,$$

where $W_F(.)$ denotes the Fock space Weyl operator on $\Gamma(X \oplus \overline{X})$ and $\Omega \in \Gamma(X \oplus \overline{X})$ denotes the Fock vacuum.

The following facts are well known:

(i) The map $W(x) \mapsto W_{\omega,\mathrm{l/r}}(x) \in U(\mathcal{H}_\omega)$ defines a regular CCR representations;
(ii) $[W_{\omega,\mathrm{l}}(x), W_{\omega,\mathrm{r}}(y)] = 0$ for $x, y \in X_0$;
(iii) $(\Omega_\omega, W_{\omega,\mathrm{l}}(x)\Omega_\omega) = \omega(W(x))$ for $x \in X_0$;
(iv) Let $L := \mathrm{d}\Gamma(\mathrm{a} \oplus -\overline{\mathrm{a}})$ act on $\mathcal{H}_\omega$. Then

$$\mathrm{e}^{-\mathrm{i}tL}\Omega_\omega = \Omega_\omega,\ \mathrm{e}^{\mathrm{i}tL}W_{\omega,\mathrm{l}}(x)\mathrm{e}^{-\mathrm{i}tL} = W_{\omega,\mathrm{l}}(\mathrm{e}^{\mathrm{i}t\mathrm{a}}x),\ x \in X_0;$$

(v) The vector Ω is cyclic for the representations $W_{\omega,\mathrm{l/r}}(.)$.

In particular, the Araki-Woods representation is the GNS representation for the KMS system $(\mathfrak{W}(X,\sigma), \tau^\circ, \omega)$ and L is the associated Liouvillean.

We will only consider the left Araki-Woods representation, thus will forget the subscript l and write $W_\omega(x) := W_{\omega,\mathrm{l}}(x)$, $x \in X$. The creation-annihilation operators associated to $W_\omega(.)$ are

$$\begin{aligned} a^*_\omega(x) &= a^*_F((1+\rho)^{\frac{1}{2}}x \oplus 0) + a_F(0 \oplus \overline{\rho}^{\frac{1}{2}}\overline{x}), \\ a_\omega(x) &= a_F((1+\rho)^{\frac{1}{2}}x \oplus 0) + a^*_F(0 \oplus \overline{\rho}^{\frac{1}{2}}\overline{x}). \end{aligned}$$

4.3 Field algebras

We recall that a *conjugation* on a Hilbert space X is an anti-unitary involution on X. Let us assume that X is equipped with a conjugation κ. To κ we associate the real vector space $X_\kappa := \{x \in X \mid \kappa x = x\}$. Let ω be the quasi-free state associated to a selfadjoint operator a, as defined in Subsection 4.1, and let $\mathcal{H}_\omega$ be the Araki-Woods space introduced in Subsection 4.2.

We will denote by $\mathcal{W} \subset \mathcal{B}(\mathcal{H}_\omega)$ the *field algebra*, i.e., the von Neumann algebra generated by the $\{W_\omega(x) \mid x \in X\}$ and by $\mathcal{W}_\kappa \subset \mathcal{B}(\mathcal{H}_\omega)$ the von Neumann algebra generated by $\{W_\omega(x) \mid x \in X_\kappa\}$. Since the symplectic form σ vanishes on X_κ, the algebra $\mathcal{W}_\kappa$ is abelian.

Lemma 4.1 *Assume that* $\mathrm{a} = \mathrm{h} - \mu q$*, where* h *and* q *are selfadjoint operators such that* $[\mathrm{h}, q] = 0$*,* $q^2 = 1$*,* $\mathrm{h} \geq m > 0$ *and* $|\mu| < m$*. Let* κ *be a conjugation on* X *such that* $[\mathrm{h}, \kappa] = 0$*. Then* $\mathcal{W}$ *is the von Neumann algebra generated by*

$\{\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} \mid t \in \mathbb{R},\ A \in \mathcal{W}_\kappa\}$.

Proof. Clearly $\{\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} \mid t \in \mathbb{R},\ A \in \mathcal{W}_\kappa\} \subset \mathcal{W}$, so it suffices to prove the converse inclusion. Using the CCR, the facts that $(1+\rho)^{\frac{1}{2}}$ and $\rho^{\frac{1}{2}}$ are bounded, and the fact that the map

$$X \oplus \overline{X} \ni x_1 \oplus \overline{x_2} \mapsto W_F(x_1 \oplus \overline{x_2}) \in \mathcal{B}(\mathcal{H}_\omega)$$

is continuous for the strong topology, it suffices to verify that

$$E = \mathrm{Vect}_{\mathbb{R}}\{\mathrm{e}^{\mathrm{i}t(\mathrm{h}-\mu q)} x,\ t \in \mathbb{R},\ x \in X_\kappa\} \text{ is dense in } X. \tag{4.8}$$

Clearly $\overline{E}$ contains X_κ, and by differentiating with respect to t, we see that $\overline{E}$ contains also $\{\mathrm{i}(\mathrm{h}-\mu q)x \mid x \in X_\kappa \cap \mathcal{D}(\mathrm{h})\}$. We now claim that for each $x \in X$ there exists $x_1 \in X_\kappa$ and $x_2 \in X_\kappa \cap \mathcal{D}(\mathrm{h})$ such that

$$x = x_1 + \mathrm{i}(\mathrm{h}-\mu q)x_2.$$

This will imply (4.8). In fact, the $\mathbb{R}$-linear map $r = \frac{1}{2}\mu q \mathrm{h}^{-1}(1-\kappa)$ has norm less than $|\mu| m^{-1} < 1$, so for $x \in X$ we can find $y \in X$ such that $y - ry = x$. If $x_1 = \frac{1}{2}(y + \kappa y)$ and $x_2 = \frac{1}{2}(\mathrm{i}\mathrm{h})^{-1}(y - \kappa y)$, then both are elements of X_κ, since $[\mathrm{h}, \kappa] = 0$. Now

$$x_1 + \mathrm{i}(\mathrm{h}-\mu q)x_2 = y - \frac{\mathrm{i}}{2}\mu q \mathrm{h}^{-1}(y - \kappa y) = y - ry = x\square.$$

4.4 Observable algebras

The gauge transformations α_t on $\mathfrak{W}(X_0, \sigma)$ can be unitarily implemented in the Araki-Woods representation:

$$\alpha_t(W_\omega(x)) = \mathrm{e}^{\mathrm{i}tQ} W_\omega(x) \mathrm{e}^{-\mathrm{i}tQ},$$

where $Q := \mathrm{d}\Gamma(q \oplus -\overline{q})$.

We denote by $\mathcal{A}$ the *observable algebra*

$$\mathcal{A} := \{A \in \mathcal{W} \mid \mathrm{e}^{\mathrm{i}tQ} A \mathrm{e}^{-\mathrm{i}tQ} = A,\ t \in [0, 2\pi[\}$$

and by $\mathcal{A}_\kappa$ the *abelian observable algebra* $\mathcal{A}_\kappa := \mathcal{A} \cap \mathcal{W}_\kappa$.

Lemma 4.2 *Assume that* $\mathrm{h} \geq m > 0$ *and* $|\mu| < m$. *Let* κ *be a conjugation on* X *such that* $[\mathrm{h}, \kappa] = 0$. *Then* $\mathcal{A}$ *is the von Neumann algebra generated by* $\{\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} \mid t \in \mathbb{R},\ A \in \mathcal{A}_\kappa\}$.

Proof. Clearly $\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} \in \mathcal{A}$, if $A \in \mathcal{A}_\kappa$, since $[L, Q] = 0$. Conversely, let $A \in \mathcal{A}$. By Lemma 4.1 there exists a net $\{A_i\}_{i \in I}$ in the algebra generated by $\{\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL},\ t \in \mathbb{R},\ A \in \mathcal{A}_\kappa\}$ such that $A = \text{s-}\lim A_i$. For $R \in \mathcal{B}(\mathcal{H}_\omega)$, let $R^{\mathrm{av}} := (2\pi)^{-1} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}tQ} R \mathrm{e}^{-\mathrm{i}tQ} \mathrm{d}t$ be the average of R with respect to the gauge group. Then by dominated convergence $\text{s-}\lim A_i^{\mathrm{av}} = A^{\mathrm{av}} = A$. Since $[L, Q] = 0$, we have $(\mathrm{e}^{\mathrm{i}tL} R \mathrm{e}^{-\mathrm{i}tL})^{\mathrm{av}} = \mathrm{e}^{\mathrm{i}tL} R^{\mathrm{av}} \mathrm{e}^{-\mathrm{i}tL}$, which implies the lemma $\square$.

Lemma 4.3 *We have* $\overline{\mathcal{A}\Omega_\omega} = \{u \in \mathcal{H}_\omega \mid Qu = 0\}$.

Proof. Since $Q\Omega_\omega = 0$ we have $\overline{\mathcal{A}\Omega_\omega} \subset \mathrm{Ker}Q$. Let now $u \in \mathrm{Ker}Q$. If $\{A_i \in \mathcal{W}\}_{i \in I}$ is a net such that $\lim A_i \Omega_\omega = u$, then

$$u = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}tQ} u \,\mathrm{d}t = \lim \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}tQ} A_i \mathrm{e}^{-\mathrm{i}tQ} \Omega_\omega \,\mathrm{d}t = \lim_{n \to \infty} A_i^{\mathrm{av}} \Omega_\omega,$$

which proves the lemma since $A_i^{\mathrm{av}} \in \mathcal{A}$ $\square$.

4.5 Stochastic positivity

In this subsection we give a criterion for the stochastic positivity of a quasi-free KMS system.

The following lemma is due to Klein and Landau [KL2].

Lemma 4.4 *Let* $\mathrm{a} \geq 0$ *be a selfadjoint operator on a Hilbert space* X. *Let* $\mathbb{R} \ni s \to r(s) \in \mathcal{B}(X)$ *be the* β*-periodic operator-valued function defined by*

$$r(s) = \frac{\mathrm{e}^{-s\mathrm{a}} + \mathrm{e}^{(s-\beta)\mathrm{a}}}{1 - \mathrm{e}^{-\beta\mathrm{a}}},\ 0 \leq s < \beta.$$

Then, for $x_i \in X$ *and* $s_i \in \mathbb{R}$, *one has*

$$\sum_{i,j} (x_i, r(s_j - s_i) x_j) \geq 0.$$

Proof. Using the spectral decomposition of a, we can assume that $x_i \in \mathbb{C}$ and $\mathrm{a} \geq 0$ is a positive real number. Hence it is sufficient to verify that $r(s)$ is a distribution of positive type. But this follows from Bochner's theorem and the fact that the Fourier transform of r is $\sum_{n\in\mathbb{Z}} r_n\delta(.-2\pi/n)$, where $r_n = \frac{2\mathrm{a}}{\mathrm{a}^2+(2\pi n/\beta)^2} \geq 0$ $\square$.

Theorem 4.5 *Let X be a Hilbert space equipped with a conjugation κ and $\mathrm{a} \geq m > 0$ a selfadjoint operator on X such that $[\mathrm{a},\kappa] = 0$. Let $X_\kappa \subset X$ be the real vector space associated to κ.*

Let $(\mathcal{W},\tau^\circ,\omega)$ be the quasi-free KMS system associated to a *and let $\mathcal{W}_\kappa \subset \mathcal{W}$ be the abelian von Neumann algebra generated by $\{W_\omega(x) \mid x \in X_\kappa\}$. Then the KMS system $(\mathcal{W},\mathcal{W}_\kappa,\tau^\circ,\omega)$ is stochastically positive.*

Proof. We start by computing the Euclidean Green's functions. Using the CCR we get, for $x_j \in X$ and $1 \leq j \leq n$,

$$\prod_1^n W(x_j) = \prod_{1\leq i\leq j\leq n} \mathrm{e}^{-\frac{\mathrm{i}}{2}\sigma(x_i,x_j)} W(\sum_1^n x_j).$$

We denote by

$$G(t_1,\ldots,t_n;W(x_1),\ldots,W(x_n)) = \omega(\prod_{j=1}^n W(\mathrm{e}^{\mathrm{i}t_j\mathrm{a}}x_j))$$

the Green's functions for the Weyl operators $W(x_j)$, $1 \leq j \leq n$. Now

$$\begin{aligned} & G(t_1,\ldots,t_n;W(x_1),\ldots,W(x_n)) \\ = & \prod_{1\leq i<j\leq n} \mathrm{e}^{-\mathrm{i}\mathrm{Im}(x_i,\mathrm{e}^{\mathrm{i}(t_j-t_i)\mathrm{a}}x_j)} \mathrm{e}^{-\frac{1}{4}(\sum_1^n \mathrm{e}^{\mathrm{i}t_j\mathrm{a}}x_j,(1+2\rho)\sum_1^n \mathrm{e}^{\mathrm{i}t_j\mathrm{a}}x_j)} \\ = & \prod_1^n \mathrm{e}^{-\frac{1}{4}(x_i,(1+2\rho)x_i)} \prod_{1\leq i<j\leq n} \mathrm{e}^{-\frac{1}{2}R(t_j-t_i)(x_i,x_j)}, \end{aligned}$$

where

$$R(t)(x,y) = (x,(1-\mathrm{e}^{-\beta\mathrm{a}})^{-1}\mathrm{e}^{\mathrm{i}t\mathrm{a}}y) + (y,\mathrm{e}^{-\beta\mathrm{a}}(1-\mathrm{e}^{-\beta\mathrm{a}})^{-1}\mathrm{e}^{\mathrm{i}t\mathrm{a}}x).$$

For $x,y \in X$ the function $t \mapsto R(t)(x,y)$ has an holomorphic extension to $0 < \mathrm{Im}z < \beta$ such that the function $(t_1,\ldots,t_n) \mapsto G(t_1,\ldots,t_n;W(x_1),\ldots,W(x_n))$ is holomorphic in the set I_β^{n+} defined in (3.4) and continuous on $\overline{I_\beta^{n+}}$ with holomorphic extension

$$(\zeta_1,\ldots,\zeta_n) \mapsto \prod_1^n \mathrm{e}^{-\frac{1}{4}(x_i,(1+2\rho)x_i)} \prod_{1\leq i<j\leq n} \mathrm{e}^{-\frac{1}{2}R(\zeta_j-\zeta_i)(x_i,x_j)}.$$

Hence the euclidean Green's functions

$$
{}^{E}G(s_1,\dots,s_n;W(x_1),\dots,W(x_n)) = \prod_{1}^{n} \mathrm{e}^{-\frac{1}{2}C(0)(x_i,x_i)} \prod_{1\le i<j\le n} \mathrm{e}^{-C(s_j-s_i)(x_i,x_j)},
$$

where

$$
C(s)(x,y) := \frac{1}{2}(x,(1-\mathrm{e}^{-\beta \mathrm{a}})^{-1}\mathrm{e}^{-s\mathrm{a}}y) + \frac{1}{2}(y,(1-\mathrm{e}^{-\beta \mathrm{a}})^{-1}\mathrm{e}^{(s-\beta)\mathrm{a}}x).
$$

Using the fact that $\kappa \mathrm{a} = \mathrm{a}\kappa$ we get

$$
C(s)(x,y) = \frac{1}{2}\Big(x, \frac{\mathrm{e}^{-s\mathrm{a}} + \mathrm{e}^{(s-\beta)\mathrm{a}}}{1-\mathrm{e}^{-\beta\mathrm{a}}}y\Big), \text{ for } x,y \in X_\kappa.
$$

Thus, for $x_j \in X_\kappa$ and $1 \le j \le n$,

$$
{}^{E}G(s_1,\dots,s_n;W(x_1),\dots,W(x_n)) = \prod_{1\le i,j\le n} \mathrm{e}^{-\frac{1}{2}C(|s_i-s_j|)(x_i,x_j)}. \tag{4.9}
$$

We will now prove the stochastic positivity. We will use the Araki-Woods representation described in Subsection 4.2. The operators of the form $F(\phi_\omega(x_1),\dots,\phi_\omega(x_n))$ for $x_i \in X_\kappa$ and $F \in C_0^\infty(\mathbb{R}^n)$ (resp. $F \in C_0^\infty(\mathbb{R}^n)$ and $F \ge 0$) are strongly dense in $\mathcal{W}_\kappa$ (resp. in $\mathcal{W}_\kappa^+$). We have to show that if $(s_1,\dots,s_n)$ is a n-tuple such that $s_1 \le \dots \le s_n$ and $s_n - s_1 \le \beta$, and $A_i \in \mathcal{W}_\kappa^+$, then

$$
{}^{E}G(s_1,\dots,s_n;A_1,\dots,A_n) \ge 0. \tag{4.10}
$$

By [KL1, Thm. 2.2] and a density argument it suffices to prove (4.10) for A_i of the form given above.

Let now $m \in \mathbb{N}$, $m \ge 1$, $k_i \in \mathbb{N}$ with $k_i \ge 1$ for $1 \le i \le n$ and $\sum_1^n k_i = m$, $l_i := \sum_{j\le i-1} k_j$. For $\mathfrak{t} = (t_1,\dots,t_m) \in \mathbb{R}^m$, $x_1,\dots,x_m \in X_\kappa$, and $F_i \in C_0^\infty(\mathbb{R}^{k_i})$ with $F_i \ge 0$ we set $\mathfrak{t}_i = (t_{l_i},\dots,t_{l_{i+1}}) \in \mathbb{R}^{k_i}$ and take

$$
\begin{aligned}
A_i &= F_i(\phi_\omega(x_{l_i}),\dots,\phi_\omega(x_{l_{i+1}})) \\
&= (2\pi)^{-k_i} \int \hat{F}_i(t_{l_i},\dots,t_{l_{i+1}}) W_\omega(\textstyle\sum_{l_i}^{l_{i+1}} t_j x_j)\, \mathrm{d}t_{l_i}\dots\mathrm{d}t_{l_{i+1}}.
\end{aligned}
$$

Now set $f_i(\mathfrak{t}_i) = \sum_{l_i}^{l_{i+1}} t_j x_j$. It follows that

$$
\begin{aligned}
&{}^{E}G(s_1, \dots, s_n; A_1, \dots, A_n) \\
&= (2\pi)^{-m} \int \prod_1^n \mathrm{d}\mathfrak{t}_i \, \hat{F}_i(\mathfrak{t}_i) G(\mathrm{i}s_1, \dots, \mathrm{i}s_n; W(f_1(\mathfrak{t}_1)), \dots, W(f_n(\mathfrak{t}_n))).
\end{aligned}
$$

We recall that by (4.9)

$$
\begin{aligned}
&{}^{E}G(s_1, \dots, s_n; W(f_1(\mathfrak{t}_1)), \dots, W(f_n(\mathfrak{t}_n))) \\
&= \prod_{1 \le i,j \le n} \mathrm{e}^{-\frac{1}{2}C(|s_i - s_j|)(f_i(\mathfrak{t}_i), f_j(\mathfrak{t}_j))} =: \mathrm{e}^{-Q(\mathfrak{t}_1, \dots, \mathfrak{t}_m)},
\end{aligned}
$$

where $Q(\mathfrak{t}_1, \dots, \mathfrak{t}_m)$ is a quadratic form. Applying Lemma 4.4, we see that Q is a positive quadratic form, and hence the inverse Fourier transform $\mathcal{F}^{-1}(\mathrm{e}^{-Q(\dots)})$ is a positive function. This implies that

$$
{}^{E}G(s_1, \dots, s_n; A_1, \dots, A_n) = (F_1 \otimes \dots \otimes F_n) * \mathcal{F}^{-1}(\mathrm{e}^{-Q})(0)
$$

is positive as the value at 0 of the convolution of two positive functions $\Box$.

4.6 Markov property

In this subsection we show a result which implies that the generalized path space associated to the quasi-free KMS system $(\mathcal{W}, \mathcal{W}_\kappa, \tau^\circ, \omega)$ considered in Subsection 4.1 has the *Markov property* (see Subsection 6.5).

Lemma 4.6 *Let X be a Hilbert space equipped with a conjugation κ and $\mathrm{a} \ge m > 0$ a selfadjoint operator on X such that $[\mathrm{a}, \kappa] = 0$. Let $X_\kappa \subset X$ be the real vector space associated to κ.*

Let $(\mathcal{W}(X), \tau^\circ, \omega)$ be the quasi-free KMS system associated to a and let $\mathcal{W}_\kappa \subset \mathcal{W}$ be the abelian von Neumann algebra generated by $\{W_\omega(x) \mid x \in X_\kappa\}$. Let $(\mathcal{H}_\omega, L, \Omega_\omega)$ be the Araki-Woods objects defined in Subsection 4.2. Then the space $\{A\mathrm{e}^{-\frac{\beta}{2}L} B\Omega,\ A, B \in \mathcal{W}_\kappa\}$ is dense in $\mathcal{H}_\omega$.

Proof. The function

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}tL} W_{\omega,l}(y)\Omega_\omega &= W_{\omega,l}(\mathrm{e}^{\mathrm{i}t\mathrm{a}} y)\Omega_\omega \\
&= W_F((1+\rho)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}t\mathrm{a}} y \oplus (\overline{\rho})^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}t\overline{\mathrm{a}}} \overline{y}) \\
&= \mathrm{e}^{\mathrm{i}a_F^*\left((1+\rho)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}t\mathrm{a}} y \oplus (\overline{\rho})^{\frac{1}{2}} \mathrm{e}^{-it\overline{\mathrm{a}}} \overline{y}\right)} \mathrm{e}^{-\frac{1}{2}(y, (1+2\rho)y)} \Omega_\omega
\end{aligned}
$$

is analytic in $\{0 < \mathrm{Imz} < \frac{\beta}{2}\}$ and continuous on $\{0 \leq \mathrm{Imz} \leq \frac{\beta}{2}\}$, and

$$\begin{aligned}
\mathrm{e}^{-\beta L/2} W_{\omega,l}(y)\Omega_\omega &= \mathrm{e}^{ia_F^*\left((1+\rho)^{\frac{1}{2}}\mathrm{e}^{-\beta \mathrm{a}/2}y \oplus (\overline{\rho})^{\frac{1}{2}}\mathrm{e}^{\beta\overline{\mathrm{a}}/2}\overline{y}\right)}\mathrm{e}^{-\frac{1}{2}(y,(1+2\rho)y)}\Omega_\omega \\
&= W_{\omega,r}(y)\Omega_\omega.
\end{aligned}$$

Hence, for $A = W_{\omega,\mathrm{l}}(x)$ and $B = W_{\omega,\mathrm{r}}(y)$, one has

$$\begin{aligned}
&A\mathrm{e}^{-\frac{\beta}{2}L}B\Omega \\
&= W_{\omega,\mathrm{l}}(x)W_{\omega,\mathrm{r}}(y)\Omega = W_F((1+\rho)^{\frac{1}{2}}x \oplus \overline{\rho x})W_F(\rho^{\frac{1}{2}}y \oplus (1+\overline{\rho})^{\frac{1}{2}}\,\overline{y})\Omega.
\end{aligned} \tag{4.11}$$

Let $\mathcal{M}$ be the von Neumann algebra generated by $\{W_{\omega,\mathrm{l}}(x), W_{\omega,\mathrm{r}}(y) \mid x, y \in X_\kappa\}$. By (4.11) the von Neumann algebra generated by $\{W_F((1+\rho)^{\frac{1}{2}}x + \rho^{\frac{1}{2}}y \oplus \overline{\rho}^{\frac{1}{2}}\overline{x} + (1+\overline{\rho})^{\frac{1}{2}}\overline{y}) \mid x, y \in X_\kappa\}$ is equal to $\mathcal{M}$. Since $[a, \kappa] = 0$, the operator

$$\begin{pmatrix} (1+\rho)^{\frac{1}{2}} & \rho^{\frac{1}{2}} \\ \rho^{\frac{1}{2}} & (1+\rho)^{\frac{1}{2}} \end{pmatrix} : X \oplus X \to X \oplus X$$

sends $X_\kappa \oplus X_\kappa$ into itself. It is invertible with inverse

$$\begin{pmatrix} (1+\rho)^{\frac{1}{2}} & -\rho^{\frac{1}{2}} \\ -\rho^{\frac{1}{2}} & (1+\rho)^{\frac{1}{2}} \end{pmatrix}.$$

Thus $\mathcal{M}$ is equal to the von Neumann algebra generated by $\{W_F(x \oplus \overline{y}), x, y \in X_\kappa\}$. It is well known that if $\mathfrak{h}$ is a Hilbert space and c is a conjugation on $\mathfrak{h}$, then the vacuum vector Ω is cyclic in the Fock space $\Gamma(\mathfrak{h})$ for the algebra generated by $\{W_F(h) | \mathrm{c}h = h\}$ (see e.g. [DG, Sect. 5.2] and references therein). We apply this result to $\mathfrak{h} = X \oplus \overline{X}$, $\mathrm{c} = \kappa \oplus \overline{\kappa}$ and obtain the lemma. □

5 Generalized path spaces

In [KL1] a canonical isomorphism is constructed between a stochastically positive β-KMS system $(\mathcal{W}, \mathcal{W}_\kappa, \tau^\circ, \omega)$ and a β-periodic *stochastic process* (Q, Σ, μ, X_t) indexed by the circle S_β of length β, with values in the compact Hausdorff space $K = \mathrm{Sp}\,(\mathcal{W}_\kappa)$, the spectrum of $\mathcal{W}_\kappa$.

We recall that a *stochastic process* (Q, Σ, μ, X_t) indexed by an interval $I \subset \mathbb{R}$ with values in a topological space K consists of

(i) a probability space (Q, Σ, μ);
(ii) a family $\{X_t\}_{t \in I}$ of measurable functions $X_t : Q \to K$.

Typically it is required that the map $I \in t \mapsto X_t$ is continuous in measure.

The stochastic process (Q, Σ, μ, X_t) associated to a stochastically positive β-KMS system in [KL1] satisfies four important properties: *stationarity*, *symmetry*, *β-periodicity* and *Osterwalder-Schrader positivity* (see [KL1, Sect. 4]).

It turns out that the only really important feature of such a stochastic process is the underlying *generalized path space*, which consists of the sub σ-algebra Σ_0 generated by the functions $F(X_0)$ for $F \in C(K)$, the automorphism group $U(t)$ of $L^\infty(Q, \Sigma, \mu)$ generated by the time translations $U(t): F(X_{t_1}, \dots, X_{t_t}) \mapsto F(X_{t_1+t}, \dots, X_{t_n+t})$ for $F \in C(K^n)$ and the automorphism R of $L^\infty(Q, \Sigma, \mu)$ generated by $R: F(X_{t_1}, \dots, X_{t_t}) \mapsto F(X_{-t_1}, \dots, X_{-t_n})$.

In particular the detailed knowledge of the random variables X_t and of the topological space K is not necessary.

(Note that time translations on the path space will correspond to imaginary time translations on the physical Hilbert space).

The analog of the constructions of [KL1] for $\beta = \infty$ done by Klein in [K] is formulated in terms of generalized path spaces. Using this essentially equivalent formulation turns out to be more convenient in applications. We now proceed to a more precise description of this structure, taken from [KL1] and [K].

If Ξ_i, for i in an index set I, is a family of subsets of a set Q, we denote by $\bigvee_{i \in I} \Xi_i$ the σ-algebra generated by $\bigcup_{i \in J} U_i$, where $U_i \in \Xi_i$ and J are countable subsets of I.

Definition 5.1 *A* generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ *consists of*

(i) *a probability space (Q, Σ, μ);*
(ii) *a distinguished sub σ-algebra Σ_0;*
(iii) *a one-parameter group $\mathbb{R} \ni t \mapsto U(t)$ of measure preserving *-automorphisms of $L^\infty(Q, \Sigma, \mu)$, which is strongly continuous in measure;*
(iv) *a measure preserving *-automorphism R of $L^\infty(Q, \Sigma, \mu)$ such that $RU(t) = U(-t)R$, $R^2 = \mathbb{1}$, $RE_0 = E_0 R$, where E_0 is the conditional expectation w.r.t. the σ-algebra Σ_0.*

Moreover one requires that
(v) $\Sigma = \bigvee_{t \in \mathbb{R}} U(t) \Sigma_0$.

It follows from (iii) and (iv) that $U(t)$ extends to a strongly continuous group of isometries of $L^p(Q,\Sigma,\mu)$, and R extends to an isometry of $L^p(Q,\Sigma,\mu)$, for $1 \leq p < \infty$.

We say that the path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is *β-periodic* for $\beta > 0$ if $U(\beta) = 1\!\!1$. On a β-periodic path space we can consider the one-parameter group $U(t)$ as indexed by the circle $S_\beta = [-\beta/2, \beta/2]$.

For $I \subset \mathbb{R}$ we denote by E_I the conditional expectation with respect to the σ-algebra $\Sigma_I := \bigvee_{t\in I} U(t)\Sigma_0$.

Definition 5.2 (0-temperature case): *A path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is* OS-positive *if $E_{[0,+\infty[}RE_{[0,+\infty[} \geq 0$ as an operator on $L^2(Q,\Sigma,\mu)$.*

(Positive temperature case:) *A β-periodic path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is* OS-positive *if $E_{[0,\beta/2]}RE_{[0,\beta/2]} \geq 0$ as an operator on $L^2(Q,\Sigma,\mu)$.*

In order to simplify the notation we set $E_0 = E_{\{0\}}$, $\Sigma_+ = \Sigma_{[0,+\infty[}$, $E_+ = E_{[0,+\infty[}$, $\Sigma_- = \Sigma_{]-\infty,0]}$ and $E_- = E_{]-\infty,0]}$. If the path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is β-periodic, we set $\Sigma_+ = \Sigma_{[0,\beta/2]}$, $E_+ = E_{[0,\beta/2]}$, $\Sigma_- = \Sigma_{[-\beta/2,0]}$ and $E_- = E_{[-\beta/2,0]}$.

Definition 5.3 *A path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is a* Markov path space *if it has the*

(i) reflection property: $RE_0 = E_0$ *(resp. $RE_{\{0,\beta/2\}} = E_{\{0,\beta/2\}}$);*
(ii) Markov property: $E_+E_- = E_+E_0E_-$ *(resp. $E_+E_- = E_+E_{\{0,\beta/2\}}E_-$).*

It follows that $E_+RE_+ = E_-E_+ = E_+E_- = E_0$ (resp. $E_+RE_+ = E_-E_+ = E_+E_- = E_{\{0,\beta/2\}}$) .

A Markov path space is OS-positive because E_0 (resp. $E_{\{0,\beta/2\}}$) is positive as an orthonormal projection. An OS-positive path space satisfies the reflection property (see [K, Prop. 1.6]).

Let $(\mathcal{F},\mathcal{U},\tau,\omega)$ be a stochastically positive β-KMS system. Let $K := \mathrm{Sp}(\mathcal{U})$ be the spectrum of the abelian C^*-algebra $\mathcal{U}$, which equipped with the weak topology is a compact Hausdorff space. Let $Q := K^{[-\beta/2,\beta/2]}$ be equipped with the product topology and let Σ be the Baire σ-algebra on Q.

Theorem 5.4 [KL1]. *Let $(\mathcal{F},\mathcal{U},\tau,\omega)$ be a stochastically positive β-KMS system. Then there exists a Baire probability measure μ on Q, a sub σ-algebra $\Sigma_0 \subset \Sigma$, a measure preserving group $U(t)$ of *-automorphisms of $L^\infty(Q,\Sigma,\mu)$ and a measure preserving automorphism R of $L^\infty(Q,\Sigma,\mu)$ such that $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ is an OS-positive β-periodic generalized path space.*

A more precise relationship between the β-KMS system and the generalized path space will be given in Theorem 6.7.

6 Reconstruction theorems

In this section we recall reconstruction theorems of Klein [K] and Klein and Landau [KL1] which associate a stochastically positive β-KMS system to an OS-positive generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$.

To simplify notation, we allow the parameter β to take values in $]0, +\infty]$. The case $\beta = +\infty$ corresponds to the 0-temperature case. If $\beta < \infty$, then the OS-positive path spaces will be assumed to be β-periodic.

6.1 Physical Hilbert space

Set $\mathcal{H}_{OS} := L^2(Q, \Sigma_+, \mu)$ and

$$(F, G) := \int_Q R(\overline{F}) G \mathrm{d}\mu, \ F, G \in \mathcal{H}_{OS}.$$

By OS-positivity

$$0 \leq (F, F) \leq \|F\|^2_{\mathcal{H}_{OS}}.$$

If we set $\mathcal{N} := \mathrm{Ker} E_+ R E_+$, then $(\cdot, \cdot)$ is a positive definite sesquilinear form on $\mathcal{H}_{OS}/\mathcal{N}$.

The *physical Hilbert space*, denoted by $\mathcal{H}_{\mathrm{phys}}$ (or simply by $\mathcal{H}$) is

$$\mathcal{H} := \text{ completion of } \mathcal{H}_{OS}/\mathcal{N} \text{ for } (\cdot, \cdot).$$

If we denote by $\mathcal{V}: \mathcal{H}_{OS} \to \mathcal{H}_{OS}/\mathcal{N}$ the canonical projection, then $\mathcal{V}$ extends uniquely to a contraction with dense range: $\mathcal{H}_{OS} \to \mathcal{H}$. In fact

$$(\mathcal{V}F, \mathcal{V}F) = (F, F) \leq \|F\|^2_{\mathcal{H}_{OS}}.$$

In the physical Hilbert space $\mathcal{H}$ we find a *distinguished vector*:

$$\Omega := \mathcal{V}(1).$$

The 0-temperature case

Proposition 6.1 [K, Thm. 1.7]. *Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be an OS-positive generalized path space. For $t \geq 0$ the time evolution $U(t)$ maps $\mathcal{N} \to \mathcal{N}$. Hence the linear operator*

$$P(t)\colon \mathcal{H}_{\mathrm{OS}}/\mathcal{N} \ni \mathcal{V}(F) \mapsto \mathcal{V}(U(t)F) \in \mathcal{H}_{\mathrm{OS}}/\mathcal{N}$$

is well defined for $t \geq 0$.

The family $\{P(t)\}_{t\geq 0}$ uniquely extends to a strongly continuous selfadjoint semigroup of contractions $\{\mathrm{e}^{-tH}\}_{t\geq 0}$ on $\mathcal{H}$, where H is a positive selfadjoint operator such that $H\Omega = 0$.

The positive temperature case

We first recall the definition of a local symmetric semigroup ([KL3], [Fr1]):

Definition 6.2 *Let $\mathcal{H}$ be a Hilbert space and $T > 0$. A* local symmetric semigroup *$(P(t), \mathcal{D}_t, T)$ is a family $\{P(t), \mathcal{D}_t\}_{t\in[0,T]}$ of linear operators $P(t)$ and vector subspaces $\mathcal{D}_t$ of $\mathcal{H}$ such that*

(i) *$D_0 = \mathcal{H}$, $\mathcal{D}_t \supset \mathcal{D}_s$ if $0 \leq t \leq s \leq T$ and $\mathcal{D} = \cup_{0<t\leq T} D_t$ is dense in $\mathcal{H}$;*
(ii) *$P(t)\colon \mathcal{D}_t \to \mathcal{H}$ is a symmetric linear operator with $P(0) = \mathbb{1}$, $P(s)\mathcal{D}_t \subset \mathcal{D}_{t-s}$ for $0 \leq s \leq t \leq T$ and $P(t)P(s) = P(t+s)$ on $\mathcal{D}_{t+s}$ for $t, s, t+s \in [0, T]$.*
(iii) *$t \mapsto P(t)$ is weakly continuous, i.e., for $u \in \mathcal{D}_s$ and $0 \leq t \leq s$ the map $t \mapsto (u, P(t)u)$ is continuous.*

The following theorem was shown in [KL3] and [Fr1].

Theorem 6.3 *Let $(P(t), \mathcal{D}_t, T)$ be a local symmetric semigroup on $\mathcal{H}$. Then there exists a unique selfadjoint operator L on $\mathcal{H}$ such that*

(i) *$\mathcal{D}_t \subset \mathcal{D}(\mathrm{e}^{-tL})$, $\mathrm{e}^{-tL}_{|\mathcal{D}_t} = P(t)$ for $0 \leq t \leq T$;*
(ii) *$\mathcal{D}_{]0,T']} := \cup_{0<t\leq T'} \cup_{0<s<t} P(s)\mathcal{D}_t$ is a core for L for $0 < T' \leq T$.*

Proposition 6.4 [KL1, Lemma 8.3]. *Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be a β-periodic OS-positive path space. Set $\mathcal{M}_t := L^2(Q, \Sigma_{[0,\beta/2-t]}, \mu)$ for $0 \leq t \leq \beta/2$. Then*

(i) $U(s)\colon \mathcal{M}_t \cap \mathcal{N} \to \mathcal{M}_{t-s} \cap \mathcal{N}$ *for* $0 \le s \le t \le \beta/2$. *If* $D_t := \mathcal{V}(\mathcal{M}_t)$, *then the linear operator*

$$\begin{array}{rccc} P(s)\colon & \mathcal{D}_t & \to & \mathcal{D}_{t-s}, \\ & \mathcal{V}(F) & \mapsto & \mathcal{V}(U(s)F) \end{array}$$

is well defined;

(ii) $(P(t), \mathcal{D}_t, \beta/2)$ *is a local symmetric semigroup.*

By Theorem 6.3 there exists a unique selfadjoint operator L *such that* $P(t)_{|\mathcal{D}_t} = \mathrm{e}^{-tL}$. *Moreover* $L\Omega = 0$.

6.3 Algebras of operators

Abelian C^*-algebra $\mathcal{U}$

Let $f \in L^\infty(Q, \Sigma_0, \mu)$. Since $\Sigma_0 \subset \Sigma_+$, f acts as a multiplication operator on $\mathcal{H}_{OS}$, which we will still denoted by f.

Proposition 6.5 [KL1, Lemma 2.2]. *For* $f \in L^\infty(Q, \Sigma_0, \mu)$ *the multiplication operator* f *preserves* $\mathcal{N}$. *Hence*

$$\tilde{f}\mathcal{V}(F) := \mathcal{V}(fF)$$

defines a unique element of $\mathcal{B}(\mathcal{H})$ *with* $\|\tilde{f}\| = \|f\|_\infty$. *Let* $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ *be defined by*

$$\mathcal{U} := \{\tilde{f} \mid f \in L^\infty(Q, \Sigma_0, \mu)\}.$$

Then $\mathcal{U}$ *is a von Neumann algebra isomorphic to* $L^\infty(Q, \Sigma_0, \mu)$ *and* Ω *is a separating vector for* $\mathcal{U}$.

We will denote by $\mathcal{U}^+$ the set of positive elements in $\mathcal{U}$.

Full algebra $\mathcal{F}$ and automorphism group

Definition 6.6 *Let* $\mathcal{F} \subset \mathcal{B}(\mathcal{H})$ *denote the von Neumann algebra generated by* $\{\mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH} \mid A \in \mathcal{U}, t \in \mathbb{R}\}$ *for* $\beta = \infty$ *(resp.* $\{\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL} \mid A \in \mathcal{U}, t \in \mathbb{R}\}$ *for* $\beta < \infty$*). We denote by* $\{\tau_t\}_{t\in\mathbb{R}}$ *the strongly continuous group of automorphisms of* $\mathcal{F}$ *defined by* $\tau_t(B) = \mathrm{e}^{\mathrm{i}tH} B \mathrm{e}^{-\mathrm{i}tH}$ *for* $B \in \mathcal{F}$, $t \in \mathbb{R}$ *and* $\beta = \infty$ *(resp.* $\tau_t(B) = \mathrm{e}^{\mathrm{i}tL} B \mathrm{e}^{-\mathrm{i}tL}$ *for* $B \in \mathcal{F}$, $t \in \mathbb{R}$ *and* $\beta < \infty$*).*

In case $\beta < \infty$ one can associate to a β-periodic OS positive path space a stochastically positive β-KMS system (see [KL1]). (The analog object in case $\beta = \infty$ is called a *positive semigroup structure* [K]). Let, for $n \in \mathbb{N}$ and $\beta > 0$,

$$J_\beta^{n+} := \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_i \geq 0,\ t_1 + \cdots + t_n \leq \beta/2\}.$$

Theorem 6.7 [KL1]. *Let L be the selfadjoint operator associated to the local symmetric semigroup $(P(t), \mathcal{D}_t, \beta/2)$. It follows that*

(i) $\Omega \in \mathcal{D}(L)$ *and* $L\Omega = 0$;

(ii) *if* $n \in \mathbb{N}$, $(t_1, \dots, t_n) \in J_\beta^{n+}$ *and* $A_1, \dots, A_n \in \mathcal{U}$, *then* $A_n(\prod_{n-1}^1 \mathrm{e}^{-t_j L} A_j)\Omega \in \mathcal{D}(\mathrm{e}^{-t_n L})$. *The vector span of these vectors is dense in* $\mathcal{H}$;

(iii) *if* $f_1, \dots, f_n \in L^\infty(Q, \Sigma_0, \mu)$ *and* $0 \leq s_1 \leq \cdots \leq s_n \leq \beta/2$, *then*

$$\mathcal{V}(\prod_1^n U(s_j) f_j) = \mathrm{e}^{-s_1 L} \tilde{f}_1 (\prod_2^n \mathrm{e}^{-(s_j - s_{j-1})L} \tilde{f}_j)\Omega,$$

where $\tilde{f}_j$ *is defined in Proposition 6.5.*

(iv) *if* $n \in \mathbb{N}$, $(t_1, \dots, t_n) \in J_\beta^{n+}$ *and* $A_1, \dots, A_n$, $B_1, \dots, B_n \in \mathcal{U}^+$, *then*

$$\Big(A_n(\prod_{n-1}^1 \mathrm{e}^{-t_j L} A_j)\Omega\, ,\ B_n(\prod_{n-1}^1 \mathrm{e}^{-t_j L} B_j)\Omega\Big) \geq 0;$$

(v) $\|\mathrm{e}^{-\beta/2L} A\Omega\| = \|A^*\Omega\|$ *for all* $A \in \mathcal{U}$.

Theorem 6.8 [KL1]. *Let ω_Ω be the state on $\mathcal{F}$ defined by $\omega_\Omega(B) = (\Omega, B\Omega)$. Then $(\mathcal{F}, \mathcal{U}, \tau, \omega_\Omega)$ is a stochastically positive β-KMS system.*

Finally let J be the modular conjugation associated to the KMS system $(\mathcal{F}, \tau, \omega_\Omega)$.

Proposition 6.9 [KL1]. *The modular conjugation J is the unique extension of*

$$J\mathcal{V}(F) = \mathcal{V}(R_{\beta/4}\overline{F}), \tag{6.12}$$

where

$$R_{\beta/4} := U(\beta/4) R U(-\beta/4) = R U(-\beta/2) = U(\beta/2) R$$

is the reflection at $t = \beta/4$ in $\mathcal{H}_{OS}$.

6.5 Markov property for β-periodic path spaces

We recall a characterization of the Markov property for a β-periodic path space in terms of the associated stochastically positive β-KMS system due to Klein and Landau [KL1].

Theorem 6.10 *A β-periodic OS-positive path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ satisfies the Markov property iff the vectors $A\mathrm{e}^{-\frac{\beta}{2}L}B\Omega$ for $A, B \in \mathcal{U}$ are dense in $\mathcal{H}$. In this case*

$$\mathcal{H} = L^2(Q, \Sigma_{\{0,\beta/2\}}, \mu).$$

Proof. The first statement of the theorem is shown in [KL1, Thm. 11.2]. The second statement is obvious: it follows from the Markov property that $E_{[0,\beta/2]}RE_{[0,\beta/2]} = E_{\{0,\beta/2\}}$ is a projection, hence $\mathcal{H}_{OS}/\mathcal{N}$ is canonically identified with $E_{\{0,\beta/2\}}\mathcal{H}_{OS} = L^2(Q, \Sigma_{\{0,\beta/2\}}, \mu)$.

Theorem 6.11 *Let $(\mathcal{W}, \mathcal{W}_\kappa, \tau^\circ, \omega_\beta)$ be the quasi-free KMS system associated to a selfadjoint operator* a ≥ 0 *and a conjugation κ with $[a, \kappa] = 0$. Then the OS-positive generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ associated to $(\mathcal{W}(X), \mathcal{W}_\kappa(X), \tau^\circ, \omega_\beta)$ satisfies the Markov property.*

Proof. Stochastic positivity of the quasi-free KMS system $(\mathcal{W}, \mathcal{W}_\kappa, \tau^\circ, \omega_\beta)$ was shown in Theorem 4.5. The Markov property follows from Lemma 4.6 and Theorem 6.10 □.

7 Perturbations of generalized path spaces

In this section we recall some results concerning perturbations of OS-positive path spaces.

7.1 FKN kernels

Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be an OS-positive path space.

Definition 7.1 *A* Feynman-Kac-Nelson *(FKN)* kernel *is a family $\{F_{[a,b]}\}$ of real measurable functions on (Q, Σ, μ) such that, for $0 \leq b - a \leq \beta$,*

(i) $F_{[a,b]} > 0$ *μ-a.e.;*

(ii) $F_{[a,b]} \in L^1(Q,\Sigma,\mu)$ *and* $F_{[a,b]}$ *is continuous in* $L^1(Q,\Sigma,\mu)$ *as a function of b;*

(iii) $F_{[a,b]}F_{[b,c]} = F_{[a,c]}$ *for* $a \le b \le c$, $c-a \le \beta$*;*

(iv) $U(s)F_{[a,b]} = F_{[a+s,b+s]}$ *for* $s \in \mathbb{R}$*;*

(v) $RF_{[a,b]} = F_{[-b,-a]}$.

The main examples of FKN kernels are those associated to a selfadjoint operator V affiliated to $\mathcal{U}$. In [KL1] and [K] perturbations associated to more general FKN kernels are considered. However, the present case is sufficient for our applications.

Let V be a selfadjoint operator affiliated to $\mathcal{U}$. Since by Proposition 6.5 the algebra $\mathcal{U}$ is isomorphic to $L^\infty(Q,\Sigma_0,\mu)$, we can uniquely associate to V a real function on Q, measurable with respect to Σ_0, which we will still denote by V.

Proposition 7.2 *Let* $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$ *be a β-periodic OS-positive path space and let V be a selfadjoint operator affiliated to $\mathcal{U}$ such that* $V \in L^1(Q,\Sigma_0,\mu)$, *and* $\mathrm{e}^{-TV} \in L^1(Q,\Sigma_0,\mu)$ *for some* $T > 0$ *if* $\beta = \infty$ *or* $\mathrm{e}^{-\beta V} \in L^1(Q,\Sigma_0,\mu)$ *if* $\beta < \infty$*. Then*

(i) *the family of functions* $F_{[a,b]} := \mathrm{e}^{-\int_a^b U(t)V\mathrm{d}t}$ *for* $0 \le b-a \le \inf(T,\beta)/2$ *is a FKN kernel;*

(ii) $F_{[0,s]} \in L^2(Q,\Sigma_{[0,s]},\mu)$ *for* $0 \le s \le \inf(T,\beta)/2$ *and the map* $s \mapsto F_{[0,s]}$ *is continuous in* $L^2(Q,\Sigma_{[0,\beta/2]},\mu)$.

Proof. All properties required in Definition 7.1 except from property (ii) follow directly from the definition of $U(t)$ and the properties of the path space $(Q,\Sigma,\Sigma_0,U(t),R,\mu)$. Let us now verify (ii). Writing $V = V_+ - V_-$, where $V_\pm$ is the positive/negative part of V, we have $F_{[a,b]} \le \exp(\int_a^b U(t)V_-\mathrm{d}t)$, and hence $F_{[0,s]}^2 \le \exp(2\int_0^{\beta/2} U(t)V_-\mathrm{d}t)$. Since μ is a probability measure, we have V_-, $\mathrm{e}^{\beta V_-} \in L^1(Q,\Sigma_0,\mu)$. We recall the following bound from [KL4, Thm. 6.2 (i)]:

$$\|\mathrm{e}^{-\int_a^b U(t)V\mathrm{d}t}\|_{L^p(Q,\Sigma,\mu)} \le \|\mathrm{e}^{-(b-a)V}\|_{L^p(Q,\Sigma,\mu)},\ 1 \le p < \infty. \tag{7.13}$$

This yields

$$\|F_{[0,s]}^2\|_{L^1(Q,\Sigma,\mu)} \le \|\mathrm{e}^{2\int_0^{\beta/2} U(t)V_-\mathrm{d}t}\|_{L^1(Q,\Sigma,\mu)} \le \|\mathrm{e}^{\beta V_-}\|_{L^1(Q,\Sigma,\mu)} < \infty.$$

Hence $F_{[0,s]} \in L^2(Q,\Sigma_{[0,\beta/2]},\mu)$ for $0 \le s \le \inf(T,\beta)/2$. The continuity w.r.t. to s follows from the dominated convergence theorem. This completes the proof of (ii).

The proof of property (ii) from Definition 7.1 for $0 \leq a$ follows from (ii) and the fact that $L^2(Q, \Sigma, \mu) \subset L^1(Q, \Sigma, \mu)$. The case $b \leq 0$ is reduced to the case $a \geq 0$ using property (v). Finally the case $a < 0 < b$ follows from the identity $F_{[a,b]} = F_{[a,0]} F_{[0,b]}$ $\square$.

7.2 Selfadjoint operator associated to a FKN kernel

In this subsection we recall a result of Klein and Landau [KL1], allowing us to construct a selfadjoint operator starting from a FKN kernel associated to a selfadjoint operator V, which is affiliated to $\mathcal{U}$. To keep the exposition compact, we will use the convention for the parameter β explained at the beginning of Section 6.

Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be an OS positive path space and V a selfadjoint operator affiliated to $\mathcal{U}$ such that $V \in L^1(Q, \Sigma, \mu)$ and $\mathrm{e}^{-TV} \in L^1(Q, \Sigma_0, \mu)$ for some $T > 0$. Let $F_{[a,b]}$ be the associated FKN kernel.

Let, for $0 < t < T/2$, $\mathcal{M}_t$ be the linear span $\bigcup_{0 \leq s \leq T/2 - t} F_{[0,s]} L^\infty(Q, \Sigma_{[0,T/2-t]}, \mu)$. Set

$$U_V(s)\colon \begin{array}{ccc} \mathcal{M}_t & \to & L^2(Q, \Sigma_+, \mu) \\ \psi & \mapsto & F_{[0,s]} U(s)\psi, \end{array} \quad 0 \leq s \leq t.$$

Lemma 7.3

(i) *For $\psi \in \mathcal{M}_t$ the map*

$$[0, t] \ni s \mapsto U_V(s)\psi \in L^2(Q, \Sigma_+, \mu)$$

is continuous on $[0, t]$.

(ii) $U_V(s)\colon \mathcal{M}_t \cap \mathcal{N} \to \mathcal{N}$ *for* $0 \leq s \leq t < T/2$.

Proof. Using the definition of $\mathcal{M}_t$ and the properties of the FKN kernel $F_{[a,b]}$ it suffices to show that for $\psi \in L^\infty(Q, \Sigma_{[0,T/2-t]}, \mu)$ the map $s \to U_V(s)\psi$ is continuous at $s = s'$, $0 < s' \leq t < T/2$. For $0 \leq s, s' \leq t < T/2$ we have

$$U_V(s')\psi - U_V(s)\psi = F_{[0,s']}(U(s')\psi - U(s)\psi) + (F_{[0,s']} - F_{[0,s]})U(s)\psi.$$

Hence

$$
\begin{aligned}
&\|U_V(s')\psi - U_V(s)\psi\|_2^2\\
\le\ & \textstyle\int_Q F_{[0,s']}^2 |U(s')\psi - U(s)\psi|^2 \mathrm{d}\mu + \int_Q (F_{[0,s]} - F_{[0,s']})^2 |U(s)\psi|^2 \mathrm{d}\mu\\
\le\ & \textstyle\int_{\{|U(s')\psi - U(s)\psi|(q)>\epsilon\}} F_{[0,s']}^2 |U(s')\psi - U(s)\psi|^2 \mathrm{d}\mu\\
&+ \textstyle\int_{\{|U(s')\psi - U(s)\psi|(q)\le\epsilon\}} F_{[0,s']}^2 |U(s')\psi - U(s)\psi|^2 \mathrm{d}\mu\\
&+ \|F_{[0,s']} - F_{[0,s]}\|_2^2 \|\psi\|_\infty^2.
\end{aligned}
$$

The last term on the r.h.s. tends to 0 if $s \to s'$ as a consequence of Proposition 7.2. The second term on the r.h.s. is less than $\epsilon^2 \|F_{[0,s']}\|_2^2$. To estimate the first term, we write the function $f := F_{[0,s']}^2$ as $f 1\!\!1_{\{|f(q)|\le M\}} + f 1\!\!1_{\{|f(q)|>M\}}$. It follows that

$$
\begin{aligned}
&\textstyle\int_{\{|U(s')\psi - U(s)\psi|(q)>\epsilon\}} f |U(s')\psi - U(s)\psi|^2 \mathrm{d}\mu\\
\le\ & 4M\|\psi\|_\infty^2 \textstyle\int 1\!\!1_{\{|U(s')\psi - U(s)\psi|(q)>\epsilon\}} \mathrm{d}\mu + 4\|f 1\!\!1_{\{|f(q)|>M\}}\|_1 \|\psi\|_\infty^2.
\end{aligned}
$$

Since $f \in L^1(Q, \Sigma_+, \mu)$, the second term tends to 0 as $M \to \infty$. Since $U(t)$ is strongly continuous in measure, the first term tends to 0 as $s \to s'$. Picking first $\epsilon \ll 1$, then $M \gg 1$ and finally $|s - s'| \ll 1$ we obtain (i).

Let us now prove (ii). Let $0 \le s \le t < T/2$. Note that $U_V(s)$ sends $\mathcal{M}_s$ into $L^2(Q, \Sigma_+, \mu)$. Let us fix $\psi \in \mathcal{M}_t$. First we consider the case $s < t$. For $0 < r \le s$ and $s + r \le t$ we have

$$
\begin{aligned}
(U_V(s)\psi, U_V(s)\psi) &= \textstyle\int_Q F_{[0,s]} U(s)\overline{\psi} R F_{[0,s]} U(s)\psi \mathrm{d}\mu\\
&= \textstyle\int_Q F_{[-r,s-r]} U(s-r)\overline{\psi} U(-r) R F_{[0,s]} U(s)\psi \, \mathrm{d}\mu\\
&= \textstyle\int_Q F_{[-r,s-r]} U(s-r)\overline{\psi} R F_{[r,s+r]} U(s+r)\psi \, \mathrm{d}\mu\\
&= \textstyle\int_Q F_{[0,s-r]} U(s-r)\overline{\psi} R F_{[0,s+r]} U(s+r)\psi \, \mathrm{d}\mu\\
&= (U_V(s-r)\psi, U_V(s+r)\psi).
\end{aligned}
$$

Since $(\,.\, , \,.\,)$ is positive, the Cauchy-Schwartz inequality implies

$$
\begin{aligned}
&(U_V(s)\psi, U_V(s)\psi)\\
\le\ & (U_V(s-r)\psi, U_V(s-r)\psi)^{\frac{1}{2}} (U_V(s+r)\psi, U_V(s+r)\psi)^{\frac{1}{2}}.
\end{aligned}
$$

Thus, by induction,

$$
\begin{aligned}
&(U_V(s)\psi, U_V(s)\psi) \\
&\le \|U_V(s-nr)\psi\| \textstyle\prod_{j=0}^{n-1} (U_V(s-(j-1)r)\psi, U_V(s-(j-1)r)\psi)^{\frac{1}{2}}.
\end{aligned}
$$

If we pick $0 < r < s$, $s = nr$, such that $s + r \le t$, then $(\psi, \psi) = 0$ implies $(U(s)\psi, U(s)\psi) = 0$. Finally, (ii) for $s = t$ follows from (ii) for $s < t$ and (i) $\Box$.

Theorem 7.4 *Let $0 < t < T/2$, $\mathcal{D}_t = \mathcal{V}(\mathcal{M}_t)$ and $0 \le s \le t$. Then*

$$
\begin{array}{cccc}
P_V(s)\colon & \mathcal{D}_t & \to & \mathcal{H} \\
 & \mathcal{V}(\psi) & \mapsto & \mathcal{V}(F_{[0,s]}U(s)\psi)
\end{array}
$$

is a well defined linear operator, and $(\mathcal{D}_t, P_V(t), T/2)$ is a local symmetric semigroup on $\mathcal{H}$. We denote by H_V the associated selfadjoint operator.

Proof. The fact that $P_V(s)$ is well defined follows from Lemma 7.3 (ii). Property (ii) of Definition 6.2 follows from the properties of the FKN kernel $F_{[a,b]}$. Monotonicity of the family $\{\mathcal{D}_t\}$ w.r.t. inclusions is immediate. That $\mathcal{D} = \cup_{0<t\le T} D_t$ is dense in $\mathcal{H}$ follows from the fact that $\mathcal{D}$ contains $\mathcal{V}(L^\infty(Q, \Sigma_+, \mu))$. Finally property (iii) follows from the continuity property stated in Lemma 7.3 $\Box$.

Theorem 7.5 [KL1, Thm. 16.4]. *Let V be a selfadjoint operator affiliated to $\mathcal{U}$ such that $V \in L^1(Q, \Sigma_0, \mu)$ and $\mathrm{e}^{-TV} \in L^1(Q, \Sigma_0, \mu)$ for some $T > 0$. Assume in addition that either $V \in L^{2+\epsilon}(Q, \Sigma_0, \mu)$ for $\epsilon > 0$ or that $V \in L^2(Q, \Sigma_0, \mu)$ and $V \ge 0$. Let, for $\beta = \infty$, H (resp. L for $\beta < \infty$) denote the selfadjoint generator of the unperturbed semi-group $t \mapsto P(t)$. Then $H + V$ (resp. $L + V$) is essentially selfadjoint and the operator H_V (for both cases) constructed in Theorem 7.4 is equal to $\overline{H+V}$ (resp. $\overline{L+V}$).*

7.3 Perturbations in the positive temperature case

The following theorem is shown in [KL1]:

Theorem 7.6 [KL1]. *Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be a β-periodic OS-positive path space, V a selfadjoint operator on $\mathcal{H}$ affiliated to $\mathcal{U}$, which satisfies the hypotheses of Proposition 7.2. Let $F = \{F_{[a,b]}\}$ be the associated β-periodic FKN kernel. Then the path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_V)$, where*

$$
\mathrm{d}\mu_V := \frac{F_{[-\beta/2,\beta/2]}\mathrm{d}\mu}{\int_Q F_{[-\beta/2,\beta/2]}\mathrm{d}\mu}\,,
$$

is a β-periodic OS-positive path space.

By the reconstruction theorem recalled in Section 6.5, one can associate to the perturbed path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_V)$ a physical Hilbert space $\mathcal{H}_V$, a distinguished vector Ω_V, an abelian von Neumann algebra $\mathcal{U}_V$, a selfadjoint operator L_V and a von Neumann algebra $\mathcal{F}_V$. If ω_V and τ_V are the state and W^*-dynamics associated to Ω_V and L_V, then $(\mathcal{F}_V, \mathcal{U}_V, \tau_V, \omega_V)$ is a stochastically positive β-KMS system.

Our next aim is to construct canonical identifications between the perturbed objects and perturbations of the original objects associated to the path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$.

Identification of the physical Hilbert spaces

We first show that there is a canonical unitary operator between $\mathcal{H}_V$ and $\mathcal{H}$.

Proposition 7.7 *Assume that* V, $\mathrm{e}^{-\beta V} \in L^1(Q, \Sigma_0, \mu)$. *Set*

$$\begin{array}{llcl} \hat{I}\colon & L^\infty(Q, \Sigma_+, \mu)/\mathcal{N}_V & \to & \mathcal{H}_{OS}/\mathcal{N} \\ & \mathcal{V}_V(\psi) & \mapsto & \frac{\mathcal{V}(F_{[0,\beta/2]}\psi)}{\left(\int_Q F_{[-\beta/2,\beta/2]}\mathrm{d}\mu\right)^{\frac{1}{2}}}. \end{array}$$

Then $\hat{I}$ is a well defined isometry from $\mathcal{H}_{OS,V}/\mathcal{N}_V$ into $\mathcal{H}_{OS}/\mathcal{N}$ with dense range and domain. Hence $\hat{I}$ uniquely extends to a unitary map $\hat{I}\colon \mathcal{H}_V \to \mathcal{H}$.

Proof. Note that μ_V is absolutely continuous w.r.t. μ. Thus $L^\infty(Q, \Sigma, \mu_V) = L^\infty(Q, \Sigma, \mu)$. If $\psi \in L^\infty(Q, \Sigma, \mu) \cap \mathcal{N}_V$, then $\int_Q R\overline{\psi}\psi \mathrm{d}\mu_V = \int_Q \mathrm{d}\mu R\overline{F_{[0,\beta/2]}\psi}F_{[0,\beta/2]}\psi = 0$. Hence $F_{[0,\beta/2]}\psi \in \mathcal{N}$. Consequently $\hat{I}$ is well defined. $\hat{I}$ is clearly isometric since

$$(\mathcal{V}_V\psi, \mathcal{V}_V\psi)_V = \frac{\int_Q R\overline{\psi}\psi \mathrm{d}\mu_V}{\int_Q F_{[-\beta/2,\beta/2]}\mathrm{d}\mu} = \frac{\int_Q R\overline{F_{[0,\beta/2]}\psi}F_{[0,\beta/2]}\psi \mathrm{d}\mu}{\int_Q F_{[-\beta/2,\beta/2]}\mathrm{d}\mu} = (\hat{I}\mathcal{V}_V\psi, \hat{I}\mathcal{V}_V\psi).$$

$\hat{I}$ is densely defined since $L^\infty(Q, \Sigma_+, \mu)$ is dense in $\mathcal{H}_{OS,V}$. Since $\mathcal{V}_V$ is a contraction, $L^\infty(Q, \Sigma_+, \mu)/\mathcal{N}_V$ is dense in $\mathcal{H}_{OS,V}/\mathcal{N}_V$ and hence in $\mathcal{H}_V$. Finally, we note that $\mathrm{Ran}\hat{I}$ contains $\mathcal{V}(F_{[0,\beta/2]}L^\infty(Q, \Sigma_+, \mu))$. Since $F_{[0,\beta/2]} > 0$ a.e., $F_{[0,\beta/2]}L^\infty(Q, \Sigma, \mu)$ is dense in $\mathcal{H}_{OS}$ and hence its image under $\mathcal{V}$ is dense in $\mathcal{H}$ $\Box$.

Identification of the abelian algebra

Proposition 7.8 *For $f \in L^\infty(Q, \Sigma_0, \mu)$ one has*

$$\hat{I}\tilde{f}\psi = \tilde{f}\hat{I}\psi, \ \psi \in \mathcal{H}_V,$$

and, consequently, $\hat{I}\mathcal{U}_V = \mathcal{U}\hat{I}$.

Proof. This follows immediately from the definitions of $\tilde{f}$ in Proposition 6.5 and $\hat{I}$ in Proposition 7.7 □.

Identification of the C^*-dynamics

Applying Theorem 7.4 we obtain a selfadjoint operator H_V from the FKN kernel associated to V. It will be called the *pseudo-Liouvillean* generated by V.

Proposition 7.9 *One has*

(i) $\hat{I}\Omega_V = \|\mathrm{e}^{-\beta H_V/2}\Omega\|^{-1}\mathrm{e}^{-\beta H_V/2}\Omega$;
(ii) *for* $0 \le s_1 \le \cdots \le s_n \le \beta/2$ *and* $A_1, \ldots, A_n \in \mathcal{U}$

$$\begin{aligned}&\hat{I}\mathrm{e}^{-s_1 L_V} A_1 (\textstyle\prod_2^n \mathrm{e}^{(s_{j-1}-s_j)L_V} A_j)\Omega_V \\ &= \frac{\mathrm{e}^{-s_1 H_V} A_1 \left(\prod_2^n \mathrm{e}^{(s_{j-1}-s_j)H_V} A_j\right)\mathrm{e}^{(s_n-\beta/2)H_V}\Omega}{\|\mathrm{e}^{-\beta H_V/2}\Omega\|} \ ;\end{aligned}$$

(iii) *for* $t_1, \ldots, t_n \in \mathbb{R}$, $A_1, \ldots, A_n \in \mathcal{U}$ *and* $\psi \in \mathcal{H}_V$

$$\hat{I}(\prod_1^n \mathrm{e}^{\mathrm{i}t_j L_V} A_j \mathrm{e}^{-\mathrm{i}t_j L_V})\psi = (\prod_1^n \mathrm{e}^{\mathrm{i}t_j H_V} A_j \mathrm{e}^{-\mathrm{i}t_j H_V})\hat{I}\psi \,;$$

(iv) $\hat{I}J_V = J\hat{I}$.

Note that in (ii) and (iii) we identify $\mathcal{U}$ with $L^\infty(Q, \Sigma_0, \mu)$.

Identification of the observable algebras

We recall that the observable algebra and the dynamics associated to the perturbed path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_V)$ are the von Neumann algebra $\mathcal{F}_V$ generated by $\{\mathrm{e}^{\mathrm{i}tL_V} A \mathrm{e}^{-\mathrm{i}tL_V} \mid A \in \mathcal{U}_V, \ t \in \mathbb{R}\}$ and the automorphism group $\tau_V: t \mapsto \tau_V(t)$, $t \in \mathbb{R}$, where

$$\tau_V(t)(B) = \mathrm{e}^{\mathrm{i}tL_V} B \mathrm{e}^{-\mathrm{i}tL_V},\ B \in \mathcal{F}_V.$$

Proposition 7.10

(i) $\hat{I}\tau_V(t)(B)\hat{I}^{-1} = \mathrm{e}^{\mathrm{i}tH_V}\hat{I}B\hat{I}^{-1}\mathrm{e}^{-\mathrm{i}tH_V}$ *for* $B \in \mathcal{F}_V$ *and* $t \in \mathbb{R}$;

(ii) *Assume that either* $V \in L^{2+\epsilon}(Q, \Sigma_0, \mu)$ *for* $\epsilon > 0$ *or that* $V \in L^2(Q, \Sigma_0, \mu)$ *and* $V \geq 0$. *It follows that* $\hat{I}\mathcal{F}_V\hat{I}^{-1} = \mathcal{F}$.

Proof. (i) follows from Proposition 7.9 (iii). To prove (ii) we recall from Theorem 7.5 that, under the assumptions of the proposition, $L + V$ is essentially selfadjoint on $\mathcal{D}(L) \cap \mathcal{D}(V)$ and $H_V = \overline{L + V}$. Hence, by Trotter's formula,

$$\mathrm{e}^{\mathrm{i}tH_V} = \text{s-}\lim_{n\to\infty}(\mathrm{e}^{\mathrm{i}tL/n}\mathrm{e}^{\mathrm{i}tV/n})^n.$$

Thus

$$\mathrm{e}^{\mathrm{i}tH_V} A \mathrm{e}^{-\mathrm{i}tH_V} = \mathrm{w} - \lim_{n\to+\infty}(\mathrm{e}^{\mathrm{i}tL/n}\mathrm{e}^{\mathrm{i}tV/n})^n A (\mathrm{e}^{-\mathrm{i}tV/n}\mathrm{e}^{-\mathrm{i}tL/n})^n.$$

Since $\mathrm{e}^{\mathrm{i}sV} \in \mathcal{U} \subset \mathcal{F}$, $A \in \mathcal{F}$ implies that $\mathrm{e}^{\mathrm{i}sV} A \mathrm{e}^{-\mathrm{i}sV} \in \mathcal{F}$. Moreover, $\mathrm{e}^{\mathrm{i}sL} A \mathrm{e}^{-isL} \in \mathcal{F}$ by definition. So $\mathrm{e}^{\mathrm{i}tH_V} A \mathrm{e}^{-\mathrm{i}tH_V} \in \mathcal{F}$, if $A \in \mathcal{U}$, and hence

$$\hat{I}\mathcal{F}_V\hat{I}^{-1} \subset \mathcal{F}.$$

According to Tomita's theorem (see, e.g., [BR]) $\mathcal{F}' = J\mathcal{F}J$ and $\mathcal{F}_V' = J_V\mathcal{F}_V J_V$. Thus using Proposition 7.9(iv):

$$(\hat{I}\mathcal{F}_V\hat{I}^{-1})' = \hat{I}\mathcal{F}_V'\hat{I}^{-1} = \hat{I}J_V\mathcal{F}_V J_V\hat{I}^{-1} = J\hat{I}\mathcal{F}_V\hat{I}^{-1}J \subset J\mathcal{F}J = \mathcal{F}'.$$

Taking commutants we obtain

$$\mathcal{F} = \mathcal{F}'' \subset (\hat{I}\mathcal{F}_V\hat{I}^{-1})'' = \hat{I}\mathcal{F}_V\hat{I}^{-1}.$$

Hence $\mathcal{F} = \hat{I}\mathcal{F}_V\hat{I}^{-1}\square$.

The results in this section are summarized in the following theorem.

Theorem 7.11 *Let* $(\mathcal{F}, \mathcal{U}, \tau, \omega)$ *be a stochastically positive* β*-KMS system. Let* $\mathcal{H}, \Omega, L$ *be the associated GNS Hilbert spaces, GNS vector and Liouvillean. Let* V *be a selfadjoint operator on* $\mathcal{H}$*, affiliated to* $\mathcal{U}$*, such that*

$$V,\ \mathrm{e}^{-\beta V} \in L^1(Q, \Sigma_0, \mu) \text{ and } \quad \begin{array}{ll} \text{either} & V \in L^{2+\epsilon}(Q, \Sigma_0, \mu),\ \epsilon > 0, \\ \text{or} & V \in L^2(Q, \Sigma_0, \mu) \text{ and } V \geq 0. \end{array}$$

Then

(i) $L+V$ *is essentially selfadjoint on* $\mathcal{D}(L)\cap\mathcal{D}(V)$*;*

(ii) $\Omega\in\mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V})$*, where* $H_V=\overline{L+V}$*;*

(iii) $(\mathcal{F},\mathcal{U},\tau_V,\omega_V)$ *is a stochastically positive* β*-KMS system for* $\tau_{V,t}(A)=\mathrm{e}^{\mathrm{i}tH_V}A\mathrm{e}^{-\mathrm{i}tH_V}$, $\omega_V(A)=\|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|^{-2}(\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega, A\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega)$, $A\in\mathcal{F}$.

Perturbed Liouvillean

In the next theorem, we identify the Liouvillean for the perturbed system.

Theorem 7.12 *Assume that* V *is a selfadjoint operator affiliated to* $\mathcal{U}$ *such that*

$$\mathrm{e}^{-\beta V}\in L^1(Q,\Sigma_0,\mu) \tag{7.14}$$

and

$$\begin{aligned}&V\in L^p(Q,\Sigma_0,\mu),\ \mathrm{e}^{-\frac{\beta}{2}V}\in L^q(Q,\Sigma_0,\mu)\ \text{for}\ p^{-1}+q^{-1}=\tfrac{1}{2},\ 2<p,\ q<\infty,\\ &\text{or}\ V\in L^2(Q,\Sigma_0,\mu)\ \text{and}\ V\geq 0.\end{aligned} \tag{7.15}$$

Let L_V *be the Liouvillean associated to the* β*-KMS system* $(\mathcal{F},\tau_V,\omega_V)$*. Then* H_V-JVJ *is essentially selfadjoint on* $\mathcal{D}(H_V)\cap\mathcal{D}(JVJ)$ *and* $L_V=\overline{H_V-JVJ}$*.*

Lemma 7.13 *For* $A\in\mathcal{U}$ *one has* $JA\Omega_V=\|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|^{-1}\mathrm{e}^{-\frac{\beta}{2}H_V}A^*\Omega$*.*

Proof. Let us set $c=\|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|^{-1}$. Then $A\Omega_V=c\mathcal{V}(AF_{[0,\beta/2]})$. Moreover, $JA\Omega_V=c\mathcal{V}(U(\beta/2)A^*F_{[0,\beta/2]})$, since $F_{[0,\beta/2]}$ is invariant under $R_{\beta/4}$. Since A^* belongs to the space $\mathcal{M}_{\beta/2}=L^\infty(Q,\Sigma_0,\mu)$ defined in Section 7.2, $\mathcal{V}(A^*)=A\Omega\in\mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V})$ and

$$c\mathrm{e}^{-\frac{\beta}{2}H_V}A^*\Omega=c\mathcal{V}(U(\beta/2)A^*F_{[0,\beta/2]})=JA\Omega_V\ \Box.$$

Lemma 7.14 *Let* f_1 *be a real function in* $L^2(Q,\Sigma_0,\mu)$ *such that* $f_1F_{[0,\beta/2]}\in L^2(Q,\Sigma_{[0,\beta/2]},\mu)$*. Then* Ω_V *and* Ω *are vectors in* $\mathcal{D}(f_1)$*. The vector* $f_1\Omega$ *is in* $\mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V})$ *and satisfies* $Jf_1\Omega_V=\|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|^{-1}\mathrm{e}^{-\frac{\beta}{2}H_V}f_1\Omega$*.*

Proof. Since $f_1\in L^2(Q,\Sigma_0,\mu)$, we have $\Omega\in\mathcal{D}(f_1)$. Now $f_1F_{[0,\beta/2]}\in L^2(Q,\Sigma_{[0,\beta/2]},\mu)$, thus $\Omega_V\in\mathcal{D}(f_1)$. Let $f_n=f_1 1\!\!1_{\{|f_1|\leq n\}}$. By dominated convergence $f_nF_{[0,\beta/2]}\to f_1F_{[0,\beta/2]}$ in $L^2(Q,\Sigma_{[0,\beta/2]},\mu)$, i.e.,

$$f_1\Omega_V=\mathcal{V}(f_1F_{[0,\beta/2]})=\lim_{n\to\infty}\mathcal{V}(f_nF_{[0,\beta/2]})=\lim_{n\to\infty}f_n\Omega_V.$$

Applying Lemma 7.13 to $A = f_n$ we obtain, for $u \in \mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V})$,

$$(\mathrm{e}^{-\frac{\beta}{2}H_V}u, f_1\Omega) = \lim_{n\to\infty}(\mathrm{e}^{-\frac{\beta}{2}H_V}u, f_n\Omega)$$

$$= \lim_{n\to\infty}(u, \mathrm{e}^{-\frac{\beta}{2}H_V}f_n\Omega) = \lim_{n\to\infty}(u, Jf_n\Omega_V) = (u, Jf_1\Omega_V).$$

This shows that $f_1\Omega \in \mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V})$ and $\mathrm{e}^{-\frac{\beta}{2}H_V}f_1\Omega = Jf_1\Omega_V$ $\Box$.

Lemma 7.15 *Assume that V is a selfadjoint operator, affiliated to $\mathcal{U}$, which satisfies (7.15). Then*

$$\Omega_V \in \mathcal{D}(H_V) \cap \mathcal{D}(V) \text{ and } (H_V - JVJ)\Omega_V = (H_V - JV)\Omega_V = 0.$$

Proof. We first verify that V satisfies the hypotheses of Lemma 7.14, i.e., that

$$V\mathrm{e}^{-\int_0^{\beta/2}U(t)V\mathrm{d}t} \in L^2(Q, \Sigma_{[0,\beta/2]}, \mu). \tag{7.16}$$

Let $2 \leq p, q \leq \infty$ be as in (7.15). If $p = 2$, then $V \geq 0$ a.e., thus (7.16) is clearly satisfied. If $q < \infty$, then, applying Hölder's inequality, it suffices to prove that

$$V \in L^p(Q, \Sigma, \mu) \text{ and } \mathrm{e}^{-\int_0^{\beta/2}U(t)V\mathrm{d}t} \in L^q(Q, \Sigma, \mu).$$

Applying (7.13) we find

$$\|\mathrm{e}^{-\int_0^{\beta/2}U(t)V\mathrm{d}t}\|_{L^q(Q,\Sigma,\mu)} \leq \|\mathrm{e}^{-\frac{\beta}{2}V}\|_q < \infty.$$

Let $u \in \mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V}) \cap \mathcal{D}(H_V) \cap \mathcal{D}(H_V\mathrm{e}^{-\frac{\beta}{2}H_V})$ and set $c := \|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|^{-1}$. Then

$$(H_Vu, \Omega_V) = c(\mathrm{e}^{-\frac{\beta}{2}H_V}H_Vu, \Omega) = c(\mathrm{e}^{-\frac{\beta}{2}H_V}u, H_V\Omega) = c(\mathrm{e}^{-\frac{\beta}{2}H_V}u, V\Omega),$$

since $\Omega \in \mathcal{D}(V) \cap \mathcal{D}(L)$ and $H_V\Omega = L\Omega + V\Omega = V\Omega$. Applying Lemma 7.14 to $f_1 = V$ we obtain

$$c(\mathrm{e}^{-\frac{\beta}{2}H_V}u, V\Omega) = c(u, \mathrm{e}^{-\frac{\beta}{2}H_V}V\Omega) = (u, JV\Omega_V).$$

This implies, together with $J\Omega_V = \Omega_V$, that $\Omega_V \in \mathcal{D}(H_V)$ and $H_V\Omega_V = JV\Omega_V = JVJ\Omega_V$ $\Box$.

Proof of Theorem 7.12. Let $\mathcal{F}_1$ be the set of $A \in \mathcal{F}$ such that $t \mapsto \tau_{V,t}(A)$ is C^1 for the strong topology and let $A \in \mathcal{F}_1$. Since H_V implements the dynamics $\tau_{V,t}$, we see that $A \in C^1(H_V)$. By [ABG], this implies that $A\colon \mathcal{D}(H_V) \to \mathcal{D}(H_V)$. Since $\Omega_V \in \mathcal{D}(H_V)$, the vector $A\Omega_V \in \mathcal{D}(H_V)$. Since JVJ is affiliated to $\mathcal{F}'$, Lemma 7.15 implies

$$\begin{aligned} L_V A\Omega_V &= \mathrm{i}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\tau_{V,t}(A)\Omega_V\,{}_{|t=0} = H_V A\Omega_V - AH_V\Omega_V \\ &= H_V A\Omega_V - AJVJ\Omega_V = H_V A\Omega_V - JVJA\Omega_V. \end{aligned}$$

This yields $L_V u = H_V u - JVJu$ for $u \in \mathcal{F}_1\Omega_V$. By Proposition 3.1, we know that $\mathcal{F}_1\Omega_V$ is a core for L_V. This implies that L_V is the closure of $H_V - JVJ$ on $\mathcal{F}_1\Omega_V$ and hence also the closure of $H_V - JVJ$ on $\mathcal{D}(H_V) \cap \mathcal{D}(JVJ)$ $\square$.

7.4 *Markov property for perturbed of path spaces*

In this subsection we show that the Markov property of a path space is preserved by the perturbations described in Subsection 7.1.

Proposition 7.16 *Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be a generalized path space satisfying the Mar-kov property and let $\{F_{[a,b]}\}$ be a FKN kernel. Then the path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_F)$ satisfies the Markov property.*

Proof. Let (Q, Σ, μ) be a probability space, $F \in L^1(Q, \Sigma, \mu)$ with $F > 0$ μ-a.e. and set $\mathrm{d}\mu_F = (\int F \mathrm{d}\mu)^{-1} F \mathrm{d}\mu$.

If $B \subset \Sigma$ is a σ-algebra and f is Σ-measurable, then we denote by $E_B(f)$, (resp. $E_B^F(f)$) the conditional expectation of f w.r.t. B for the measure μ (resp. μ_F). Then (see [Lo, Sect. 2.4])

$$E_B(fg) = E_B(f)g,\ E_B^F(fg) = E_B^F(f)g \ \ \mu\text{-a.e. if } g \text{ is } B\text{-measurable} \tag{7.17}$$

and

$$E_B^F(f) = \frac{E_B(Ff)}{E_B(F)} \ \ \mu\text{-a.e.} \tag{7.18}$$

To simplify the notation, let us set $E_0 = E_{\{0\}}$ if $\beta = +\infty$ and $E_0 = E_{\{0,\beta/2\}}$ if $\beta < \infty$. Set $F_+ = F_{[0,\beta/2]}$ and $F_- = F_{[-\beta/2,0]}$, so that $F = F_- F_+$. Set $E_+^{(F)} = E_{[0,\beta/2]}^{(F)}$ and $E_-^{(F)} = E_{[-\beta/2,0]}^{(F)}$. Finally set $E_0^{(F)} = E_{\{0\}}^{(F)}$ if $\beta = +\infty$ and $E_0^{(F)} = E_{\{0,\beta/2\}}^{(F)}$ if $\beta < \infty$.

Let now f be Σ-measurable. Then

$$E_+^F(f) = \frac{E_+(Ff)}{E_+(F)} = \frac{E_+(F_-F_+f)}{E_+(F_-F_+)} = \frac{E_+(F_-f)}{E_+(F_-)},$$

using (7.18), (7.17) and the fact that F_+ is $\Sigma_{[0,\beta/2]}$-measurable. Next

$$\frac{E_+(F_-f)}{E_+(F_-)} = \frac{E_+(F_-f)}{E_+E_-(F_-)} = \frac{E_+(F_-f)}{E_0(F_-)},$$

by the Markov property for (Q, Σ, μ) and the fact that F_- is $\Sigma_{[-\beta/2,0]}$-measurable. Since $E_0(F_-)$ is $\Sigma_{[-\beta/2,0]}$-measurable, we have, by (7.18) and (7.17),

$$E_-^F E_+^F(f) = \frac{E_-(FE_+(F_-f))}{E_0(F_-)E_-(F)} = \frac{E_-(F_-F_+E_+(F_-f))}{E_0(F_-)E_-(F_-F_+)} = \frac{E_-(F_+E_+(F_-f))}{E_0(F_-)E_-(F_+)},$$

since F_- is $\Sigma_{[-\beta/2,0]}$-measurable.

Now

$$\frac{E_-(F_+E_+(F_-f))}{E_0(F_-)E_-(F_+)} = \frac{E_0(Ff)}{E_0(F_+)E_0(F_-)},$$

by the Markov property for (Q, Σ, μ) and the fact that F_+ is $\Sigma_{[0,\beta/2]}$-measurable. Finally

$$\begin{aligned} E_0(F_-)E_0(F_+) &= E_+E_-(F_-)E_0(F_+) = E_+(F_-E_0(F_+)) \\ &= E_+(F_-E_-(F_+)) = E_+E_-(F_-F_+) = E_0(F). \end{aligned}$$

This yields $E_-^F E_+^F(f) = E_0^F(f)$ μ-a.e. and completes the proof $\square$.

8 Free Klein-Gordon fields at positive temperature

In this section we recall some results about the complex Klein-Gordon field and show that it provides an example of a charge symmetric Kähler structure.

The classical Klein-Gordon equation describing a charged particle of mass m is

$$\partial_t^2\Phi - \partial_x^2\Phi + m^2\Phi = 0, \ (t, x) \in \mathbb{R}^{d+1},$$

where $\Phi\colon \mathbb{R}^{d+1} \to \mathbb{C}$ is a complex valued function. For later use we recall the discrete symmetries of the Klein-Gordon equation, namely the *parity* p, *time reversal* θ and *charge conjugation* c:

$$\mathrm{p}\Phi(t,x) := \Phi(t,-x),\ \theta\Phi(t,x) = \overline{\Phi}(-t,x) \text{ and } \mathrm{c}\Phi(t,x) = \overline{\Phi}(t,x).$$

In particular, real solutions of the Klein-Gordon equation without external field describe neutral scalar particles. In the sequel only time-reversal and charge conjugation will play a role.

8.1 The complex Klein-Gordon field

Let us now describe the abstract Klein-Gordon equation that we will consider in the sequel.

Abstract Klein-Gordon equation

Let $\mathfrak{h}$ be a Hilbert space. We denote by i the complex structure on $\mathfrak{h}$ and by $(\,.\,,\,.\,)_{\mathfrak{h}}$ the scalar product on $\mathfrak{h}$. We assume that $\mathfrak{h}$ is equipped with a conjugation denoted by $\Phi \to \overline{\Phi}$.

Let

$$\epsilon \geq m > 0 \tag{8.19}$$

be a real selfadjoint operator on $\mathfrak{h}$, i.e., such that $\overline{\epsilon\Phi} = \epsilon\overline{\Phi}$.

For $0 \leq s \leq 1$ we denote by $\mathfrak{h}_s$ the Hilbert space $\mathcal{D}(\epsilon^s)$ with complex structure i and scalar product $v, u \mapsto (v, \epsilon^{2s}u)_{\mathfrak{h}}$ and by $\mathfrak{h}_{-s}$ the completion of $(\mathfrak{h}, \mathrm{i})$ for the norm $(v, \epsilon^{-2s}v)_{\mathfrak{h}}$. The space $\mathfrak{h}_{-s}$ can be identified with the anti-dual of $\mathfrak{h}_s$ using the sesquilinear form $\langle v, u\rangle = (v,u)_{\mathfrak{h}}$ for $v \in \mathfrak{h}_{-s}$ and $u \in \mathfrak{h}_s$.

We consider the abstract Klein-Gordon equation

$$\text{(KG)}\ \ (\partial_t^2\Phi)(t) + \epsilon^2\Phi(t) = 0,$$

where $\Phi(t)$ is a function of $t \in \mathbb{R}$ with values in $\mathfrak{h}$. This (complex) KG equation describes a classical field of scalar charged particles.

The complex structure on $\mathfrak{h}$ yields a complex structure on the space of solutions of (KG), associated to the $U(1)$ gauge group. Following the convention of

Subsection 2.1 this 'charge' complex structure will be denoted by j. It is defined by

$$(\mathrm{j}\Phi)(t) := \mathrm{i}\Phi(t) \text{ for } \Phi \text{ a solution of (KG) and } t \in \mathbb{R}.$$

The following quantity does not depend on t:

$$q(\Psi, \Phi) := \mathrm{i}(\Psi(t), (\partial_t\Phi)(t))_{\mathfrak{h}} - \mathrm{i}((\partial_t\Psi)(t), \Phi(t))_{\mathfrak{h}}.$$

Hence it defines a symmetric sesquilinear form on the space of solutions of (KG). The following transformations preserve the solutions of (KG):

– *gauge transformations* $\Phi(t) \mapsto \mathrm{e}^{\mathrm{i}\alpha}\Phi(t) = (\mathrm{e}^{\mathrm{j}\alpha}\Phi)(t)$, $\alpha \in [0, 2\pi]$;

– *time-reversal* $\theta\colon \Phi(t) \mapsto \overline{\Phi}(-t)$;

– *charge conjugation* $\mathrm{c}\colon \Phi(t) \mapsto \overline{\Phi(t)}$.

Energy space

It is convenient to identify a solution of (KG) with its Cauchy data at $t = 0$,

$$f := (\Phi(0), (\partial_t\Phi)(0)) \in \mathfrak{h} \times \mathfrak{h}.$$

To do so one introduces the *energy space* $\mathcal{E} := \mathfrak{h}_1 \oplus \mathfrak{h}$ equipped with the norm

$$(f, f)_{\mathcal{E}} = (f_1, \epsilon^2 f_1)_{\mathfrak{h}} + (f_2, f_2)_{\mathfrak{h}},$$

where we set $f = (f_1, f_2)$. Note that the complex structure j becomes $\mathrm{i} \oplus \mathrm{i}$ on $\mathcal{E}$. Setting $f_t = (\Phi(t), (\partial_t\Phi)(t))$ one can rewrite the Klein-Gordon equation as the first order system:

$$\mathrm{j}(\partial_t f)_t = L f_t \text{ for } L = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i}\epsilon^2 & 0 \end{pmatrix}.$$

It is convenient to diagonalize L using the unitary map

$$\begin{array}{rccc} U_0\colon & \mathcal{E} & \to & \mathfrak{h} \oplus \mathfrak{h} \\ & f & \mapsto & u = (u_1, u_2), \end{array}$$

where

$$U_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon & \mathrm{i} \\ \epsilon & -\mathrm{i} \end{pmatrix} \text{ and } U_0^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1} \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}.$$

It follows that

$$U_0 L U_0^* = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}.$$

In particular, L is selfadjoint on $\mathcal{E}$ with domain $U^{-1}(\mathfrak{h}_1 \times \mathfrak{h}_1)$ and the evolution $\mathbb{R} \ni t \mapsto \mathrm{e}^{-\mathrm{j}tL}$ is a strongly continuous unitary group. Therefore the space of solutions of (KG) can be identified with $\mathcal{E}$. On $\mathcal{E}$ the symmetric form q is

$$q(g, f) = \mathrm{i}(g_1, f_2)_{\mathfrak{h}} - \mathrm{i}(g_2, f_1)_{\mathfrak{h}}.$$

Charged Kähler space structure

On $\mathcal{E}$ we put the 'energy' complex structure $\mathrm{i} := \mathrm{j}\frac{L}{|L|}$.

Proposition 8.1 *The space $(\mathcal{E}, \mathrm{j}, \mathrm{i}, q)$ is a charged Kähler space.*

Proof. Clearly $[\mathrm{i}, \mathrm{j}] = 0$. We have to prove that

$$(g, f) := \mathrm{Im} q(g, \mathrm{i} f) + \mathrm{i} \mathrm{Im} q(g, f)$$

is a positive definite symmetric sesquilinear form on $(\mathcal{E}, \mathrm{i})$. If $U_0 f = (u_1, u_2)$ and $U_0 g = (v_1, v_2)$, then

$$q(g, f) = -(v_2, \epsilon^{-1} u_2)_{\mathfrak{h}} + (v_1, \epsilon^{-1} u_1)_{\mathfrak{h}},$$

$$q(g, \mathrm{i} f) = -(v_2, -\mathrm{i}\epsilon^{-1} u_2)_{\mathfrak{h}} + (v_1, \mathrm{i}\epsilon^{-1} u_1)_{\mathfrak{h}} = \mathrm{i}(v_1, \epsilon^{-1} u_1)_{\mathfrak{h}} + \mathrm{i}(v_2, \epsilon^{-1} u_2)_{\mathfrak{h}},$$

and consequently

$$(g, f) = (v_1, \epsilon^{-1} u_1)_{\mathfrak{h}} + \overline{(v_2, \epsilon^{-1} u_2)_{\mathfrak{h}}}. \tag{8.20}$$

□

Definition 8.2 *We denote by $(\mathcal{E}_{\mathrm{q}}, \mathrm{i}, (\,.\,,\,.\,))$ the completion of $(\mathcal{E}, \mathrm{i})$ for the scalar product $(\,.\,,\,.\,)$.*

Proposition 8.3 *The space $\mathcal{E}_{\rm q}$ is equal to the space $\mathfrak{h}_{\frac{1}{2}} \oplus \mathfrak{h}_{-\frac{1}{2}}$ equipped with the complex structure*

$$\mathrm{i} = \begin{pmatrix} 0 & -\epsilon^{-1} \\ \epsilon & 0 \end{pmatrix}$$

and the scalar product $(g, f) = \mathrm{Re}(g_1, \epsilon f_1)_{\mathfrak{h}} + \mathrm{Re}(g_2, \epsilon^{-1} f_2)_{\mathfrak{h}} + \mathrm{i}(\mathrm{Re}(g_1, f_2)_{\mathfrak{h}} - \mathrm{Re}(g_2, f_1)_{\mathfrak{h}})$.

Standard form of the complex Klein-Gordon field

It is convenient to introduce the map

$$U_{\rm q}(f_1, f_2) := \frac{1}{\sqrt{2}}(\epsilon^{\frac{1}{2}} f_1 + \mathrm{i}\epsilon^{-\frac{1}{2}} f_2, \epsilon^{\frac{1}{2}} \overline{f}_1 + \mathrm{i}\epsilon^{-\frac{1}{2}} \overline{f}_2) =: (u_1, u_2).$$

Using (8.20) we obtain that $U_{\rm q}$ extends to a unitary map

$$U_{\rm q}\colon (\mathcal{E}_{\rm q}, \mathrm{i}, (\cdot, \cdot)) \to (\mathfrak{h}, \mathrm{i}) \oplus (\mathfrak{h}, \mathrm{i}).$$

Let us describe the various objects after conjugation by $U_{\rm q}$. We will denote by the same letter an object acting on $\mathcal{E}_{\rm q}$ and its conjugation by $U_{\rm q}$ acting on $\mathfrak{h} \oplus \mathfrak{h}$.

– *symmetric form*: after conjugation by $U_{\rm q}$ the symmetric form $q(g, f)$ becomes

$$q((v_1, v_2), (u_1, u_2)) = (v_1, u_1) - (u_2, v_2).$$

– *'charge' complex structure*: after conjugation by $U_{\rm q}$ the complex structure j becomes

$$\mathrm{j} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

- *Hamiltonian*: the infinitesimal generator of $\mathbb{R} \ni t \mapsto \mathrm{e}^{-\mathrm{j}tL}$ on $(\mathcal{E}_{\mathrm{q}}, \mathrm{i}, (\,.\,,\,.\,))$ is the *Hamiltonian*, denoted by h. After conjugation by U_{q},

$$\mathrm{h} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}.$$

  In particular h is positive.
- *Gauge transformations:* the infinitesimal generator of $[0, 2\pi] \ni \alpha \mapsto \mathrm{e}^{-\mathrm{j}\alpha}$ on $(\mathcal{E}_{\mathrm{q}}, \mathrm{i}, (\,.\,,\,.\,))$ is the *charge operator* q. After conjugation by U_{q},

$$\mathrm{q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  We have $\mathrm{q} = -\mathrm{ij}$. Hence q is a charge operator in the sense of Subsection 2.4.
- *Time reversal:* we have $\theta(f_1, f_2) = (\overline{f}_1, -\overline{f}_2)$, and after conjugation by U_{q},

$$\theta(u_1, u_2) = (\overline{u}_1, \overline{u}_2).$$

- *charge conjugation:* we have $\mathrm{c}(f_1, f_2) = (\overline{f}_1, \overline{f}_2)$, and after conjugation by U_{q},

$$\mathrm{c}(u_1, u_2) = (u_2, u_1).$$

  We see that $(\mathcal{E}_{\mathrm{q}}, \mathrm{j}, \mathrm{i}, q, \mathrm{c})$ is a charge-symmetric Kähler space.

From now on we will set $X := \mathfrak{h} \oplus \mathfrak{h}$ with elements $x = (x^+, x^-)$ and equip X with the complex structures

$$\mathrm{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \text{ and } \mathrm{j} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},$$

with the symmetric form and the scalar product

$$q(y,x) = (y^+, x^+) - (x^-, y^-) \text{ and } (y,x) := (y^+, x^+) + (y^-, x^-),$$

the Hamiltonian and the charge operator

$$\mathrm{h} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \text{ and } \mathrm{q} = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix},$$

and the time-reversal and the charge conjugation

$$\theta(x^+, x^-) = (\overline{x^+}, \overline{x^-}) \text{ and } \mathrm{c}(x^+, x^-) = (x^-, x^+).$$

From the discussion above we obtain the following theorem.

Theorem 8.4 *The map $U_{\mathrm{q}}: (\mathcal{E}_{\mathrm{q}}, \mathrm{j}, \mathrm{i}, q, \mathrm{c}) \to (X, \mathrm{j}, \mathrm{i}, q, \mathrm{c})$ is unitary between $(\mathcal{E}_{\mathrm{q}}, \mathrm{i}, (\,.\,,\,.\,))$ and $(X, \mathrm{i}, (\,.\,,\,.\,))$, and isometric between $(\mathcal{E}_{\mathrm{q}}, \mathrm{j}, q)$ and (X, j, q). It satisfies*

$$U_{\mathrm{q}} a U_{\mathrm{q}}^{-1} = a \textit{ for } a = \mathrm{h},\ \mathrm{q},\ \mathrm{t},\ \mathrm{c}.$$

For later use we set $\kappa := \theta\mathrm{c}$ and $X_\kappa := \{x \in X | \kappa x = x\} = \{(x^+, \overline{x}^+),\, x^+ \in \mathfrak{h}\}$. Note that in terms of solutions of (KG) we have $\kappa\Phi(t,x) = \Phi(-t,x)$ and an element of X_κ corresponds to a solution of (KG) with Cauchy data $(u, 0)$, where $u \in \mathfrak{h}_{\frac{1}{2}}$.

We see that κ is a conjugation on $(X, \mathrm{i}, (\,.\,,\,.\,))$ and hence $\mathrm{Im}(\,.\,,\,.\,)$ vanishes on X_κ. Since $[\kappa, \mathrm{j}] = 0$, the vector space X_κ is a complex vector space for the complex structure j.

For comparison with the physics literature, let us consider the case $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}x)$ and $\epsilon = (-\Delta_x + m^2)^{\frac{1}{2}}$. Then $\mathfrak{h}_{-\frac{1}{2}}$ is the Sobolev space $H^{-\frac{1}{2}}(\mathbb{R}^d)$. In the physics literature one defines for $u \in C_0^\infty(\mathbb{R}^d)$ the time-zero field $\phi_{\mathrm{p}}(u)$ to be the Hermitian field associated with the solution of (KG) with Cauchy data $(\frac{1}{2\pi}\epsilon^{-1}u, 0)$.

After the unitary transformation U_{q}, $(\frac{1}{2\pi}\epsilon^{-1}u, 0)$ becomes the element

$$\frac{1}{\sqrt{2}2\pi}(\epsilon^{-\frac{1}{2}}u, \epsilon^{-\frac{1}{2}}\overline{u}) \in L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d),$$

i.e.,

$$\phi_{\mathrm{p}}(u) = \frac{1}{\sqrt{2}2\pi}\phi(\epsilon^{-\frac{1}{2}}u, \epsilon^{-\frac{1}{2}}\overline{u}).$$

In the physics litterature one also considers the *complex time-zero field* $\varphi_{\mathrm{p}}(u)$ defined as $\phi_{\mathrm{p}}(u) + \mathrm{i}\phi_{\mathrm{p}}(\mathrm{i}u)$, i.e.,

$$\varphi_{\mathrm{p}}(u) = \frac{1}{2\pi}\varphi(\epsilon^{-\frac{1}{2}}u, \epsilon^{-\frac{1}{2}}\overline{u}).$$

8.2 The real Klein-Gordon field

We now quickly discuss the real Klein-Gordon field.

Abstract real Klein-Gordon equation

Let $\mathfrak{h}_{\mathbb{R}}$ be a real Hilbert space. Let $\epsilon \geq m > 0$ be a selfadjoint operator on $\mathfrak{h}_{\mathbb{R}}$. We consider the Klein-Gordon equation:

$$\partial_t^2\Phi(t) + \epsilon^2\Phi(t) = 0,$$

where Φ is a function of $t \in \mathbb{R}$ with values in $\mathfrak{h}_{\mathbb{R}}$. The real Klein-Gordon equation describes a classical field of scalar neutral particles.

Let us denote by $\mathfrak{h} := \mathbb{C}\mathfrak{h}_{\mathbb{R}}$ the complexification of $\mathfrak{h}_{\mathbb{R}}$ with its canonical scalar product $(\cdot,\cdot)_{\mathfrak{h}}$. The space $\mathfrak{h}$ is equipped with the canonical conjugation $\mathfrak{h} \ni \Phi \mapsto \overline{\Phi}$, $\Phi \in \mathfrak{h}$.

On the space of real solutions of the Klein-Gordon equation, the charge conjugation c acts as identity and the time-reversal θ takes the form $\theta\colon \Phi(t) \mapsto \Phi(-t)$. We will still denote by ϵ the complexification of ϵ acting on $\mathfrak{h}$. We can now apply the results of Subsection 8.1 to the Hilbert space $\mathfrak{h}$.

The real energy space is $\mathcal{E}_{\mathbb{R}} := \mathcal{E} \cap \mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}$. The image of $\mathcal{E}_{\mathbb{R}}$ under the transformation U is

$$U\mathcal{E}_{\mathbb{R}} =: \mathcal{S}_{\mathbb{R}} = \{(u_1, u_2) \in \mathfrak{h} \oplus \mathfrak{h} | u_2 = \overline{u_1}\}.$$

Note that $\mathrm{e}^{-\mathrm{j}tL}$ preserves $\mathcal{E}_{\mathbb{R}}$. More general, if $F\colon \mathbb{R} \to \mathbb{C}$ is a bounded measurable function such that $\overline{F}(\lambda) = F(-\lambda)$ then $F(L)$ preserves $\mathcal{E}_{\mathbb{R}}$. Therefore i preserves $\mathcal{E}_{\mathbb{R}}$ and hence defines a complex structure on $\mathcal{E}_{\mathbb{R}}$. The space $(\mathcal{E}_{\mathbb{R}}, \mathrm{i}, q)$ is a Kähler space.

Definition 8.5 *We denote by* $(\mathcal{E}_{\mathrm{q},\mathbb{R}}, \mathrm{i}, (\,.\,,\,.\,))$ *the closure of* $(\mathcal{E}_{\mathbb{R}}, \mathrm{i})$ *for the scalar product* $(\,.\,,\,.\,)$.

Proposition 8.6 *The space* $\mathcal{E}_{\mathrm{q},\mathbb{R}}$ *is equal to* $\mathfrak{h}_{\frac{1}{2},\mathbb{R}} \oplus \mathfrak{h}_{-\frac{1}{2},\mathbb{R}}$ *equipped with the complex structure*

$$\mathrm{i} = \begin{pmatrix} 0 & -\epsilon^{-1} \\ \epsilon & 0 \end{pmatrix}$$

and the scalar product $(g, f) = (g_1, \epsilon f_1)_{\mathfrak{h}} + (g_2, \epsilon^{-1} f_2)_{\mathfrak{h}} + \mathrm{i}((g_1, f_2)_{\mathfrak{h}} - (g_2, f_1)_{\mathfrak{h}})$.

Standard form of the real Klein-Gordon field

We set

$$\begin{array}{rccc} U_{\mathbb{R}}\colon & \mathcal{E}_{\mathbb{R}} & \to & \mathfrak{h} \\ & f & \mapsto & (\epsilon^{\frac{1}{2}} f_1 + \mathrm{i}\epsilon^{-\frac{1}{2}} f_2). \end{array}$$

Then $U_{\mathbb{R}}$ extends to a unitary map between $(\mathcal{E}_{\mathrm{q},\mathbb{R}}, \mathrm{i}, (\,.\,,\,.\,))$ and $\mathfrak{h}$. Let us describe the various objects after conjugation by $U_{\mathbb{R}}$:

- *Hamiltonian*: The infinitesimal generator of $\mathbb{R} \ni t \mapsto \mathrm{e}^{-\mathrm{j}tL}$ on $(\mathcal{E}_{\mathrm{q},\mathbb{R}}, \mathrm{i}, (\cdot,\cdot))$ is the *Hamiltonian* denoted by h. After conjugation by $U_{\mathbb{R}}$,

$$\mathrm{h} = \epsilon.$$

  In particular, h is positive.
- *Time reversal:* We have $\theta(f_1, f_2) = (f_1, -f_2)$. After conjugation by $U_{\mathbb{R}}$, one finds $\theta u_1 = \overline{u}_1$.

From the discussion above we obtain the following theorem.

Theorem 8.7 *There exist a map* $U_{\mathbb{R}}$ *between* $(\mathcal{E}_{\mathrm{q},\mathbb{R}}, \mathrm{i}, q, \theta)$ *and* $(\mathfrak{h}, \mathrm{j}, q, \theta)$ *which is unitary between* $(\mathcal{E}_{\mathrm{q},\mathbb{R}}, \mathrm{i}, (\,.\,,\,.\,))$ *and* $(\mathfrak{h}, \mathrm{j}, (\,.\,,\,.\,))$, *and satisfies*

$$U_{\mathrm{q},\mathbb{R}} a U_{\mathrm{q},\mathbb{R}}^{-1} = a \text{ for } a = \mathrm{h},\ \mathrm{t}.$$

For later use we set $\kappa := \theta$ and $\mathfrak{h}_{\kappa} := \{h \in \mathfrak{h} \mid h = \overline{h}\}$.

We can now apply the results of Section 4 to the real and complex Klein-Gordon fields.

In the complex case we set $X = \mathfrak{h} \oplus \mathfrak{h}$, $\mathrm{h} = \epsilon \oplus \epsilon$, $\mathrm{q} = \mathbb{1} \oplus -\mathbb{1}$ and introduce for $|\mu| < m$ the state $\omega_{\beta,\mu}$ on $\mathfrak{W}(X)$ defined by the functional

$$\omega_{\beta,\mu}(W(x)) := \mathrm{e}^{-\frac{1}{4}(x,(1+2\rho)x)},\ x \in X,$$

where $\rho = (\mathrm{e}^{\beta \mathrm{a}} - 1)^{-1}$ and $\mathrm{a} = \mathrm{h} - \mu \mathrm{q}$. As recalled in Section 4, $\omega_{\beta,\mu}$ is a (τ, β)-KMS state for the dynamics $\tau_t(W(x)) = W(\mathrm{e}^{\mathrm{i}t\mathrm{a}}x)$, which is invariant under the gauge transformations $\alpha_t(W(x)) = W(\mathrm{e}^{\mathrm{i}t\mathrm{q}}x)$. For $\mu = 0$ the state $\omega_{\beta,\mu}$ will be denoted by ω_β.

In the real case we set $X = \mathfrak{h}$, $\mathrm{h} = \epsilon$ and consider the state on $\mathfrak{W}(X)$ defined by the functional

$$\omega_\beta(W(x)) := \mathrm{e}^{-\frac{1}{4}(x,(1+2\rho)x)},\ x \in X,$$

where $\rho = (\mathrm{e}^{-\beta\epsilon} - 1)^{-1}$. It is a (τ, β)-KMS state for the dynamics $\tau_t(W(x)) = W(\mathrm{e}^{\mathrm{i}t\epsilon}x)$.

In both cases we denote by $\mathcal{F}$ and $\mathcal{U}$ the algebras defined in Subsection 4.3; note that $\mathcal{U}$ is defined w.r.t. the appropriate conjugation κ.

Applying Theorem 4.5 we obtain that the KMS system $(\mathcal{F}, \mathcal{U}, \tau, \omega_\beta)$ is stochastically positive both for real and complex Klein-Gordon fields. Moreover, by Lemma 4.6 and Theorem 6.10, the stochastic process associated to $(\mathcal{F}, \mathcal{U}, \tau, \omega_\beta)$ satisfies the Markov property.

In the next lemma we show that for $\mu \neq 0$, the KMS system $(\mathcal{F}, \mathcal{U}, \tau, \omega_{\beta,\mu})$ is *not* stochastically positive. The same is true, if we restrict the KMS state $\omega_{\beta,\mu}$ to gauge invariant observables (see Subsection 4.4).

The physical reason for this fact is that a system of charged particles is only invariant under the combination of time reversal and charge conjugation. A nonzero chemical potential introduces a disymmetry between particles of positive and negative charge and hences breaks time reversal invariance, which is a necessary property shared by all stochastically positive KMS systems, as we have seen in Proposition 3.4.

Lemma 8.8 *For $\mu \neq 0$ the KMS systems $(\mathcal{F}, \mathcal{U}, \tau, \omega_{\beta,\mu})$and $(\mathcal{A}, \mathcal{A}_\kappa, \tau, \omega_{\beta,\mu})$ are not stochastically positive.*

Proof. Using the results of Subsection 2.4 we have:

$$\varphi_\omega(x) = a_\omega(x^+) + a_\omega^*(x^-),\ \varphi_\omega^*(x) = a_\omega^*(x^+) + a_\omega(x^-),$$

which, by an easy computation using the results recalled in Subsection 4.2, implies

$$\begin{aligned}\varphi_\omega^*(x)\varphi_\omega(x)\Omega_{\beta,\mu} &= a_F^*((1+\rho)^{\frac{1}{2}}x^+ \oplus \overline{\rho}^{\frac{1}{2}}\overline{x^-})a_F^*((1+\rho)^{\frac{1}{2}}x^- \oplus \overline{\rho}^{\frac{1}{2}}\overline{x^+})\Omega_{\beta,\mu} \\ &\quad +((x^-, (1+\rho)x^-) + (x^+, \rho x^+))\Omega_{\beta,\mu}.\end{aligned}$$

Set $H = \mathrm{d}\Gamma(\mathrm{h} \oplus -\mathrm{h})$ and $Q = \mathrm{d}\Gamma(\mathrm{q} \oplus -\mathrm{q})$, so that $L = H - \mu Q$. Then

$$\begin{aligned}&\mathrm{e}^{-sL}\varphi_\omega^*(x)\varphi_\omega(x)\Omega_{\beta,\mu} = \mathrm{e}^{-sH}\varphi_\omega^*(x)\varphi_\omega(x)\Omega_{\beta,\mu} \\ &= a_F^*((1+\rho)^{\frac{1}{2}}\mathrm{e}^{-s\mathrm{h}}x^+ \oplus \overline{\rho}^{\frac{1}{2}}\mathrm{e}^{s\overline{\mathrm{h}}}\overline{x^-})a_F^*((1+\rho)^{\frac{1}{2}}\mathrm{e}^{-s\mathrm{h}}x^- \oplus \overline{\rho}^{\frac{1}{2}}\mathrm{e}^{s\overline{\mathrm{h}}}\overline{x^+})\Omega_{\beta,\mu} \\ &\quad +((x^-, (1+\rho)x^-) + (x^+, \rho x^+))\Omega_{\beta,\mu}.\end{aligned}$$

Thus, for $x, y \in X$,

$$\begin{aligned}&(\varphi^*(y)\varphi(y)\Omega_{\beta,\mu}, \mathrm{e}^{-sL}\varphi^*(x)\varphi(x)\Omega_{\beta,\mu}) \\ =&((1+\rho)^{\frac{1}{2}}y^+ \oplus \overline{\rho}^{\frac{1}{2}}\overline{y^-}, (1+\rho)^{\frac{1}{2}}\mathrm{e}^{-s\mathrm{h}}x^+ \oplus \overline{\rho}^{\frac{1}{2}}\mathrm{e}^{s\overline{\mathrm{h}}}\overline{x^-}) \\ &\times((1+\rho)^{\frac{1}{2}}y^- \oplus \overline{\rho}^{\frac{1}{2}}\overline{y^+}, (1+\rho)^{\frac{1}{2}}\mathrm{e}^{-s\mathrm{h}}x^- \oplus \overline{\rho}^{\frac{1}{2}}\mathrm{e}^{s\overline{\mathrm{h}}}\overline{x^+}) \\ &+((x^-, (1+\rho)x^-) + (x^+, \rho x^+))((y^-, (1+\rho)y^-) + (y^+, \rho y^+)).\end{aligned}$$

Let us now restrict ourselves to $x, y \in X_\kappa$, i.e., $x = (u, \overline{u})$, $y = (v, \overline{v})$, $u, v \in \mathfrak{h}$. We obtain $x^+ = u$, $x^- = \overline{u}$, $y^+ = v$ and $y^- = \overline{v}$. If we set $\rho^\pm = (\mathrm{e}^{\beta(\epsilon\mp\mu)} - 1)^{-1}$, then

$$\begin{aligned}&(\varphi^*(y)\varphi(y)\Omega_{\beta,\mu}\,,\ \tau_t(\varphi^*(x)\varphi(x))\Omega_{\beta,\mu})_{|t=is} \\ &= (v, (\mathrm{e}^{-s\epsilon}(1+\rho^+) + \mathrm{e}^{s\epsilon}\rho^-)u) \times (u, (\mathrm{e}^{-s\epsilon}(1+\rho^-) + \mathrm{e}^{s\epsilon}\rho^+)v) \\ &\quad +(u, (1+\rho^+ + \rho^-)u)(v, (1+\rho^+ + \rho^-)v)).\end{aligned}$$

This quantity is not real if $s \neq 0$ and $\mu \neq 0$. Since $\varphi_\omega^*(x)\varphi_\omega(x)$ is a positive operator affiliated to $\mathcal{A}_\kappa$ this shows that the KMS systems $(\mathcal{F}, \mathcal{U}, \tau, \omega_{\beta,\mu})$ and $(\mathcal{A}, \mathcal{A}_\kappa, \tau, \omega_{\beta,\mu})$ are not stochastically positive □.

9 Scalar quantum fields at positive temperature with spatially cutoff interactions

In this section we present the main results of this paper, namely the construction of scalar quantum fields at positive temperature in one space dimension with spatially cutoff interactions. For the real scalar quantum field the two kinds of interactions that we will consider are the spatially cutoff $P(\phi)_2$ and ${\rm e}^{\alpha\phi}{}_2$ models (the later one is known as the Høegh-Krohn model). The first model is specified by the formal interaction $\int g({\rm x})P(\phi(x)){\rm dx}$, where $P(\lambda)$ is a real polynomial, which is bounded from below. The second model is specified by $\int g({\rm x}){\rm e}^{\alpha\phi({\rm x})}{\rm dx}$ for $|\alpha| < \sqrt{2\pi}$. In both cases g is a positive function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

For the complex scalar field we will consider the spatially cutoff $P(\varphi^*\varphi)_2$ interaction, specified by the formal interaction term $\int g({\rm x})P(\varphi^*({\rm x})\varphi({\rm x})){\rm dx}$.

9.1 Some preparations

In this subsection we prove some auxiliary results, which we will need to prove some properties of the interaction terms later on. We first recall a result of Klein and Landau [KL1].

Lemma 9.1 *Let $(\mathcal{F}, \mathcal{U}, \tau, \omega)$ be a stochastically positive KMS system and let $\mathcal{H}_1$ be the closure of $\mathcal{U}\Omega$. Let $\mathcal{U}_1 := \mathcal{U}_{|\mathcal{H}_1}$. Then Ω is a cyclic and separating vector for $\mathcal{U}_1$, and $\mathcal{U}_1$ and $\mathcal{U}$ are isomorphic as C^*-algebras.*

Lemma 9.2 *Let $(\mathcal{F}, \mathcal{U}, \tau, \omega)$ be the stochastically positive KMS system introduced in Section 4.5. Let X_ρ be the vector space X equipped with the scalar product $(x, x)_\rho = (x, (1 + 2\rho)x)$ and set*

$$
\begin{array}{rccc}
j\colon & X_\rho & \to & X \oplus \overline{X} \\
& x & \mapsto & (1+\rho)^{\frac{1}{2}}x \oplus \overline{\rho}^{\frac{1}{2}}\overline{\kappa x}.
\end{array}
$$

Then

(i) $\Gamma(j)$ *is an isometry from* $\Gamma(X_\rho)$ *into* $\Gamma(X \oplus \overline{X})$ *such that*

$$\Gamma(j){\rm e}^{{\rm i}\phi(x)} = W_\omega(x)\Gamma(j),\ x \in X_\kappa;$$

(ii) $\mathcal{H}_1 = \Gamma(j)\Gamma(X_\rho) \equiv L^2(Q, \Sigma_0, \mu)$.

Proof. The map $x \to \overline{\kappa x}$ is $\mathbb{C}$-linear from X to $\overline{X}$, hence j is $\mathbb{C}$-linear. From the results recalled in Subsection 4.2 and the functional properties of $\Gamma(j)$ we obtain that $\Gamma(j)\mathrm{e}^{\mathrm{i}\phi(x)} = W_F(jx)\Gamma(j)$. Now $W_F(jx) = W_\omega(x)$ for $x \in X_\kappa$, and this proves (i).

Let us now prove (ii). The fact that $\mathcal{H}_1$ is isomorphic to $L^2(Q, \Sigma_0, \mu)$ follows from the definition of $\mathcal{U}$ in Subsection 6.3. To prove the second equality, we note that κ extends to a conjugation on X_ρ, since $[\kappa, \rho] = 0$. By a well-known result on Fock spaces, which we already recalled in the proof of Lemma 4.6, the vacuum vector $\Omega \in \Gamma(X_\rho)$ is cyclic for $\{W(x) \mid x \in X_\rho, \kappa x = x\}$.

Let now $u \in \Gamma(X_\rho)$. Because of the result recalled above we find

$$u = \lim_{n\to\infty} u_n,\ u_n = \sum_1^N \lambda_j W(x_j)\Omega,\ x_j \in X_\rho,\ \kappa x_j = x_j.$$

It follows that

$$\Gamma(j)u = \lim_{n\to\infty} v_n,\ v_n = \sum_1^N \lambda_j W_\omega(x_j)\Omega.$$

Since $v_n \in \mathcal{U}\Omega$ we have $\Gamma(j)u \in \mathcal{H}_1$ and hence $\Gamma(j)\Gamma(X_\rho) \subset \mathcal{H}_1$. Let us now prove the converse inclusion: let $v \in \mathcal{H}_1$ with

$$v = \lim_{n\to\infty} v_n,\ v_n = \sum_1^N \lambda_j W_\omega(x_j)\Omega,\ x_j \in X,\ \kappa x_j = x_j.$$

Then

$$v_n = \Gamma(j)u_n \text{ for } u_n = \sum_1^N \lambda_j W(x_j)\Omega.$$

Since $\Gamma(j)$ is isometric, $u_n \to u \in \Gamma(X_\rho)$ and $v = \Gamma(j)u$. This shows that $\mathcal{H}_1 \subset \Gamma(j)\Gamma(X_\rho)$ $\Box$.

9.2 Wick ordering

We recall some well known facts concerning the Wick ordering of Gaussian random variables. Let (Q, Σ_0, μ) be a probability space, F a real vector space equipped with a positive quadratic form $f \mapsto c(f, f)$, called a *covariance*. Let $F \ni f \mapsto \phi(f)$ be a $\mathbb{R}$-linear map from F to the space of real measurable functions on Q.

The *Wick ordering* $:\phi(f)^n:$ with respect to the covariance c is defined using a generating series:

$$:\mathrm{e}^{\alpha\phi(f)}:_c := \sum_0^\infty \frac{\alpha^n}{n!} :\phi(f)^n:_c = \mathrm{e}^{\alpha\phi(f)}\mathrm{e}^{-\frac{\alpha^2}{2}c(f,f)}. \tag{9.21}$$

Thus

$$:\phi(f)^n:_c = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m!)}\phi(f)^{n-2m}\Big(-\frac{1}{2}c(f,f)\Big)^m. \tag{9.22}$$

If now c_1, c_2 are two covariances on F, then

$$:\mathrm{e}^{\alpha\phi(f)}:_{c_2} = :\mathrm{e}^{\alpha\phi(f)}:_{c_1} \mathrm{e}^{-\frac{\alpha^2}{2}(c_2-c_1)(f,f)}. \tag{9.23}$$

This implies the following *Wick reordering identities* (see e.g. [GJ]):

$$:\phi(f)^n:_{c_2} = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m!)} :\phi(f)^{n-2m}:_{c_1} \Big(-\frac{1}{2}(c_2-c_1)(f,f)\Big)^m. \tag{9.24}$$

9.3 The spatially cutoff $P(\phi)_2$ interaction

We recall from Section 8.2 that the real Klein-Gordon field in one space dimension is described by the Weyl algebra $\mathfrak{W}(\mathfrak{h})$, where $\mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}k)$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a real cutoff function with $\int_{\mathbb{R}} \chi(\mathrm{x})\mathrm{dx} = 1$. For $\mathrm{x} \in \mathbb{R}$ and $\Lambda \in [1, +\infty[$ an ultraviolet cutoff parameter, we define $f_{\Lambda,\mathrm{x}} \in \mathfrak{h}$ by

$$f_{\Lambda,\mathrm{x}}(k) := \frac{1}{(4\pi)^{\frac{1}{2}}}\mathrm{e}^{-\mathrm{i}k.\mathrm{x}}\hat{\chi}\Big(\frac{k}{\Lambda}\Big)\epsilon(k)^{-\frac{1}{2}}.$$

We set

$$\phi_\Lambda(\mathrm{x}) := \sqrt{2}\phi_\omega(f_{\Lambda,\mathrm{x}}) = a_\omega^*(f_{\Lambda,\mathrm{x}}) + a_\omega(f_{\Lambda,\mathrm{x}}),\ \mathrm{x} \in \mathbb{R}.$$

Note that $f_{\Lambda,\mathrm{x}} \in \mathfrak{h}_\kappa$, so $\phi_\Lambda(\mathrm{x})$ is affiliated to $\mathcal{U}$; i.e., $\phi_\Lambda(\mathrm{x})$ can be considered as a measurable function on (Q, Σ_0, μ).

In order to define the spatially cut-off $P(\phi)_2$ interaction we fix a real polynomial of degree $2n$, which is bounded from below, namely

$$P(\lambda) = \sum_{j=0}^{2n} a_j\lambda^j \text{ with } a_{2n} > 0, \tag{9.25}$$

and a real function $g \in L^1_{\mathbb{R}}(\mathbb{R}, \mathrm{d}x) \cap L^2(\mathbb{R}, \mathrm{d}x)$ with $g \geq 0$.

We set

$$V_\Lambda = \int g(\mathrm{x}) :P(\phi_\Lambda(\mathrm{x})):_0 \mathrm{dx},$$

where $: :_0$ denotes the Wick ordering with respect to the covariance at temperature 0 given by $c_0(f,f) = \frac{1}{2}(f,f)_{\mathfrak{h}}$.

For technical reasons we will also need to consider similar UV cutoff interactions with the Wick ordering done with respect to the covariance at inverse temperature β given by $c_\beta(f,f) = \frac{1}{2}(f,f)_\rho = \frac{1}{2}(f,(1+2\rho)f)$, $f \in \mathfrak{h}$. We set

$$V_{\Lambda,\beta} = \int g(\mathrm{x}) :P(\phi_\Lambda(\mathrm{x})):_\beta \mathrm{dx},$$

where $::_\beta$ denotes Wick ordering with respect to c_β. Note that V_Λ and $V_{\Lambda,\beta}$ are affiliated to $\mathcal{U}$. We first collect some properties of these auxiliary interactions.

Lemma 9.3 *The family $\{V_{\Lambda,\beta}\}$ is Cauchy in all spaces $L^p(Q,\Sigma_0,\mu)$ for $1 \leq p < \infty$ and converges when $\Lambda \to \infty$ to a function $V_\beta \in L^p(Q,\Sigma_0,\mu)$, $1 \leq p < \infty$, which satisfies $\mathrm{e}^{-tV_\beta} \in L^1(Q,\Sigma_0,\mu)$ for all $t > 0$. We set*

$$V_\beta =: \int g(\mathrm{x}) :P(\phi(\mathrm{x})):_\beta \mathrm{dx}.$$

Proof. We use the identification of $L^2(Q,\Sigma_0,\mu)$ with $\Gamma(\mathfrak{h}_\rho)$ presented in Lemma 9.2. Then Wick ordering with respect to c_β coincides with Wick ordering with respect to the Fock vacuum on $\Gamma(\mathfrak{h}_\rho)$. By exactly the same arguments as those used in the 0-temperature case (see e.g. [S-H.K] or [DG, Sect. 6] for a recent survey) we obtain that, for $0 \leq p \leq 2n$, the cuttoff interaction $V_{\Lambda,\beta}$ is a linear combination of Wick monomials of the form

$$\sum_{r=0}^{p} \binom{p}{r} \int w_{p,\Lambda}(k_1,\dots,k_r,k_{r+1},\dots,k_p) a^*(k_1)\cdots a^*(k_r) a(-k_{r+1})\cdots a(-k_p) \mathrm{d}k_1\cdots \mathrm{d}k_p,$$

where

$$w_{p,\Lambda}(k_1,\cdots,k_p) = \hat{g}(\sum_1^p k_i) \prod_1^p \hat{\chi}\Big(\frac{k_i}{\Lambda}\Big)\epsilon(k_i)^{-\frac{1}{2}}.$$

Recalling that $1+2\rho = \frac{1+\mathrm{e}^{-\beta\epsilon}}{1-\mathrm{e}^{-\beta\epsilon}}$ we see that

$$w_{p,\Lambda} \in \otimes^p \mathfrak{h}_\rho = L^2\Big(\mathbb{R}^p, \prod_1^p \frac{1+\mathrm{e}^{-\beta\epsilon(k_i)}}{1-\mathrm{e}^{-\beta\epsilon(k_i)}} \mathrm{d}k_1 \ldots, \mathrm{d}k_p\Big).$$

The sequence $\{w_{p,\Lambda}\}$ is Cauchy in this space. Consequently $w_{p,\Lambda} \to w_{p,\infty}$ when $\Lambda \to \infty$, where

$$w_{p,\infty}(k_1, \cdots, k_p) = \hat{g}(\sum_1^p k_i) \prod_1^p \epsilon(k_i)^{-\frac{1}{2}}.$$

We can now apply these Wick monomials to the Fock vacuum and conclude that $V_{\Lambda,\beta}\Omega$ converges to a vector $V_\beta\Omega$ in $\Gamma(\mathfrak{h}_\rho)$, or equivalently that $V_{\Lambda,\beta}$ converges to V_β in $L^2(Q, \Sigma_0, \mu)$. Since $V_{\lambda,\beta}\Omega$ is a finite particle vector, it follows from a standard argument (see e.g. [Si2, Thm. 1.22] or [DG, Lemma 5.12]) that $V_{\Lambda,\beta} \to V_\beta \in L^p(Q, \Sigma_0, \mu)$ for all $1 \leq p < \infty$.

We will now prove that $\mathrm{e}^{-tV_\beta} \in L^1(Q, \Sigma_0, \mu)$. We argue as in the 0-temperature case: we first verify that $\|w_{p,\Lambda} - w_{p,\infty}\| \leq C\Lambda^{-\epsilon_0}$ for some $\epsilon_0 > 0$ and therefore $\|V_{\Lambda,\beta} - V_\beta\|_{L^2(Q,\Sigma_0,\mu)} \leq C\Lambda^{-\epsilon_0}$. Applying again [DG, Lemma 5.12] we find

$$\|V_{\Lambda,\beta} - V_\beta\|_{L^p(Q,\Sigma_0,\mu)} \leq C(p-1)^n \Lambda^{-\epsilon_0},\ p > 1. \tag{9.26}$$

Using the Wick ordering identities (9.22) we obtain as identities between functions on K (see, e.g., [DG, Lemma 6.6]):

$$: P(\phi_\Lambda(\mathrm{x})) :_\beta \geq -C(\|\phi_\Lambda(\mathrm{x})\Omega\|^{2n} + 1).$$

Now $\|\phi_\Lambda(\mathrm{x})\Omega\| = C\|\epsilon^{-1}\hat{\chi}\big(\frac{\cdot}{\Lambda}\big)\|_{\mathfrak{h}_\rho} \leq C(\ln(\Lambda))^{\frac{1}{2}}$. This yields

$$V_{\Lambda,\beta} \geq -C\ln(\Lambda)^n. \tag{9.27}$$

Applying now [Si2, Lemma V.5] we deduce from (9.26) and (9.27) that $\mathrm{e}^{-tV_\beta} \in L^1(Q, \Sigma_0, \mu)$ for all $t > 0$ $\Box$.

Proposition 9.4 *The family $\{V_\Lambda\}$ is Cauchy in all spaces $L^p(Q, \Sigma_0, \mu)$ for $1 \leq p < \infty$ and converges when $\Lambda \to \infty$ to a function $V \in L^p(Q, \Sigma_0, \mu)$, $1 \leq p < \infty$, which satisfies $\mathrm{e}^{-tV} \in L^1(Q, \Sigma_0, \mu)$ for all $t > 0$. We set*

$$V =: \int g(\mathrm{x}) : P(\phi(\mathrm{x})) :_0 \mathrm{d}\mathrm{x}.$$

Proof. With the help of the Wick reordering identity (9.24) we find, for $f \in \mathfrak{h}_\kappa$,

$$\begin{aligned}
:P(\phi_\omega(f)):_0 &= \textstyle\sum_{j=0}^{2n} a_j \,:\phi_\omega(f)^n:_0 \\
&= \textstyle\sum_{j=0}^{2n}\sum_{m=0}^{[j/2]} a_j \frac{j!}{m!(j-2m!)} :\phi(f)^{j-2m}:_\beta (-\frac{1}{2}(c_0-c_\beta)(f,f))^m.
\end{aligned}$$

For $f = f_{\Lambda,x}$

$$\begin{aligned}
r_\Lambda &:= (c_\beta - c_0)(f_{\Lambda,x}, f_{\Lambda,x}) = (f_{\Lambda,0}, \rho f_{\Lambda,0}) \\
&= \textstyle\int \mathrm{e}^{-\beta\epsilon(k)}\hat{\chi}(\frac{k}{\Lambda})\mathrm{d}k = r_\infty + O(\Lambda^{-\infty}),
\end{aligned}$$

where $r_\infty = \int \mathrm{e}^{-\beta\epsilon(k)}\mathrm{d}k$.

On the other hand,

$$\int\limits_Q |\phi_\omega(f_{\Lambda,\mathrm{x}})|^p \mathrm{d}\mu \in O(|c_\beta(f_{\Lambda,\mathrm{x}}, f_{\Lambda,\mathrm{x}})|^p) \in O(\ln(\Lambda)^p).$$

Therefore

$$:P(\phi_\Lambda(\mathrm{x})):_0 = :\tilde{P}(\phi_\Lambda(\mathrm{x})):_\beta + O(\ln(\Lambda)^{2n}\Lambda^{-\infty}) \text{ uniformly for } \mathrm{x} \in \operatorname{supp} g,$$

where

$$\tilde{P}(\lambda) = \sum_{j=0}^{2n}\sum_{m=0}^{[j/2]} a_j \frac{j!}{m!(j-2m!)}\lambda^{j-2m}(\frac{1}{2}r_\infty)^m.$$

We see that $\tilde{P}(\lambda) - P(\lambda)$ is of degree less than $2n-1$. Applying Lemma 9.3 to $\tilde{P}$ this yields

$$\lim_{\Lambda\to\infty}\int g(\mathrm{x}) :P(\phi_\Lambda(\mathrm{x})):_0 \mathrm{dx} = \lim_{\Lambda\to\infty}\int g(\mathrm{x}) :\tilde{P}(\phi_\Lambda(\mathrm{x})):_\beta \mathrm{dx} = \int g(\mathrm{x}) :\tilde{P}(\phi(\mathrm{x})):_\beta \mathrm{dx},$$

which completes the proof of the proposition □.

9.4 The spatially cutoff $\mathrm{e}^{\alpha\phi}{}_2$ *interaction*

As in Subsection 9.3 we set, for $|\alpha| < \sqrt{2\pi}$,

$$V_\Lambda = \int g(\mathrm{x}) :\mathrm{e}^{\alpha\phi_\Lambda(\mathrm{x})}:_0 \mathrm{dx}$$

and

$$V_{\Lambda,\beta} = \int g(\mathrm{x}) :\mathrm{e}^{\alpha\phi_\Lambda(\mathrm{x})}:_\beta \mathrm{dx}.$$

Note that, as above, V_Λ and $V_{\Lambda,\beta}$ are affiliated to $\mathcal{U}$.

Lemma 9.5 *For $|\alpha| < \sqrt{2\pi}$ the family $\{V_{\Lambda,\beta}\}$ is Cauchy in $L^2(Q,\Sigma_0,\mu)$ and converges when $\Lambda \to \infty$ to a positive function $V_\beta \in L^2(Q,\Sigma_0,\mu)$. We set*

$$V_\beta =: \int g(x) :\mathrm{e}^{\alpha\phi(\mathrm{x})}:_\beta \mathrm{dx}.$$

Proof. The proof is completely similar to the 0-temperature case where $\rho = 0$ (see e.g. [Si2], [H-K2]). For completeness we will give an outline. Note first that by (9.21) $:\mathrm{e}^{\alpha\phi_\Lambda(x)}:_\beta$ is a positive function on Q, hence the same holds for $V_{\Lambda,\beta}$ as $g \geq 0$. We now show that $V_{\Lambda,\beta}$ converges in $L^2(Q,\Sigma_0,\mu)$, and we will identify $V_{\Lambda,\beta}$ with $V_{\Lambda,\beta}\Omega$. We have

$$1\!\!1_{\{n\}}(N)V_{\Lambda,\beta} = \frac{\alpha^n}{n!}\int g(\mathrm{x}) :\phi^n_\Lambda(\mathrm{x}): \Omega\mathrm{dx} = \frac{\alpha^n}{(4\pi)^{n/2}\sqrt{n!}}\hat{g}(\sum_1^n k_i)\prod_1^n \hat{\chi}\Big(\frac{k_i}{\Lambda}\Big)\frac{1}{\epsilon(k_i)^{\frac{1}{2}}}.$$

Hence

$$\begin{aligned}\|1\!\!1_{\{n\}}(N)V_{\Lambda,\beta}\|^2 &= \tfrac{1}{n!}(\tfrac{\alpha^2}{4\pi})^n \int |\hat{g}(\textstyle\sum_1^n k_i)|^2 \prod_1^n |\hat{\chi}(\tfrac{k_i}{\Lambda})|^2 \tfrac{1+2\rho(k_i)}{\epsilon(k_i)}\mathrm{d}k_1\ldots\mathrm{d}k_n\\ &\leq \tfrac{1}{n!}(\tfrac{\alpha^2}{4\pi})^n \int |\hat{g}(\textstyle\sum_1^n k_i)|^2 \prod_1^n \tfrac{1+2\rho(k_i)}{\epsilon(k_i)}\mathrm{d}k_1\ldots\mathrm{d}k_n =: \epsilon_n.\end{aligned}$$

Next we find

$$\epsilon_n = \frac{1}{n!}\Big(\frac{\alpha^2}{2\pi}\Big)^n \int g(\mathrm{x})g(\mathrm{y})K_\beta(\mathrm{x}-\mathrm{y})^n\mathrm{dxdy}$$

for

$$K_\beta(\mathrm{x}) = \frac{1}{2}\int \mathrm{e}^{\mathrm{i}k\mathrm{x}}\frac{1+2\rho(k)}{\epsilon(k)}\mathrm{d}k.$$

We claim now that

$$\mathrm{e}^{\frac{\alpha^2}{2\pi}|K_\beta(\mathrm{x})|} \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \text{ for } |\alpha| < \sqrt{2\pi}. \tag{9.28}$$

This implies that

$$\sum_{n=0}^\infty \epsilon_n \leq \int g(\mathrm{x})g(\mathrm{y})\mathrm{e}^{\frac{\alpha^2}{2\pi}|K_\beta|(\mathrm{x}-\mathrm{y})}\mathrm{dxdy} < \infty. \tag{9.29}$$

If we set

$$K_0(\mathrm{x}) = \frac{1}{2}\int \mathrm{e}^{\mathrm{i}k\mathrm{x}}\frac{1}{\epsilon(k)}\mathrm{d}k,$$

then because of the rapid decay of $\rho(k)$ when $|k|\to\infty$, we have $K_0 - K_\beta \in L^\infty(\mathbb{R})$, and (see [H-K2, equ. (2.4)]) $K_0(\mathrm{x}) \in O(1)$ in $|\mathrm{x}| \geq 1$, $K_0(\mathrm{x}) = -\ln(\mathrm{x}) + O(1)$ in $|\mathrm{x}| \leq 1$. This implies (9.28).

Now by the arguments in the proof of Lemma 9.3, we see that

$$\lim_{\Lambda\to\infty} 1\!\!1_{\{n\}}(N)V_{\Lambda,\beta} = \frac{\alpha^n}{n!}\int g(\mathrm{x}) :\phi(\mathrm{x})^n : \Omega\mathrm{dx}.$$

Since $1\!\!1_{\{n\}}(N)V_{\Lambda,\beta} \to V_n$ in $L^2(Q,\Sigma_0,\mu)$ for each n and $\sup_\Lambda \|1\!\!1_{\{n\}}(N)V_{\Lambda,\beta}\|^2 \leq \epsilon_n$ with $\sum \epsilon_n < \infty$, we see that $V_{\Lambda,\beta}$ converges to some element $V \in L^2(Q,\Sigma_0,\mu)$, which is a.e. positive as a limit of positive functions $\Box$.

Proposition 9.6 *For $|\alpha| < \sqrt{2\pi}$, the family $\{V_\Lambda\}$ is Cauchy in $L^2(Q,\Sigma_0,\mu)$ and converges to a positive function $V \in L^2(Q,\Sigma_0,\mu)$. We set*

$$V =: \int g(x) :\mathrm{e}^{\alpha\phi(\mathrm{x})}:_0 \mathrm{dx}.$$

Proof. By the Wick reordering identity (9.23) we have

$$:\mathrm{e}^{\alpha\phi_{\Lambda,x}}:_0 =: \mathrm{e}^{\alpha\phi_{\Lambda,x}}:_\beta \mathrm{e}^{\frac{\alpha^2}{2}r_\Lambda},$$

Hence $V_\Lambda = \mathrm{e}^{\frac{\alpha^2}{2}r_\Lambda}V_{\Lambda,\beta}$, which implies, using Lemma 9.5, that V_Λ converges in $L^2(Q,\Sigma_0,\mu)$ to the positive function $\mathrm{e}^{\frac{\alpha^2}{2}r_\infty}V_\beta$ $\Box$.

9.5 The spatially cutoff $P(\varphi^*\varphi)_2$ interaction

We consider now the complex Klein-Gordon field in one space dimension which is described by the Weyl algebra $\mathfrak{W}(X)$ for $X = \mathfrak{h}\oplus\mathfrak{h}$, $\mathfrak{h} = L^2(\mathbb{R},\mathrm{d}k)$. We recall that the Gibbs state at inverse temperature β is given by $\omega(W(x)) = \mathrm{e}^{\frac{1}{4}(x,(1+2\rho x))}$, where $\rho = (\mathrm{e}^{\beta\mathrm{h}}-1)^{-1}$ and $\mathrm{h} = \epsilon\oplus\epsilon$.

We set

$$\varphi_\Lambda(\mathrm{x}) = \varphi_\omega(f_{\Lambda,\mathrm{x}}\oplus f_{\Lambda,\mathrm{x}}),\ \varphi^*_\Lambda(\mathrm{x}) = \varphi^*_\omega(f_{\Lambda,\mathrm{x}}\oplus f_{\Lambda,\mathrm{x}}),\ \mathrm{x}\in\mathbb{R}.$$

Note that $f_{\Lambda,\mathrm{x}}$ is invariant under the conjugation $h \to \overline{h}$. This implies that $\varphi_\Lambda(x)$ is affiliated to $\mathcal{U}$, since $f_{\Lambda,\mathrm{x}} \oplus f_{\Lambda,\mathrm{x}} \in X_\kappa$. Moreover, $\varphi^*_\Lambda(\mathrm{x})\varphi_\Lambda(\mathrm{x}) = \frac{1}{2}(\phi^2_\omega(f_{\Lambda,\mathrm{x}} \oplus f_{\Lambda,\mathrm{x}}) + \phi^2_\omega(\mathrm{i}f_{\Lambda,\mathrm{x}} \oplus -\mathrm{i}f_{\Lambda,\mathrm{x}}))$.

For P a real polynomial of degree $2n$, which is bounded from below, and g a positive function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we set

$$V_\Lambda = \int g(\mathrm{x}) : P(\varphi^*_\Lambda(\mathrm{x})\varphi_\Lambda(\mathrm{x})) :_0 \mathrm{dx},$$

where $: :_0$ denotes Wick ordering with respect to the 0-temperature covariance $c_0(x, x) = \frac{1}{2}(x, x)$, and

$$V_{\Lambda,\beta} = \int g(\mathrm{x}) : P(\varphi^*_\Lambda(\mathrm{x})\varphi_\Lambda(\mathrm{x})) :_\beta \mathrm{dx},$$

where $: :_\beta$ denotes Wick ordering with respect to the covariance at inverse temperature β specified by $c_\beta(x, x) = \frac{1}{2}(x, (1+2\rho)x)$. The following two results can be shown by exactly the same methods as in Subsection 9.3.

Lemma 9.7 *The family $\{V_{\Lambda,\beta}\}$ is Cauchy in all $L^p(Q, \Sigma_0, \mu)$ spaces and converges, when $\Lambda \to \infty$, to a function $V_\beta \in L^p(Q, \Sigma_0, \mu)$, $1 \le p < \infty$, which satisfies $\mathrm{e}^{-tV_\beta} \in L^1(Q, \Sigma_0, \mu)$ for all $t > 0$. We set*

$$V_\beta =: \int g(\mathrm{x}) : P(\varphi^*(\mathrm{x})\varphi(\mathrm{x})) :_\beta \mathrm{dx}.$$

Proposition 9.8 *The family $\{V_\Lambda\}$ is Cauchy in all spaces $L^p(Q, \Sigma_0, \mu)$ and converges, when $\Lambda \to \infty$, to a function $V \in L^p(Q, \Sigma_0, \mu)$, $1 \le p < \infty$, which satisfies $\mathrm{e}^{-tV} \in L^1(Q, \Sigma_0, \mu)$ for all $t > 0$. We set*

$$V =: \int g(\mathrm{x}) : P(\varphi^*(\mathrm{x})\varphi(\mathrm{x})) :_0 \mathrm{dx}.$$

9.6 Scalar quantum fields at positive temperature with spatially cutoff interactions

To construct the space-cutoff $P(\phi)_2$ and $\mathrm{e}^{\alpha\phi}{}_2$ models at positive temperature, we apply the general results of Subsection 7.3. Note that by Subsections 9.3 and 9.4, the interactions terms $V = \int g(\mathrm{x}) : P(\phi(\mathrm{x})) :_0 \mathrm{dx}$ and $V = \int g(\mathrm{x}) : \mathrm{e}^{\alpha\phi(\mathrm{x})} :_0 \mathrm{dx}$ for $|\alpha| < \sqrt{2\pi}$ satisfy all the hypotheses of Subsection 7.3. Consequently we obtain the following theorem:

Theorem 9.9 *Let $(\mathcal{W}, \mathcal{W}_\kappa, \tau^\circ, \omega)$ be the quasi-free β-KMS system describing the free neutral Klein-Gordon field in one space dimension at temperature β^{-1},* *described in Subsection 8.3. Let $\mathcal{H}, L, \Omega$ be the associated GNS objects described in Subsection 4.2. Let V be the selfadjoint operator on $\mathcal{H}$ affiliated to $\mathcal{W}_\kappa$ equal either to $\int g(\mathrm{x}) :P(\phi(\mathrm{x})):_0 \mathrm{dx}$ or to $\int g(\mathrm{x}) :\mathrm{e}^{\alpha\phi(\mathrm{x})}:_0 \mathrm{dx}$. Then the following statements hold true:*

(i) $L+V$ *is essentially selfadjoint and* $\Omega \in \mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V})$, *where* $H_V := \overline{L+V}$.

(ii) Let $\tau_V(t)$ be the W^*-dynamics generated by H_V and ω_V be the vector state induced by $\Omega_V = \|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|^{-1}\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega$. Then τ_V is a group of *-automorphisms of $\mathcal{W}$, continuous for the strong operator topology such that $(\mathcal{W}, \mathcal{W}_\kappa, \tau_V, \omega_V)$ is a stochastically positive β-KMS system.

(iii) The generalized path space associated to $(\mathcal{W}, \mathcal{W}_\kappa, \tau_V, \omega_V)$ satisfies the Markov property.

(iv) Let L_V, J_V be the perturbed Liouvillean and modular conjugation associated to $(\mathcal{W}, \mathcal{W}_\kappa, \tau_V, \omega_V)$. Then $J_V = J$ and $L_V = \overline{H_V - JVJ}$.

Finally we state the corresponding result for the charged Klein-Gordon field:

Theorem 9.10 *Let $(\mathcal{W}, \mathcal{W}_\kappa, \tau^\circ, \omega)$ be the quasi-free β-KMS system describing the free charged Klein-Gordon field in one space dimension at temperature β^{-1} and zero chemical potential, described in Subsection 8.3. Let $\mathcal{H}, L, \Omega$ be the associated GNS objects described in Subsection 4.2. Let V be the selfadjoint operator on $\mathcal{H}$ affiliated to $\mathcal{W}_\kappa$ equal to $\int g(\mathrm{x}) :P(\overline{\varphi}(\mathrm{x})\varphi(\mathrm{x})):_0 \mathrm{dx}$. Then the following statements hold true:*

(i) $L+V$ *is essentially selfadjoint and* $\Omega \in \mathcal{D}(\mathrm{e}^{-\frac{\beta}{2}H_V})$, *where* $H_V := \overline{L+V}$.

(ii) Let $\tau_V(t)$ be the W^*-dynamics generated by H_V and ω_V be the vector state induced by $\Omega_V = \|\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega\|^{-1}\mathrm{e}^{-\frac{\beta}{2}H_V}\Omega$. Then τ_V is a group of *-automorphisms of $\mathcal{W}$, continuous for the strong operator topology such that $(\mathcal{W}, \mathcal{W}_\kappa, \tau_V, \omega_V)$ is a stochastically positive β-KMS system.

(iii) The generalized path space associated to $(\mathcal{W}, \mathcal{W}_\kappa, \tau_V, \omega_V)$ satisfies the Markov property.

(iv) Let L_V, J_V be the perturbed Liouvillean and modular conjugation associated to $(\mathcal{W}, \mathcal{W}_\kappa, \tau_V, \omega_V)$. Then $J_V = J$ and $L_V = \overline{H_V - JVJ}$.

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