Spectral and scattering theory of spatially cut-off $P(\varphi)_2$ Hamiltonians

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Abstract

We study spatially cut-off $P(\varphi)_2$ Hamiltonians. We show the local finiteness of the pure point spectrum outside of thresholds, the limiting absorption principle and asymptotic completeness of scattering for such Hamiltonians. Our results imply the absence of singular continuous spectrum.

1 Introduction

1.1 $P(\varphi)_2$ models in quantum field theory

Models of quantum field theory used by physicists to describe basic interactions, although very successful experimentally, are defined only in a formal and perturbative way. In year 1952 Wightman and Gårding formulated a set of axioms, which, at least at that time, seemed to constitute a rather general mathematical framework for a physically acceptable QFT of basic interactions. In particular, these axioms satisfied the requirements of relativistic covariance and causal locality. It was hoped that physically realistic models of QFT can be interpreted in a mathematically consistent way and that they can be shown to satisfy axioms similar to Wightman axioms. At that time no examples of theories satisfying Wightman axioms were known except for free fields, which are in a sense trivial both from the physical and mathematical point of view.

It is not difficult to give a list of non-trivial QFT models which on a formal level satisfy Wightman axioms. These models can be ordered according to their difficulty and physically realistic models in 3+1 dimensions are quite high on this list. Wightman proposed to construct these models one by one and check whether they satisfy the axioms he formulated, starting with the easiest (but unfortunately, non-physical) ones. Thus began one of the most famous chapters of mathematical physics – constructive quantum field theory.

The simplest class of models in the Wightman program were the so-called $P(\varphi)_2$ models, that is the models of self-interacting bosons in 2 space-time dimensions with the interaction given by a semibounded polynomial $P(\varphi)$ of degree at least 4. The construction of these models was one of the early successes of constructive field theory. A number of different constructions were given. One of the approaches (in fact, the one that was used in the earliest works) started with considering a spatially cutoff $P(\varphi)_2$ interaction, where the cutoff is defined using a positive coupling function $g(x)$, which decays sufficiently fast at infinity. One can then define the Hamiltonian

$$H := H_0 + \int g(x) :P(\varphi(x)): \mathrm{d}x, \tag{1.1}$$

as a semibounded self-adjoint operator on Fock space $\Gamma(L^2(\mathbb{R}))$, where $H_0 = \mathrm{d}\Gamma(\sqrt{k^2+m^2})$ is the free Hamiltonian. (1.1) is called the *spatially cut-off $P(\varphi)$ Hamiltonian.* The next step is to show that, as $g(x) \to 1$, one obtains a limiting dynamics which acts in a different, renormalized Hilbert space and satisfies the Wightman axioms.

The Hamiltonian H will be the main subject of our paper. We will always assume that $g \in L^1(\mathbb{R})$; for most results we will also need some additional assumptions on the decay and differentiability of g.

The program of constructive field theory has not attained its original goal of constructing a physically realistic and mathematically rigorous model satisfying the conditions of covariance and locality in 4 space-time dimensions. To our knowledge, the models that have been constructed, including $P(\varphi)_2$ do not describe any real physical systems. Nevertheless, we believe that the heritage of constructive field theory is a source of models and techniques which are very interesting both physically and mathematically.

One could ask what are the reasons to look at the Hamiltonians (1.1). One of them is historic – as we tried to sketch above, these Hamiltonians played an important role in the development of constructive field theory and there is a considerable literature on this subject. Unfortunately, (1.1) is not relativistic, since it is not even translation invariant. Nevertheless, it has a certain remarkable property: it satisfies the axiom of the causal locality, more precisely, if one defines a net of local algebras in the sense of Haag–Kastler with help of H, then this net is causally local.

Another reason is that spatially cutoff $P(\varphi)_2$ Hamiltonians can be viewed as examples of Schrödinger operators in infinite dimension. Studying such Hamiltonians is a good occasion to test various advanced tools of functional analysis and sheds light on the mathematical structure of quantum field theory. This point of view was advocated in Simon's survey [Si2], where a number of mathematical questions concerning the spectral theory of $P(\varphi)_2$ Hamiltonians are formulated.

1.2 Content of this paper

In the present paper we extend methods developed for N-body Schrödinger operators to study spatially cut-off $P(\varphi)_2$ Hamiltonians. Our results include
1) local finiteness of the pure point spectrum outside of the thresholds,
2) the limiting absorption principle,
3) asymptotic completeness.
Note that 2) and 3) imply the absence of the singular continuous spectrum. (The properties 2) and 3) are proven under different assumptions, neither of which implies the other).

Recently a number of papers appeared that study other models that belong to a broadly understood QFT [AH, BFS, BFSS, DG1, Ge, HuSp1, HuSp2, JP1, JP2, Sk, Sp1, Sp2]. The Hamiltonians studied in these papers are sometimes called Pauli-Fierz Hamiltonians. They are non-relativistic, non-local and they have little to do with the Wightman program. Nevertheless, they are physically relevant and of a significant mathematical interest.

One of these papers – [DG1] – can be viewed as a predecessor of this paper. It is devoted to massive Pauli-Fierz Hamiltonians and it contains results similar to those contained in this paper (except for the limiting absorption principle, which, however, could be easily shown in the context of [DG1]). It should be noted that there are a lot of analogies between this paper and [DG1]. Both massive Pauli-Fierz Hamiltonians and spatially cut-off $P(\varphi)_2$ models share a lot of common characteristics, in particular the basic framework of scattering theory is essentially the same. Both classes are examples of QFT Hamiltonians with localized interactions.

Nevertheless, the technical difficulties of this paper are more serious than those of [DG1]. This is partly due to the fact that it is much more difficult to define a $P(\varphi)_2$ Hamiltonian than a Pauli-Fierz Hamiltonian. In the case of Pauli-Fierz Hamiltonians considered in [DG1], the perturbation is relatively bounded, which is not true in the case of (1.1). These problems become especially apparent when one considers the Mourre estimate, which requires a rather careful treatment and in many respects is more difficult than in the case of Schrödinger operators. In fact, the original theory of Mourre [Mo] does not seem to be applicable in the case of H and we need to apply its more sophisticated version contained in [ABG]. The key idea of the approach of [ABG] is the property of $C^1(A)$ regularity of an Hamiltonian H with respect to a unitary group $\mathrm{e}^{\mathrm{i}sA}$, which fortunately can be verified in the case of $P(\varphi)_2$ Hamiltonians.

The main tools that we use in the study of H is considering the interaction as an operator of multiplication in the Q-representation and the higher order estimates due to Rosen. These tools were developed in the early years of constructive field theory [Ne, Ro1, Ro2, Se, S-H.K]. Note that these tools are not needed in the case of technically simpler Pauli-Fierz Hamiltonians considered in [DG1].

Another difference between this paper and [DG1] is a significant simplification of the proof of asymptotic completeness and a different proof of the Fock property of asymptotic fields.

Our paper can be divided into two parts. The first part, which consists of Sects. 2, 3, 4, 5 describes the general formalism of CAR representations, bosonic Fock spaces and $Q-$space representation. Our presentation is quite general and at some points its generality goes beyond what we need in the case of the Hamiltonians (1.1). Actually, when one considers some other models of QFT (such as those with massless particles) one needs the formalism in its more general form (see for instance Theorem 4.3, which allows for a non-Fock component of CCR representations). Most of the material of these sections can be found in the literature, notably Sects. 2 and 4 follow quite closely [BR] and Sect. 5 follows [S-H.K, Si1]. Nevertheless, our presentation has some modifications and improvements as compared for example to that in [BR] and we believe that the reader will find it useful, especially since it is compact and essentially self-contained.

A considerable effort has been devoted to develop a concise notation for operators in Fock spaces. Some elements of this notation are standard (due in particular to I. Segal), others were introduced in [DG1]. In [DG1] we did not need to consider Wick polynomials, which play an important role in this paper. We devote a special attention to the properties of Wick polynomials in Subsect. 3.12, and also in the context of the Q-representation in Subsection 5.2. Note that the calculus and notation in the literature on QFT can be quite cumbersome and ad hoc, which we wanted to avoid.

The second part of our paper is devoted to the study of spatially cutoff $P(\varphi)_2$ Hamiltonians. In Section 6 we introduce spatially cut-off $P(\varphi)_2$ Hamiltonians and we describe their basic properties, following eg. [S-H.K].

One of the most difficult results about such Hamiltonians are the so-called *higher order estimates* due to Rosen. They are described with some of their consequences in Sect. 7. Strictly speaking, their proof contained in [Ro2] does not cover the class of Hamiltonians that we consider. Therefore, we indicate how to modify the arguments of [Ro2] to cover our class of coupling functions g.

In Section 8 we study the commutator of H with the second quantized generator of dilations A. The operator A will play the role of a conjugate operator in the Mourre theory. The abstract framework of this section is based on [ABG], where a theory of the $C^1(A)$ property is developed. Such a careful treatment of this question was not needed neither in [DG1] nor in the case of $N-$particle Schrödinger operators.

The case of the φ_2^4 model, ie the case when P is a polynomial of degree 4 is simpler. For example the construction of the space cutoff φ_2^4 model can be done without using the $Q-$representation (see [GJ2]). Similarly the Mourre theory in the φ_2^4 case can be treated in a simpler way, under weaker conditions on the cutoff function g.

Sect. 9 is devoted to the spectral theory of $P(\varphi)_2$ Hamiltonians. The analog of the HVZ theorem is proven in Subsect. 9.1. This result was first proven in [GJ4, S-H.K]; we give a different proof (analogous to the one given in [DG1]), which is essentially a by-product of the techniques which we develop in our paper for other purposes.

In Subsection 9.2 we prove the Mourre estimate for H. The proof is similar to the one contained in [DG1]. The Mourre estimate implies the local finiteness of the pure point spectrum outside of the thresholds. The set of thresholds is defined as $\{\lambda+nm \mid n=1.2,\dots,\ \lambda\in\sigma_{\rm pp}(H)\}$, where $\sigma_{\rm pp}(H)$ denotes the pure point spectrum of H. Note that this result implies that the pure point spectrum of H is contained in a closed set of measure zero.

Under stronger conditions on the coupling function, we can also show the limiting absorption principle, which implies the absence of singular continuous spectrum. More precisely, we show the existence of the boundary value of the resolvent on the real line

$$\lim_{\epsilon\downarrow 0}\langle A\rangle^{-\mu}(\lambda+{\rm i}\epsilon-H)^{-1}\langle A\rangle^{-\mu},$$

where A is the conjugate operator and $\mu>\frac{1}{2}$.

In Section 10 we study the scattering theory for spatially cut-off $P(\varphi)_2$ Hamiltonians. The basic construction of scattering theory in the context of this paper are asymptotic fields, that is the limits of the field operators in the interaction picture:

$$a^{\pm,\#}(g):=\text{s-}\lim_{t\to\pm\infty}{\rm e}^{{\rm i}tH}a^{\#}(g_t){\rm e}^{-{\rm i}tH},$$

where $g_t:={\rm e}^{-{\rm i}t\sqrt{k^2+m^2}}g$. We prove the existence of the asymptotic fields and show that they realize a CCR representation satisfying the Fock property. This result was first proven in [HK]. Up to technical details due to a more singular character of the interaction, the proof of the existence of asymptotic fields follows the proof of the analogous result in [DG1]. The proof of the Fock property is based on the general theory of CCR representations. Its main ingredient is the concept of the number operator associated to regular CCR representation described in

Section 4. Note that the proof of the Fock property contained in [DG1] was different – it was closer to the original argument of [HK].

With the CCR representations given by $a^{\pm,\#}(g)$ one can associate the spaces of asymptotic vacua $\mathcal{K}^{\pm}$, that is the states annihilated by asymptotic annihilation operators. The Fock property is equivalent to saying that vectors of the form $a^{\pm*}(g_1)\cdots a^{\pm*}(g_n)\psi$, where $\psi \in \mathcal{K}^{\pm}$, span the whole Hilbert space $\mathcal{H}$. It is easy to see that the bound states are contained in the spaces of the asymptotic vacua $\mathcal{K}^{\pm}$. The property of *asymptotic completeness* means that the spaces $\mathcal{K}^{\pm}$ are equal to the space of bound states of H. This property is formulated at the end of Sect. 10. Among its consequences are the fact that the asymptotic vacua at $t = -\infty$ and at $t = +\infty$ coincide and the justification of the formalism of asymptotic states commonly used by physicists.

In Section 11 we describe various propagation estimates. Their proofs do not differ substantially from the Pauli-Fierz case and we refer to [DG1] for most of them.

In Sect. 12 we prove asymptotic completeness, that is, we show that the space of asymptotic vacua equals the pure point subspace of H. In principle, in this section we could repeat almost verbatim the arguments of the analogous section of [DG1]. Nevertheless, we simplify substantially the arguments of [DG1]. The major difference is that in [DG1] we used operators P_k, Q_k and their asymptotic counterparts. In this paper we avoid using them and the main role is played by the operators $\Gamma^+(q)$ (describing something similar to the asymptotic velocity) and inverse wave operators $W^+(j)$ (both of these objects were also used in [DG1]). Note in parenthesis that the absence of the operators P_k in the present paper has its price – it seems that one needs them to show a certain interesting intermediate result concerning the inverse wave operators $W^+(j)$ (see [DG1, Thm 7.13]), which, fortunately, is not needed for the proof of asymptotic completeness itself.

The methods of this paper can be applied to other models of QFT with a localized interaction and a massive dispersion relation. In particular, one can use ideas of this paper to simplify some of the arguments of [DG1].

As we mentioned earlier, the $P(\varphi)_2$ models are the simplest nontrivial models considered in constructive field theory. Still, their treatment requires a lot of care and involves a number of various techniques, which go beyond the problems usually encountered in quantum mechanics and PDE's. Even more difficult and more interesting problems arise when one considers other models of constructive field theory such as Y_2 or $\lambda\varphi_3^4$. It would be interesting to extend our results to spatially cut-off versions of these models. We believe that it is feasible, since they are also models with a localized interaction and a massive dispersion relation. In particular, the framework of scattering theory for these models is essentially the same as the one considered in this paper. The main new difficulty would be the various renormalization procedures needed to define these Hamiltonians.

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2 CCR Representations

We recall some standard facts on CCR representations.

2.1 Weyl operators

Let $\mathcal{H}$ be a Hilbert space. Let $\mathfrak{g}$ be a real vector space equipped with an antisymmetric form σ. A map

$$\mathfrak{g} \ni h \mapsto W_\pi(h) \in B(\mathcal{H}) \tag{2.1}$$

is a *representation of the canonical commutation relations* (in short a CCR representation) over $\mathfrak{g}$ in $\mathcal{H}$ if

$$\begin{aligned} W_\pi(h_1)W_\pi(h_2) &= \mathrm{e}^{-\mathrm{i}\sigma(h_1,h_2)/2}W_\pi(h_1+h_2), \\ W_\pi^*(h) = W_\pi(-h), &\qquad W_\pi(0) = 1\!\!1. \end{aligned} \tag{2.2}$$

Note that as a consequence $W_\pi(h)$ are unitary and we have

$$W_\pi(h_1)W_\pi(h_2) = \mathrm{e}^{-\mathrm{i}\sigma(h_1,h_2)}W_\pi(h_2)W_\pi(h_1). \tag{2.3}$$

2.2 Field operators

We say that the CCR representation (2.1) is *regular*, if

$$t \mapsto W_\pi(th) \quad \text{is strongly continuous for any } h \in \mathfrak{g}.$$

From now on we assume that we are given a regular representation.

By the Stone theorem, for any $h \in \mathfrak{g}$ we can define the corresponding field operator

$$\phi_\pi(h) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W_\pi(th)\Big|_{t=0}.$$

The following proposition is well known (see eg [BR]).

Proposition 2.1 *i) In the sense of a quadratic form on $\mathcal{D}(\phi_\pi(h_1)) \cap \mathcal{D}(\phi_\pi(h_2))$ the Heisenberg commutation relations are satisfied:*

$$[\phi_\pi(h_1), \phi_\pi(h_2)] = \mathrm{i}\sigma(h_1, h_2). \tag{2.4}$$

ii) $W_\pi(g)$ leaves invariant $\mathcal{D}(\phi_\pi(h))$ and

$$[\phi_\pi(h), W_\pi(g)] = \mathrm{i}\sigma(g, h)W_\pi(g). \tag{2.5}$$

iii) Let $\mathfrak{f}$ be a finite dimensional subspace of $\mathfrak{g}$. Then

$$\mathfrak{f} \ni f \mapsto W_\pi(h + f)$$

is strongly continuous for any $h \in \mathfrak{g}$.
iv) If $\mathfrak{f}$ is a finite dimensional subspace of $\mathfrak{g}$, then the intersection of $\mathcal{D}(\phi_\pi(h_p)\cdots\phi_\pi(h_1))$, $h_1, \dots, h_p \in \mathfrak{f}$, $p \in \mathbb{N}$ is dense in $\mathcal{H}$.

2.3 Creation and annihilation operators

From now on we assume that $\mathfrak{g}$ is equipped with a complex structure (that is an $\mathbb{R}-$linear operator $\mathrm{i} : \mathfrak{g} \to \mathfrak{g}$ with $\mathrm{i}^2 = -1$). We assume that σ and i are compatible in the following sense:

$$\sigma(\mathrm{i}h_1, h_2) + \sigma(h_1, \mathrm{i}h_2) = 0,$$
$$\sigma(h, \mathrm{i}h) > 0, \quad h \neq 0.$$

(In particular, this forces σ to be non-degenerate). Then

$$(h_1|h_2) := \sigma(h_1, \mathrm{i}h_2) + \mathrm{i}\sigma(h_1, h_2)$$

defines a positive definite scalar product. From now on we will treat $\mathfrak{g}$ as a complex space equipped with this scalar product. One defines the creation and annihilation operators as

$$\begin{aligned} a_\pi^*(h) &= \tfrac{1}{\sqrt{2}}(\phi_\pi(h) - \mathrm{i}\phi_\pi(\mathrm{i}h)), \\ a_\pi(h) &= \tfrac{1}{\sqrt{2}}(\phi_\pi(h) + \mathrm{i}\phi_\pi(\mathrm{i}h)). \end{aligned} \tag{2.6}$$

Clearly,

$$\phi_\pi(h) := \frac{1}{\sqrt{2}}(a_\pi^*(h) + a_\pi(h)), \ h \in \mathfrak{g}. \tag{2.7}$$

Proposition 2.2 *i) The operators $a_\pi^*(h)$ and $a_\pi(h)$ with domain $\mathcal{D}(\phi_\pi(h)) \cap \mathcal{D}(\phi_\pi(\mathrm{i}h))$ are closed. (By Proposition 2.1 iii) this domain is dense in $\mathcal{H}$).*
ii) The commutation relations are true in the sense of a quadratic form:

$$\begin{aligned} [a_\pi(h_1), a_\pi^*(h_2)] &= (h_1|h_2)\mathbb{1}, \\ [a_\pi(h_2), a_\pi(h_1)] &= [a_\pi^*(h_2), a^*(h_1)] = 0. \end{aligned} \tag{2.8}$$

iii) $W_\pi(g)$ leaves invariant $\mathcal{D}(a_\pi^\#(h))$ and

$$\begin{aligned} [a_\pi(h), W_\pi(g)] &= \tfrac{\mathrm{i}}{\sqrt{2}}\overline{(g,h)}W_\pi(g), \\ [a_\pi^*(h), W_\pi(g)] &= -\tfrac{\mathrm{i}}{\sqrt{2}}(g,h)W_\pi(g). \end{aligned} \tag{2.9}$$

iv) If $\mathfrak{f}$ is a finite dimensional subspace of $\mathfrak{g}$, then the intersection of $\mathcal{D}(a_\pi^\#(h_p)\cdots a_\pi^\#(h_1))$, $h_1, \dots, h_p \in \mathfrak{f}$, $p \in \mathbb{N}$ is dense in $\mathcal{H}$.

3 Operators in bosonic Fock spaces

We recall various constructions on bosonic Fock spaces.

3.1 Bosonic Fock spaces

Let $\mathfrak{h}$ be a Hilbert space, which we will call the 1-particle space. Let $\otimes_{\mathrm{s}}^n\mathfrak{h}$ denote the symmetric nth tensor power of $\mathfrak{h}$. Let $\mathcal{S}_n$ denote the orthogonal projection of $\otimes^n\mathfrak{h}$ onto $\otimes_{\mathrm{s}}^n\mathfrak{h}$. If $u \in \otimes_{\mathrm{s}}^p\mathfrak{h}$ and $v \in \otimes_{\mathrm{s}}^q\mathfrak{h}$, then we will write

$$u \otimes_{\mathrm{s}} v := \mathcal{S}_{p+q} u \otimes v \in \otimes_{\mathrm{s}}^{p+q}\mathfrak{h}.$$

If $a \in B(\otimes_{\rm s}^p\mathfrak{h}, \otimes_{\rm s}^r\mathfrak{h})$ and $b \in B(\otimes_{\rm s}^q\mathfrak{h}, \otimes_{\rm s}^s\mathfrak{h})$, then we will write

$$a \otimes_{\rm s} b := \mathcal{S}_{r+s} a \otimes b \in B(\otimes_{\rm s}^{p+q}\mathfrak{h}, \otimes_{\rm s}^{r+s}\mathfrak{h}).$$

We define the bosonic Fock space over $\mathfrak{h}$ to be the direct sum

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes_{\rm s}^n \mathfrak{h}.$$

Ω will denote the vacuum vector – the vector $1 \in \mathbb{C} = \otimes_{\rm s}^0\mathfrak{h}$. The number operator N is defined as

$$N\Big|_{\bigotimes_{\rm s}^n \mathfrak{h}} = n\mathbb{1}.$$

For a selfadjoint operator A on $\Gamma(\mathfrak{h})$, we denote by $\mathcal{H}_{\rm comp}(A)$ the space

$$\mathcal{H}_{\rm comp}(A) = \{u \in \mathcal{H} \,|\chi(A)u = u, \chi \in C_0^\infty(\mathbb{R})\}.$$

We define the space of finite particle vectors and finite particle operators:

$$\Gamma_{\rm fin}(\mathfrak{h}) = \mathcal{H}_{\rm comp}(N) := \{u \in \Gamma(\mathfrak{h}) \mid \text{for some } \; n \in \mathbb{N}, \;\; \mathbb{1}_{[0,n]}(N)u = u\},$$

$$B_{\rm fin}(\Gamma(\mathfrak{h})) := \{B \in B(\Gamma(\mathfrak{h})) \mid \text{for some } \; n \in \mathbb{N}, \;\; \mathbb{1}_{[0,n]}(N)B\mathbb{1}_{[0,n]}(N) = B\}.$$

3.2 Creation and annihilation operators

There exists a natural representation of CCR over $\mathfrak{h}$ in $\Gamma(\mathfrak{h})$ (where $\mathfrak{h}$ is equipped with the symplectic form $Im(\cdot|\cdot)$). To construct this representation it is natural to proceed in the reverse order from the one used in Sect. 2: first one constructs creation/annihilation operators, then field operators and then Weyl operators. If $h \in \mathfrak{h}$, we define the creation operator $a^*(h)$ by setting

$$a^*(h) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}),$$
$$a^*(h)u := \sqrt{n+1}\, h \otimes_{\rm s} u, \qquad u \in \otimes_{\rm s}^n \mathfrak{h}.$$

$a(h)$ denotes the adjoint of $a^*(h)$, and is called the annihilation operator. Both $a^*(h)$ and $a(h)$ are defined on $\Gamma_{\rm fin}(\mathfrak{h})$ and can be extended to densely defined closed operators on $\Gamma(\mathfrak{h})$. By writing $a^\sharp(h)$ we will mean both $a^*(h)$ and $a(h)$.

Creation and annihilation operators $a^\#(h)$ on a Fock space satisfy (2.8) .

In our paper we will usually have

$$\mathfrak{h} = L^2(\mathbb{R}, {\rm d}k). \tag{3.1}$$

Then we will often write (as is customary in the literature)

$$a^*(h) = \int a^*(k)h(k){\rm d}k, \quad a(h) = \int a(k)\bar{h}(k){\rm d}k,$$

where $a^*(k)$ and $a(k)$, $k \in \mathbb{R}$, have the meaning of operator valued distributions.

3.3 Field operators

We define the field operator

$$\phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)),\ h \in \mathfrak{h}.$$

The operators $\phi(h)$ are essentially selfadjoint on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ and can be extended to self-adjoint operators on $\Gamma(\mathfrak{h})$. Field operators $\phi(h)$ on a Fock space satisfy (2.4). In the case of (3.1), one can also write

$$\phi(h) = \int h(k)\phi(k)\mathrm{d}k,$$

where $\phi(k)$ is an operator valued distribution

$$\phi(k) := \frac{1}{\sqrt{2}}(a^*(k) + a(k)),\ k \in \mathbb{R}.$$

3.4 Weyl operators

We introduce also the Weyl operators:

$$W(h) := \mathrm{e}^{\mathrm{i}\phi(h)}, \quad h \in \mathfrak{h}.$$

The map $\mathfrak{h} \ni h \mapsto W(h)$ is a regular representation of CCR over $\mathfrak{h}$ in $\Gamma(\mathfrak{h})$. Moreover, Weyl operators in a Fock space have the following properties:

Proposition 3.1 *i) the map*

$$\mathbb{R} \ni s \mapsto W(sh)(N+1)^{-\frac{1}{2}}$$

is C^1 in the strong topology and the map

$$\mathbb{R} \ni s \mapsto W(sh)(N+1)^{-\frac{1}{2}-\epsilon}$$

is C^1 in the norm topology. More precisely,

$$\lim_{s\to 0} \sup_{\|h\|\leq C} s^{-1}\Big\|(W(sh) - \mathbb{1} - \mathrm{i}s\phi(h))(N+1)^{-1/2-\epsilon}\Big\| = 0.$$

ii)

$$\|(W(h_1) - W(h_2))u\| \leq C_\epsilon \|h_1 - h_2\|^\epsilon \Big((\|h_1\|^2 + \|h_2\|^2)^{\frac{\epsilon}{2}}\|u\| + \|(N+1)^{\frac{\epsilon}{2}}u\|\Big).$$

3.5 Operator dΓ

If b is an operator on $\mathfrak{h}$, we define the operator

$$\begin{aligned} \mathrm{d}\Gamma(b) \quad &: \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}), \\ \mathrm{d}\Gamma(b)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}} \quad &:= \sum_{j=1}^n \mathbb{1}^{\otimes(j-1)} \otimes b \otimes \mathbb{1}^{\otimes(n-j)} \\ &= nb \otimes_{\mathrm{s}} \mathbb{1}^{\otimes(n-1)}. \end{aligned}$$

An important example is the number operator

$$N := \mathrm{d}\Gamma(1).$$

Lemma 3.2 *i) Heisenberg derivatives:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\Gamma(b) = \mathrm{d}\Gamma(\tfrac{\mathrm{d}}{\mathrm{d}t}b),$$
$$[\mathrm{d}\Gamma(b_1), \mathrm{d}\Gamma(b_2)] = \mathrm{d}\Gamma([b_1, b_2]).$$

ii) Commutation properties:

$$[\mathrm{d}\Gamma(b), a^*(h)] = a^*(bh),$$
$$[\mathrm{d}\Gamma(b), a(h)] = -a(b^*h),$$
$$[\mathrm{d}\Gamma(b), \mathrm{i}\phi(h)] = \phi(\mathrm{i}bh), \ \textit{if } b = b^*,$$
$$W(h)\mathrm{d}\Gamma(b)W(-h) = \mathrm{d}\Gamma(b) - \phi(\mathrm{i}bh) - \tfrac{1}{2}Re(bh|h) \ \textit{if } b = b^*.$$

iii) Estimates:

$$b_1 \leq b_2 \ \ \textit{implies} \ \ \mathrm{d}\Gamma(b_1) \leq \mathrm{d}\Gamma(b_2),$$
$$\|N^{-\frac{1}{2}}\mathrm{d}\Gamma(b)u\| \leq \|\mathrm{d}\Gamma(b^*b)^{\frac{1}{2}}u\|,$$
$$\mathrm{d}\Gamma(b)^\alpha \leq N^{1-\alpha}\mathrm{d}\Gamma(b^\alpha), \quad \textit{if } b \geq 0, \ \ 1 \leq \alpha,$$
$$\mathrm{d}\Gamma(ab) \leq \mathrm{d}\Gamma(a^p)^{\frac{1}{p}}\mathrm{d}\Gamma(b^q)^{\frac{1}{q}}, \qquad \textit{if } a \geq 0, \ b \geq 0, \ ab = ba, \ \ p^{-1} + q^{-1} = 1.$$

3.6 Functor Γ

Let $\mathfrak{h}_i, i = 1, 2$ be Hilbert spaces. Let $q : \mathfrak{h}_1 \mapsto \mathfrak{h}_2$ be a bounded linear operator. We define

$$\Gamma(q) : \Gamma(\mathfrak{h}_1) \mapsto \Gamma(\mathfrak{h}_2)$$
$$\Gamma(q)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}_1} = q \otimes \cdots \otimes q.$$

The Γ functor has the following properties:

Lemma 3.3 *i) Relationship with* $\mathrm{d}\Gamma$*: assume* $\mathfrak{h}_1 = \mathfrak{h}_2$*. Then*

$$\mathrm{e}^{\mathrm{d}\Gamma(b)} = \Gamma(\mathrm{e}^b).$$

ii) Intertwining properties:

$$\Gamma(q)a^*(h_1) = a^*(qh_1)\Gamma(q), \quad h_1 \in \mathfrak{h}_1,$$
$$\Gamma(q)a(q^*h_2) = a(h_2)\Gamma(q), \quad h_2 \in \mathfrak{h}_2.$$

If q is isometric, that is $q^*q = 1$, then

$$\Gamma(q)a^\sharp(h_1) = a^\sharp(qh_1)\Gamma(q),$$
$$\Gamma(q)\phi(h_1) = \phi(qh_1)\Gamma(q).$$

If q is unitary, then

$$\Gamma(q)a^\sharp(h_1)\Gamma(q^{-1}) = a^\sharp(qh_1),$$

$$\Gamma(q)\phi(h_1)\Gamma(q^{-1}) = \phi(qh_1).$$

iii) If $\|q\| \leq 1$, then

$$\|\Gamma(q)\| = 1.$$

3.7 Operator $\mathrm{d}\Gamma(q, r)$

Let q, r be operators from $\mathfrak{h}_1$ to $\mathfrak{h}_2$. We define

$$\begin{aligned}
\mathrm{d}\Gamma(q, r) \quad & : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2), \\
\mathrm{d}\Gamma(q, r)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}_1} \quad & := \sum_{j=1}^{n} q^{\otimes(j-1)} \otimes r \otimes q^{\otimes(n-j)} \\
& = n r \otimes_{\mathrm{s}} q^{\otimes(n-1)}.
\end{aligned}$$

Lemma 3.4 *i) Relationship with $\mathrm{d}\Gamma$ and Γ:*

$$\mathrm{d}\Gamma(1, r) = \mathrm{d}\Gamma(r), \quad \mathrm{d}\Gamma(r, r) = N\Gamma(r).$$

ii) Heisenberg derivatives of $\Gamma(q)$:

$$\mathrm{d}\Gamma(b_2)\Gamma(q) = \mathrm{d}\Gamma(q, b_2 q), \quad \Gamma(q)\mathrm{d}\Gamma(b_1) = \mathrm{d}\Gamma(q, q b_1),$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Gamma(q) = d\Gamma(q, \tfrac{\mathrm{d}}{\mathrm{d}t} q).$$

iii) Intertwining properties:

$$a(h_2)\mathrm{d}\Gamma(q, r) = \mathrm{d}\Gamma(q, r)a(q^* h_2) + \Gamma(q)a(r^* h_2),$$

$$\mathrm{d}\Gamma(q, r)a^*(h_1) = a^*(q h_1)\mathrm{d}\Gamma(q, r) + a^*(r h_1)\Gamma(q).$$

iv) Estimates:

$$0 \leq r, \quad \textit{and} \quad 0 \leq q \leq 1 \textit{ implies} \quad \mathrm{d}\Gamma(q, r) \leq \mathrm{d}\Gamma(r),$$

$$|(u_2|\mathrm{d}\Gamma(q, r_2 r_1)u_1)| \leq \|\mathrm{d}\Gamma(r_2 r_2^*)^{\frac{1}{2}} u_2\| \|\mathrm{d}\Gamma(r_1^* r_1)^{\frac{1}{2}} u_1\|, \quad \|q\| \leq 1,$$

$$\|N^{-\frac{1}{2}}\mathrm{d}\Gamma(q, r)u\| \leq \|\mathrm{d}\Gamma(r^* r)^{\frac{1}{2}} u\|, \quad \|q\| \leq 1.$$

3.8 Tensor product of Fock spaces

We will adopt the following convention for tensor products: $E \otimes F$ will denote the algebraic tensor product of E and F, except when E, F are both Hilbert spaces, in which case it will denote the hilbertian tensor product. Let $\mathfrak{h}_i$, $i = 1, 2$ be two Hilbert spaces. Let i_1, i_2 the injections of $\mathfrak{h}_1, \mathfrak{h}_2$ into $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. We define $U : \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \to \Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ as follows:

$$U u \otimes v := \sqrt{\frac{(p+q)!}{p!q!}}\Gamma(i_1)u \otimes_{\mathrm{s}} \Gamma(i_2)v, \quad u \in \otimes_{\mathrm{s}}^p \mathfrak{h}_1, \ u \in \otimes_{\mathrm{s}}^q \mathfrak{h}_2. \tag{3.1}$$

Proposition 3.5 *i) U is unitary,*
ii) $U\Omega\otimes\Omega=\Omega$,
iii)

$$a^\sharp(h_1\oplus h_2)U=U(a^\sharp(h_1)\otimes 1\!\!1+1\!\!1\otimes a^\sharp(h_2)),\ h_1\in\mathfrak{h}_1,\quad h_2\in\mathfrak{h}_2,$$

$$\phi(h_1\oplus h_2)U=U(\phi(h_1)\otimes 1\!\!1+1\!\!1\otimes\phi(h_2)),\ h_1\in\mathfrak{h}_1,\quad h_2\in\mathfrak{h}_2.$$

iv)

$$\mathrm{d}\Gamma(b_1\oplus b_2)U=U(\mathrm{d}\Gamma(b_1)\otimes 1\!\!1+1\!\!1\otimes\mathrm{d}\Gamma(b_2)), \tag{3.2}$$

$$U\Gamma(q_1)\otimes\Gamma(q_2)=\Gamma(q_1\oplus q_2)U.$$

3.9 Scattering identification operator I

Along with the space $\Gamma(\mathfrak{h})$ we will consider the space $\Gamma(\mathfrak{h}\oplus\mathfrak{h})\simeq\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$. We will use the notation

$$N_0:=N\otimes 1\!\!1,\qquad N_\infty:=1\!\!1\otimes N.$$

We will also write

$$a_0^\sharp(h):=a^\sharp(h)\otimes 1\!\!1,\quad a_\infty^\sharp(h):=1\!\!1\otimes a^\sharp(h).$$

Following [HuSp1], we define the *scattering identification operator*

$$I:\Gamma_{\mathrm{fin}}(\mathfrak{h})\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h})\to\Gamma_{\mathrm{fin}}(\mathfrak{h}),$$

$$Iu\otimes v:=\sqrt{\frac{(p+q)!}{p!q!}}u\otimes_{\mathrm{s}}v,\ u\in\otimes_s^p\mathfrak{h},\ v\in\otimes_s^q\mathfrak{h}. \tag{3.3}$$

Another formula defining I is

$$I:=\Gamma(i)U$$

where $U:\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})\to\Gamma(\mathfrak{h}\oplus\mathfrak{h})$ is the unitary operator introduced in (3.1) for $\mathfrak{h}_1=\mathfrak{h}_2=\mathfrak{h}$ and

$$i:\mathfrak{h}\oplus\mathfrak{h}\to\mathfrak{h},$$

$$(h_0,h_\infty)\mapsto h_0+h_\infty.$$

Note that since $\|i\|=\sqrt{2}$, the operator $\Gamma(i)$ is unbounded. Therefore, I is unbounded too.
Yet another formula defining I is:

$$I\prod_{i=1}^n a^*(h_i)\Omega\otimes\prod_{i=1}^p a^*(g_i)\Omega:=\prod_{i=1}^p a^*(g_i)\prod_{i=1}^n a^*(h_i)\Omega,\qquad h_i,g_i\in\mathfrak{h}. \tag{3.4}$$

If $\mathfrak{h}=L^2(\mathbb{R},\mathrm{d}k)$, then we can write still another formula for I:

$$Iu\otimes\psi=\frac{1}{(p!)^{\frac12}}\int\psi(k_1,\cdots,k_p)a^*(k_1)\cdots a^*(k_p)u\mathrm{d}k_1\cdots\mathrm{d}k_r,\quad u\in\Gamma(\mathfrak{h}),\ \psi\in\otimes_{\mathrm{s}}^p\mathfrak{h}. \tag{3.5}$$

Proposition 3.6 *i) Let b, q be operators on $\mathfrak{h}$. Then*

$$\mathrm{d}\Gamma(b)I = I(\mathrm{d}\Gamma(b)\otimes 1\!\!1 + 1\!\!1\otimes \mathrm{d}\Gamma(b)),$$

$$\Gamma(q)I = I\Gamma(q)\otimes\Gamma(q).$$

ii) For $h\in\mathfrak{h}$

$$a(h)I = I(a_0(h) + a_\infty(h)),$$

$$a^*(h)I = Ia_0^*(h) = Ia_\infty^*(h).$$

iii)

$$I(N_0+1)^{-k/2}1\!\!1_{[0,k]}(N_\infty) \text{ is bounded.} \tag{3.6}$$

3.10 Operator $I(j)$

Let j_0, j_∞ be two operators on $\mathfrak{h}$. Set $j = (j_0, j_\infty)$. We define

$$I(j) : \Gamma_{\mathrm{fin}}(\mathfrak{h})\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h})$$

$$I(j) := I\Gamma(j_0)\otimes\Gamma(j_\infty).$$

If we identify j with the operator

$$\begin{aligned} &j : \mathfrak{h}\oplus\mathfrak{h}\to\mathfrak{h}, \\ &j(h_0\oplus h_\infty) := j_0h_0 + j_\infty h_\infty, \end{aligned} \tag{3.7}$$

then we have

$$I(j) = \Gamma(j)U.$$

Remark 3.7 *$I(j)$ equals $\check{\Gamma}(j^*)^*$ in the notation of [DG1].*

Note that $I = I(1,1)$. Other formulas defining $I(j)$ are

$$I(j)\prod_{i=1}^{n} a^*(h_i)\Omega\otimes\prod_{i=1}^{p} a^*(g_i)\Omega := \prod_{i=1}^{p} a^*(j_0g_i)\prod_{i=1}^{n} a^*(j_\infty h_i)\Omega, \qquad h_i, g_i\in\mathfrak{h}, \tag{3.8}$$

$$I^*(j)\Pi_{i=1}^{n}a^*(h_i)\Omega := \Pi_{i=1}^{n}\left(a_0^*(j_0^*h_i) + a_\infty^*(j_\infty^*h_i)\right)\Omega\otimes\Omega, h_i\in\mathfrak{h}. \tag{3.9}$$

Lemma 3.8 *i)*

$$I(\tilde{j})I^*(j) = \Gamma(\tilde{j}_0j_0^* + \tilde{j}_\infty j_\infty^*).$$

In particular, if $j_0^* + j_\infty^* = 1\!\!1$, then

$$II^*(j) = 1\!\!1.$$

ii) Intertwining properties: For $h\in\mathfrak{h}$

$$a(h)I(j) = I(j)(a_0(j_0^*h) + a_\infty(j_\infty^*h)),$$

$$a^*(j_0h)I(j) = I(j)a_0^*(h),$$

$$a^*(j_\infty h)I(j) = I(j)a_\infty^*(h).$$

iv) $I(j)$ is bounded iff $\|j_0^*j_0 + j_\infty^*j_\infty\|\leq 1$, and then

$$\|I(j)\| = 1.$$

Let us note some additional properties of $I(j)$ in the coisometric case.

Lemma 3.9 *Assume*

$$j_0 j_0^* + j_\infty j_\infty^* = 1. \tag{3.10}$$

(This assumption implies that j is coisometric, that is $jj^* = 1$). Then
i)

$$I(j)I^*(j) = \mathbb{1}.$$

ii) Intertwining properties:

$$a^\#(h)I(j) = I(j)\left(a_0^\#(j_0^* h) + a_\infty^\#(j_\infty^* h)\right),$$

$$\phi(h)I(j) = I(j)\left(\phi_0(j_0^* h) + \phi_\infty(j_\infty^* h)\right).$$

iii) If in addition j_0, j_∞ are self-adjoint, then

$$\mathrm{d}\Gamma(b) = I(j)\left(\mathrm{d}\Gamma(b)\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{d}\Gamma(b)\right)I^*(j) + \tfrac{1}{2}\mathrm{d}\Gamma(\mathrm{ad}_{j_0}^2 b + \mathrm{ad}_{j_\infty}^2 b).$$

3.11 Operator $\mathrm{d}I(j,k)$

Let $j = (j_0, j_\infty)$, $k = (k_0, k_\infty)$ be pairs of maps from $\mathfrak{h}$ to $\mathfrak{h}$. We define

$$\mathrm{d}I(j,k) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}) \to \Gamma_{\mathrm{fin}}(\mathfrak{h})$$

as follows:

$$\mathrm{d}I(j,k) := I(\mathrm{d}\Gamma(j_0,k_0)\otimes\Gamma(j_\infty) + \Gamma(j_0)\otimes\mathrm{d}\Gamma(j_\infty,k_\infty)).$$

Equivalently, treating j and k as maps from $\mathfrak{h}\oplus\mathfrak{h}$ to $\mathfrak{h}$, as in (3.7), we can write

$$\mathrm{d}I(j,k) := \mathrm{d}\Gamma(j,k)U.$$

Remark 3.10 $\mathrm{d}I(j,k)$ *equals* $\mathrm{d}\check{\Gamma}(j^*,k^*)^*$ *in the notation of [DG1].*

Lemma 3.11 *i) Heisenberg derivative of $I(j)$:*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}I(j) = \mathrm{d}I(j, \tfrac{\mathrm{d}}{\mathrm{d}t}j),$$

$$I(j)\left(\mathrm{d}\Gamma(b_0)\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{d}\Gamma(b_\infty)\right) = \mathrm{d}I(j,k),$$

$$\mathrm{d}\Gamma(b)I(j) = \mathrm{d}I(j, bj).$$

Here b, b_0, b_∞ are operators on $\mathfrak{h}$ and $k = (j_0 b_0, j_\infty b_\infty)$.
ii) Intertwining properties:

$$\mathrm{d}I(j,k)a_0^*(h) = a^*(j_0 h)\mathrm{d}I(j,k) + a^*(k_0 h)I(j),$$

$$\mathrm{d}I(j,k)(a_0(j_0^* h) + a_\infty(j_\infty^* h)) + I(j)(a_0(k_0^* h) + a_\infty(k_\infty^* h)) = a(h)\mathrm{d}I(j,k).$$

iv) If $j_0 j_0^* + j_\infty j_\infty^* \leq 1$, k_0, k_∞ are self-adjoint, we have the estimate:

$$\begin{aligned}|(u_2|\mathrm{d}I^*(j,k)u_1)| &\leq \|\mathrm{d}\Gamma(|k_0|)^{\frac12}\otimes 1\!\!1 u_2\|\|\mathrm{d}\Gamma(|k_0|)^{\frac12}u_1\| \\ &\quad +\|1\!\!1\otimes \mathrm{d}\Gamma(|k_\infty|)^{\frac12}u_2\|\|\mathrm{d}\Gamma(|k_\infty|)^{\frac12}u_1\|,\ u_1\in\Gamma(\mathfrak{h}),\ u_2\in\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h}).\end{aligned}$$

v) If $j_0 j_0^* + j_\infty j_\infty^* \leq 1$, then

$$\|(N_0+N_\infty)^{-\frac12}\mathrm{d}I^*(j,k)u\| \leq \|\mathrm{d}\Gamma(k_0k_0^*+k_\infty k_\infty^*)^{\frac12}u\|,\ u\in\Gamma(\mathfrak{h}).$$

3.12 Wick polynomials

Let $w\in B(\otimes_{\mathrm{s}}^p\mathfrak{h},\otimes_{\mathrm{s}}^q\mathfrak{h})$. We define the operator

$$\mathrm{Wick}(w):\Gamma_{\mathrm{fin}}(\mathfrak{h})\to\Gamma_{\mathrm{fin}}(\mathfrak{h})$$

as follows:

$$\mathrm{Wick}(w)\Big|_{\bigotimes_{\mathrm{s}}^n\mathfrak{h}} := \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}w\otimes_{\mathrm{s}} 1\!\!1^{\otimes(n-p)}. \tag{3.11}$$

This definition extends to $w\in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$ by linearity. The operator $\mathrm{Wick}(w)$ is called a *Wick polynomial.* The operator w is called the *symbol* of the Wick polynomial $\mathrm{Wick}(w)$.

Before we describe properties of Wick polynomials, let us introduce more definitions. If u is an element of a Hilbert space $\mathcal{H}$, we denote by $(u|$ the map $\mathcal{H}\ni v\mapsto (u,v)\in\mathbb{C}$ and by $|u):\mathbb{C}\to\mathcal{H}$ its adjoint.

If $u\in\otimes_{\mathrm{s}}^m\mathfrak{h}$, $v\in\otimes_{\mathrm{s}}^n\mathfrak{h}$, $w\in B(\otimes_{\mathrm{s}}^p\mathfrak{h},\otimes_{\mathrm{s}}^q\mathfrak{h})$ with $m\leq p$, $n\leq q$, then, to simplify the notation, we will introduce the 'contracted' symbols

$$\begin{aligned}(v|w &:= \Big((v|\otimes_{\mathrm{s}} 1\!\!1^{\otimes(q-n)}\Big)w\ \in B(\otimes_{\mathrm{s}}^p\mathfrak{h},\otimes_{\mathrm{s}}^{q-n}\mathfrak{h}),\\ w|u) &:= w\Big(|u)\otimes_s 1\!\!1^{\otimes(p-m)}\Big)\ \in B(\otimes_{\mathrm{s}}^{p-m}\mathfrak{h},\otimes_{\mathrm{s}}^q\mathfrak{h}),\\ (v|w|u) &:= \Big((v|\otimes_{\mathrm{s}} 1\!\!1^{\otimes(q-n)}\Big)w\Big(|u)\otimes_s 1\!\!1^{\otimes(p-m)}\Big)\ \in B(\otimes_{\mathrm{s}}^{p-m}\mathfrak{h},\otimes_{\mathrm{s}}^{q-n}\mathfrak{h}).\end{aligned}$$

Theorem 3.12 *i) Case $p=q=0$. Let $\lambda\in\mathbb{C}=B(\otimes_{\mathrm{s}}^0\mathfrak{h},\otimes_{\mathrm{s}}^0 h)$. Then* $\mathrm{Wick}(\lambda)=\lambda 1\!\!1$.

ii) Case $p=q=1$. (Note that $\otimes_{\mathrm{s}}^1\mathfrak{h}=\mathfrak{h}$) For $b\in B(\mathfrak{h})$ we have

$$\mathrm{Wick}(b)=\mathrm{d}\Gamma(b).$$

iii) Cases $q=0$, $p=1$ and $q=1$, $p=0$. For $h\in\mathfrak{h}$ we have

$$\mathrm{Wick}(|h))=a^*(h),\quad \mathrm{Wick}((h|)=a(h).$$

iv) Let $u\in\otimes_{\mathrm{s}}^m\mathfrak{h}$, $v\in\otimes_{\mathrm{s}}^n\mathfrak{h}$, $w\in B(\otimes_{\mathrm{s}}^p\mathfrak{h},\otimes_{\mathrm{s}}^q\mathfrak{h})$. Then

$$\mathrm{Wick}\Big(|v)\otimes_{\mathrm{s}} w\otimes_{\mathrm{s}}(u|\Big)=\mathrm{Wick}(|v))\mathrm{Wick}(w)\mathrm{Wick}((u|).$$

v) Let $h_1',\dots,h_p',h_1,\dots,h_q\in\mathfrak{h}$. Then

$$\mathrm{Wick}\Big(|\ h_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}} h_q)(h_p'\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}} h_1')|\Big)=a^*(h_1)\cdots a^*(h_q)a(h_p')\cdots a(h_1').$$

vi) Let $w \in B(\otimes_{\rm s}^p\mathfrak{h}, \otimes_{\rm s}^q\mathfrak{h})$*. Then*

$$\mathrm{Wick}(w)^* = \mathrm{Wick}(w^*).$$

Note that *v)* of the above theorem (which follows immediately from *iv)*), justifies the name Wick polynomials.

If we fix a basis $\{h_i\}_{i\in I}$ of $\mathfrak{h}$, then any operator in $B(\otimes_{\rm s}^p\mathfrak{h}, \otimes_{\rm s}^q\mathfrak{h})$ can be written as a sum (convergent for the weak topology):

$$w = \sum_{i_1,\dots,i_q,i'_1,\dots,i'_p} w_{i_1,\dots,i_q,i'_p,\dots,i'_1} |h_{i_1} \otimes_{\rm s} \cdots \otimes_{\rm s} h_{i_q})(h_{i'_p} \otimes_{\rm s} \cdots \otimes_{\rm s} h_{i'_1}|$$

where we can assume that $w_{i_1,\dots,i_q,i'_p,\dots,i'_1}$ is separately symmetric wrt the first q and the last p indices. Then, writing $a_i^\#$ for $a^\#(h_i)$, we have

$$\mathrm{Wick}(w) = \sum_{i_1,\dots,i_q,i'_p,\dots,i'_1} w_{i_1,\dots,i_q,i'_p,\dots,i'_1} a^*_{i_1} \cdots a^*_{i_q} a_{i'_p} \cdots a_{i'_1}.$$

In later sections, we will consider the case when $\mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}k)$. Any operator w from $\mathcal{S}(\mathbb{R}^p)$ to $\mathcal{S}'(\mathbb{R}^q)$, in particular, any $w \in B(\otimes_{\rm s}^p\mathfrak{h}, \otimes_{\rm s}^q\mathfrak{h})$, has a distributional kernel

$$w(k_1, \dots, k_q, k'_p, \dots, k'_1) \in S'(\mathbb{R}^{p+q}), \tag{3.12}$$

where we can assume that the kernel w in (3.12) is separately symmetric wrt the first q and the last p variables. The following formal expression is then commonly used to denote the Wick polynomial Wick(w):

$$\int w(k_1, \dots, k_q, k'_p, \dots, k'_1) a^*(k_1) \cdots a^*(k_q) a(k'_p) \cdots a(k'_1) \mathrm{d}k_1 \cdots \mathrm{d}k_q \mathrm{d}k'_p \cdots \mathrm{d}k'_1. \tag{3.13}$$

Although the definition (3.11) always makes sense, let us describe a few cases in which a rigorous meaning can be attached to the formal expression (3.13).

First of all if $u \in \Gamma_{\rm fin}(S(\mathbb{R}))$, $a(k_1)\dots a(k_p)u$ is well defined as an element of $S(\mathbb{R}^p) \otimes \Gamma(\mathfrak{h})$. This shows that the expression

$$\begin{aligned} \int \;& w(k_1, \dots, k_q, k'_p, \dots, k'_1) \\ & \times (a(k_1)\dots a(k_q)u | a(k'_1)\dots a(k_p)u)_{\Gamma(\mathfrak{h})} \, \mathrm{d}k_1, \cdots \mathrm{d}k_q \mathrm{d}k'_p \cdots \mathrm{d}k'_1 \end{aligned} \tag{3.14}$$

is well defined for $u \in \Gamma_{\rm fin}(S(\mathbb{R}))$, $w \in S'(\mathbb{R}^{p+q})$. Hence if $w \in S'(\mathbb{R}^{p+q})$, (3.13) always makes sense as a quadratic form on $\Gamma_{\rm fin}(S(\mathbb{R}))$.

If $u \in \mathcal{D}(N^{p/2})$, $a(k_1)\dots a(k_p)u$ is also well defined as an element of $\bigotimes_{\rm s}^p L^2(\mathbb{R}) \otimes \Gamma(L^2(\mathbb{R}))$. and the expression (3.14) is well defined for $u \in \mathcal{D}(N^{\sup(p,q)/2})$, $w \in L^2(\mathbb{R}^{p+q})$. Hence if $w \in L^2(\mathbb{R}^{p+q})$, (3.12) makes sense as a quadratic form on $\mathcal{D}(N^{\sup(p,q)/2})$.

The following proposition summarizes basic properties of Wick polynomials.

Proposition 3.13 *i) If* $w \in B(\otimes_{\rm s}^p\mathfrak{h}, \otimes_{\rm s}^q\mathfrak{h})$ *and* $k + m \geq \frac{p+q}{2}$*, then*

$$\|(N+1)^{-k}\mathrm{Wick}(w)(N+1)^{-m}\| \leq \|w\|.$$

If moreover, $\text{s-}\lim w_n = w$*, then*

$$\text{s-}\lim_{n\to\infty}(N+1)^{-k}\mathrm{Wick}(w_n)(N+1)^{-m} = (N+1)^{-k}\mathrm{Wick}(w)(N+1)^{-m}.$$

ii) Identities:

Let $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$*,* $b \in B(\mathfrak{h})$*. Then*

$$[\mathrm{d}\Gamma(b), \mathrm{Wick}(w)] = \mathrm{Wick}([\mathrm{d}\Gamma(b), w]). \tag{3.15}$$

Let $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}_2), \Gamma(\mathfrak{h}_1))$*,* $q \in B(\mathfrak{h}_1, \mathfrak{h}_2)$*. Then*

$$\Gamma(q)\mathrm{Wick}(w\Gamma(q)) = \mathrm{Wick}(\Gamma(q)w)\Gamma(q). \tag{3.16}$$

Let $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}_1), \Gamma(\mathfrak{h}_1))$*,* $q \in B(\mathfrak{h}_1, \mathfrak{h}_2)$*. Then*

$$\Gamma(q)\mathrm{Wick}(w) = \mathrm{Wick}(\Gamma(q)w\Gamma(q^*))\Gamma(q), \text{ for isometric } q, \tag{3.17}$$

$$\Gamma(q)\mathrm{Wick}(w)\Gamma(q^{-1}) = \mathrm{Wick}(\Gamma(q)w\Gamma(q^{-1})), \quad \text{for unitary } q. \tag{3.18}$$

Let $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$*,* $h \in \mathfrak{h}$*. Then*

$$[\mathrm{Wick}(w), a^*(h)] = p\mathrm{Wick}(w|h)), \quad [\mathrm{Wick}(w), a(h)] = q\mathrm{Wick}((h|w), \tag{3.19}$$

$$W(h)\mathrm{Wick}(w)W(-h) = \sum_{s=0}^{p}\sum_{r=0}^{q}\frac{p!}{s!}\frac{q!}{r!}(\frac{i}{\sqrt{2}})^{p+q-r-s}\mathrm{Wick}(w_{s,r}), \tag{3.20}$$

where

$$w_{s,r} = (h^{\otimes(q-r)}|w|h^{\otimes(p-s)}). \tag{3.21}$$

Proof. The first part of *i)* is a particular case of the well known N_τ estimates (see eg [GJ1]). It follows directly from the definition (3.11) and the fact that

$$\frac{(n!(n+q-p)!)^{\frac{1}{2}}}{n-p!} \le (n+q-p)^{q/2}n^{p/2}.$$

The second part of *i)* follows similarly from (3.11). All identities of *ii)* are easy, except for the last one which follows for example from Thm. 3.12 *v)* and the identity

$$W(h)a^*(f)W(-h) = a^*(f) + \frac{i}{\sqrt{2}}(h|f).$$

□

Let now $\mathcal{K}$ be an additional Hilbert space. If $w \in B_{\mathrm{fin}}(\mathcal{K}\otimes\Gamma(\mathfrak{h}))$, then we can also define $\mathrm{Wick}(w)$ as an operator acting on $\mathcal{K}\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h})$, by

$$\mathrm{Wick}(w)\Big|_{\mathcal{K}\otimes\otimes_{\mathrm{s}}^n\mathfrak{h}} := \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\mathbb{1}_{\mathcal{K}}\otimes S_{n+q-p}w\otimes\mathbb{1}^{\otimes(n-p)},$$

for $w \in B(\mathcal{K}\otimes\otimes_{\mathrm{s}}^p\mathfrak{h}, \mathcal{K}\otimes\otimes_{\mathrm{s}}^q\mathfrak{h})$. This construction can be used for example if one considers generalizations of Pauli-Fierz Hamiltonians with more general interactions than those considered in [DG1].

In particular, this additional space can be also a Fock space. Then if

$$w \in B(\otimes_{\mathrm{s}}^{p_1}\mathfrak{h}_1 \otimes \otimes_{\mathrm{s}}^{p_2}\mathfrak{h}_2, \otimes_{\mathrm{s}}^{q_1}\mathfrak{h}_1 \otimes \otimes_{\mathrm{s}}^{q_2}\mathfrak{h}_2),$$

we define:

$$\begin{aligned}&\mathrm{Wick}\otimes\mathrm{Wick}(w)\Big|_{\otimes_{\mathrm{s}}^{n_1}\mathfrak{h}_1\otimes\otimes_{\mathrm{s}}^{n_2}\mathfrak{h}_2}\\ &:= \tfrac{\sqrt{n_1!(n_1+q_1-p_1)!}}{(n_1-p_1)!}\tfrac{\sqrt{n_2!(n_2+q_2-p_2)!}}{(n_2-p_2)!}(S_{n_1+q_1-p_1}\otimes S_{n_2+q_2-p_2})\mathbb{1}^{\otimes(n_1-p_1)}\otimes w\otimes \mathbb{1}^{\otimes(n_2-p_2)}.\end{aligned}$$

We extend this definition to $w \in B(\Gamma_{\mathrm{fin}}(\mathfrak{h}_1)\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h}_2))$ by linearity. The following proposition is completely similar to Prop. 3.13.

Proposition 3.14 *i) For* $w_i \in B(\Gamma_{\mathrm{fin}}(\mathfrak{h}_i)), i = 1,2$ *then*

$$\mathrm{Wick}\otimes\mathrm{Wick}(w_1\otimes w_2) = \mathrm{Wick}(w_1)\otimes\mathrm{Wick}(w_2).$$

ii) If

$$w = |\overset{q_1}{\underset{1}{\otimes_{\mathrm{s}}}} h_{i,1})(\overset{p_1}{\underset{1}{\otimes_{\mathrm{s}}}} g_{i,1}| \otimes |\overset{q_2}{\underset{1}{\otimes_{\mathrm{s}}}} h_{i,2})(\overset{p_2}{\underset{1}{\otimes_{\mathrm{s}}}} g_{i,2}|$$

then

$$\mathrm{Wick}\otimes\mathrm{Wick}(w) = \Big(\Pi_1^{q_1}a^*(h_{i,1})\otimes\Pi_1^{q_2}a^*(h_{i,2})\Big)\Big(\Pi_1^{p_1}a(g_{i,1})\otimes\Pi_1^{p_2}a(g_{i,2})\Big).$$

iii) For $j_i \in B(\mathfrak{h}_i), i = 1,2$, $w \in B(\Gamma_{\mathrm{fin}}(\mathfrak{h}_1)\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h}_2))$

$$\Gamma(j_1)\otimes\Gamma(j_2)\mathrm{Wick}\otimes\mathrm{Wick}(w\Gamma(j_1)\otimes\Gamma(j_2)) = \mathrm{Wick}\otimes\mathrm{Wick}(\Gamma(j_1)\otimes\Gamma(j_2)w)\Gamma(j_1)\otimes\Gamma(j_2).$$

iv) for $w \in B(\bigotimes_{\mathrm{s}}^{p_1}\mathfrak{h}_1\otimes\bigotimes_{\mathrm{s}}^{p_2}\mathfrak{h}_2, \bigotimes_{\mathrm{s}}^{q_1}\mathfrak{h}_1\otimes\bigotimes_{\mathrm{s}}^{q_2}\mathfrak{h}_2)$ *and* $k_i + m_i \geq \frac{p_i+q_i}{2}$, $i = 1,2$ *we have:*

$$\|(N+1)^{-k_1}\otimes(N+1)^{-k_2}\mathrm{Wick}\otimes\mathrm{Wick}(w)(N+1)^{-m_1}\otimes(N+1)^{-m_2}\| \leq \|w\|.$$

In the next proposition we describe additional properties of Wick polynomials which have a nice formulation if one uses the Wick ⊗ Wick notation.

Proposition 3.15 *i) Let* $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}_1\oplus\mathfrak{h}_2))$. *Then*

$$U^*\mathrm{Wick}(w)U = \mathrm{Wick}\otimes\mathrm{Wick}(\tilde{U}^*w\tilde{U}).$$

Here the map $\tilde{U} : \Gamma_{\mathrm{fin}}(\mathfrak{h}_1)\otimes\Gamma_{\mathrm{fin}}(\mathfrak{h}_2) \to \Gamma_{\mathrm{fin}}(\mathfrak{h}_1\oplus\mathfrak{h}_2)$ *is defined as follows (recall that* i_1, i_2 *are the injections of* $\mathfrak{h}_1, \mathfrak{h}_2$ *into* $\mathfrak{h}_1\oplus\mathfrak{h}_2$*):*

$$\tilde{U}u_1\otimes u_2 := \frac{(p+q)!}{p!q!}\Gamma(i_1)u_1\otimes_{\mathrm{s}}\Gamma(i_2)u_2, \quad u_1\in\otimes_{\mathrm{s}}^p\mathfrak{h}_1,\ u_2\in\otimes_{\mathrm{s}}^q\mathfrak{h}_2. \tag{3.22}$$

ii) Let $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$. *Then*

$$\mathrm{Wick}(w)I = I\mathrm{Wick}\otimes\mathrm{Wick}(Pw\tilde{I}).$$

Here $\tilde{I}:\Gamma_{\rm fin}(\mathfrak{h})\otimes\Gamma_{\rm fin}(\mathfrak{h})\to\Gamma_{\rm fin}(\mathfrak{h})$ *is defined as follows:*

$$\tilde{I}u\otimes v:=\frac{(p+q)!}{p!q!}u\otimes_{\rm s}v,\quad u\in\otimes_{\rm s}^{p}\mathfrak{h},\ v\in\otimes_{\rm s}^{q}\mathfrak{h}. \tag{3.23}$$

and $P:\Gamma_{\rm fin}(\mathfrak{h})\to\Gamma_{\rm fin}(\mathfrak{h})\otimes\Gamma_{\rm fin}(\mathfrak{h})$ *is defined as*

$$Pu:=u\otimes\Omega,\quad u\in\Gamma_{\rm fin}(\mathfrak{h}).$$

iii) Let us keep the notation of ii) and let j *be as in Subsection 3.10. Then*

$$\mathrm{Wick}(\Gamma(j_0)w)I(j)=I(j)\mathrm{Wick}\otimes\mathrm{Wick}(Pw\tilde{I}\Gamma(j_0)\otimes\Gamma(j_\infty)).$$

Proof. By linearity it suffices to check the identities *i)* and *ii)* for w of rank one. One can then use the identities in Lemma 3.5 *iii)*, Thm. 3.12 *v)* and Prop. 3.14 *ii)* to verify *i)* and *ii)*. Finally *iii)* follows from *ii)*, the fact that $I(j)=I\Gamma(j_0)\otimes\Gamma(j_\infty)$ and Prop. 3.14 *iii)*.□

In the following sections, we will need to estimate some commutators between Wick polynomials and $\Gamma(q), I^*(j)$ operators. To this end we will need the identities described in the next proposition.

Proposition 3.16 *Let* $w\in B_{\rm fin}(\Gamma(\mathfrak{h}))$. *Then*

i) for $q\in B(\mathfrak{h})$:

$$[\Gamma(q),\mathrm{Wick}(w)]=\Gamma(q)\mathrm{Wick}\Big(w(1\!\!1-\Gamma(q))\Big)+\mathrm{Wick}\Big((\Gamma(q)-1\!\!1)w\Big)\Gamma(q).$$

ii) for $j=(j_0,j_\infty)\in B(\mathfrak{h}\otimes\mathfrak{h},\mathfrak{h})$:

$$\begin{array}{rl} & I^*(j)\mathrm{Wick}(w)-(\mathrm{Wick}(w)\otimes 1\!\!1)I^*(j)\\ = & I^*(j)\mathrm{Wick}\Big(w(1\!\!1-\Gamma(j_0^*))\Big)\\ + & \mathrm{Wick}\otimes\mathrm{Wick}\Big((\Gamma(j_0^*)-1\!\!1)\otimes\Gamma(j_\infty^*)\tilde{I}^*wP^*\Big)I^*(j)\\ + & \mathrm{Wick}\otimes\mathrm{Wick}\Big(1\!\!1\otimes(\Gamma(j_\infty^*)-|\Omega)(\Omega|)\tilde{I}^*wP^*\Big)I^*(j). \end{array}$$

Proof. *i)* follows directly from the identity (3.16). To prove *ii)*, we deduce from Prop. 3.15 that

$$I^*(j)\mathrm{Wick}(w)=I^*(j)\mathrm{Wick}\Big(w(1\!\!1-\Gamma(j_0^*))\Big)+\mathrm{Wick}\otimes\mathrm{Wick}\Big(\Gamma(j_0^*)\otimes\Gamma(j_\infty^*)\tilde{I}^*wP^*\Big)I^*(j).$$

We use then the identities

$$w\otimes|\Omega)(\Omega|=PwP^*,\ P=(1\!\!1\otimes|\Omega)(\Omega|)\tilde{I}^*,$$

to obtain

$$\begin{aligned}\mathrm{Wick}(w)\otimes 1\!\!1 &=\mathrm{Wick}\otimes\mathrm{Wick}\Big(w\otimes|\Omega)(\Omega|\Big)\\ &=\mathrm{Wick}\otimes\mathrm{Wick}\Big((1\!\!1\otimes|\Omega)(\Omega|)\tilde{I}^*wP^*\Big).\end{aligned}$$

Next we write

$$\begin{aligned}&\Gamma(j_0^*)\otimes\Gamma(j_\infty^*)-\mathbb{1}\otimes|\Omega)(\Omega|\\&=\Big(\Gamma(j_0^*)-\mathbb{1}\Big)\otimes\Gamma(j_\infty^*)+\mathbb{1}\otimes\Big(\Gamma(j_\infty^*)-|\Omega)(\Omega|\Big)\end{aligned}$$

to obtain *ii)*. □

Lemma 3.17 *Assume that* $\mathfrak{h}=L^2(\mathbb{R},\mathrm{d}k)$ *and that* $w\in B(\otimes_{\mathrm{s}}^p\mathfrak{h},\otimes_{\mathrm{s}}^r\mathfrak{h})$ *is given by a kernel* w *as in (3.12) with* $w\in L^2(\mathbb{R}^{p+r})$.

i) Let $q\in B(\mathfrak{h})$, $\|q\|\leq 1$. *Then for* $m+k\geq\frac{p+r}{2}$:

$$\begin{aligned}&\|(N+1)^{-m}[\Gamma(q),\mathrm{Wick}(w)](N+1)^{-k}\|\\\leq\;&C_{p,r}\sup\nolimits_{1\leq i\leq p+r}\|\mathbb{1}^{\otimes(i-1)}\otimes(\mathbb{1}-q)\otimes\mathbb{1}^{\otimes(p+r-i)}w\|_{L^2(\mathbb{R}^{p+r})}.\end{aligned}\tag{3.24}$$

ii) Let $j=(j_0,j_\infty)$ *with* $j_0,j_\infty\in B(\mathfrak{h})$, $\|j_0^*j_0+j_\infty^*j_\infty\|\leq 1$. *Then for* $m+k\geq\frac{p+r}{2}$:

$$\begin{aligned}&\|(N_0+N_\infty+1)^{-m}\Big(I^*(j)\mathrm{Wick}(w)-(\mathrm{Wick}(w)\otimes\mathbb{1})I^*(j)\Big)(N+1)^{-k}\|\\\leq\;&C_{p,r}\sup\nolimits_{1\leq i\leq p+r}\|\mathbb{1}^{\otimes(i-1)}\otimes(\mathbb{1}-j_0)\otimes\mathbb{1}^{\otimes(p+r-i)}w\|_{L^2(\mathbb{R}^{p+r})}\\+\;&C_{p,r}\sup\nolimits_{1\leq i\leq p+r}\|\mathbb{1}^{\otimes(i-1)}\otimes(j_\infty)\otimes\mathbb{1}^{\otimes(p+r-i)}w\|_{L^2(\mathbb{R}^{p+r})}.\end{aligned}\tag{3.25}$$

Proof. To prove *i)* it suffices by Prop. 3.13 *i)* to estimate the operator norm of the symbols $w(\mathbb{1}-\Gamma(q))$ and $(\Gamma(q)-\mathbb{1})w$, which are bounded by the r.h.s. of (3.24).

Similarly to prove *ii)* it suffices by Prop. 3.14 *iv)* to estimate the operator norm of the symbols $w(\mathbb{1}-\Gamma(j_0^*))$, $(\Gamma(j_0^*)-\mathbb{1})\otimes\Gamma(j_\infty^*)\tilde{I}^*wP^*$ and $(\mathbb{1}\otimes(\Gamma(j_\infty^*)-|\Omega)(\Omega|)\tilde{I}^*wP^*$ (note that $\|I^*(j)\|=1$ by Lemma 3.8 *iv)* and $I^*(j)N=(N_0+N_\infty)I^*(j)$).

The norm of the first symbol is bounded by the r.h.s. of (3.25) by the same argument as in *i)*. Since $\|\Gamma(j_\infty^*)\|=\|P^*\|=1$ the norm of the second symbol is less than $\|w\tilde{I}(\Gamma(j_0)\mathbb{1})\otimes\mathbb{1}\|$ which is bounded by the r.h.s. of (3.25). Similarly the norm of the third symbol is less than $\|w\tilde{I}\mathbb{1}\otimes(\Gamma(j_\infty)-|\Omega)(\Omega|)\|$. This is also bounded by the r.h.s. of (3.25). (Note that $\Gamma(j_\infty)-|\Omega)(\Omega|$ vanishes on the vacuum sector). □

4 Fock representations of CCR

In this section we described the construction of the Fock subrepresentation of a regular CCR representation, (see [BR, CMR]).

4.1 Construction of the Fock subrepresentation

Suppose that we are given a regular CCR representation over a pre-Hilbert space $\mathfrak{g}$ in the Hilbert space $\mathcal{H}$. Let $\mathfrak{h}$ be the completion of $\mathfrak{g}$.

We define the *space of vacua*

$$\mathcal{K}_\pi:=\{u\in\mathcal{H}\mid a_\pi(h)u=0,\ h\in\mathfrak{g}\}.$$

Proposition 4.1 *i)* $\mathcal{K}_\pi$ *is a closed space.*
ii) $\mathcal{K}_\pi$ *is contained in the set of analytic vectors of* $\phi_\pi(h)$, $h \in \mathfrak{g}$.

Proof. $\mathcal{K}_\pi$ is closed as an intersection of null spaces of closed operators. To prove *ii)* we will show that

$$(u|W_\pi(h)u) = \exp(-\|h\|^2/4),\ u \in \mathcal{K}_\pi. \tag{4.1}$$

Clearly, $\mathcal{K}_\pi \subset \mathcal{D}(\phi_\pi(h))$, hence for $u \in \mathcal{K}_\pi$

$$f(t) := (u|W_\pi(th)u)$$

is continuously differentiable. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(t) &= \mathrm{i}(u|\phi_\pi(h)W_\pi(th)u)\\ &= \frac{\mathrm{i}}{\sqrt{2}}(a_\pi(h)u|W_\pi(th)u) + \frac{\mathrm{i}}{\sqrt{2}}(u|W_\pi(th)a_\pi(h)u) - \frac{1}{2}t\|h\|^2(u|W_\pi(th)u)\\ &= -\frac{1}{2}t\|h\|^2 f(t).\end{aligned}$$

Therefore

$$f(t) = \exp(-t^2\|h\|^2/4),$$

which shows (4.1). Now the spectral theorem implies that u is an analytic vector for $\phi_\pi(h)$ and thus $t \mapsto W_\pi(th)u$ extends to an analytic function around the real axis. □

Define the space $\mathcal{H}_\pi := \mathcal{K}_\pi \otimes \Gamma(\mathfrak{h})$. We define $\Omega_\pi : \mathcal{K}_\pi \otimes \Gamma_{\mathrm{fin}}(\mathfrak{g}) \to \mathcal{H}$ by setting

$$\Omega_\pi \psi \otimes \phi(h)^p\Omega := \phi_\pi(h)^p\psi, \qquad h \in \mathfrak{g}, \quad \psi \in \mathcal{K}^+. \tag{4.2}$$

(Note that the vectors $\phi(h)^p\Omega = 2^{-p/2}a^*(h)^p\Omega$ for $h \in \mathfrak{g}$ span $\otimes_{\mathrm{s}}^p\mathfrak{h}$)

Proposition 4.2 *The map* Ω_π *extends to an isometric map*

$$\Omega_\pi : \mathcal{H}_\pi \to \mathcal{H},$$

satisfying

$$\Omega_\pi \mathbb{1} \otimes a^\#(h) = a_\pi^\#(h)\Omega_\pi,\ h \in \mathfrak{g}. \tag{4.3}$$

4.2 Number operator

We discuss now the number operator N_π associated to a regular CCR representation in the space $\mathcal{H}$. One can give two equivalent definitions of N_π. Note that a more general notion of a number operator associated to a CCR representation, (which in particular does not need to be a positive operator) has been introduced by Chaiken in [Ch1, Ch2].

The first definition uses the intertwiner Ω_π. Define $\mathcal{D}(N_\pi) := \Omega_\pi\mathcal{K}_\pi \otimes \mathcal{D}(N)$, which is a subspace of $\mathcal{H}_\pi$ whose closure is $\mathrm{Ran}\Omega_\pi$. Let N_π be the operator on $\mathcal{H}$ with the domain $\mathcal{D}(N_\pi)$ defined by

$$N_\pi := \Omega_\pi \mathbb{1} \otimes N\Omega_\pi^*.$$

(Note that N_π needs not be densely defined).

Before we give an alternative definition of N_π, let us recall some facts about quadratic forms. We will assume that a positive quadratic form is defined on the whole space $\mathcal{H}$ and takes values in $[0,\infty]$. The domain of a positive quadratic form b is defined as

$$\mathcal{D}(b) := \{u \in \mathcal{H}|\ b(u) < \infty\}.$$

If the form b is closed, then there exists a unique positive self-adjoint operator B such that

$$\mathcal{D}(b) = \mathcal{D}(B^{\frac{1}{2}}), \quad b(u) = (u|Bu).$$

If A is a closed operator, then $\|Au\|^2$ is a closed form. The sum of closed forms is a closed form, and the supremum of a family of closed forms is a closed form.

The following theorem gives an alternative definition of N_π.

Theorem 4.3 *For each finite dimensional space $\mathfrak{f} \subset \mathfrak{g}$, one defines*

$$n_{\pi,\mathfrak{f}}(u) := \sum_{i=1}^{\dim\mathfrak{f}} \|a_\pi(h_i)u\|^2,$$

where $\{h_i\}$ is an orthonormal basis of $\mathfrak{f}$. (If $u \notin \mathcal{D}(a_\pi(h_i)$ for some i, then $n_{\pi,\mathfrak{f}}(u) = \infty$). Then the quadratic form $n_{\pi,\mathfrak{f}}$ does not depend on the choice of the basis $\{h_i\}$ of $\mathfrak{f}$. The quadratic form n_π is defined by

$$n_\pi(u) := \sup_{\mathfrak{f}} n_{\pi,\mathfrak{f}}(u),\ u \in \mathcal{H}.$$

Then $\mathcal{D}(n_\pi) = \mathcal{D}((N_\pi)^{\frac{1}{2}})$ and

$$n_\pi(u) = (u|N_\pi u),\ u \in \mathcal{D}(N_\pi).$$

In particular, $\mathrm{Ran}\Omega_\pi = \overline{\mathcal{D}(n_\pi)}$.

To prepare for the proof of the above theorem, note that n_π defines a positive operator, which we denote temporarily $\tilde{N}_\pi$, such that $\mathcal{D}(n_\pi) = \mathcal{D}((\tilde{N}_\pi)^{\frac{1}{2}})$ and

$$n_\pi(u) = (u|\tilde{N}_\pi u),\ u \in \mathcal{D}(\tilde{N}_\pi). \tag{4.4}$$

Our aim is to show that $\tilde{N}_\pi = N_\pi$.

Note also that

$$\mathcal{D}(n_\pi) \subset \mathcal{D}(\phi_\pi(h)), \quad h \in \mathfrak{g}. \tag{4.5}$$

Lemma 4.4 *If $v \in \mathcal{D}(\tilde{N}_\pi^{\frac{1}{2}})$, and F is a Borel function, then*

$$a_\pi(h)F(\tilde{N}_\pi - 1)v = F(\tilde{N}_\pi)a_\pi(h)v. \tag{4.6}$$

Proof. First we note that $W_\pi(h)$ maps $\mathcal{D}(n_\pi)$ into itself and we have

$$n_\pi(W_\pi(h)u) = n_\pi(u) + (u|\phi_\pi(\mathrm{i}h)u) + \|h\|^2\|u\|^2/2. \tag{4.7}$$

In fact, using (2.5) we see that (4.7) is true if we replace n_π with $n_{\pi,\mathfrak{f}}$, where $\mathfrak{f}$ is a finite subspace of $\mathfrak{g}$ containing h. Then (4.7) follows immediately.

By the polarization identity, (4.7) has the following consequence for $u, w \in \mathcal{D}(n_\pi)$:

$$(4.8)\qquad (\tilde{N}_\pi^{\frac{1}{2}} W_\pi(h)w|\tilde{N}_\pi^{\frac{1}{2}} W_\pi(h)u) = (\tilde{N}_\pi^{\frac{1}{2}} w|\tilde{N}_\pi^{\frac{1}{2}} u) + (w|\phi_\pi(\mathrm{i}h)u) + \|h\|^2(w|u)/2.$$

Replacing w with $W_\pi(h)^* v$ and using the invariance of $\mathcal{D}(n_\pi)$ under $W_\pi(h)$ we can rewrite (4.8) as follows, for $u, v \in \mathcal{D}(n_\pi)$:

$$(4.9)\qquad \begin{aligned} &(\tilde{N}_\pi^{\frac{1}{2}} v|\tilde{N}_\pi^{\frac{1}{2}} W_\pi(h)u) \\ &= (\tilde{N}_\pi^{\frac{1}{2}} W_\pi(h)^* v|\tilde{N}_\pi^{\frac{1}{2}} u) + (W_\pi(h)^* v|\phi_\pi(\mathrm{i}h)u) + \tfrac{1}{2}\|h\|^2(W_\pi(h)^* v|u). \end{aligned}$$

Next assume in addition that $u, v \in \mathcal{D}(\tilde{N}_\pi)$. Then we can rewrite (4.9) as

$$(4.10)\qquad \begin{aligned} &(\tilde{N}_\pi v|W_\pi(h)u) \\ &= (W_\pi(h)^* v|\tilde{N}_\pi u) + (W_\pi(h)^* v|\phi_\pi(\mathrm{i}h)u) + \tfrac{1}{2}\|h\|^2(W_\pi(h)^* v|u). \end{aligned}$$

Next we set $h = tg$, for $g \in \mathfrak{g}$ and we differentiate (4.10) w.r.t. t. (Differentiating is allowed by (4.5)). We obtain

$$(4.11)\qquad (\tilde{N}_\pi v|\phi_\pi(g)u) = (\phi_\pi(g)v|\tilde{N}_\pi u) - \mathrm{i}(v|\phi_\pi(\mathrm{i}g)u).$$

Substituting $\mathrm{i}g$ for g in (4.11) we obtain

$$(4.12)\qquad (\tilde{N}_\pi v|\phi_\pi(\mathrm{i}g)u) = -(\phi_\pi(\mathrm{i}g)v|\tilde{N}_\pi u) + \mathrm{i}(v|\phi_\pi(g)u).$$

Adding up (4.11) and (4.12) we get

$$(4.13)\qquad (\tilde{N}_\pi v|a_\pi(g)u) = (a_\pi^*(g)v|\tilde{N}_\pi u) - (v|a_\pi(g)u),\ u, v \in D(\tilde{N}_\pi).$$

Next let us assume that $u \in \mathcal{D}(\tilde{N}_\pi^{\frac{3}{2}})$. Then $\tilde{N}_\pi u \in \mathcal{D}(\tilde{N}_\pi^{\frac{1}{2}}) \subset \mathcal{D}(a_\pi(g))$. Hence (4.13) implies

$$(4.14)\qquad (\tilde{N}_\pi v|a_\pi(g)u) = (v|a_\pi(g)(\tilde{N}_\pi - \mathbb{1})u).$$

Therefore, $a_\pi(g)u \in \mathcal{D}(\tilde{N}_\pi)$ and we have

$$(4.15)\qquad \tilde{N}_\pi a_\pi(g)u = a_\pi(g)(\tilde{N}_\pi - 1)u,$$

or equivalently

$$(4.16)\qquad (\tilde{N}_\pi + \lambda)a_\pi(g)u = a_\pi(g)(\tilde{N}_\pi + \lambda - 1)u.$$

Now let $v \in \mathcal{D}(\tilde{N}_\pi^{\frac{1}{2}})$ and $\lambda > 1$. Then $(\tilde{N}_\pi + \lambda - 1)^{-1}v \in \mathcal{D}(\tilde{N}_\pi^{\frac{3}{2}})$. Therefore, by (4.16)

$$(4.17)\qquad (\tilde{N}_\pi + \lambda)a_\pi(g)(\tilde{N}_\pi + \lambda - 1)^{-1}v = a_\pi(g)v.$$

Multiplying this with $(\tilde{N}_\pi + \lambda)^{-1}$ we obtain

$$(4.18)\qquad a_\pi(g)(\tilde{N}_\pi + \lambda - 1)^{-1}v = (\tilde{N}_\pi + \lambda)^{-1}a_\pi(g)v.$$

Since linear combinations of functions $(\tilde{N}_\pi + \lambda)^{-1}$ with $\lambda > 0$ are strongly dense in the Von Neumann algebra of functions of $\tilde{N}_\pi$, and $a_\pi(g)$ is closed, (4.18) implies

$$a_\pi(g)F(\tilde{N}_\pi - 1)v = F(\tilde{N}_\pi)a_\pi(g)v,\ v \in \mathcal{D}(\tilde{N}_\pi^{\frac{1}{2}})$$

for any bounded Borel function F. □

Lemma 4.5 *$\mathcal{K}_\pi = \{0\}$ implies $\mathcal{D}(n_\pi) = \{0\}$.*

Proof. Suppose that $\mathcal{D}(n_\pi) \neq \{0\}$. We know that $\tilde{N}_\pi \geq 0$. Therefore, $\sigma(\tilde{N}_\pi)$ is nonempty and bounded from below. Hence $\lambda_0 := \inf \sigma(\tilde{N}_\pi)$ is a finite number, and $\mathrm{Ran}1\!\!1_{[\lambda_0,\lambda_0+1[}(\tilde{N}_\pi) \neq \{0\}$. By Lemma 4.4, for any $h \in \mathfrak{h}$

$$a_\pi(h)1\!\!1_{[\lambda_0,\lambda_0+1[}(\tilde{N}_\pi) = 1\!\!1_{[\lambda_0-1,\lambda_0[}(\tilde{N}_\pi)a_\pi(h). \tag{4.19}$$

But

$$1\!\!1_{[\lambda_0-1,\lambda_0[}(\tilde{N}_\pi) = 0.$$

Therefore, (4.19) is zero and

$$\mathrm{Ran}1\!\!1_{[\lambda_0-1,\lambda_0[}(\tilde{N}_\pi) \subset \mathcal{K}_\pi.$$

□

The following lemma is immediate:

Lemma 4.6 *Suppose that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Suppose that*

$$\mathfrak{h} \ni h \mapsto W_\pi(h) \in \mathcal{H}$$

is a CCR representation and $W_\pi(h)$ leave $\mathcal{H}_0$ invariant. Then $W_\pi(h)$ leave also $\mathcal{H}_1$ invariant. Thus we have two CCR representations

$$\mathfrak{g} \ni h \mapsto W_\pi(h)\Big|_{\mathcal{H}_0},$$

$$\mathfrak{g} \ni h \mapsto W_\pi(h)\Big|_{\mathcal{H}_1}.$$

Let $\mathcal{K}_{\pi,i}$, $\tilde{N}_{\pi,i}$ denote the corresponding spaces of vacua and the operators defined by (4.4) for the representations $i = 0, 1$. Then

$$\mathcal{K}_\pi = \mathcal{K}_{\pi,0} \oplus \mathcal{K}_{\pi,1}, \tag{4.20}$$

$$\tilde{N}_\pi = \tilde{N}_{\pi,0} \oplus \tilde{N}_{\pi,1}. \tag{4.21}$$

Lemma 4.7 *The operators $W_\pi(h)$ preserve $\mathrm{Ran}\Omega_\pi$ for $h \in \mathfrak{g}$.*

Proof. Since Ω_π is isometric, $\Omega_\pi(\mathcal{K}_\pi \otimes \Gamma_{\mathrm{fin}}(\mathfrak{g}))$ is dense in $\mathrm{Ran}\Omega_\pi$. It is also preserved by $\phi_\pi(h)$, $h \in \mathfrak{g}$, and consists of vectors analytic for $\phi_\pi(h)$. Hence, it is also preserved by $W_\pi(h) = \mathrm{e}^{\mathrm{i}\phi_\pi(h)}$. □

Proof of Theorem 4.3. By Lemma 4.7, we are in the situation of Lemma 4.6 and we have two CCR representations in $\mathcal{H}_0 = \mathrm{Ran}\Omega_\pi$ and in $\mathcal{H}_1 = \mathcal{H}_0^\perp$.

By the definition of N_π, we have

$$N_\pi = N_{\pi,0} \oplus N_{\pi,1},$$

where $\mathcal{D}(N_{\pi,1}) = \{0\}$. We check immediately by (4.3) that $\tilde{N}_{\pi,0} = N_{\pi,0}$.

We know that $\mathcal{K}_\pi \subset \mathcal{H}_0$, hence $\mathcal{K}_{\pi,1} = \{0\}$. By Lemma 4.5, this implies $\mathcal{D}(\tilde{N}_{1,\pi}) = \{0\}$. Therefore, $\tilde{N}_\pi = N_\pi$. □

5 Gaussian random processes and the Q-space representation

In this section we describe the $Q-$space representation of Fock space and discuss the notion of Wick ordering associated to a $Q-$space representation. We also recall the notion of hypercontractivity, following [S-H.K, Si1].

5.1 Gaussian processes

Let $\mathfrak{f}$ be a real Hilbert space with the scalar product (h_1, h_2), $h_1, h_2 \in \mathfrak{f}$. Let Q be a space with a σ-algebra $\mathcal{Q}$ and a probability measure μ. Let $\mathrm{Exp}(F)$ denote $\int_Q F\mathrm{d}\mu$, for any measurable function F on Q. A linear map

$$\mathfrak{f} \ni h \mapsto \phi(h) \tag{5.1}$$

into measurable functions on Q is called a Gaussian random process if

$$\mathrm{Exp}\left(\phi(h)^{2p}\right) = 2^{-p}(h,h)^p, \quad \mathrm{Exp}\left(\phi(h)^{2p+1}\right) = 0,$$

or equivalently

$$\mathrm{Exp}\left(\mathrm{e}^{\mathrm{i}\phi(h)}\right) = \exp\left(-\frac{1}{2}(h,h)\right).$$

Proposition 5.1 *The following conditions are equivalent:*
(1) $\mathcal{Q}$ is the smallest σ-algebra for which $\phi(h)$, $h \in \mathfrak{f}$ are measurable;
(2) $L^2(Q,\mu)$ is spanned by $\phi(h)^n$, $h \in \mathfrak{f}$;
(3) $L^\infty(Q,\mu)$ is the smallest W^*-algebra containing $\mathrm{e}^{\mathrm{i}\phi(h)}$, $h \in \mathfrak{f}$.

We refer to [Si1, Lemma I.5] for the proof.

If the conditions of the above proposition are satisfied, then one says that the process (5.1) is *full*. If not, one can always make it full by choosing a smaller σ-algebra of subsets of Q. (Obviously, this procedure does not change the Hilbert space spanned by $\phi(h)$, nor the W^*-algebra spanned by $\mathrm{e}^{\mathrm{i}\phi(h)}$). Let us assume that the random process (5.1) is full.

Let P_n denote the projection onto polynomials of degree n in $\phi(h)$, $h \in \mathfrak{f}$, inside $L^2(Q,\mu)$. For any $h_1, \dots, h_n \in \mathfrak{f}$ we define

$$:\phi(h_1)\cdots\phi(h_n): \ := (1\!\!1 - P_{n-1})\phi(h_1)\cdots\phi(h_n).$$

We recall the well known *Wick identities*

$$\begin{aligned} :\phi(h)^n: &= \sum_{m=0}^{[n/2]} \tfrac{n!}{m!(n-2m)!}\phi(h)^{n-2m}(-\tfrac{1}{2}(h,h))^m, \\ \phi(h)^n &= \sum_{m=0}^{[n/2]} \tfrac{n!}{m!(n-2m)!} :\phi(h)^{n-2m}: (\tfrac{1}{2}(h,h))^m. \end{aligned} \tag{5.2}$$

5.2 $Q-$space representation of Fock space

Let $\mathfrak{h}$ be a Hilbert space with a complex conjugation c, that is an antilinear map $c : \mathfrak{h} \to \mathfrak{h}$ such that $c^2 = 1$ and $(ch|cg) = (g|h)$. We set $\mathfrak{h}_c := \{h \in \mathfrak{h} \mid ch = h\}$. Let $\mathfrak{M}_c \subset B(\Gamma(\mathfrak{h}))$ be the abelian Von Neumann algebra generated by the Weyl operators $W(h)$ for $h \in \mathfrak{h}_c$. The following basic result follows from the fact that Ω is a cyclic vector for $\mathfrak{M}_c$ (see eg [S-H.K]).

Theorem 5.2 *There exists a compact Hausdorff space Q, a probability measure μ on Q and a unitary map R such that*

$$R : \Gamma(\mathfrak{h}) \to L^2(Q, d\mu),$$

$$R\Omega = 1,$$

$$R\mathfrak{M}_c R^* = L^\infty(Q, d\mu).$$

where $1 \in L^2(Q, \mathrm{d}\mu)$ is the constant function equal to 1 on Q. Moreover,

$$R\Gamma(c)u = \overline{Ru},\ u \in \Gamma(\mathfrak{h}),$$

and $\mathfrak{h}_c \ni h \mapsto R\phi(h)R^*$ is a full Gaussian random process on Q.

The space $L^2(Q, d\mu)$ is called the *$Q-$space representation* of the Fock space $\Gamma(\mathfrak{h})$ associated to $\mathfrak{M}_c$.

The following property of the $Q-$space representation is often useful (see [Si1, Prop.1.7]).

Proposition 5.3 *Let $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_i, i = 1, 2$ are Hilbert spaces with conjugations c_i. Equip $\mathfrak{h}$ with the conjugation $c = c_1 \oplus c_2$. Then as $Q-$space representation of the Fock space $\Gamma(\mathfrak{h})$ one can take $L^2(Q, d\mu)$ for*

$$Q = Q_1 \times Q_2,\ \mu = \mu_1 \otimes \mu_2,$$

where $L^2(Q_i, d\mu_i), i = 1, 2$ is the $Q-$space representation of $\Gamma(\mathfrak{h}_i)$. We have

$$RU = R_1 \otimes R_2,$$

where $U : \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \to \Gamma(\mathfrak{h})$ is defined in Subsect. 3.8.

To simplify notation, we will often omit the unitary transformation R in the formulas. Similarly a function V on Q will be identified with the operator of multiplication by V on $\Gamma(\mathfrak{h}) \equiv L^2(Q, d\mu)$.

In particular an element of $\Gamma(\mathfrak{h}) \equiv L^2(Q, \mathrm{d}\mu)$ can be considered as a multiplication operator on $\Gamma(\mathfrak{h})$, ie as an unbounded operator affiliated to the Von Neumann algebra $\mathfrak{M}_c$. For $v \in \Gamma(\mathfrak{h})$ this operator will be denoted by $\mathrm{Wick}_c(v)$. It is the unique operator affiliated to $\mathfrak{M}_c$ such that $\mathrm{Wick}_c(v)\Omega = v$. For instance, if $h \in \mathfrak{h} = \otimes_{\mathrm{s}}^1 \mathfrak{h}$:

$$\mathrm{Wick}_c(h) = a^*(h) + a(ch).$$

One can generalize this formula to an arbitrary $v \in \Gamma_{\mathrm{fin}}(\mathfrak{h})$, by writing $\mathrm{Wick}_c(v)$ as a Wick polynomial.

Proposition 5.4 *Let $v \in \Gamma_{\mathrm{fin}}(\mathfrak{h})$. Then*

$$\mathrm{Wick}_c(v) = \mathrm{Wick}(\gamma_c(v)),$$

where

$$\gamma_c(v) : \Gamma_{\mathrm{fin}}(\mathfrak{h}) \mapsto \Gamma_{\mathrm{fin}}(\mathfrak{h})$$

is defined by

$$(u_2|\gamma_c(v)u_1) := (\tilde{I}\Gamma(c)u_1 \otimes u_2|N!^{-\frac{1}{2}}v),\ u_1, u_2 \in \Gamma_{\mathrm{fin}}(\mathfrak{h}). \tag{5.3}$$

Proof. The proposition follows from Prop. 5.6 below. In fact by linearity we may assume that $v = \Pi_1^p a^*(h_i)\Omega$. Using then the concrete expression of $\gamma_c(v)$ given in Prop. 5.6 *i)* we easily check the proposition. □

Proposition 5.5

$$i)\ \gamma_c(v)\Omega = v,$$

$$ii)\ \gamma_c(v)\Gamma(c) = \Gamma(c)\gamma_c(\Gamma(c)v),$$

$$iii)\ \text{ if } v \in \otimes_{\rm s}^p \mathfrak{h},\ \gamma_c(v) \in \mathop{\oplus}_{r=0}^{p} B(\otimes_{\rm s}^r \mathfrak{h}, \otimes_{\rm s}^{p-r}\mathfrak{h}).$$

Using the identity (3.16) (which is also true for antilinear q) we see that *ii)* in Prop. 5.5 is equivalent to the identity

$$\Gamma(c){\rm Wick}_c(v) = {\rm Wick}_c(\Gamma(c)v)\Gamma(c),$$

which in turns follows from the fact that $\Gamma(c)$ is simply the complex conjugation on $L^2(Q, {\rm d}\mu)$.

Proposition 5.6 *i) Let* $v = \Pi_1^p a^*(h_i)\Omega$. *Then:*

$${\rm Wick}_c(v) = \sum_{I\subset\{1,\dots,p\}} \Pi_{i\in I} a^*(h_i)\Pi_{i\not\in I} a(ch_i),$$

$$\gamma_c(v) = \sum_{I\subset\{1,\dots,p\}} |\mathop{\otimes_{\rm s}}_{i\in I} h_i)(\mathop{\otimes_{\rm s}}_{i\not\in I} ch_i|.$$

ii) (Wick's theorem). For $h \in \mathfrak{h}_c$*:*

$$:\phi(h)^p:= \frac{1}{2^{p/2}} \sum_0^p \begin{pmatrix} r \\ p \end{pmatrix} a^*(h)^r a(h)^{p-r}.$$

Proof. it is easy to verify that the operator in the right hand side of the first identity of *i)* commutes with $\phi(h), h \in \mathfrak{h}_c$ and maps Ω onto v. Hence it equals ${\rm Wick}_c(v)$. The second identity of *i)* follows then from Thm. 3.12 *v)*. To prove *ii)*, we first claim that $:\phi(h)^p:\Omega = \frac{1}{2^{p/2}}a^*(h)^p\Omega$. In fact the r.h.s. is orthogonal to the polynomials in $\phi(h_i), h_i \in \mathfrak{h}_c$ of order less than $p-1$ and differs from $\phi(h)^p\Omega$ by a polynomial of order less than $p-1$. Hence it equals $:\phi(h)^p:\Omega$. Now *ii)* follows from *i)* and the fact that since $:\phi(h)^p:$ is affiliated to $\mathfrak{M}_c$, ${\rm Wick}_c(:\phi(h)^p:)\Omega =:\phi(h)^p:$. □

If $\mathfrak{h} = L^2(\mathbb{R}, {\rm d}k)$ and the conjugation c is defined by $h(k) = \bar{h}(-k)$, (which will be the case in the $P(\varphi)_2$ theory), and $v \in \otimes_{\rm s}^p L^2(\mathbb{R})$, then using the notation (3.13) we can write:

$$\begin{array}{l} {\rm Wick}_c(v) \\ = \sum_{r=0}^{p} \begin{pmatrix} p \\ r \end{pmatrix} p!^{-\frac{1}{2}} \int w(k_1, \dots, k_r, k_{r+1}, \dots, k_p) \\ \times a^*(k_1)\dots a^*(k_r)a(-k_{r+1})\dots a(-k_p){\rm d}k_1\cdots{\rm d}k_p. \end{array}$$

5.3 Hypercontractive semigroups

Let (Q, μ) be a measure probability space.

Definition 5.7 *Let $H_0 \geq 0$ be a selfadjoint operator on $\mathcal{H} = L^2(Q, d\mu)$. The semigroup* e^{-tH_0} *is* hypercontractive *if*

i) e^{-tH_0} *is a contraction on $L^1(Q, d\mu)$ for all $t > 0$,*

ii) $\exists\, T, C$ such that

$$\|\mathrm{e}^{-TH_0}\psi\|_{L^4(Q,d\mu)} \leq C\|\psi\|_{L^2(Q,d\mu)}.$$

The abstract result used to construct the $P(\varphi)_2$ Hamiltonian is the following theorem, due to Segal ([Se]).

Theorem 5.8 *Let* e^{-tH_0} *be a hypercontractive semigroup. Let V be a real function on Q such that $V \in L^p(Q, d\mu)$ for some $p > 2$ and* $\mathrm{e}^{-tV} \in L^1(Q, d\mu)$ *for all $t > 0$. Let $V_n = 1\!\!1_{\{|V| \leq n\}}V$ and $H_n = H_0 + V_n$. Then the semigroups* e^{-tH_n} *converge strongly on $\mathcal{H}$ when $n \to \infty$ to a strongly continuous semigroup on $\mathcal{H}$ denoted by* e^{-tH}*. Its infinitesimal generator H has the following properties:*

i) H is the closure of $H_0 + V$ defined on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$,

ii) H is bounded below:

$$H \geq -c - \ln \|\mathrm{e}^{-V}\|_{L^p(Q,d\mu)},$$

where c and p depend only on the constants C and T in Def. 5.7.

The following technical result (see [Si1, Lemma V.5] for a proof) will be used later to show that a given function V on Q verifies $\mathrm{e}^{-tV} \in L^1(Q, d\mu)$.

Lemma 5.9 *Let for $\kappa \geq 1$, V_κ, V be functions on Q such that for some $n \in \mathbb{N}$*

$$\begin{aligned} &\|V - V_\kappa\|_{L^p(Q,d\mu)} \leq C(p-1)^n \kappa^{-\epsilon}, \\ &V_\kappa \geq -C(\ln \kappa)^n. \end{aligned} \tag{5.4}$$

Then

$$\mu\{q \in Q | V(q) \leq -2C(\ln \kappa)^n\} \leq C\mathrm{e}^{-c\kappa^\alpha}$$

for some $\alpha > 0$. Consequently $\mathrm{e}^{-tV} \in L^1(Q, d\mu)$, $\forall t > 0$.

The following theorem of Nelson (see [Si1, Thm. 1.17]) establishes a connection between contractions on $\mathfrak{h}$ and hypercontractive semigroups on $L^2(Q, d\mu)$.

Theorem 5.10 *Let $r \in B(\mathfrak{h})$ be a selfadjoint contraction commuting with c. Then*

i) $U\Gamma(r)U^*$ is a positivity preserving contraction on $L^p(Q, d\mu)$, $1 \leq p \leq \infty$.

ii) if $\|r\| \leq (p-1)^{\frac{1}{2}}(q-1)^{-\frac{1}{2}}$ for $1 < p, q < \infty$ then $U\Gamma(r)U^*$ is a contraction from $L^p(Q, d\mu)$ to $L^q(Q, d\mu)$.

Combining Thm. 5.10 with Thm. 5.8, we obtain the following result.

Theorem 5.11 *Let $\mathfrak{h}$ be a Hilbert space with a conjugation c. Let a be a selfadjoint operator on $\mathfrak{h}$ with*

$$[a, c] = 0, \ a \geq m > 0. \tag{5.5}$$

Let $L^2(Q, d\mu)$ be the $Q-$space representation of $\Gamma(\mathfrak{h})$ and let V be a real function on Q with $V \in L^p(Q, d\mu)$ for some $p > 2$ and $\mathrm{e}^{-tV} \in L^1(Q, d\mu)$ for all $t > 0$. Then:

i) the operator sum $H = d\Gamma(a) + V$ is essentially selfadjoint on $\mathcal{D}(d\Gamma(a)) \cap \mathcal{D}(V)$.

ii) $H \geq -C$, where C depends only on m and $\|\mathrm{e}^{-V}\|_{L^p(Q,d\mu)}$, for some p depending only on m.

Note that by applying Thm. 5.10 to $a = (q-1)^{-\frac{1}{2}}\mathbb{1}_{\mathfrak{h}}$ for $q > 2$, we obtain the following lemma about the L^p properties of finite vectors in $\Gamma(\mathfrak{h})$ (see [Si1, Thm. 1.22]).

Lemma 5.12 *Let $\psi \in \otimes_{\mathrm{s}}^n \mathfrak{h}$ and $q \geq 2$. Then*

$$\|R\psi\|_{L^q(Q,d\mu)} \leq (q-1)^{n/2}\|\psi\|.$$

6 The spatially cut-off $P(\varphi)_2$ Hamiltonian

In this section, we recall some standard facts about the construction of the spatially cut-off $P(\varphi)_2$ Hamiltonian. Some of these facts are presented in a slightly more general form, which will be useful later when we will consider the Mourre theory for $P(\varphi)_2$ Hamiltonians.

6.1 The spatially cut-off $P(\varphi)_2$ model

We recall now the definition of the spatially cut-off $P(\varphi)_2$ model that we will study in this paper (see for example [GJ1, S-H.K]). $P(\varphi)_2$ models describe quantum field theories in 2 space-time dimensions, which means that the $1-$particle Hilbert space $\mathfrak{h}$ is taken equal to $L^2(\mathbb{R}, \mathrm{d}k)$. We normalize the Fourier transform by

$$\mathcal{F} : L^2(\mathbb{R}, \mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}k),$$

$$\mathcal{F}\chi(k) = \hat{\chi}(k) = \textstyle\int \mathrm{e}^{-\mathrm{i}k\cdot x}\chi(x)dx.$$

The complex conjugation is the map c defined by $ch(k) = \overline{h}(-k)$, which corresponds to the usual conjugation $f \mapsto \overline{f}$ on $L^2(\mathbb{R}, dx)$ by Fourier transformation. The bosonic Fock space $\Gamma(\mathfrak{h})$ will be denoted by $\mathcal{H}$.

We fix the *dispersion relation*

$$\mathbb{R} \ni k \mapsto \omega(k) = (k^2 + m^2)^{\frac{1}{2}}, \ m > 0.$$

The kinetic energy is $H_0 = d\Gamma(\omega)$.

Let us now define the interaction term. Let

$$\varphi(x) := \int \mathrm{e}^{-\mathrm{i}k\cdot x}\,(a^*(k) + a(-k))\,\frac{\mathrm{d}k}{\omega(k)^{\frac{1}{2}}} \tag{6.1}$$

be the local relativistic field operator defined in distribution sense.

Note that the local field $\varphi(x)$ is denoted by a different variety of the letter phi than the Segal field $\phi(h)$, $h \in \mathfrak{h}$ (see Subsect. 2.2). It is useful to note that $\varphi(x)$ can be formally expressed in terms of ϕ as follows. Set

$$f(k) := \sqrt{2}\omega(k)^{-\frac{1}{2}}.$$

Let $\tau_x : L^2(\mathbb{R}, \mathrm{d}k) \to L^2(\mathbb{R}, \mathrm{d}k)$ be the translation by x, that is $\tau_x h = \mathrm{e}^{-\mathrm{i}k\cdot x}h$. Then formally

$$\varphi(x) = \phi(\tau_x f). \tag{6.2}$$

Unfortunately, $\tau_x f \notin L^2(\mathbb{R})$, so (6.2) has to be understood in distribution sense.

To remedy this one introduces UV-cutoff fields. Let $\chi \in L^1_{\mathbb{R}}(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$ with $\int \chi(x)dx = 1$ and let $\kappa \geq 1$ be a large UV-cutoff parameter. We introduce for later use the *cutoff fields*

$$\begin{aligned}\varphi_\kappa(x) &:= \kappa \int \varphi(y)\chi(\kappa(y-x))dy \\ &= \int \mathrm{e}^{-\mathrm{i}k\cdot x}\hat{\chi}(\tfrac{k}{\kappa})\left(a^*(k) + a(-k)\right)\frac{\mathrm{d}k}{\omega(k)^{\frac{1}{2}}}.\end{aligned} \tag{6.3}$$

If one sets

$$f_\kappa(k) := \sqrt{2}\omega(k)^{-\frac{1}{2}}\hat{\chi}(\frac{k}{\kappa}), \tag{6.4}$$

then one can write

$$\varphi_\kappa(x) = \phi(\tau_x f_\kappa).$$

Note that since the function $\tau_x f_\kappa(k) = \mathrm{e}^{-\mathrm{i}k\cdot x}\hat{\chi}(\frac{k}{\kappa})\omega(k)^{-\frac{1}{2}}$ belongs to $\mathfrak{h}_c$, $\varphi_\kappa(x)$ is affiliated to the algebra $\mathfrak{M}_c$. We will set $\varphi_\infty(x) := \varphi(x)$.

To define the spatially cut-off $P(\varphi)_2$ interaction, we fix a real polynomial of degree $2n$:

$$P(\lambda) = \sum_{j=0}^{2n} a_j \lambda^j, \text{ with } a_{2n} > 0, \tag{6.5}$$

and a real function $g \in L^1_{\mathbb{R}}(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$ with $g \geq 0$. We set for $\kappa < \infty$:

$$V_\kappa := \int g(x) : P(\varphi_\kappa(x)) : dx \tag{6.6}$$

which is an unbounded operator affiliated to $\mathfrak{M}_c$. We will see later that, when $\kappa \to \infty$, V_κ converges in $L^2(Q, d\mu)$ to a function V, which we will denote by

$$V = \int g(x) : P(\varphi(x)) : dx.$$

Alternatively one can view the multiplication operators V_κ and V as Wick polynomials, using the discussion in Subsect. 5.2. In fact for $p \in \mathbb{N}$, we have

$$\begin{aligned}&\int g(x) : \varphi_\kappa(x)^p : dx \\ &= \sum_{r=0}^{p} \begin{pmatrix} p \\ r \end{pmatrix} \int w_{p,\kappa}(k_1, \ldots, k_r, k_{r+1}, \ldots, k_p) \\ &\times a^*(k_1) \cdots a^*(k_r) a(-k_{r+1}) \cdots a(-k_p) \mathrm{d}k_1 \cdots \mathrm{d}k_p,\end{aligned} \tag{6.7}$$

for

$$w_{p,\kappa}(k_1,\cdots,k_p)=\hat{g}(\sum_1^p k_i)\Pi_1^p\hat{\chi}(\frac{k_i}{\kappa})\omega(k_i)^{-\frac{1}{2}}. \tag{6.8}$$

From Lemma 6.1 and Prop. 6.5 below, we deduce that V and V_κ are defined as unbounded operators with domain $\mathcal{D}(N)^n$. This implies that the operator sum H_0+V is well defined as a symmetric operator on $\mathcal{D}(H_0)\cap\mathcal{D}(N^n)$.

The construction of a unique selfadjoint extension of H_0+V is outlined in Subsect. 6.4. This Hamiltonian is denoted by

$$H=H_0+\int g(x):P(\varphi(x)):dx$$

and is called a *spatially cut-off* $P(\varphi)_2$ *Hamiltonian* or simply a $P(\varphi)_2$ *Hamiltonian.*

6.2 Assumptions on g

In this subsection, we discuss the various assumptions on the cutoff function g which will be used in our paper.

A spatially cut-off $P(\varphi)_2$ model is completely specified by the polynomial P and the cutoff g. Let us introduce the following assumptions:

$$(A)\ \ g\geq 0, g\in L^1_{\mathbb{R}}(\mathbb{R},dx)\cap L^2(\mathbb{R},dx).$$

Assumption *(A)* is a standard assumption needed to construct H as a selfadjoint operator. (This assumption can be relaxed to $g\geq 0$, $g\in L^1_{\mathbb{R}}(\mathbb{R},dx)\cap L^{1+\epsilon}(\mathbb{R},dx)$ $\epsilon>0$, see [Si1]).

$$(C)\ \ g\in H^{\frac{1}{2}}(\mathbb{R}).$$

Assumption *(C)* will be needed if $\deg P=4$ to ensure that $\mathcal{D}(H)=\mathcal{D}(H_0)\cap\mathcal{D}(V)$. This fact allows to give a simpler treatment of the Mourre theory for φ_2^4 Hamiltonians that does not use the $Q-$representation.

$$(Mm)\ \ (x\cdot\partial_x)^j g\in L^2(\mathbb{R},dx),\ \ j=1,\dots,m.$$

Assumption *(Mm)* is needed to define the commutators $\mathrm{ad}^m_{\mathrm{d}\Gamma(a)}V$ where a is the generator of dilations on $\mathfrak{h}$ as densely defined operators (a priori they are only defined as quadratic forms).

$$(Is)\ \ \langle x\rangle^s g\in L^2(\mathbb{R},dx),\ s\geq 0.$$

Assumption *(Is)* is needed for the scattering theory of spatially cut-off $P(\varphi)_2$ Hamiltonians. In particular (Is) for $s>1$ is a short-range condition, under which the asymptotic fields and the wave operators can be constructed.

$$(Bm)\ \ g(x)\leq Cg(y)\langle x-y\rangle^N,\quad |(x\cdot\partial_x)^j g(x)|\leq Cg(x),\ \ 0\leq j\leq m.$$

Assumption *(Bm)* will be needed in Section 8 in order to be able to control the commutator $\mathrm{ad}^m_{\mathrm{d}\Gamma(a)}V$. Note that *(Bm)* implies *(Mm)*.

We will always assume *(A)*. All other assumptions will be explicitly stated.

6.3 Some properties of the interaction kernel

We collect here various properties of the interaction kernels $w_{p,\kappa}$ of the Wick polynomials V_κ.

The following lemma is well known.

Lemma 6.1 *The kernels $w_{p,\kappa}$ are in $L^2(\mathbb{R}^p)$ for $1 \leq \kappa \leq \infty$ and $\|w_{p,\kappa} - w_{p,\infty}\|_{L^2(\mathbb{R}^p)} \leq C\|g\|_{L^2(\mathbb{R})}\kappa^{-\epsilon}$, $\epsilon > 0$.*

Proof. We use the bound

$$\Pi_1^p a_j \leq \sum_{i=1}^p (\Pi_{j\neq i} a_j)^{p/(p-1)}, \tag{6.9}$$

which follows from the fact that

$$(\Pi_1^p \lambda_i)^{1/p} \leq \sum_1^p \lambda_i,$$

applied to $\lambda_i = \Pi_{j\neq i} a_j^{p/(p-1)}$. Applying (6.9) to $a_i = \omega(k_i)^{-\frac{1}{2}}$ we obtain that $w_{p,\infty}$ and hence $w_{p,\kappa}$ for $\kappa < \infty$ belong to $L^2(\mathbb{R}^d)$. The bound on $\|w_{p,\kappa} - w_{p,\infty}\|$ is a direct computation, using (6.9). $\Box$

We deduce from Lemma 6.1 the following result:

Lemma 6.2 *The operators $V_\kappa(N+1)^{-n}$ are bounded on $\mathcal{H}$ for $1 \leq \kappa \leq \infty$ and $\|(V - V_\kappa)(N+1)^{-n}\| \leq C\|g\|_{L^2(\mathbb{R})}\kappa^{-\epsilon}$, $\epsilon > 0$.*

Lemma 6.3 *Let $j \in C^\infty(\mathbb{R})$ with $j \equiv 0$ near 0 and $j \equiv 1$ near infinity. Then for $1 \leq i \leq p$:*

$$\|j(\frac{x_i}{R})w_p\|_{L^2(\mathbb{R}^p)} \in \begin{cases} O(R^{-s}) \text{ under hypothesis } (Is), \\ o(R^0) \text{ under hypothesis } (A). \end{cases}$$

Proof. It suffices to prove the lemma for $i = 1$.

It follows from *(Is)* that $\hat{g}$ belongs to the Sobolev space $H^s(\mathbb{R})$. Let us first check that

$$|D_{k_1}|^s w_p \in L^2(\mathbb{R}^p). \tag{6.10}$$

By interpolation it suffices to check this for $s \in \mathbb{N}$. We see that $\partial_{k_1}^s w_p$ is a sum of terms of the form

$$\hat{g}^{(s_1)}(\sum_1^p k_i)\omega^{-\frac{1}{2}}(k_1)^{(s_2)}\Pi_2^p \omega(k_i)^{-\frac{1}{2}},$$

where $s = s_1 + s_2$ and hence $\hat{g}^{(s_1)} \in L^2(\mathbb{R})$. Since $\hat{g}^{(s_1)}$ belongs to $L^2(\mathbb{R}, dk)$, the bound (6.9) gives that $\partial_{k_1}^s w_p \in L^2(\mathbb{R}^d)$, and hence (6.10) is true. Next we note that since j vanishes near the origin, $j(x) = |x|^s j_s(x)$, where j_s is bounded. Now

$$j(\frac{x_1}{R}) = j(\frac{-D_{k_1}}{R}) = R^{-s} j_s(\frac{-D_{k_1}}{R})|D_{k_1}|^s.$$

and by the spectral theorem, $j_s(\frac{-D_{k_1}}{R})$ is a uniformly bounded operator on $L^2(\mathbb{R}^p)$. The lemma follows then from (6.10). $\Box$

6.4 Existence and basic properties

We summarize now standard results on the existence of the $P(\varphi)_2$ Hamiltonian(see [GJ1, Se, S-H.K, Ro1]).

Theorem 6.4 *Let P be a real, bounded below polynomial of degree $2n$ and $g \geq 0$, $g \in L^1_{\mathbb{R}}(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$. Then:*
i) the Hamiltonian $H = H_0 + V$ for $V = \int g(x) : P(\varphi(x)) : dx$ is essentially selfadjoint on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$.
ii) there exist $b, C \geq 0$ such that the following first order estimates *hold*

$$H_0 \leq C(H + b), \tag{6.11}$$

$$N \leq C(H + b). \tag{6.12}$$

Proof. *i)* follows from Thm. 5.11 using Lemma 6.6 below. *ii)* follows from Thm. 5.11 with a replaced by $(1 - \epsilon)a$. (6.12) follows from (6.11). $\square$

Proposition 6.5 *Let $P(\lambda)$ be a real polynomial of degree $2n$ as in (6.5) and $\tilde{P}(\lambda)$ be a real polynomial of degree $\leq 2n$. Assume that the coefficient $\tilde{a}_{2n}$ of $\tilde{P}(\lambda)$ and a_{2n} of $P(\lambda)$ satisfy $|\tilde{a}_{2n}| < a_{2n}$. Let $g, \tilde{g} \in L^1_{\mathbb{R}}(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$ be two functions with $g \geq 0, |\tilde{g}| \leq g$. Let*

$$\tilde{V} := \int \tilde{g}(x) \tilde{P}(\varphi(x)) \colon dx.$$

Then $\tilde{V}$ is a multiplication operator in the $Q-$space representation and there exist C, b such that

$$|\tilde{V}| \leq C(H + b).$$

The above theorem and proposition follow from the next lemma, where we collect some well-known properties (see eg [Ne, Se, S-H.K]) which show that the $P(\varphi)_2$ interaction is a multiplication operator in the $Q-$space representation.

Lemma 6.6 *Under the conditions of Proposition 6.5, set*

$$W := C \int g(x) : P(\varphi(x)) \colon - \tilde{g}(x) : \tilde{P}(\varphi(x)) \colon \mathrm{d}x.$$

Then W is a multiplication operator in the $Q-$space representation. Moreover $W \in L^p(Q, d\mu)$ for all $p < \infty$ and $\|\mathrm{e}^{-tW}\|_{L^1(Q,d\mu)}$ depends only on $P, \tilde{P}$ and $\|g\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}$.

Proof. Note first that since $|\tilde{g}| \leq g$, $\|\tilde{g}\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}$ is less than $\|g\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}$. Let for $\kappa \geq 1$ W_κ be the cutoff operator defined as in (6.6). Both W and W_κ are Wick polynomials. Applying Lemma 6.2 we see that $W(N+1)^{-n}$ is bounded and

$$\|(W - W_\kappa)(N+1)^{-n}\| \leq C\|g\|_{L^2(\mathbb{R})} \kappa^{-\epsilon}, \ \epsilon > 0. \tag{6.13}$$

We have seen at the end of Subsect. 6.1 that W_κ is a multiplication operator by a function W_κ on Q. Moreover $W_\kappa \in L^2(Q, d\mu)$ since $\|W_\kappa\|_{L^2(Q,d\mu)} = \|W_\kappa\Omega\|_{\mathcal{H}} < \infty$. Next it follows from (6.13) that W_κ is Cauchy in $L^2(Q, d\mu)$ and hence converges to W in $L^2(Q, d\mu)$ when $\kappa \to +\infty$. We note then that $W\Omega \in \otimes_{\rm s}^{2n}\mathfrak{h}$ since W is a Wick polynomial of degree $2n$, which by Lemma 5.12 implies that $W \in L^p(Q, d\mu)$ for all $p < \infty$.

To bound $\|{\rm e}^{-tW}\|_{L^1(Q,d\mu)}$, we will use Lemma 5.9, checking that the constants C, ϵ, n there depend only on P, $\tilde{P}$ and $\|g\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}$. The first bound of (5.4) follows from (6.13) and Lemma 5.12. To check the second bound, we use the Wick identities (5.2), which yield

$$
\begin{aligned}
:P(\varphi_\kappa(x)):= \sum_{p=0}^{2n} a_p \sum_{r=0}^{[p/2]} b_{p,r}\|\varphi_\kappa(x)\Omega\|^{2r}\varphi_\kappa(x)^{p-2r},\\
:\tilde{P}(\varphi_\kappa(x)):= \sum_{p=0}^{2n} \tilde{a}_p \sum_{r=0}^{[p/2]} b_{p,r}\|\varphi_\kappa(x)\Omega\|^{2r}\varphi_\kappa(x)^{p-2r},
\end{aligned}
\tag{6.14}
$$

for $b_{p,0} = 1$. Hence as an inequality between functions on Q we have:

$$
:P(\varphi_\kappa(x)): -|:\tilde{P}(\varphi_\kappa(x)):| \geq F_\kappa(\varphi_\kappa(x)),
$$

for

$$
F_\kappa(\lambda) = \sum_{p=0}^{2n}\sum_{r=0}^{[p/2]} c_{p,r}\|\varphi_\kappa(x)\Omega\|^{2r}\lambda^{p-2r},
\tag{6.15}
$$

and $c_{2n,0} = a_{2n} - |\tilde{a}_{2n}|$. If we apply the bound $a^{2r}b^{p-2r} \leq \epsilon b^p + C_\epsilon a^p$ to all terms in (6.15) for $p < 2n$ and use that $a_{2n} - |\tilde{a}_{2n}| > 0$, we obtain

$$
F_\kappa(\lambda) \geq -C(\|\varphi_\kappa(x)\Omega\|^{2n} + 1).
\tag{6.16}
$$

By a direct computation, we check that $\|\varphi_\kappa(x)\Omega\| = \|\varphi_\kappa(0)\Omega\| \in O(\ln \kappa^{\frac{1}{2}})$. This gives

$$
\begin{aligned}
W_\kappa &= \int g(x) :P(\varphi_\kappa(x)): -\tilde{g}(x) :\tilde{P}(\varphi_\kappa(x)): {\rm d}x\\
&\geq \int g(x) :P(\varphi_\kappa(x)): -g(x)| :\tilde{P}(\varphi_\kappa(x)): |{\rm d}x\\
&\geq -C\|g\|_{L^1(\mathbb{R})}((\ln\kappa)^n + 1),
\end{aligned}
$$

which proves the second bound of (5.4), with a constant depending only on $\|g\|_{L^1(\mathbb{R})}$. □

7 Higher order estimates

In this section we will state some *higher order estimates* which will be very important in the sequel. These higher order estimates are due to Glimm and Jaffe [GJ2] for the φ_2^4 model and (in a more general form than here) to Rosen [Ro2] for the general $P(\varphi)_2$ model. Note however that the proof in [Ro2] is only valid under the additional hypothesis that $g \in C_0^\infty(\mathbb{R})$ (see Remark 7.5). In this subsection we will explain the modifications to Rosen's proof necessary to treat the general case when $g \in L^1_{\mathbb{R}}(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$. The reader may also consult [Si2] for a review of higher order estimates.

Theorem 7.1 *Assume hypothesis (A). Then there exists $b > 0$ such that for all $\alpha \in \mathbb{N}$, the following* higher order estimates *hold:*

$$(7.1)\qquad \begin{aligned} &\|N^{\alpha}(H+b)^{-\alpha}\| < \infty,\\ &\|H_0 N^{\alpha}(H+b)^{-n-\alpha}\| < \infty,\\ &\|N^{\alpha}(H+b)^{-1}(N+1)^{1-\alpha}\| < \infty. \end{aligned}$$

In the case of the φ_2^4 model a better estimate is known.

Theorem 7.2 *Assume* degP $= 4$ *and hypothesis (C). Then there exists $b > 0$ such that*

$$\|H_0(H+b)^{-1}\| < \infty.$$

Consequently $\mathcal{D}(H) = \mathcal{D}(H_0) \cap \mathcal{D}(V)$.

Proof. The proof given in [GJ2] for $g \in C_0^{\infty}(\mathbb{R})$ is still valid under hypothesis *(C)*. In fact one first proves that if $W = \int g(x) :\varphi_{\kappa}(x)^p: \mathrm{d}x$ for $p \leq 4$ then (see [GJ2, Equ. 2.17]):

$$\|(H_0+1)^{-1}[H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, W]](H_0+1)^{-1}\| \leq C\|\omega^{\frac{1}{2}}(\sum_1^4 k_i)w_p\|_{L^2(\mathbb{R}^4)}.$$

(Note that the expressions W, $[H_0^{\frac{1}{2}}, W]$ and $[H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, W]]$ are well defined as quadratic forms on $\mathcal{S} = \Gamma_{\rm fin}(C_0^{\infty}(\mathbb{R}))$ since $H_0^{\frac{1}{2}}$ preserves $\mathcal{S}$). One uses then Jaffe's double commutator trick (see eg [Si2, Sect. 4]) to obtain that as quadratic forms on $\mathcal{S}$

$$(7.2)\qquad H^2 \geq c(H_0^2 + V^2) - d.$$

Now it follows from the higher order estimates that any core for H_0^n is a core for H, and in particular $\mathcal{S}$ is a core for H. Hence (7.2) extends to $\mathcal{D}(H)$, which proves the theorem. □

Remark 7.3 *The importance of the higher order estimates comes from the fact that the domain of H is not known explicitly. In particular, the question if $\mathcal{D}(H) = \mathcal{D}(H_0) \cap \mathcal{D}(V)$ is still an open problem, except for the φ_2^4 model, where this result was shown by Glimm and Jaffe in [GJ2]. This means that for $u \in \mathcal{D}(H)$ the identity $Hu = H_0u + Vu$ does not make sense for the general $P(\varphi)_2$ model. However a consequence of the higher order estimates is that $\mathcal{D}(H^n) \subset \mathcal{D}(V) \cap \mathcal{D}(H_0)$ so that this identity makes sense for $u \in \mathcal{D}(H^n)$.*

The proof of higher order estimates in [Ro2] is based on the *pullthrough formula*, which gives an expression for the multi-commutators of annihilation operators $a(k_i)$ with the resolvent $(H-z)^{-1}$. The technical problem is that in order to make sense of these formal computations one needs a subspace $\mathcal{D}$ of $\mathcal{H}$ which is in the domain of all powers of creation operators and on which H is essentially selfadjoint.

To circumvent this difficulty, Rosen introduced *cutoff Hamiltonians* for which the interaction acts only on a finite number of degrees of freedom. In a $Q-$space representation these cutoff Hamiltonians become differential operators, for which the construction of a subspace $\mathcal{D}$ with the above properties is easy. The higher order estimates are then shown for the cutoff Hamiltonians, with constants uniform in the cutoff parameters. The proof is then completed by taking the cutoff to infinity.

7.1 Cutoff Hamiltonians

In this subsection we introduce the U.V. cutoff Hamiltonians used in the proof of the higher order estimates.

Let $\mathfrak{h}$ be a Hilbert space equipped with a conjugation c. Let $\pi_1 : \mathfrak{h} \to \mathfrak{h}_1$ be an orthogonal projection on a closed subspace $\mathfrak{h}_1$ of $\mathfrak{h}$ with $[\pi_1, c] = 0$. Let $\mathfrak{h}_1^\perp$ be the orthogonal complement of $\mathfrak{h}_1$. In all formulas below we will consider π_1 as an element of $B(\mathfrak{h}, \mathfrak{h}_1)$. With this convention the orthogonal projection on $\mathfrak{h}_1$, considered as an element of $B(\mathfrak{h}, \mathfrak{h})$, is equal to $\pi_1^* \pi_1$.

Let $U : \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_1^\perp) \to \Gamma(\mathfrak{h})$ the unitary map defined in Subsect. 3.8. We denote by $L^2(Q_1, \mathrm{d}\mu_1)$, $L^2(Q_1^\perp, \mathrm{d}\mu_1^\perp)$ the $Q-$space representations of $\Gamma(h_1)$, $\Gamma(\mathfrak{h}_1^\perp)$. Recall that by Prop. 5.3, we may take as $Q-$space representation of $\Gamma(\mathfrak{h})$ the space $L^2(Q, \mathrm{d}\mu)$ for $Q = Q_1 \times Q_1^\perp$, $\mu = \mu_1 \otimes \mu_1^\perp$. Accordingly we denote by $(q_1, q_1^\perp)$ the elements of $Q = Q_1 \times Q_1^\perp$.

If $W \in B(\Gamma(\mathfrak{h}))$ we set:

$$B(\Gamma(\mathfrak{h})) \ni \Pi_1 W := U\Big(\Gamma(\pi_1) W \Gamma(\pi_1^*) \otimes 1\!\!1\Big) U^*.$$

Lemma 7.4 *i) If* $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$ *then*

$$\Pi_1 \mathrm{Wick}(w) = \mathrm{Wick}(\Gamma(\pi_1^* \pi_1) w \Gamma(\pi_1^* \pi_1)). \tag{7.3}$$

ii) If V *is a multiplication operator by a function in* $L^2(Q, d\mu)$ *then* $\Pi_1 V$ *is the operator of multiplication by the function*

$$\Pi_1 V(q_1) = \int_{Q_1^\perp} V(q_1, q_1^\perp) \mathrm{d}\mu_1^\perp. \tag{7.4}$$

Proof. To prove *i)* we may assume by linearity that

$$w = |\overset{q}{\underset{1}{\otimes}}{}_{\mathrm{s}} h_i)(\overset{p}{\underset{1}{\otimes}}{}_{\mathrm{s}} g_i|,$$

for which the verification of *i)* is easy. To prove *ii)* we first deduce from Prop. 3.5 that $\Gamma(\pi_1) U = 1\!\!1 \otimes |\Omega)(\Omega|$. This implies that as a multiplication operator on Q_1, $\Gamma(\pi_1) V \Gamma(\pi_1^*)$ is given by (7.4). Then one uses Prop. 5.3. $\square$

In particular if $W = \Pi_1^q a^*(h_i) \Pi_1^p a(g_i)$, then

$$\Pi_1 W = \Pi_1^q a^*(\pi_1^* \pi_1 h_i) \Pi_1^p a(\pi_1^* \pi_1 g_i). \tag{7.5}$$

Let now $\{\pi_n\}_{n \in \mathbb{N}}$ be a sequence of orthogonal projections on $\mathfrak{h}$ such that

$$\pi_n \leq \pi_{n+1},\ [\pi_n, c] = 0,\ \text{s-}\lim_{n \to +\infty} \pi_n = 1\!\!1, \tag{7.6}$$

and let Π_n the associated maps defined by (7.3). Using the representation (7.4) it is shown in [S-H.K, Prop. 4.9] that

$$\begin{aligned} &i)\ \Pi_n V \to V \text{ in } L^p(Q, d\mu), \text{ when } n \to \infty, \text{ if } V \in L^p(Q, \mathrm{d}\mu), \\ &ii)\ \|\mathrm{e}^{-t\Pi_n V}\|_{L^1(Q, d\mu)} \leq \|\mathrm{e}^{-tV}\|_{L^1(Q, d\mu)}. \end{aligned} \tag{7.7}$$

Let us now specify the particular sequence of projections corresponding to the lattice regularization and volume cutoff used in [Ro2]. For $v \geq 1$ we introduce the lattice $v^{-1}\mathbb{Z}$ and let

$$\mathbb{R} \ni k \mapsto [k]_v \in v^{-1}\mathbb{Z}$$

be the integer part of k defined by $-(2v)^{-1} < k - [k]_v \leq (2v)^{-1}$. For $\gamma \in v^{-1}\mathbb{Z}$, we denote by $e_\gamma \in \mathfrak{h}$ the vector $e_\gamma(k) = v^{\frac{1}{2}} \mathbb{1}_{]-(2v)^{-1},(2v)^{-1}]}(k-\gamma)$ and set $a^\sharp(\gamma) := a^\sharp(e_\gamma)$. For $\kappa \in [1, +\infty[$ an UV cutoff parameter, we denote by $\Gamma_{\kappa,v}$ the set $v^{-1}\mathbb{Z} \cap \{|\gamma| \leq \kappa\}$, by $\mathfrak{h}_{\kappa,v}$ the finite dimensional space spanned by $\{e_\gamma\}_{\gamma \in \Gamma_{\kappa,v}}$, and by $\pi_{\kappa,v}$ the orthogonal projection from $\mathfrak{h}$ onto $\mathfrak{h}_{\kappa,v}$.

Let us fix a sequence (κ_n, v_n) tending to (∞, ∞) in such a way that

$$\Gamma_{\kappa_n,v_n} \subset \Gamma_{\kappa_{n+1},v_{n+1}}, \tag{7.8}$$

We denote by Γ_n the finite lattice Γ_{κ_n,v_n} and choose as π_n the projection on $\mathfrak{h}_n := \mathfrak{h}_{\kappa_n,v_n}$, which satisfies (7.6).

If $V = \int g(x) :P(\varphi(x)): dx$, we define the cutoff interaction:

$$V_n := \Pi_n V.$$

Using (7.5) and (6.7), we obtain the following explicit expression for V_n:

$$V_n = \sum_{p=0}^{\deg P} a_p \int g(x) :\varphi_n(x)^p: dx, \tag{7.9}$$

where

$$:\varphi_n(x)^p: = \sum_{r=0}^{p} \left(\begin{array}{c} p \\ r \end{array}\right) \sum_{\gamma_1,\dots,\gamma_p \in \Gamma_n} a^*(\gamma_1)\cdots a^*(\gamma_r) a(-\gamma_{r+1}) \cdots a(-\gamma_p) \Pi_1^p \mu_n(x, \gamma_i), \tag{7.10}$$

and

$$\mu_n(x,\gamma) := v_n^{\frac{1}{2}} \int_{-(2v_n)^{-1}}^{+(2v_n)^{-1}} \mathrm{e}^{-\mathrm{i}\langle x, \gamma+k\rangle} \omega(\gamma+k)^{-\frac{1}{2}} dk. \tag{7.11}$$

Remark 7.5 *Our definition of the cutoff interaction is different from the one used in [Ro2]. In fact the cutoff interaction used there is obtained by replacing the orthogonal projection $\pi_n : \mathfrak{h} \to \mathfrak{h}_n$ by the unbounded operator:*

$$h \mapsto \sum_{\gamma \in \Gamma_{\kappa_n,v_n}} v^{-\frac{1}{2}} h(\gamma) e_\gamma.$$

With this convention, it is easy to see that for example $V_n\Omega$ will not converge to $V\Omega$ for an arbitrary $g \in L^1_{\mathbb{R}}(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$.

To define the cutoff kinetic energy, we set as in [Ro2]:

$$\omega_n : \mathbb{R} \to \mathbb{R},$$

$$\omega_n(k) := \omega([k]_{v_n}).$$

Since $[\omega_n, \pi_n] = 0$, the operator ω_n acts on $\mathfrak{h}_n$ and $\mathfrak{h}_n^\perp$. By Prop. 3.5, we have:

$$H_{0,n} := \mathrm{d}\Gamma(\omega_n) = U_n \left(\mathrm{d}\Gamma(\omega_n|_{\mathfrak{h}_n}) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_n|_{\mathfrak{h}_n^\perp})\right) U_n^*, \tag{7.12}$$

where U_n is the unitary operator between $\Gamma(\mathfrak{h}_n) \otimes \Gamma(\mathfrak{h}_n^\perp)$ and $\Gamma(\mathfrak{h})$. The *cutoff Hamiltonian* is then defined as:

$$H_n := H_{0,n} + V_n.$$

7.2 Properties of the cutoff Hamiltonians

In this subsection we collect some properties of the cutoff Hamiltonians which are needed to prove the higher order estimates. These properties are: existence of a suitable domain of essential selfadjointness, uniform lower bounds and finally resolvent convergence to the Hamiltonian H.

Proposition 7.6 (Ro2) *The Hamiltonians H_n^j, $j \in \mathbb{N}$ are essentially selfadjoint on*

$$\mathcal{D}_n = U(\Gamma_{\text{fin}}(\mathfrak{h}_n) \otimes \Gamma_{\text{fin}}(\mathfrak{h}_n^\perp \cap S(\mathbb{R}))).$$

Proof. As in [Ro2], we have:

$$U_n^* H_n U_n = \hat{H}_n \otimes 1\!\!1 + 1\!\!1 \otimes d\Gamma(\omega_n|_{\mathfrak{h}_n^\perp}),$$

for $\hat{H}_n = d\Gamma(\omega_n|_{\mathfrak{h}_n}) + V_n$. Since $\mathfrak{h}_n$ is finite dimensional, in the $Q-$space representation $\hat{H}_n$ becomes a differential operator $-\Delta + W(x)$ for W a bounded below polynomial, acting on $L^2(\mathbb{R}^{\dim \mathfrak{h}_n}, dx)$. By [GJ3, Thm. 2.2.6] $\hat{H}_n^j$ is essentially selfadjoint on $\Gamma_{\text{fin}}(\mathfrak{h}_n)$ for $j \in \mathbb{N}$. The arguments in [Ro2] give then the proposition. $\Box$

Proposition 7.7 *Let for $n \in \mathbb{N}$, $\tilde{g}_n \in L^1_{\mathbb{R}}(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$ with $|\tilde{g}_n| \leq Cg$ and $\tilde{P}(\lambda)$ a real polynomial of degree less than* $\deg P - 1$*. Then there exist constants $a, b > 0$ such that*

$$\begin{aligned} &H_{0,n} \leq a(H_n + b), \\ &\textstyle\int \tilde{g}_n(x) :\tilde{P}(\varphi_n(x)): dx \leq C(H_n + b). \end{aligned}$$

Proof. Let

$$W^n = \int g(x) :P(\varphi(x)): - \tilde{g}_n(x) :\tilde{P}(\varphi(x)): dx,$$

and $W_n = \Pi_n W^n$. By Lemma 6.6 $\|\mathrm{e}^{-tW^n}\|_{L^1(Q,d\mu)}$ is bounded uniformly in n. Hence by (7.7) $\|\mathrm{e}^{-tW_n}\|_{L^1(Q,d\mu)}$ is bounded uniformly in n. On the other hand $H_{0,n} = d\Gamma(\omega_n)$, where ω_n satisfies (5.5). It follows then from Thm. 5.11 that $H_{0,n} + W_n$ is bounded below uniformly in n. This shows the second bound in the proposition. The first one follows from the same argument, replacing $H_{0,n}$ by $(a-1)H_{0,n}$. $\Box$

Finally the following result is shown in [S-H.K, Prop. 4.8].

Proposition 7.8 *For $z \leq -b$:*

$$(H_n - z)^{-1} \to (H - z)^{-1} \text{ in norm}$$

when $n \to +\infty$.

7.3 Proof of the higher order estimates

We now explain the modifications to the proof of Rosen [Ro2] needed in our case. The only places where modifications are needed are the ones where the interaction V_n appears, ie in [Ro2, Lemma 4.4]. We define for $I = \{1, \ldots, s\}$, $k_i \in \mathbb{R}$, $i \in I$:

$$V_n^I := ad_{a(k_1)} \cdots ad_{a(k_s)} V_n,$$

which is well defined as an operator on $\mathcal{D}_n$. The analog of [Ro2, Lemma 4.4] is now:

Lemma 7.9 *There exist $b, c > 0$ such that for all $\lambda_1, \lambda_2 < -b$, $(H_n - \lambda_2)^{-\frac{1}{2}} V_n^I (H_n - \lambda_1)^{-\frac{1}{2}}$ defined on $(H_n - \lambda_1)^{\frac{1}{2}} \mathcal{D}_n$ extends to a bounded operator on $\mathcal{H}$ such that*

$$\|(H_n - \lambda_2)^{-\frac{1}{2}} V_n^I (H_n - \lambda_1)^{-\frac{1}{2}}\| \leq c \Pi_i^s \omega(k_i)^{-\frac{1}{2}}.$$

Proof. Using (7.10) and the commutation relation

$$[a(k), a^*(\gamma)] = e_\gamma(k) = v^{\frac{1}{2}} \delta(\gamma, [k]_v),$$

for

$$\delta(\gamma, \gamma') = \left\{ \begin{array}{l} 1 \text{ if } \gamma = \gamma', \\ 0 \text{ otherwise} \end{array} \right. ,$$

we obtain that:

$$[a(k), :\varphi_n^p(x):] = 0, \text{ if } |k| > \kappa_n,$$

and

$$\begin{array}{rl} [a(k), :\varphi_n^p(x):] & = \sum\limits_{r=0}^{p} r \left(\begin{array}{c} p \\ r \end{array} \right) \sum_{\gamma_2, \ldots, \gamma_p \in \Gamma_n} a^*(\gamma_2) \cdots a^*(\gamma_r) a(-\gamma_{r+1}) \cdots a(-\gamma_p) \\ & \times v_n^{\frac{1}{2}} \mu_n(x, [k]_{v_n}) \Pi_2^p \mu_n(x, \gamma_i), \end{array}$$

if $|k| \leq \kappa_n$.

We obtain

$$V_n^I = \Pi_1^s \mathbb{1}_{\{|k| \leq \kappa\}}(k_i) v_n^{s/2} \int g(x) \Pi_1^s \mu_n(x, [k_i]_{v_n}) :P^{(s)}(\varphi_n(x)): dx.$$

For fixed $(k_1, \ldots, k_s)$, the operator V_n^I is of the form:

$$\int \tilde{g}_n(x, k_1, \ldots, k_s) :Q(\varphi_n(x)): dx$$

for

$$\tilde{g}_n(x, k_1, \ldots, k_s) = v_n^{s/2} g(x) \Pi_1^s \mathbb{1}_{\{|k| \leq \kappa\}}(k_i) \mu_n(x, k_i),$$

and $Q(\lambda) = P^{(s)}(\lambda)$. Since

$$\sup_{\gamma \in \Gamma_v, |k| \leq 1} \frac{\omega(\gamma)}{\omega(\gamma + k)} \leq c_0,$$

we have:

$$|\mu_n(x,k)| \leq c_0\omega(k)^{-\frac{1}{2}}v_n^{-\frac{1}{2}},$$

which yields

$$|\tilde{g}_n(x,k_1,\dots,k_s)| \leq c_0 g(x)\Pi_1^s\omega(k_i)^{-\frac{1}{2}}.$$

Applying then Prop. 7.7, we obtain that

$$\|(H_n+b)^{-\frac{1}{2}}V_n^I(H_n+b)^{-\frac{1}{2}}\| \leq C\Pi_1^s\omega(k_i)^{-\frac{1}{2}}, \text{ uniformly in } n,$$

which implies the lemma. □

7.4 Number energy estimates

In this subsection we state a consequence of the higher order estimates which will be used in the sequel. We will denote by H^{ext} the Hamiltonian $H\otimes \mathbb{1} + \mathbb{1}\otimes H_0$, acting on the extended Hilbert space $\mathcal{H}^{\text{ext}} = \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$.

We will use the following notation: let an operator $B(t)$ depending on some parameter t map $\cap_n\mathcal{D}(N^n)\subset\mathcal{H}$ into itself. We will write

(7.13) $$B(t)\in(N+1)^m O_N(t^p) \text{ for } m\in\mathbb{R} \text{ if}$$

$$\|(N+1)^{-m-k}B(t)(N+1)^k\| \leq C_k\langle t\rangle^p, \quad k\in\mathbb{Z}.$$

If (7.13) holds for any $m\in\mathbb{R}$, then we will write

$$B(t)\in(N+1)^{-\infty}O_N(t^p).$$

Likewise, for an operator $C(t)$ that maps $\cap_n\mathcal{D}(N^n)\subset\mathcal{H}$ into $\cap_n\mathcal{D}((N_0+N_\infty)^n)\subset\mathcal{H}^{\text{ext}}$ we will write

(7.14) $$C(t)\in(N+1)^m\check{O}_N(t^p) \text{ for } m\in\mathbb{R} \text{ if}$$

$$\|(N_0+N_\infty)^{-m-k}C(t)(N+1)^k\| \leq C_k\langle t\rangle^p, \quad k\in\mathbb{Z}.$$

If (7.14) holds for any $m\in\mathbb{R}$, then we will write

$$B(t)\in(N+1)^{-\infty}\check{O}_N(t^p).$$

The notation $(N+1)o_N(t^p)$, $(N+1)^m\check{o}_N(t^p)$ are defined similarly.

Proposition 7.10 *The following properties hold:*
i) uniformly for z in a compact set of $\mathbb{C}$ we have:

$$(H-z)^{-1}\in(N+1)^{-1}O_N(|Imz|^{-1}), \quad m\in\mathbb{R}.$$

ii) for $\chi\in C_0^\infty(\mathbb{R})$ we have:

$$\|N^m\chi(H)N^p\| < \infty, m,p\in\mathbb{N}.$$

Proof. *ii)* follows directly from (7.1). To prove *i)*, it is enough to prove that for $m \in \mathbb{N}$

$$(N+1)^m(H-z)^{-1}(N+1)^{1-m} \in O(|Imz|^{-1}), \quad m \in \mathbb{R}.$$

We use an induction on m. For $m=0$, *i)* follows from (7.1). Assume that *i)* holds for $m-1$. Then we write:

$$\begin{aligned} &N^m(H-z)^{-1}(N+1)^{1-m} \\ &= N^m(H+b)^{-1}(N+1)^{1-m}(N+1)^{m-1}(H+b)(H-z)^{-1}(N+1)^{1-m} \\ &= N^m(H+b)^{-1}(N+1)^{1-m}(N+1)^{m-1}(\mathbb{1}+(b+z)(H-z)^{-1})(N+1)^{1-m}. \end{aligned}$$

So *i)* for m follows from (7.1) and the induction hypothesis.□

7.5 Commutator estimates

In this subsection we estimate commutators between operators $\Gamma(q)$, $I(j)$ and the Hamiltonians H and $H^{\rm ext}$. These estimates rely on the identities of Subsect. 3.12 and the higher order estimates.

The following lemma is analogous to [DG1, Lemma 3.3].

Lemma 7.11 *Let $q \in C_0^\infty(\mathbb{R}^d)$, $0 \leq q \leq 1$, $q = 1$ near 0. Set for $R \geq 1$, where $q^R(x) = q(\frac{x}{R})$. Then for $\chi \in C_0^\infty(\mathbb{R})$:*

$$[\Gamma(q^R), \chi(H)] \in \begin{cases} (N+1)^{-\infty}O_N(R^{-\inf(s,1)}) \text{ under hypothesis } (Is), \\ (N+1)^{-\infty}o_N(R^0) \text{ under hypothesis } (A). \end{cases}$$

Proof. Let us prove the lemma under hypothesis *(Is)*, the proof under hypothesis *(A)* being similar. We have $[\Gamma(q^R), N] = 0$, hence $\Gamma(q^R)$ preserves $\mathcal{D}(N^n)$. By Lemma 3.4 *ii)*

$$[H_0, \Gamma(q^R)] = {\rm d}\Gamma(q^R, [\omega, q^R]), \tag{7.15}$$

$[\omega, q^R]$ is bounded and hence $[H_0, \Gamma(q^R)](H_0+1)^{-1}$ is bounded. Therefore, $\Gamma(q^R)$ preserves $\mathcal{D}(H_0)$.

Using that on $\mathcal{D}(H_0) \cap \mathcal{D}(N^n)$ we have $H = H_0 + V$ and $\Gamma(q^R)$ preserves $\mathcal{D}(H_0) \cap \mathcal{D}(N^n)$, the following identity is valid as an operator identity on $\mathcal{D}(H_0) \cap \mathcal{D}(N^n)$:

$$[H, \Gamma(q^R)] = [H_0, \Gamma(q^R)] + [V, \Gamma(q^R)] =: T.$$

Using (7.15) and the fact that $[\omega, q^R] \in O(R^{-1})$, we get

$$[\Gamma(q^R), H_0] \in (N+1)O_N(R^{-1}),$$

and using Lemma 3.17 and Lemma 6.3, we have

$$[\Gamma(q^R), V] \in (N+1)^n O_N(R^{-s}),$$

which gives

$$T \in (N+1)^n O(R^{-\inf(s,1)}). \tag{7.16}$$

Now let

$$R(z) := [\Gamma(q^R), (z-H)^{-1}] = -(z-H)^{-1}[\Gamma(q^R), H](z-H)^{-1}.$$

We want to show that

$$N^m R(z)(H+b)^{-n-m} \in |Imz|^{-2}O(R^{-\inf(s,1)}), \quad m \geq 0, \tag{7.17}$$

$$(H+b)^{-n-m} R(z) N^m \in |Imz|^{-2}O(R^{-\inf(s,1)}), \quad m \geq 0. \tag{7.18}$$

By the higher order estimates (7.1) $\mathcal{D}(H^n) \subset \mathcal{D}(H_0) \cap \mathcal{D}(N^n)$, so the following operator identity holds on $\mathcal{D}(H^{n-1})$:

$$R(z) = (z-H)^{-1} T (z-H)^{-1}.$$

Now

$$\|N^m R(z)(H+b)^{-n-m}\| \leq \|N^m (z-H)^{-1}(N+1)^{-m+1}\|$$
$$\times \|(N+1)^{m-1} T (N+1)^{-n-m+1}\| \|(N+1)^{n+m-1}(H+b)^{-n-m+1}\| \|(H+b)(z-H)^{-1}\|,$$

which proves (7.17), and (7.18) follows then by taking the adjoint.

If $\chi \in C_0^\infty(\mathbb{R})$, we denote by $\tilde{\chi} \in C_0^\infty(\mathbb{C})$ an almost analytic extension of χ, satisfying

$$\tilde{\chi}_{|\mathbb{R}} = \chi,$$
$$|\partial_{\overline{z}} \tilde{\chi}(z)| \leq C_n |Imz|^n, \quad n \in \mathbb{N}.$$

We use the following functional calculus formula (see [HS, DG2]) for $\chi \in C_0^\infty(\mathbb{R})$:

$$\chi(A) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{\chi}(z)(z-A)^{-1} \mathrm{d}z \wedge \mathrm{d}\,\overline{z}. \tag{7.19}$$

Let now $\chi_1 \in C_0^\infty(\mathbb{R})$ with $\chi_1 \chi = \chi$ and $\tilde{\chi}_1$ an almost analytic extension of χ_1. We write:

$$\begin{aligned}
& N^m [\chi(H), \Gamma(q^R)] N^p \\
&= N^m \chi_1(H)[\chi(H), \Gamma(q^R)] N^p + N^m [\chi_1(H), \Gamma(q^R)] \chi(H) N^p \\
&= \tfrac{\mathrm{i}}{2\pi} \textstyle\int_{\mathbb{C}} \partial_{\overline{z}} \tilde{\chi}(z) N^m \chi_1(H) R(z) N^p \mathrm{d}z \wedge \mathrm{d}\,\overline{z} \\
&+ \tfrac{\mathrm{i}}{2\pi} \textstyle\int_{\mathbb{C}} \partial_{\overline{z}} \tilde{\chi}_1(z) N^m R(z) \chi(H) N^p \mathrm{d}z \wedge \mathrm{d}\,\overline{z}.
\end{aligned}$$

Using Prop. 7.10 *ii)*, (7.17) and (7.18), we see that $N^m [\chi(H), \Gamma(q^R)] N^p$ is $O(R^{-\inf(s,1)})$, as claimed. $\Box$

The following lemma is analogous to [DG1, Lemma 3.4].

Lemma 7.12 *Let $j_0 \in C_0^\infty(\mathbb{R}^d)$, $j_\infty \in C^\infty(\mathbb{R}^d)$, $0 \leq j_0$, $0 \leq j_\infty$, $j_0^2 + j_\infty^2 \leq 1$, $j_0 = 1$ near 0 (and hence $j_\infty = 0$ near 0). Set $j := (j_0, j_\infty)$ and for $R \geq 1$ $j^R = (j_0^R, j_\infty^R)$. Then for $\chi \in C_0^\infty(\mathbb{R})$:*

$$\chi(H^{\mathrm{ext}}) I^*(j^R) - I^*(j^R)\chi(H) \in \begin{cases} (N+1)^{-\infty} \check{O}(R^{-\inf(s,1)}) & \text{under hypothesis } (Is), \\ (N+1)^{-\infty} \check{o}(R^0) & \text{under hypothesis } (A). \end{cases}$$

Proof. Again we will only prove the lemma under hypothesis *(Is)*. We have by Lemma 3.11 *i)*

$$H_0^{\rm ext} I^*(j^R) - I^*(j^R) H_0 \in (N+1)\check{O}_N(R^{-1}). \tag{7.20}$$

This implies that $I^*(j^R)$ sends $\mathcal{D}(H_0)$ into $\mathcal{D}(H_0^{\rm ext})$, and since $I^*(j^R)N = (N_0 + N_\infty)I^*(j^R)$, $I^*(j^R)$ sends also $\mathcal{D}(N^n)$ into $\mathcal{D}((N_0+N_\infty)^n)$.

Next by Lemma 3.17 and Lemma 6.3 we obtain

$$(V\otimes 1\!\!1)I^*(j^R) - I^*(j^R)V \in (N+1)^n \check{O}_N(R^{-s}). \tag{7.21}$$

This and (7.20) show that as an operator identity on $\mathcal{D}(H_0)\cap\mathcal{D}(N^n)$ we have

$$H^{\rm ext} I^*(j^R) - I^*(J^R) H \in (N+1)^n \check{O}_N(R^{-\min(1,s)}). \tag{7.22}$$

Using then the higher order estimates (7.1) and the fact that $I^*(j^R)$ sends $\mathcal{D}(H_0)$ into $\mathcal{D}(H_0^{\rm ext})$ and $\mathcal{D}(N^n)$ into $\mathcal{D}((N_0+N_\infty)^n)$, we obtain the following operator identity on $\mathcal{D}(H^n)$:

$$\begin{aligned} R(z) &= (z-H^{\rm ext})^{-1} I^*(j^R) - I^*(j^R)(z-H)^{-1} \\ &= (z-H^{\rm ext})^{-1}\Big(I^*(j^R)H - H^{\rm ext} I^*(j^R)\Big)(z-H)^{-1}. \end{aligned}$$

Using Prop. 7.10, we see that

$$(N_0+N_\infty)^m R(z)(H+b)^{-m-n} \in O(|Imz|^{-2})R^{-\inf(s,1)}, \tag{7.23}$$

$$(H^{\rm ext}+b)^{-m-n} R(z) N^m \in O(|Imz|^{-2})R^{-\inf(s,1)}. \tag{7.24}$$

Let us again pick $\chi_1 \in C_0^\infty(\mathbb{R})$ with $\chi_1\chi = \chi$. We have:

$$\begin{aligned} &(N_0+N_\infty)^m \chi(H^{\rm ext}) I^*(j^R) - I^*(j^R)\chi(H)N^m \\ &= (N_0+N_\infty)^m \chi_1(H^{\rm ext})\Big(\chi(H^{\rm ext})I^*(j^R) - I^*(j^R)\chi(H)\Big)N^m \\ &+ (N_0+N_\infty)^m\Big(\chi_1(H^{\rm ext})I^*(j^R) - I^*(j^R)\chi_1(H)\Big)\chi(H)N^m \\ &= \tfrac{\rm i}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(N_0+N_\infty)^m \chi_1(H^{\rm ext})R(z)N^m {\rm d}z\wedge {\rm d}\,\overline{z} \\ &+ \tfrac{\rm i}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi_1}(z)(N_0+N_\infty)^m R(z)\chi(H)N^m {\rm d}z\wedge {\rm d}\,\overline{z}. \end{aligned}$$

Using Prop. 7.10 *ii)*, (7.23) and (7.24), the above operator is $O(R^{-\inf(s,1)})$, as claimed. □

8 A conjugate operator for $P(\varphi)_2$ Hamiltonians

8.1 Introduction

This section is devoted to the study of a conjugate operator for $P(\varphi)_2$ Hamiltonians. The central point of the construction of a conjugate operator A for a Hamiltonian H is the proof that the quadratic form $[H, {\rm i}A]$ defined on $\mathcal{D}(H)\cap\mathcal{D}(A)$ extends as a bounded operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$ which is locally positive, ie such that $1\!\!1_\Delta(H)[H,{\rm i}A]1\!\!1_\Delta(H) \geq c_0 1\!\!1_\Delta(H) + K$, for $c_0>0$, $\Delta\subset\mathbb{R}$ an open interval and K a compact operator. However the local positivity of the quadratic form $[H,{\rm i}A]$ is not sufficient to apply the Mourre method. Additional conditions on

H, A are needed. It seems that the weakest property one can impose is the property that H is of class $C^1(A)$, introduced in the book [ABG].

Let us recall the precise definition. First let us define this property for a bounded operator B on $\mathcal{H}$. Let A be a self-adjoint operator on $\mathcal{H}$. We say that $B \in C^1(A)$ if the map

$$s \mapsto \mathrm{e}^{\mathrm{i}sA} B \mathrm{e}^{-\mathrm{i}sA}$$

is C^1 for the strong topology. The condition that $B \in C^1(A)$ can be characterized in terms of the commutator $[B, A]$. Namely (see [ABG, Lemma 6.2.9]) $B \in C^1(A)$ if and only if the quadratic form on $\mathcal{D}(A)$

$$Q(v, v) = (Av, Bv) - (B^* v, Av),\ v \in \mathcal{D}(A)$$

satisfies

$$|Q(v, v)| \leq C\|v\|^2, v \in \mathcal{D}(A). \tag{8.1}$$

Let us note the following consequence of the $C^1(A)$ property, proven in [ABG]:

Proposition 8.1 *Let B be bounded and $B \in C^1(A)$. Then B maps $\mathcal{D}(A)$ into itself, so that the expression $[A, \mathrm{i}B]$ makes sense as an operator on $\mathcal{D}(A)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{\mathrm{i}sA} B \mathrm{e}^{-\mathrm{i}sA} = [A, \mathrm{i}B].$$

For an arbitrary self-adjoint operator H [ABG] propose a different definition (which is equivalent to the one above for bounded H): $H \in C^1(A)$ if for some $z \notin \sigma(H)$ the map

$$s \mapsto \mathrm{e}^{\mathrm{i}sA}(z - H)^{-1}\mathrm{e}^{-\mathrm{i}sA}$$

is C^1 for the strong topology. These are some of the consequences of the fact that $H \in C^1(A)$ (see [ABG, Thm. 6.2.10, Prop. 7.2.10]):

Proposition 8.2 *Let A, H be self-adjoint and $H \in C^1(A)$. Then the following is true:*
i) For $z \notin \sigma(H)$, $(z - H)^{-1}$ maps $\mathcal{D}(A)$ into itself, the space $(z - H)^{-1}\mathcal{D}(A)$ does not depend on z and is a dense subspace of $\mathcal{D}(A) \cap \mathcal{D}(H)$.
ii) $\mathcal{D}(A) \cap \mathcal{D}(H)$ is dense both in $\mathcal{D}(A)$ and in $\mathcal{D}(H)$.
iii) The quadratic form $[H, \mathrm{i}A]$ on $\mathcal{D}(A) \cap \mathcal{D}(H)$ extends uniquely to a bounded operator $[H, \mathrm{i}A]_0$ from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$.
iv) For any $\lambda \in \mathbb{R}$, the virial relation $\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]_0 \mathbb{1}_{\{\lambda\}}(H) = 0$ holds.

In our case the application of the Mourre method to $P(\varphi)_2$ Hamiltonians runs into two related problems. The first problem is that (except for the ϕ_2^4 model) the domain of H is not explicitly known. (This indicates that it is unlikely that the stronger set of conditions introduced in the original paper of Eric Mourre [Mo], which require in particular that $\mathrm{e}^{\mathrm{i}sA}$ preserves $\mathcal{D}(H)$, can be checked for $P(\varphi)_2$ Hamiltonians).

The second problem is that the actual computation of $[H, \mathrm{i}A]$, needed to prove its positivity, cannot be done easily on $\mathcal{D}(A) \cap \mathcal{D}(H)$, since the identity $H = H_0 + V$ needed to do this computation does not make sense on $\mathcal{D}(H)$. For general $P(\varphi)_2$ Hamiltonians, these two problems will be addressed in Thms. 8.4 and 8.7 below. If $\deg P = 4$, a simpler argument, using the Wick calculus instead of the $Q-$representation can be used to show that $H \in C^1(A)$. This is done in Thm. 8.8

8.2 Analysis of $[H, \mathrm{i}A]$ part I

Let $a = \frac{1}{2}(x \cdot D_x + D_x \cdot x) = -\frac{1}{2}(k \cdot D_k + D_k \cdot k)$ be the generator of dilations on $L^2(\mathbb{R}, dk)$. We denote by A the second quantized generator of dilations

$$A := \mathrm{d}\Gamma(a).$$

We put

$$\begin{aligned} H_0^{(1)} &:= [H_0, \mathrm{i}A] \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(H_0), \\ V^{(1)} &:= [V, \mathrm{i}A], \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(N^n). \\ H^{(1)} &:= [H, \mathrm{i}A], \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(H^n). \end{aligned} \tag{8.2}$$

Remark 8.3 *It is important here to define the quadratic form* $[H, \mathrm{i}A]$ *on* $\mathcal{D}(A) \cap \mathcal{D}(H^n)$*. In fact from the higher order estimates (7.1), we know that* $\mathcal{D}(H^n) \subset \mathcal{D}(H_0) \cap \mathcal{D}(N^n)$ *so that on* $\mathcal{D}(A) \cap \mathcal{D}(H^n)$ *we have* $[H, \mathrm{i}A] = [H_0, \mathrm{i}A] + [V, \mathrm{i}A]$.

By a direct computation we see that $H_0^{(1)}$ extends uniquely as a bounded operator from $\mathcal{D}(H_0)$ to $\mathcal{H}$ (still denoted by $H_0^{(1)}$), equal to $d\Gamma(k \cdot \nabla\omega(k))$.

Note that the operator $\mathrm{e}^{\mathrm{i}sa}$ commutes with the conjugation c. Therefore, since V is a multiplication operator on Q-space, so is $V^{(1)}$. In this subsection we will study properties of $V^{(1)}$ which follow from the assumption *(M1)* and its expression as a Wick polynomial.

It is convenient to introduce the notation $H_s = \mathrm{e}^{\mathrm{i}sA} H \mathrm{e}^{-\mathrm{i}sA}$, $H_{0,s} = \mathrm{e}^{\mathrm{i}sA} H_0 \mathrm{e}^{-\mathrm{i}sA}, V_s = \mathrm{e}^{\mathrm{i}sA} V \mathrm{e}^{-\mathrm{i}sA}$. We have $H_{0,s} = \mathrm{d}\Gamma(\omega_s)$, for $\omega_s(k) = \omega(\mathrm{e}^s k)$, and using (3.18), we see that V_s is a Wick polynomial with the kernels $w_{p,s} = \Gamma(\mathrm{e}^{\mathrm{i}sa}) w_p$.

Theorem 8.4 *Assume hypothesis (M1). Then*

i) the form $V^{(1)}$ *extends to a bounded operator from* $\mathcal{D}(N^n)$ *to* $\mathcal{H}$*. It is a multiplication operator on Q-space by a function in* $\bigcap_{p<\infty} L^p(Q, \mathrm{d}\mu)$.

ii) the form $H^{(1)}$ *extends uniquely as an operator still denoted by* $H^{(1)}$ *bounded from* $\mathcal{D}(H^n)$ *into* $\mathcal{H}$.

iii) for all $z \notin \sigma(H)$*, for* $r \geq 2n$*,* $(z - H)^{-r} \in C^1(A)$ *and hence the following identity holds as an identity between bounded operators from* $\mathcal{D}(A)$ *to* $\mathcal{H}$*:*

$$A(H - z)^{-r} = (H - z)^{-r} A + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}(H_s - z)^{-r}_{|s=0}, \tag{8.3}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}s}(H_s - z)^{-r}_{|s=0} = \sum_{j=0}^{r-1} (H - z)^{-r+j} (H_0^{(1)} + V^{(1)})(H - z)^{-j-1} \tag{8.4}$$

is a bounded operator on $\mathcal{H}$.

iv) Assume in addition that $\deg P = 4$ *and that hypothesis (C) holds. Then* $(z - H)^{-1} \in C^1(A)$ *and the following identity holds as an identity between bounded operators from* $D(A)$ *to* $\mathcal{H}$*:*

$$A(H - z)^{-1} = (H - z)^{-1} A + (H - z)^{-1} (H_0^{(1)} + V^{(1)})(H - z)^{-1}.$$

In the next subsection we will need to approximate V by V_κ. Let us set

$$V_\kappa^{(1)} := [V_\kappa, \mathrm{i}A], \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(N^n).$$

Proposition 8.5 *Assume hypothesis (M1). Then*
i) the form $V_\kappa^{(1)}$ *extends to a bounded operator from* $\mathcal{D}(N^n)$ *to* $\mathcal{H}$. *It is a multiplication operator on Q-space by a function in* $\bigcap_{p<\infty} L^p(Q, \mathrm{d}\mu)$.
ii) as bounded operators from $\mathcal{D}(N^n)$ *to* $\mathcal{H}$, *we have:*

$$V^{(1)} = \lim_{\kappa \to +\infty} V_\kappa^{(1)}.$$

iii) For some $\epsilon > 0$

$$\|V^{(1)} - V_\kappa^{(1)}\|_{L^p(Q,d\mu)} \leq C(p-1)^n \kappa^{-\epsilon}, \ \epsilon > 0.$$

Lemma 8.6 *Assume hypothesis (M1). Then*
i) uniformly in κ,

$$d\Gamma(a) w_{p,\kappa} \in L^2(\mathbb{R}^p), \ 1 \leq \kappa \leq \infty,$$

ii) there exists $\epsilon > 0$ *such that:*

$$\|\mathrm{d}\Gamma(a)(w_{p,\kappa} - w_{p,\infty})\|_{L^2(\mathbb{R}^p)} \in O(\kappa^{-\epsilon}).$$

Proof. We compute

$$\begin{aligned} & d\Gamma(a) w_{p,\infty} \\ &= \Big(a\hat{g}(\textstyle\sum_1^p k_i)\Big)\Pi_1^p \omega(k_i)^{-\frac{1}{2}} + \sum_1^p \hat{g}(\sum_1^p k_i)\Big(a\omega(k_i)^{-\frac{1}{2}}\Big)\Pi_{j\neq i}\omega(k_j)^{-\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} & d\Gamma(a) w_{p,\kappa} \\ &= \Big(a\hat{g}(\textstyle\sum_1^p k_i)\Big)\Pi_1^p \hat{\chi}(\frac{k_i}{\kappa})\omega(k_i)^{-\frac{1}{2}} + \sum_1^p \hat{g}(\sum_1^p k_i)\Big(a\omega(k_i)^{-\frac{1}{2}}\hat{\chi}(\frac{k_i}{\kappa})\Big)\Pi_{j\neq i}\omega(k_j)^{-\frac{1}{2}}\hat{\chi}(\frac{k_j}{\kappa}). \end{aligned}$$

Using *(M1)* and the bound (6.9), one sees that $d\Gamma(a) w_{p,\kappa} \in L^2(\mathbb{R}^p)$, $1 \leq \kappa \leq \infty$.
Then one checks that

(8.5) $$|a^\alpha (1 - \hat{\chi}(\frac{k}{\kappa}))\omega(k)^{-\frac{1}{2}}| \leq C|k|^{-\frac{1}{2}+\epsilon}\kappa^{-\epsilon}, \text{ for some } \epsilon > 0 \text{ and } \alpha = 0, 1.$$

Using again (6.9) we deduce from (8.5) statement *ii)* in the lemma. □

Proof of Theorem 8.4. Applying Prop. 3.13, we obtain the following identity between quadratic forms on $\mathcal{D}(A) \cap \mathcal{D}(N^n)$:

(8.6) $$\begin{aligned} V_p^{(1)} &:= [\textstyle\int g(x) :\varphi(x)^p: \mathrm{d}x, \mathrm{i}A] \\ &= \sum_{r=0}^p \begin{pmatrix} p \\ r \end{pmatrix} \int w_p^{(1)}(k_1, \dots, k_r, k_{r+1}, \dots, k_p) \\ &\quad \times a^*(k_1) \cdots a^*(k_r) a(-k_{r+1}) \cdots a(-k_p) \mathrm{d}k_1 \cdots \mathrm{d}k_p, \end{aligned}$$

where

$$w_p^{(1)} = \mathrm{d}\Gamma(a)w_p.$$

By Lemma 8.6 with $\kappa=\infty$, $w_p^{(1)} \in L^2(\mathbb{R}^p)$. Hence, the rhs of (8.6) defines a bounded operator from $\mathcal{D}(N^n)$ to $\mathcal{H}$. Next we note that $V^{(1)}\Omega \in \oplus_{p=0}^{2n} \otimes_{\mathrm{s}}^p \mathfrak{h}$, since $V^{(1)}$ is a Wick polynomial of degree $2n$. Hence by Lemma 5.12 $V^{(1)}\Omega \in \bigcap\limits_{p<\infty} L^p(Q,\mathrm{d}\mu)$. Therefore, as a multiplication operator $V^{(1)} \in \bigcap\limits_{p<\infty} L^p(Q,\mathrm{d}\mu)$. This ends the proof of *i)*.

Let us prove *iii)*. For $r\in\mathbb{N}, z\in\mathbb{C}\backslash\sigma(H)$, the following identity makes sense (all terms are bounded operators):

$$(H_s-z)^{-r}-(H-z)^{-r}=\sum_{j=0}^{r-1}(H_s-z)^{-r+j}(H-H_s)(H-z)^{-j-1}. \tag{8.7}$$

We deduce easily from the explicit form of $H_{0,s}$ that

$$\|H_{0,s}(H_0+1)^{-1}\|\leq C,\ \text{uniformly for } |s|\leq 1, \tag{8.8}$$

Since $H_{0,s}$ and H_0 commute, this implies that $D(H^r_{0,s})=D(H^r_0)$ for $r\in\mathbb{N}$. Since on the other hand $\mathrm{e}^{\mathrm{i}sA}$ preserves N, we have

$$(N+1)^{\alpha}(H_0+1)^2\leq C(H_s+b)^{2n+\alpha},\ \alpha\geq 0,\ |s|\leq 1, \tag{8.9}$$

ie Rosen's higher order estimates are uniformly valid for $H_s, |s|\leq 1$.

We will first show that for $r\geq 2n$

$$s^{-1}((H_s-z)^{-r}-(H-z)^{-r})\ \text{is uniformly bounded for } |s|\leq 1. \tag{8.10}$$

To prove (8.10) it suffices to show that

$$\|((H_s-z)^{-r}-(H-z)^{-r})u\|\leq Cs\|u\|,\ u\in D(H^n),\ |s|\leq 1. \tag{8.11}$$

By the higher order estimates, $D(H^n)\subset D(H_0)\cap D(N^n)$ and hence $D(H^n)\subset D(H_0)\cap D(V)$. Hence we can write:

$$\begin{array}{rl} & ((H_s-z)^{-r}-(H-z)^{-r})(H+b)^{-n} \\ = & \sum_{j=0}^{r-1}(H_s-z)^{-r+j}(H_0-H_{0,s}+V-V_s)(H-z)^{-j-1}(H+b)^{-n}. \end{array} \tag{8.12}$$

We note that $(\mathrm{i}s)^{-1}(H_{0,s}-H_0)=\mathrm{d}\Gamma((\mathrm{i}s^{-1})(\omega_s-\omega))$ and that

$$\begin{array}{rl} & \text{s-}\lim_{s\to 0}(\mathrm{i}s)^{-1}(H_{0,s}-H_0)(H_0+1)^{-1} \\ = & H_0^{(1)}(H_0+1)^{-1}, \\ & \|(H_{0,s}-H_0)(H_0+1)^{-1}\|\leq C|s|. \end{array} \tag{8.13}$$

The same result holds for $(\mathrm{i}s)^{-1}(H_0+1)^{-1}(H_{0,s}-H_0)$.

For $j+1\geq n$, we write:

$$\begin{array}{rl} & (\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(H_{0,s}-H_0)(H-z)^{-j-1} \\ = & (\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(H_{0,s}-H_0)(H_0+1)^{-1}(H_0+1)(H-z)^{-j-1}, \end{array}$$

and for $r-j\geq n$ we write:

$$
\begin{aligned}
& (\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(H_{0,s}-H_0)(H-z)^{-j-1}\\
= \; & (\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(H_0+1)(H_0+1)^{-1}(H_{0,s}-H_0)(H-z)^{-j-1},
\end{aligned}
$$

Since $r\geq 2n$, if $0\leq j\leq r-1$ we have either $j+1\geq n$ or $r-j\geq n$. Using (8.9) we obtain that

$$\|(\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(H_{0,s}-H_0)(H-z)^{-j-1}\|\leq C. \tag{8.14}$$

Next it follows from Lemma 8.6 *i)* for $\kappa=\infty$ that the map

$$s\mapsto w_{p,s}=\hat{g}\left(\mathrm{e}^s\sum_{i=1}^{p}k_i\right)\Pi_{i=1}^{p}\omega(\mathrm{e}^s k_i)^{-\frac{1}{2}}$$

is $C^1(\mathbb{R},L^2(\mathbb{R}^p))$ with derivative $d\Gamma(a)w_p$. This implies that

$$(\mathrm{i}s)^{-1}(N+1)^{-r_1}(V_s-V)(N+1)^{-r_2}\to \mathrm{i}(N+1)^{-r_1}V^{(1)}(N+1)^{-r_2} \tag{8.15}$$

in operator norm when $s\to 0$, for $r_1+r_2\geq n$.

We write

$$
\begin{aligned}
& (\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(V_s-V)(H-z)^{-j-1}\\
= \; & (\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(N+1)^{r-j}(N+1)^{j-r}(V_s-V)(N+1)^{-j-1}(N+1)^{j+1}(H-z)^{-j-1}.
\end{aligned}
$$

Using (8.9), we obtain that

$$\|(\mathrm{i}s)^{-1}(H_s-z)^{-r+j}(V_s-V)(H-z)^{-j-1}\|\leq C. \tag{8.16}$$

Combining (8.16), (8.14) and (8.12), we obtain (8.11) and hence (8.10).

By (8.10), to prove that $(H-z)^{-r}\in C^1(A)$ if suffices to show the convergence of $(\mathrm{i}s)^{-1}((H_s-z)^{-r}-(H-z)^{-r})$ on a dense subspace of $\mathcal{H}$. But by (8.13) and (8.15) this convergence holds on $D(H^n)$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(H_s-z)^{-r}_{|s=0}=\sum_{j=0}^{r-1}(H-z)^{-r+j}(H_0^{(1)}+V^{(1)})(H-z)^{-j-1}.$$

This completes the proof of *iii)*.

Let us now prove *iv)*. We assume hence that $\deg P=4$ and hypothesis *(C)* holds. By Thm. 7.2 and (8.8), the Glimm-Jaffe estimate holds uniformly in $|s|\leq 1$

$$
\begin{aligned}
& H_0^2\leq C(H_s+b)^2,\\
& N^2+C(H_s+b)^2.
\end{aligned} \tag{8.17}
$$

Another consequence of the fact that $\deg P=4$ is that

$$(\mathrm{i}s)^{-1}(N+1)^{-1}(V_s-V)(N+1)^{-1}\to \mathrm{i}(N+1)^{-1}V^{(1)}(N+1)^{-1}, \tag{8.18}$$

since we may take $n=2$ in (8.15). Next if we use (8.13), (8.18) as before and (8.17) instead of (8.9), we see that the proof of *iii)* extends to the case $r=1$.

Finally let us prove *ii)*. By Remark 8.3 the quadratic form $[H, \mathrm{i}A]$ on $\mathcal{D}(A) \cap \mathcal{D}(H^n)$ is equal to $[H_0, \mathrm{i}A] + [V, \mathrm{i}A]$. We have seen that $[H_0, \mathrm{i}A]$ extends as an operator $H_0^{(1)}$ such that $H_0^{(1)}(H_0 + 1)^{-1}$ is bounded, which by the higher order estimates implies that $H_0^{(1)}(H + b)^{-n}$ is bounded. In *i)* we have seen that $[V, \mathrm{i}A]$ extends as an operator $V^{(1)}$ such that $V^{(1)}(N + 1)^{-n}$ is bounded, which again by the higher order estimates implies that $V^{(1)}(H + b)^{-n}$ is bounded. It remains to check that the extension of $[H, \mathrm{i}A]$ to an operator $H^{(1)}$ with domain $\mathcal{D}(H^n)$ is unique, ie that $\mathcal{D}(A) \cap \mathcal{D}(H^n)$ is dense in $\mathcal{D}(H^n)$. For $u = (H + b)^{-n}v \in \mathcal{D}(H^n)$, we set $u_\epsilon = (H + b)^{-n}(1 + i\epsilon A)^{-1}v$. Clearly $u_\epsilon \in \mathcal{D}(H^n)$ and u_ϵ tends to u in $\mathcal{D}(H^n)$ when $\epsilon \to 0$. It follows then from *iii)* that $u_\epsilon \in \mathcal{D}(A)$, which completes the proof of *ii)*. $\Box$

Proof of Proposition 8.5. *i)* follows from Lemma 8.6.

To prove *ii)* we note that

$$\|(V_p^{(1)} - V_{p,\kappa}^{(1)})(N + 1)^{-n}\| \leq C\|w_p - w_{p,\kappa}\|_{L^2(\mathbb{R}^p)} \leq C\kappa^{-\epsilon}.$$

To show *iii)* we note that

$$\|V_p^{(1)} - V_{p,\kappa}^{(1)}\|_{L^2(Q,\mathrm{d}\mu)} = \|w_p - w_{p,\kappa}\|_{L^2(\mathbb{R}^p)} \leq C\kappa^{-\epsilon}.$$

Then we use Lemma 5.12. $\Box$

8.3 Analysis of $[V, \mathrm{i}A]$ part II

In this subsection we continue our study of $[H, \mathrm{i}A]$. The main new ingredient is the use of hypothesis *(B1)*, which will allow to dominate $|V^{(1)}|$, as a function on $Q-$space, by H, using hypercontractivity arguments.

Theorem 8.7 *Assume hypothesis (B1).*
i) There exists $c_0, b > 0$ *such that*

$$|(u, H^{(1)}u)| \leq c_0(u, (H + b)u), \ u \in \mathcal{D}(H^n),$$

and hence $H^{(1)}$ *extends uniquely as an operator bounded from* $\mathcal{D}(H^{\frac{1}{2}})$ *to* $\mathcal{D}(H^{\frac{1}{2}})^*$.
ii) $H \in C^1(A)$.
iii) The operator $[H, \mathrm{i}A]_0$ *from* $\mathcal{D}(H)$ *into* $\mathcal{D}(H)^*$*, associated to* $[H, \mathrm{i}A]$ *by Proposition 8.2, coincides with* $H^{(1)}$ *and hence is bounded from* $\mathcal{D}(H^{\frac{1}{2}})$ *to* $\mathcal{D}(H^{\frac{1}{2}})^*$.
iv) The virial relation holds

$$\mathbb{1}_{\{\lambda\}}(H)[H, \mathrm{i}A]_0\mathbb{1}_{\{\lambda\}}(H) = 0, \ \lambda \in \mathbb{R}.$$

Thm. 8.7 is the main result of this section. Property *ii)* allows in particular to justify the virial relation *iv)*. Property *iii)* allows to actually compute $[H, \mathrm{i}A]_0$ which will be important to prove its positivity in Subsect. 9.2.

In the φ_2^4 case a similar result holds under weaker hypotheses on g.

Theorem 8.8 *Assume* $\deg P = 4$ *and hypotheses (M1), (C).*
i) There exists $c_0, b > 0$ *such that*

$$|(Hu, Au) - (Au, Hu)| \leq c_0 \|(H+b)u\|^2,\ u \in \mathcal{D}(A) \cap \mathcal{D}(H),$$

and hence $H^{(1)}$ *extends uniquely as an operator bounded from* $\mathcal{D}(H)$ *to* $\mathcal{D}(H)^*$.
ii) $H \in C^1(A)$.
iii) The operator $[H, \mathrm{i}A]_0$ *from* $\mathcal{D}(H)$ *into* $\mathcal{D}(H)^*$ *associated to* $[H, \mathrm{i}A]$ *coincides with* $H^{(1)}$.
iv) The virial relation holds

$$1\!\!1_{\{\lambda\}}(H)[H, \mathrm{i}A]_0 1\!\!1_{\{\lambda\}}(H) = 0,\ \lambda \in \mathbb{R}.$$

It is convenient to make a specific choice of the cutoff χ used to define the cutoff interactions V_κ. Namely we will fix for this section

$$\chi(x) = \frac{1}{2}\mathrm{e}^{-|x|},\ \hat{\chi}(k) = (1+k^2)^{-1}. \tag{8.19}$$

We denote by $u \star v$ the convolution

$$u \star v(x) = \int u(y) v(x-y) \mathrm{d}y,$$

so that $\mathcal{F}(u \star v) = \mathcal{F}(u)\mathcal{F}(v)$. Recall that the function f_κ was defined in (6.4).

Lemma 8.9

$$\tau_x(\mathrm{i}a f_\kappa) = 2\tau_x f_\kappa - \alpha_\kappa \star \tau_x f_\kappa,$$

where

$$\alpha_\kappa(x) = \kappa \mathrm{e}^{-\kappa|x|} + \frac{m}{4}\mathrm{e}^{-m|x|}.$$

Proof. After conjugation by the Fourier transformation, we are reduced to check that

$$(-k\partial_k - \frac{1}{2})\hat{f}_\kappa = (2\delta_0 - \hat{\alpha}_\kappa)\hat{f}_\kappa.$$

This is a direct computation, using (8.19). □

Lemma 8.10 *Assume hypothesis (B0). Then there exists* C_0 *such that for* $\kappa \geq m$*:*

$$|\alpha_\kappa \star g(x)| \leq C_0 g(x). \tag{8.20}$$

An analogous estimate is true if we replace $\alpha_\kappa(x)$ *with* $x\partial_x \alpha_\kappa(x)$ *or* $\alpha_\kappa \star \alpha_\kappa(x)$.

Proof. It is sufficient to show the estimate replacing α_κ with a function $\kappa\psi(\kappa x)$ for $\psi \in \mathcal{S}(\mathbb{R})$. Now by *(B0)*:

$$\int g(x - x')\psi(\kappa x')\kappa \mathrm{d}x' \quad \leq C g(x) \int \psi(\tfrac{x'}{\kappa})\langle x'\rangle \kappa \mathrm{d}x'.$$

□

Lemma 8.11 *We have:*

$$
\begin{aligned}
&[A, \mathrm{i}\textstyle\int g(x)\varphi_\kappa(x)^p dx] \\
&= \textstyle\int (2p + \partial_x x) g(x)\varphi_\kappa(x)^p \mathrm{d}x \\
&- p \textstyle\int\int g(x)\alpha_\kappa(x'-x)\varphi_\kappa(x')(\varphi_\kappa(x))^{p-1}\mathrm{d}x\mathrm{d}x'.
\end{aligned} \tag{8.21}
$$

Proof. Using

$$\varphi_\kappa(x) = \phi(\tau_x f_\kappa) \tag{8.22}$$

we obtain as a quadratic form on $\mathcal{D}(A)\cap\mathcal{D}(N^{\frac{1}{2}})$:

$$[A, \mathrm{i}\varphi_\kappa(x)] = \phi(\mathrm{i}a\tau_x f_\kappa).$$

Now

$$
\begin{aligned}
\mathrm{i}a\tau_x f_\kappa(y) &= y\partial_y f_\kappa(y-x) + \tfrac{1}{2} f_\kappa(y-x) \\
&= (y-x)\partial_y f_\kappa(y-x) + \tfrac{1}{2} f_\kappa(y-x) - x\partial_x f_\kappa(y-x) \\
&= (2 - x\partial_x)\tau_x f_\kappa(y) - \textstyle\int \alpha_\kappa(x'-x)\tau_{x'} f_\kappa(y)\mathrm{d}x',
\end{aligned}
$$

by Lemma 8.9. This gives

$$[A, \mathrm{i}\varphi_\kappa(x)] = (2 - x\partial_x)\varphi_\kappa(x) - \int \alpha_\kappa(x'-x)\varphi_\kappa(x')\mathrm{d}x'. \tag{8.23}$$

Since $\mathrm{i}a = x\partial_x + \frac{1}{2}$ preserves $L^2_{\mathbb{R}}(\mathbb{R}, dx)$, we have $[\phi(\mathrm{i}a\tau_x f), \phi(\tau_x f)] = 0$ and hence

$$
\begin{aligned}
[A, \mathrm{i}\varphi_\kappa(x)^p] &= p\varphi_\kappa(x)^{p-1}[A, \mathrm{i}\varphi_\kappa(x)] \\
&= 2p\varphi_\kappa(x)^p - x\partial_x\varphi_\kappa^p(x) - p\textstyle\int \alpha_\kappa(x'-x)\varphi_\kappa(x')\varphi_\kappa^{p-1}(x)\mathrm{d}x',
\end{aligned} \tag{8.24}
$$

as a quadratic form on $\mathcal{D}(A)\cap\mathcal{D}(N^{p/2})$. The lemma follows then using (8.24) and integration by parts. □

As a consequence of Lemmas 8.9, 8.11, we have the following inequality, which should be understood as an inequality between functions on Q-space:

Lemma 8.12 *Assume hypothesis (B1). Then for* $p \in \mathbb{N}$

$$|[A, \mathrm{i}\int g(x)\varphi_\kappa(x)^p \mathrm{d}x]| \le C_p \int g(x)|\varphi_\kappa(x)|^p \mathrm{d}x, \text{ uniformly for } \kappa \ge m. \tag{8.25}$$

Proof. Let us denote by I_1, I_2 the terms in the r.h.s. of (8.21). We will estimate separately I_1 and I_2.

Estimate of I_1:

Since by *(B1)* $|x\partial_x g(x)| \le Cg(x)$, we see that

$$|I_1| \le C\int g(x)|\varphi_\kappa(x)|^p \mathrm{d}x. \tag{8.26}$$

Estimate of I_2:

We have:

$$|I_2| = p|\int g(x)\alpha_\kappa(x'-x)\varphi_\kappa(x')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x'|$$
$$\leq C\int g(x)|\alpha_\kappa(x'-x)||\varphi_\kappa(x')|^p\mathrm{d}x\mathrm{d}x'$$
$$+C\int g(x)|\alpha_\kappa(x'-x)||\varphi_\kappa(x)|^p\mathrm{d}x\mathrm{d}x',$$

using the fact that $ab^{p-1}\leq C_p(a^p+b^p)$. This yields

$$|I_2|\leq C_0c_\kappa\int g(x)|\varphi_\kappa(x)|^p\mathrm{d}x + C_0\int g_\kappa(x)|\varphi_\kappa(x)|^p\mathrm{d}x,$$

for

$$c_\kappa = \int \alpha_\kappa(x')\mathrm{d}x',\quad g_\kappa = g\star\alpha_\kappa.$$

We have used here the fact that g and α_κ are positive functions. Clearly

$$|c_\kappa|\leq C,\ \text{uniformly in}\ \kappa, \tag{8.27}$$

and from Lemma 8.10, we get that

$$|g_\kappa(x)|\leq C_0g(x),\ \text{uniformly in}\ \kappa\geq m. \tag{8.28}$$

From (8.27) and (8.28), we obtain:

$$|I_2|\leq C_0\int g(x)|\varphi_\kappa(x)|^p\mathrm{d}x,\ \text{uniformly in}\ \kappa\geq m. \tag{8.29}$$

□

Proposition 8.13 *Assume hypotheses (B1). Then there exists* $c>0$ *such that for* $t>0$*:*

$$\mathrm{e}^{-t(cV-|V^{(1)}|)}\in L^1(Q,d\mu).$$

Proof. Set

$$W := cV-|V^{(1)}|.$$

To check that $\mathrm{e}^{-tW}\in L^1(Q,d\mu)$, we use Lemma 5.9. The first bound of (5.4) follows from Prop. 8.5 *iii)*. Let us now check the second bound.

Using Wick identities (5.2) and Lemma 8.12 we obtain

$$|V_\kappa^{(1)}|\leq C_0\sum_{p=0}^{2n}|a_p|\sum_{r=0}^{[p/2]}\frac{p!2^{-r}}{(p-2r)!r!}\|\varphi_\kappa(x)\Omega\|^{2r}\int g(x)|\varphi_\kappa(x)|^{p-2r}\mathrm{d}x,$$

so that

$$cV_\kappa - V_\kappa^{(1)}\geq\int g(x)F_\kappa(\varphi_\kappa(x))\mathrm{d}x,$$

where $F_\kappa(\lambda)$ is a function as in (6.15), with $c_{2n,0}=ca_{2n}-c_0|a_{2n}|>0$ for $c>c_0$. Using (6.16) we obtain

$$cV_\kappa-|V_\kappa^{(1)}|\geq -c_2\|g\|_{L^1(\mathbb{R})}(\|\varphi_\kappa(x)\Omega\|^{2n}+1), \tag{8.30}$$

which since $\|\varphi_\kappa(x)\Omega\| = O(\ln \kappa^{\frac{1}{2}})$ completes the proof of the second bound in (5.4). Applying now Lemma 5.9, we get that there exists c large enough such that $e^{-tW} \in L^1(Q, d\mu)$ for all $t > 0$. This completes the proof of the proposition. □

Proof of Thm. 8.7. For large enough C we have $C\omega - \omega^{(1)} \geq C_0 > 0$. Therefore, we can apply Thm. 5.11 to $a = C\omega - \omega^{(1)}$ and $W = CV - |V^{(1)}|$ and show that $CH_0 - H_0^{(1)} + CV - |V^{(1)}|$ is bounded from below on $\mathcal{D}(CH_0 - H_0^{(1)}) \cap \mathcal{D}(N^n)$. But $\mathcal{D}(CH_0 - H_0^{(1)}) = \mathcal{D}(H_0)$ and $\mathcal{D}(H_0) \cap \mathcal{D}(N^n)$ contains $\mathcal{D}(H^n)$ by the higher order estimates and hence is dense in $\mathcal{D}(H)$. Therefore the inequality

$$H_0^{(1)} + V^{(1)} \leq C(H_0 + V + b) \text{ on } \mathcal{D}(H^n)$$

extends as the inequality

$$H^{(1)} \leq C(H + b) \text{ on } \mathcal{D}(H).$$

Likewise, we prove

$$-H^{(1)} \leq C(H + b) \text{ on } \mathcal{D}(H).$$

This proves *i)*.

Next, let us show *ii)*. To prove that $H \in C^1(A)$, we check condition (8.1). Let us first prove that $(H+b)^{-1}$ preserves $D(A)$. By Thm. 8.4, we have the following identity on $D(A)$, for $s \geq 0$:

$$A(H + b + s)^{-2n} = (H + b + s)^{-2n}A + \mathrm{i} \sum_{j=0}^{2n-1} (H + b + s)^{-2n-j} H^{(1)}(H + b + s)^{-j-1}.$$

By *i)* $(H + b)^{-\frac{1}{2}} H^{(1)}(H + b)^{-\frac{1}{2}}$ is bounded. Using then the bound

$$\|(H + b + s)^{-1}(H + b)^{\frac{1}{2}}\| \leq c(b + s)^{-\frac{1}{2}},$$

we obtain that $(H + b + s)^{-2n}$ has a norm $O(\langle s\rangle^{-2n})$ in $B(D(A))$. We use then the formula

$$(H + b)^{-1} = c_n \int_0^{+\infty} s^{2n-2}(H + b + s)^{-2n} \mathrm{d}s.$$

The integrand has a norm $O(s^{2n-2}\langle s\rangle^{-2n})$ in $B(D(A))$ hence the integral converges in norm. This implies that $(H + b)^{-1}$ is a bounded operator on $D(A)$.

Since $(H + b)^{-1}$ preserves $\mathcal{D}(A)$, we can write for $v \in \mathcal{D}(A)$:

$$\begin{aligned} Q(v, v) &= (v, A(H + b)^{-1}v) - (A(H + b)^{-1}v, v) \\ &= (Hu, Au) - (Au, Hu), \text{ for } u = (H + b)^{-1}v. \end{aligned}$$

Since $(H + b)^{-1}\mathcal{D}(A) \subset \mathcal{D}(A) \cap \mathcal{D}(H)$, (8.1) is implied by

$$|(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2),\ u \in \mathcal{D}(A) \cap \mathcal{D}(H). \tag{8.31}$$

We know by *i)* that (8.31) holds for $u \in \mathcal{D}(A)\cap\mathcal{D}(H^n)$. So to prove (8.31) it suffices to show that $\mathcal{D}(A)\cap\mathcal{D}(H^n)$ is dense in $\mathcal{D}(A)\cap\mathcal{D}(H)$ for the intersection topology. Let hence $u \in \mathcal{D}(A)\cap\mathcal{D}(H)$ and $u_\epsilon = (1 + i\epsilon H)^{-2n}u$. Clearly $u_\epsilon \in \mathcal{D}(A) \cap \mathcal{D}(H^n)$, $u_\epsilon \to u, Hu_\epsilon \to Hu$. Now from (8.3), we get:

$$\begin{aligned} &A(1 + i\epsilon H)^{-2n}u = (1 + i\epsilon H)^{-2n}Au - i\epsilon \textstyle\sum_{j=0}^{2n-1}(1 + i\epsilon H)^{-2n+j} H^{(1)}(1 + i\epsilon H)^{-j-1}u \\ &=: (1 + i\epsilon H)^{-2n}Au - R_\epsilon u. \end{aligned}$$

We claim that

$$\text{s-}\lim_{\epsilon\to 0} R_\epsilon = 0, \tag{8.32}$$

which will imply that $Au_\epsilon \to Au$ when $\epsilon \to 0$. In fact we write

$$\begin{aligned}&(1+i\epsilon H)^{-2n+j}H^{(1)}(1+i\epsilon H)^{-j-1}\\ &= (1+i\epsilon H)^{-2n+j}(H+b)^{\frac{1}{2}}(H+b)^{-\frac{1}{2}}H^{(1)}(H+b)^{-\frac{1}{2}}(H+b)^{\frac{1}{2}}(1+i\epsilon H)^{-j-1}.\end{aligned}$$

Using *i)* and the bound $\|(1+i\epsilon H)^{-j}(H+b)^{\frac{1}{2}}\| \in O(\epsilon^{-\frac{1}{2}})$ for $j \geq 1$, we get that $R_\epsilon \in O(1)$. So it suffices to prove (8.32) on a dense subset of $\mathcal{H}$. For $u \in \mathcal{D}(H^n)$, we have:

$$(1+i\epsilon H)^{-r+j}H^{(1)}(1+i\epsilon H)^{-j-1}u = (1+i\epsilon H)^{-r+j}H^{(1)}(H+b)^{-n}(1+i\epsilon H)^{-j-1}(H+b)^n u,$$

which by Thm. 8.4 *ii)* shows that $R_\epsilon u \to 0$ when $\epsilon \to 0$. This proves that H is of class $C^1(A)$.

To prove *iii)*, we note that both $[H, \mathrm{i}A]_0$ and $H^{(1)}$ are extensions of $[H, \mathrm{i}A]$ on $\mathcal{D}(A)\cap\mathcal{D}(H)$ and $\mathcal{D}(A)\cap\mathcal{D}(H^n)$ respectively. Since $\mathcal{D}(A)\cap\mathcal{D}(H^n)$ is dense in $\mathcal{D}(A)\cap\mathcal{D}(H)$ these two extensions coincide.

Finally the fact that the virial theorem is true follows also from the $C^1(A)$ property (see [ABG, Prop. 7.2.10]). □

Proof of Thm. 8.8. We will prove that

$$(Hu, Au) - (Au, Hu) = \mathrm{i}^{-1}(u, H_0^{(1)}u) + (u, V^{(1)}u),\ u \in \mathcal{D}(H)\cap\mathcal{D}(A). \tag{8.33}$$

We have seen in Subsect. 8.2 that

$$|(u, H_0^{(1)}u)| \leq C\|H_0^{\frac{1}{2}}u\|^2, u \in \mathcal{D}(H_0),$$

and

$$|(u, V^{(1)}u)| \leq \|(N+1)u\|^2, u \in \mathcal{D}(N),\ \text{since } \deg P = 4.$$

Hence *i)* follows from (8.33) and the higher order estimates. Hence we have shown (8.31) which, as we have seen in the proof of Thm. 8.7, implies that $H \in C^1(A)$, ie that *ii*) holds. Finally *iv)* follows from *ii)* and *iii)* follows from (8.33). So it suffices to prove (8.33).

Since by Thm. 7.2 $\mathcal{D}(H) = \mathcal{D}(H_0)\cap\mathcal{D}(V)$, we have:

$$(Hu, Au) - (Au, Hu) = (H_0u, Au) - (Au, H_0u) + (Vu, Au) - (Au, Vu),\ u \in \mathcal{D}(H)\cap\mathcal{D}(A).$$

By definition

$$(H_0u, Au) - (Au, H_0u) = \mathrm{i}^{-1}(u, H_0^{(1)}u),$$

so it remains to justify the identity

$$(Vu, Au) - (Au, Vu) = \mathrm{i}^{-1}(u, V^{(1)}u),\ u \in \mathcal{D}(H)\cap\mathcal{D}(A). \tag{8.34}$$

Note that while (8.34) holds for example on $\mathcal{D}(N^2)\cap\mathcal{D}(A)$, it is not obvious that it extends to $u \in \mathcal{D}(H)\cap\mathcal{D}(A)$. In fact the expression of V as a Wick polynomial of order 4 needed to prove (8.34) is meaningful on $\mathcal{D}(N^2)$, but not on $\mathcal{D}(V)$. To justify (8.34) we use an approximation argument similar to the one used in the proof of Thm. 8.3.

So let $u \in \mathcal{D}(H) \cap \mathcal{D}(A)$, and $u_\epsilon = (1 + \mathrm{i}\epsilon N)^{-1}u$. Since by the higher order estimates $\mathcal{D}(H) \subset \mathcal{D}(N)$, $u_\epsilon \in \mathcal{D}(N^2) \cap \mathcal{D}(A)$ and hence (8.34) holds for u_ϵ. So it remains to show that $Au_\epsilon \to Au, Vu_\epsilon \to Vu$ when $\epsilon \to 0$. The first convergence is obvious. To prove the second one it suffices to show that

$$V(1 + \mathrm{i}\epsilon N)^{-1}u - (1 + \mathrm{i}\epsilon N)^{-1}Vu \to 0, \text{ when } \epsilon \to 0. \tag{8.35}$$

Note that since $u \in \mathcal{D}(N)$, we can write V as a Wick polynomial to prove (8.35). If

$$W = \int w(k_1, \cdots, k_4)a^*(k_1) \cdots a^*(k_r)a(-k_{r+1}) \cdots a(-k_4)\mathrm{d}k_1 \cdots \mathrm{d}k_4,$$

for $w \in L^2(\mathbb{R}^4)$, then

$$\begin{aligned} &W(1 + \mathrm{i}\epsilon N)^{-1} \\ &= \int (1 + \mathrm{i}\epsilon N + \mathrm{i}\epsilon(4 - 2r))^{-1}w(k_1, \cdots, k_4)a^*(k_1) \cdots a^*(k_r)a(-k_{r+1}) \cdots a(-k_4)\mathrm{d}k_1 \cdots \mathrm{d}k_4. \end{aligned}$$

Using the first resolvent formula and the bound $\|N(1 + \mathrm{i}\epsilon N)^{-1}\| \leq \epsilon^{-1}$, we obtain that

$$\begin{aligned} &\|(W(1 + \mathrm{i}\epsilon N)^{-1} - (1 + \mathrm{i}\epsilon N)^{-1}W)u\| \leq C\|(N + 1)u\|, \\ &\|(W(1 + \mathrm{i}\epsilon N)^{-1} - (1 + \mathrm{i}\epsilon N)^{-1}W)u\| \leq C\epsilon\|(N + 1)^2u\|. \end{aligned} \tag{8.36}$$

The first inequality in (8.36), it suffices to prove (8.35) for u in a dense subspace of $\mathcal{D}(N)$. By the second inequality in (8.36), (8.35) holds for $u \in \mathcal{D}(N^2)$.□

8.4 Analysis of $[[H, \mathrm{i}A], \mathrm{i}A]$.

The aim of this subsection is to show that, under hypothesis *(B2)*, $H \in C^2(A)$. The structure of the argument is parallel to the arguments used in Subsects 8.2 and 8.3.

We put

$$\begin{aligned} H_0^{(2)} &:= [H_0^{(1)}, \mathrm{i}A] \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(H_0), \\ V^{(2)} &:= [V^{(1)}, \mathrm{i}A], \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(N^n), \\ H^{(2)} &:= [H^{(1)}, \mathrm{i}A], \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(H^n). \end{aligned} \tag{8.37}$$

By a direct computation we see that $H_0^{(2)}$ extends uniquely as a bounded operator from $\mathcal{D}(H_0)$ to $\mathcal{H}$ (still denoted by $H_0^{(2)}$), equal to $\mathrm{d}\Gamma((k \cdot \nabla)^2\omega(k))$.

Proposition 8.14 *Assume hypothesis (M2). Then*
i) the form $V^{(2)}$ extends to a bounded operator from $\mathcal{D}(N^n)$ to $\mathcal{H}$. It is a multiplication operator on Q-space by a function in $\bigcap_{p<\infty} L^p(Q, \mathrm{d}\mu)$.
ii) the form $H^{(2)}$ extends uniquely to a bounded operator from $\mathcal{D}(H^n)$ to $\mathcal{H}$.

Proof. The proof is analogous to the proof of Thm. 8.4 *i)* and *ii)*. We write $V^{(2)}$ as a Wick polynomial using the fact that $a^2g \in L^2(\mathbb{R})$, which follows from hypothesis *(M2)*. □

Then we set

$$V_\kappa^{(2)} := [V_\kappa^{(1)}, \mathrm{i}A] \text{ as a quadratic form on } \mathcal{D}(A) \cap \mathcal{D}(N^n).$$

The following proposition is analogous to Prop. 8.5.

Proposition 8.15 *Assume hypothesis (M2). Then*
i) the form $V_\kappa^{(2)}$ *extends to a bounded operator from* $\mathcal{D}(N^n)$ *to* $\mathcal{H}$*. It is a multiplication operator on Q-space by a function in* $\bigcap_{p<\infty} L^p(Q, \mathrm{d}\mu)$.
ii) As bounded operators from $\mathcal{D}(N^n)$ *to* $\mathcal{H}$*, we have:*

$$V^{(2)} = \lim_{\kappa\to+\infty} V_\kappa^{(2)}.$$

iii) For some $\epsilon > 0$

$$\|V^{(2)} - V_\kappa^{(2)}\|_{L^p(Q,d\mu)} \leq C(p-1)^n\kappa^{-\epsilon},\ \epsilon > 0.$$

Theorem 8.16 *Assume hypothesis (B2).*
i) There exists $c_0, b > 0$ *such that*

$$|(u, H^{(2)}u)| \leq c_0(u, (H+b)u),\ u \in \mathcal{D}(H^n).$$

Hence $H^{(2)}$ *extends uniquely to a bounded operator from* $\mathcal{D}((H+b)^{\frac{1}{2}})$ *to* $\mathcal{D}((H+b)^{\frac{1}{2}})^*$.
ii) $H \in C^2(A)$.

The proof of this theorem will be similar to the proof of Theorem 8.7. The main difference is the following lemma, which is used instead of Lemma 8.12.

Lemma 8.17 *Assume hypothesis (B2). Then for* $p \in \mathbb{N}$*:*

$$|[A, \mathrm{i}[A, \mathrm{i}\int g(x)\varphi_\kappa(x)^p\mathrm{d}x]]| \leq C\int g(x)|\varphi_\kappa(x)|^p\mathrm{d}x,\ \text{uniformly for } \kappa \geq m.$$

Proof. We recall from Lemma 8.11 that $[A, \mathrm{i}\int g(x)\varphi_\kappa(x)^p\mathrm{d}x] = I_1 + I_2$, for

$$I_1 = \int(2pg(x) + \partial_x x g(x))\varphi_\kappa(x)^p\mathrm{d}x,$$

$$I_2 = -p\int\int g(x)\alpha_\kappa(x'-x)\varphi_\kappa(x')(\varphi_\kappa(x))^{p-1}\mathrm{d}x\mathrm{d}x',$$

The terms $[A, \mathrm{i}I_1]$ is completely analogous to $[A, \mathrm{i}\int g(x)\varphi_\kappa(x)^p\mathrm{d}x]$, with g replaced by $g_1 = 2pg + \partial_x xg$. It follows from hypothesis *(B2)* that $|x\partial_x g_1(x)| \leq g(x)$. The argument used in the proof of Lemma 8.12 shows then that $|[A, \mathrm{i}I_1]| \leq C\int g(x)|\varphi_\kappa(x)|^p\mathrm{d}x$.

We consider next $[A, \mathrm{i}I_2]$. Using the identity (8.23), we obtain:

$$\begin{aligned}
[A, \mathrm{i}I_2] &= p\int g(x)\alpha_\kappa(x'-x)x'\partial_{x'}\varphi_\kappa(x')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x' \\
&+p\int g(x)\alpha_\kappa(x'-x)\varphi_\kappa(x')x\partial_x\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x' \\
&-4p\int g(x)\alpha_\kappa(x'-x)\varphi_\kappa(x')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x' \\
&-p\int g(x)\alpha_\kappa(x'-x)\alpha_\kappa(x''-x')\varphi_\kappa(x'')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x'\mathrm{d}x'' \\
&-p(p-1)\int g(x)\alpha_\kappa(x'-x)\alpha_\kappa(x''-x)\varphi_\kappa(x')\varphi_\kappa(x'')\varphi_\kappa(x)^{p-2}\mathrm{d}x\mathrm{d}x'\mathrm{d}x'' \\
&= R_1 + \cdots + R_5.
\end{aligned} \tag{8.38}$$

The term R_3 is equal to $4I_2$ and hence bounded by $C\int g(x)|\varphi_\kappa(x)|^p\mathrm{d}x$. Integrating by parts, we have:

$$\begin{aligned}
R_1 + R_2 &= -2p\int g(x)\alpha_\kappa(x'-x)\varphi_\kappa(x')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x' \\
&-p\int x\partial_x g(x)\alpha_\kappa(x'-x)\varphi_\kappa(x')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x' \\
&-p\int g(x)(x'-x)\partial_{x'}\alpha_\kappa(x'-x)\varphi_\kappa(x')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x'.
\end{aligned} \tag{8.39}$$

The first term in (8.39) equals $2I_2$. The second term is similar to I_2, with g replaced by $x\partial_x g$. Note that it follows from *(B1)* that $|x\partial_x g(x)| \leq cg(x)$. The third term is also similar to I_2, with α_κ replaced by $x\partial_x\alpha_\kappa$. By the argument in Lemma 8.12, we obtain that $R_1 + R_2$ is bounded by $C\int g(x)|\varphi_\kappa(x)|^p \mathrm{d}x$.

The term R_4 is equal to

$$-p\int g(x)\rho_\kappa(x''-x)\varphi_\kappa(x'')\varphi_\kappa(x)^{p-1}\mathrm{d}x\mathrm{d}x'', \text{ for } \rho_\kappa = \alpha_\kappa \star \alpha_\kappa.$$

Again the argument in Lemma 8.12 shows that R_4 is bounded by $C\int g(x)|\varphi_\kappa(x)|^p \mathrm{d}x$.

Finally to estimate R_5, we use the fact that $abc^{p-2} \leq C(a^p + b^p + c^p)$, and get:

$$\begin{array}{rl} R_5 & \leq C\int g(x)\alpha_\kappa(x'-x)\alpha_\kappa(x''-x)|\varphi_\kappa(x'')|^p \mathrm{d}x\mathrm{d}x'\mathrm{d}x'' \\ & +C\int g(x)\alpha_\kappa(x'-x)\alpha_\kappa(x''-x)|\varphi_\kappa(x')|^p \mathrm{d}x\mathrm{d}x'\mathrm{d}x'' \\ & +C\int g(x)\alpha_\kappa(x'-x)\alpha_\kappa(x''-x)|\varphi_\kappa(x)|^p \mathrm{d}x\mathrm{d}x'\mathrm{d}x'' \\ \leq & 2Cc_\kappa \int g_\kappa(x)|\varphi_\kappa(x)|^p \mathrm{d}x + Cc_\kappa^2 \int g(x)|\varphi_\kappa(x)|^p \mathrm{d}x, \end{array}$$

for $c_k = \int \alpha_\kappa(x)\mathrm{d}x$, $g_\kappa = g \star \alpha_\kappa$. Using (8.27), (8.28), we obtain that R_5 is bounded by $C\int g(x)|\varphi_\kappa(x)|^p \mathrm{d}x$. This completes the proof of the lemma. □

Proof of Theorem 8.16. *i)* is shown exactly as the analogous statement of Theorem 8.7, using Lemma 8.17 instead of Lemma 8.12.

Let us prove that $H \in C^2(A)$. It follows first from the fact that $H \in C^1(A)$ that the following identity holds as quadratic forms on $\mathcal{D}(A)$:

$$[(H+b)^{-1}, \mathrm{i}A] = -(H+b)^{-1}H^{(1)}(H+b)^{-1},$$

(see [ABG, Thm. 6.2.10]). To show that $H \in C^2(A)$, we have to check that

$$(u|[(H+b)^{-1}H^{(1)}(H+b)^{-1}, A]u) \leq C\|u\|^2, \quad u \in \mathcal{D}(A). \tag{8.40}$$

We have remarked in the proof of Theorem 8.4 (see (8.32)) that $\mathcal{D}(H^n) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(A)$ for the graph topology. So it is enough to show (8.40) for $u \in \mathcal{D}(H^n) \cap \mathcal{D}(A)$.

For $u \in \mathcal{D}(H^n) \cap \mathcal{D}(A)$, we have

$$\begin{aligned} & ((H+b)^{-1}H^{(1)}(H+b)^{-1}u, Au) - (Au, (H+b)^{-1}H^{(1)}(H+b)^{-1}u) \\ & = (H^{(1)}(H+b)^{-1}u, (H+b)^{-1}Au) - ((H+b)^{-1}Au, H^{(1)}(H+b)^{-1}u) \\ & = (H^{(1)}(H+b)^{-1}u, A(H+b)^{-1}u) - (A(H+b)^{-1}u, H^{(1)}(H+b)^{-1}u) \\ & +i(H^{(1)}(H+b)^{-1}u, (H+b)^{-1}H^{(1)}(H+b)^{-1}u) \\ & +i((H+b)^{-1}H^{(1)}(H+b)^{-1}u, H^{(1)}(H+b)^{-1}u). \end{aligned} \tag{8.41}$$

We use the fact that $H^{(1)}(H+b)^{-n}$ is bounded by Thm. 8.4 *ii)* to justify the first equality in (8.41). Then we note that since $(H+b)^{-1}$ preserves $\mathcal{D}(A)$, the following identity is valid as bounded operators from $\mathcal{D}(A)$ to $\mathcal{H}$:

$$(H+b)^{-1}A = A(H+b)^{-1} + i(H+b)^{-1}H^{(1)}(H+b)^{-1}. \tag{8.42}$$

and we use the identity (8.42) in the second equality of (8.41).

Applying Thm. 8.7 *i)*, we see that the last two terms of (8.41) are less than $C\|u\|^2$. This shows that as quadratic forms on $\mathcal{D}(H^n)\cap\mathcal{D}(A)$, we have:

$$[(H+b)^{-1}H^{(1)}(H+b)^{-1},\mathrm{i}A]=(H+b)^{-1}[H^{(1)},\mathrm{i}A](H+b)^{-1}+R,$$

where R is bounded for the topology of $\mathcal{H}$. By *i)*, also the first term on the rhs is bounded for the topology of $\mathcal{H}$. $\Box$

9 Spectral analysis of $P(\varphi)_2$ Hamiltonians

This section is devoted to the spectral theory of $P(\varphi)_2$ Hamiltonians. We first show an HVZ type result. Note that the $\subset$ part of the HVZ theorem is well known (see [GJ3], [S-H.K]), although our proof is different. We then prove the Mourre estimate, which implies the local finiteness of point spectrum and under additional hypotheses, the limiting absorption principle.

9.1 HVZ theorem and existence of a ground state

Theorem 9.1 *We have*

$$\sigma_{\rm ess}(H)=[\inf\sigma(H)+m,+\infty[.$$

Consequently $\inf\sigma(H)$ *is a discrete eigenvalue of* H.

Let us pick functions $j_0,j_\infty\in C^\infty(\mathbb{R})$ with $0\leq j_0\leq 1, j_0\in C_0^\infty(\mathbb{R})$, $j_0=1$ near 0 and $j_0^2+j_\infty^2=1$. For $R\geq 1$, j^R is defined as in Subsect. 7.5. We will also set $q^R=(j_0^R)^2$.

Proof. We prove first the $\subset$ part of the theorem. Let $\chi\in C_0^\infty(]-\infty,\inf\sigma(H)+m[)$. Because of supp$\chi$ we have:

$$\chi(H^{\rm ext})=\chi(H^{\rm ext})1\!\!1_{\{0\}}(N_\infty).$$

Hence using twice Lemma 7.12 we have

$$\begin{aligned}&\chi(H)=\chi(H)I(j^R)I^*(j^R)=I(j^R)\chi(H^{\rm ext})I^*(j^R)+o(R^0)\\&=I(j^R)\chi(H^{\rm ext})1\!\!1_{\{0\}}(N_\infty)I^*(j^R)+o(R^0)=I(j^R)1\!\!1_{\{0\}}(N_\infty)I^*(j^R)\chi(H)+o(R^0).\end{aligned}$$

We claim that the operator $I(j^R)1\!\!1_{\{0\}}(N_\infty)I^*(j^R)\chi(H)=\Gamma(q^R)\chi(H)$ is compact. In fact since $(H_0+1)^{\frac12}(H+b)^{-\frac12}$ is bounded by the first order estimates (6.11), it suffices to verify that $\Gamma(q^R)(H_0+1)^{-\frac12}$ is compact, which is easy (see eg [DG1, Lemma 4.2]). Hence $\Gamma(q^R)\chi(H)$ is compact as a norm limit of compact operators.

Let us now prove the $\supset$ part of the theorem. Note that it follows from the $\subset$ part of the theorem that H admits a ground state. Let $\lambda>\inf\sigma(H)+m$. Let u be a ground state of H. Let $h\in C_0^\infty(\mathbb{R})$ with $\int h(k)\mathrm{d}k=1$ and let $x_0\in\mathbb{R}, x_0\neq 0$, $k_0\in\mathbb{R}, k_0\neq 0, \omega(k_0)=\lambda-\inf\sigma(H)$. Choose a sequence R_j such that $\lim_{j\to\infty}j^{-1}R_j=\infty$ and define $h_j\in C_0^\infty(\mathbb{R})$ by setting

$$h_j(k)=j^{d/2}h(j(k-k_0))\mathrm{e}^{\mathrm{i}R_jk\cdot x_0}.$$

Then $\|h_j\|=1$, $\mathrm{w}-\lim_{j\to\infty}h_j=0$ and $\lim_{j\to\infty}(\omega(k)-\omega(k_0))h_j=0$. Let

$$u_j=a^*(h_j)u.$$

We have $\lim_{j\to\infty}\|u_j\| = 1$ and $\mathrm{w}-\lim_{j\to\infty} u_j = 0$. Note that $u \in \mathcal{D}(H^m)$ for any m, so it belongs to $\mathcal{D}(H_0 N^m)$ for any $m \in \mathbb{R}$. Therefore, $u_j \in \mathcal{D}(H_0) \cap \mathcal{D}(N^n) \subset \mathcal{D}(H)$ and

$$\begin{aligned}(H-\lambda)u_j &= (H_0 + V - \lambda)u_j \\ &= a^*(h_j)(H-\lambda)u + a^*(\omega(k)h_j)u + [V, a^*(h_j)]u \\ &= a^*\Big((\omega(k)-\omega(k_0))h_j\Big)u + [V, a^*(h_j)]u.\end{aligned}$$

It is easy to see that

$$\| \lim_{i\to\infty} (h_i|w_p\|_{L^2(\mathbb{R}^{p-1})} = 0.$$

Therefore, by Proposition 3.13 we get that $[a^*(h_j), V](N+1)^{-n+1/2} \to 0$, when $j \to \infty$. This implies that $(H-\lambda)u_j \to 0$ when $j \to \infty$, and since u_j tends weakly to 0, u_j is a Weyl sequence for λ. □

9.2 The Mourre estimate and its consequences

We denote by τ the set of *thresholds*

$$\tau := \sigma_{\mathrm{pp}}(H) + m\mathbb{N}^*.$$

For $\lambda \in \mathbb{R}$, $\epsilon > 0$, let $I(\lambda,\epsilon)$ denote $[\lambda-\epsilon, \lambda+\epsilon]$. Likewise, for a subset $\Theta \subset \mathbb{R}$, let $I(\Theta,\epsilon)$ denote the set $\{k \in \mathbb{R} \ : \ \mathrm{dist}(\Theta, k) \le \epsilon\}$.

Theorem 9.2 *Assume hypothesis (B1), or if* $\mathrm{degP} = 4$, *hypotheses (C), (M1). Then*
i) let $\lambda \in \mathbb{R}\backslash\tau$. *Then there exists* $\epsilon > 0, c_0 > 0$ *and a compact operator* K *such that*

$$\mathbb{1}_{I(\lambda,\epsilon)}(H)[H, \mathrm{i}A]\mathbb{1}_{I(\lambda,\epsilon)}(H) \ge c_0 \mathbb{1}_{I(\lambda,\epsilon)}(H) + K.$$

ii) for all $[\lambda_1, \lambda_2]$ *such that* $[\lambda_1,\lambda_2] \cap \tau = \emptyset$, *one has*

$$\dim \mathbb{1}^{\mathrm{pp}}_{[\lambda_1,\lambda_2]}(H) < \infty.$$

Consequently $\sigma_{\mathrm{pp}}(H)$ *can accumulate only at* τ, *which is a closed countable set.*
iii) Let $\lambda \in \mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H))$. *Then there exists* $\epsilon > 0, c_0 > 0$ *such that*

$$\mathbb{1}_{I(\lambda,\epsilon)}(H)[H, \mathrm{i}A]\mathbb{1}_{I(\lambda,\epsilon)}(H) \ge c_0 \mathbb{1}_{I(\lambda,\epsilon)}(H).$$

Remark 9.3 *There is an example due to Simon [Si3] of a* $P(\varphi)_2$ *Hamiltonian with eigenvalues embedded in* $[\Sigma+m, \Sigma+2m[$.

Theorem 9.4 *Assume hypothesis (B2). Then the strong limiting absorption principle holds:*

$$\mathrm{w}-\lim_{\epsilon\to\pm 0}(1+|A|)^{-r}(H-\lambda-i\epsilon)(1+|A|)^{-r} \ \textit{exists}$$

locally uniformly on $\sigma(H)\backslash(\tau\cup\sigma_{\mathrm{pp}}(H))$, *for* $r > \frac{1}{2}$. *Consequently* H *has no singular continuous spectrum.*

Thm. 9.4 is a consequence of Thm. 8.16 and the abstract Mourre theory (see [Mo], [PSS], [ABG, Thm. 7.4.1]).

Proof of Theorem 9.2. The proof will be very similar to that of [DG1, Thm. 4.3]. Let us set $\tilde{\omega}(k) := k\cdot\nabla\omega(k) = k^2(k^2+m^2)^{-\frac{1}{2}}$. Let

$$\begin{aligned} d(\lambda) &= \inf_{p=1}^{\infty}\inf_{k_1,\dots,k_p\in\mathbb{R}}\Big\{\sum_{i=1}^{p}\tilde{\omega}(k_i)\;\Big|\;\lambda-\sum_{i=0}^{p}\omega(k_i)\in\sigma_{\rm pp}(H)\Big\},\\ \tilde{d}(\lambda) &= \inf_{p=0}^{\infty}\inf_{k_1,\dots,k_p\in\mathbb{R}}\Big\{\sum_{i=1}^{p}\tilde{\omega}(k_i)\;\Big|\;\lambda-\sum_{i=1}^{p}\omega(k_i)\in\sigma_{\rm pp}(H)\Big\}.\end{aligned}$$

Let us note that

$$\tilde{d}(\lambda) := \begin{cases} d(\lambda),\ \lambda\notin\sigma_{\rm pp}(H),\\ 0,\ \lambda\in\sigma_{\rm pp}(H).\end{cases}$$

We introduce also "smeared out" versions of the functions $d(\lambda)$ and $\tilde{d}(\lambda)$. We set

$$\begin{aligned} d^{\kappa}(\lambda) &:= \inf_{\mu\in I(\lambda,\kappa)} d(\mu),\\ &= \inf_{p=1}^{\infty}\inf_{k_1,\dots,k_p\in\mathbb{R}}\Big\{\sum_{i=1}^{p}\tilde{\omega}(k_i)\;\Big|\;\lambda-\sum_{i=1}^{p}\omega(k_i)\in I(\sigma_{\rm pp}(H),\kappa)\Big\},\\ \tilde{d}^{\kappa}(\lambda) &:= \inf_{\mu\in I(\lambda,\kappa)} \tilde{d}(\mu)\\ &= \inf_{p=0}^{\infty}\inf_{k_1,\dots,k_p\in\mathbb{R}}\Big\{\sum_{i=1}^{p}\tilde{\omega}(k_i)\;\Big|\;\lambda-\sum_{i=1}^{p}\omega(k_i)\in I(\sigma_{\rm pp}(H),\kappa)\Big\}.\end{aligned}$$

Note that the following equality holds

$$\inf_{p=1}^{\infty}\inf_{k_1,\dots,k_p\in\mathbb{R}^p}\left(\tilde{d}^{\kappa}\Big(\lambda-\sum_{i=1}^{p}\omega(k_i)\Big)+\sum_{i=1}^{p}\tilde{\omega}(k_i)\right) = d^{\kappa}(\lambda). \tag{9.1}$$

We will use an induction with respect to $n\in\mathbb{N}$. Let us first list the statements that we will show. We put $E_0 := \inf\sigma(H)$.
$H_1(n)$: Let $\epsilon>0$ and $\lambda\in[E_0, E_0+nm[$. Then there exists a compact operator K, an interval $\Delta\ni\lambda$ such that

$$1\!\!1_{\Delta}(H)[H,{\rm i}A]1\!\!1_{\Delta}(H)\geq(d(\lambda)-\epsilon)1\!\!1_{\Delta}(H)+K.$$

$H_2(n)$: Let $\epsilon>0$ and $\lambda\in[E_0, E_0+nm[$. Then there exists an interval $\Delta\ni\lambda$ such that

$$1\!\!1_{\Delta}(H)[H,{\rm i}A]1\!\!1_{\Delta}(H)\geq(\tilde{d}(\lambda)-\epsilon)1\!\!1_{\Delta}(H).$$

$H_3(n)$: Let $\kappa>0$, $\epsilon_0>0$ and $\epsilon>0$. Then there exists $\delta>0$ such that for all $\lambda\in[E_0, E_0+nm-\epsilon_0]$, one has

$$1\!\!1_{I(\lambda,\delta)}(H)[H,{\rm i}A]1\!\!1_{I(\lambda,\delta)}(H)\geq(\tilde{d}^{\kappa}(\lambda)-\epsilon)1\!\!1_{I(\lambda,\delta)}(H).$$

$S_1(n)$: τ is a closed countable set in $[E_0, E_0+nm]$.
$S_2(n)$: for all $\lambda_1\leq\lambda_2\leq E_0+nm$ with $[\lambda_1,\lambda_2]\cap\tau=\emptyset$, we have $\dim 1\!\!1^{\rm pp}_{[\lambda_1,\lambda_2]}(H)<\infty$.

We will prove, for all $n \in \mathbb{N}$, the following implications:

$$H_1(n) \Rightarrow H_2(n),$$

$$H_2(n) \Rightarrow H_3(n),$$

$$H_1(n) \Rightarrow S_2(n),$$

$$S_2(n-1) \Rightarrow S_1(n),$$

$$S_1(n) \text{ and } H_3(n-1) \Rightarrow H_1(n).$$

Note first that the statements $H_1(1)$ and $S_1(1)$ are immediate since the spectrum of H is discrete in $[E_0, E_0+m[$. Note also that the implication $S_2(n-1) \Rightarrow S_1(n)$ is obvious. The proof of the implications $H_1(n) \Rightarrow H_2(n)$, $H_2(n) \Rightarrow H_3(n)$ is a standard argument which adapt directly to the present setting (see [FH], [CFKS]). The proof of the implication $H_1(n) \Rightarrow S_2(n)$ is also standard and based on the virial relation, which holds here by Thm. 8.7.

It remains to prove that $S_1(n)$ and $H_3(n-1) \Rightarrow H_1(n)$.

Recall that the Hamiltonian H^{ext} acting on $\mathcal{H}\otimes\mathcal{H}$ was introduced in Subsect. 7.4. We also set $A^{\mathrm{ext}} = A\otimes \mathbb{1} + \mathbb{1}\otimes A$, acting on $\mathcal{H}\otimes\mathcal{H}$. Let us first show that for all $\lambda \in [E_0, E_0+nm-\epsilon_0[$, there exists $\delta > 0$ such that

$$\begin{aligned}
&\mathbb{1}_{I(\lambda,\delta)}(H^{\mathrm{ext}}))[H^{\mathrm{ext}}, \mathrm{i}A^{\mathrm{ext}}]\mathbb{1}_{I(\lambda,\delta)}(H^{\mathrm{ext}})\mathbb{1}_{[1,\infty[}(N_\infty)\\
&\geq (d(\lambda) - \tfrac{2\epsilon}{3})\mathbb{1}_{I(\lambda,\delta)}(H^{\mathrm{ext}})\mathbb{1}_{[1,\infty[}(N_\infty).
\end{aligned} \tag{9.2}$$

To simplify, let us write $\mathrm{d}\Gamma_\infty(\omega)$, $\mathrm{d}\Gamma_\infty(\tilde{\omega})$, instead of $\mathbb{1}\otimes \mathrm{d}\Gamma(\omega)$, $\mathbb{1}\otimes \mathrm{d}\Gamma(\tilde{\omega})$. We will also write B instead of $B\otimes\mathbb{1}$. Using the closedness of τ in $[E_0, E_0+nm]$, i.e the induction hypothesis $S_1(n)$, we see that

$$d(\lambda) = \sup_{\kappa>0} d^\kappa(\lambda),$$

for $\lambda \in [E_0, E_0+nm[$. So we may choose κ small enough so that $d^\kappa(\lambda) \geq d(\lambda) - \epsilon/3$. Next using $H_3(n-1)$ we choose δ such that for $\lambda_1 \in [E_0, E_0+(n-1)m-\epsilon_0[$, we have

$$\mathbb{1}_{I(\lambda_1,\delta)}(H)[H, \mathrm{i}A]\mathbb{1}_{I(\lambda_1,\delta)}(H) \geq \left(\tilde{d}^\kappa(\lambda_1) - \frac{\epsilon}{3}\right)\mathbb{1}_{I(\lambda_1,\delta)}(H).$$

Replacing λ_1 with $\lambda - \mathrm{d}\Gamma(\omega(k))$, we obtain for $\lambda \in [E_0, E_0+nm-\epsilon_0[$ the following estimate:

$$\begin{aligned}
&\mathbb{1}_{I(\lambda,\delta)}\Big(H + \mathrm{d}\Gamma_\infty(\omega)\Big)\Big([H, \mathrm{i}A] + \mathrm{d}\Gamma_\infty(\tilde{\omega})\Big)\mathbb{1}_{I(\lambda,\delta)}\Big(H + \mathrm{d}\Gamma_\infty(\omega)\Big)\mathbb{1}_{[1,\infty[}(N_\infty)\\
&\geq \mathbb{1}_{I(\lambda,\delta)}\Big(H + \mathrm{d}\Gamma_\infty(\omega)\Big)\Big(\tilde{d}^\kappa(\lambda - \mathrm{d}\Gamma_\infty(\omega)) + \mathrm{d}\Gamma_\infty(\tilde{\omega}) - \tfrac{\epsilon}{3}\Big)\mathbb{1}_{[1,\infty[}(N_\infty)\\
&\geq (d^\kappa(\lambda) - \tfrac{\epsilon}{3})\mathbb{1}_{I(\lambda,\delta)}\Big(H + \mathrm{d}\Gamma_\infty(\omega)\Big)\mathbb{1}_{[1,\infty[}(N_\infty)\\
&\geq (d(\lambda) - \tfrac{2\epsilon}{3})\mathbb{1}_{I(\lambda,\delta)}\Big(H + \mathrm{d}\Gamma_\infty(\omega)\Big)\mathbb{1}_{[1,\infty[}(N_\infty),
\end{aligned}$$

which yields (9.2).

Now let $\chi \in C_0^\infty(\mathbb{R})$. As in the proof of Theorem 9.1

$$\begin{aligned}
\chi^2(H) &= I(j^R)\mathbb{1}_{\{0\}}(N_\infty)I^*(j^R)\chi^2(H) + I(j^R)\mathbb{1}_{[1,\infty[}(N_\infty)I^*(j^R)\chi^2(H)\\
&= \Gamma(q^R)\chi^2(H) + I(j^R)\mathbb{1}_{[1,\infty[}(N_\infty)\chi^2(H^{\mathrm{ext}})I^*(j^R) + o(R^0).
\end{aligned} \tag{9.3}$$

The first term of (9.3) is compact as in the proof of Thm. 9.1.

Next we use that $[H, \mathrm{i}A]$ equals $H^{(1)}$ and that on $\mathcal{D}(H^n)$ $H^{(1)}$ can be written as $H_0^{(1)} + V^{(1)}$. So on $\mathcal{D}(H^n)$ $[H, \mathrm{i}A]$ is similar to H with ω replaced by $\tilde{\omega}$ and V replaced by the Wick polynomial $V^{(1)}$. It is then easy to see that the analog of (7.22) holds ie as an operator identity on $\mathcal{D}(H^n)$ we have:

(9.4) $$[H^{\mathrm{ext}}, \mathrm{i}A^{\mathrm{ext}}]\check{\Gamma}(j^R) - \check{\Gamma}(j^R)[H, \mathrm{i}A] \in (N+1)^n \check{o}_N(R^0),$$

for $[H^{\mathrm{ext}}, \mathrm{i}A^{\mathrm{ext}}] = [H, \mathrm{i}A] \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\tilde{\omega})$. Using also Lemma 7.12, we obtain

(9.5) $$\begin{aligned} \chi(H)[H, \mathrm{i}A]\chi(H) &= \Gamma(q^R)\chi(H)[H, \mathrm{i}A]\chi(H) \\ &\quad + I(j^R)\mathbb{1}_{\{[1,\infty[\}}(N_\infty)\chi(H^{\mathrm{ext}})[H^{\mathrm{ext}}, \mathrm{i}A^{\mathrm{ext}}]\chi(H^{\mathrm{ext}})I^*(j^R) + o(R^0), \end{aligned}$$

where the first term on the right is again compact.

Now (9.2), (9.3) and (9.5) for $\operatorname{supp}\chi \subset [\lambda - \delta, \lambda + \delta]$, yield

$$\chi(H)[H, \mathrm{i}A]\chi(H) \geq (d(\lambda) - 2\epsilon/3)\chi^2(H) + K_1 + o(R^0),$$

where K_1 is compact. Picking R large enough, this proves $H_1(n)$. □

10 Scattering theory of $P(\varphi)_2$ Hamiltonians

This section is devoted to the scattering theory of $P(\varphi)_2$ Hamiltonians. In quantum field theory, the scattering theory is usually based on the construction of the asymptotic fields, which is done in Subsect. 10.1. The unitarity of the wave operator (a result originally due to Høgh-Krohn) is shown in Subsect. 10.2, using general properties of regular CCR representations shown in Sect. 4. The asymptotic completeness property is formulated in Subsect. 10.3 and will be shown in Sect. 12.

10.1 Asymptotic fields

In all this section, we will assume the conditions *(A)*, *(Is)* for $s > 1$.

For $h \in \mathfrak{h}$ we set $h_t := \mathrm{e}^{-\mathrm{i}t\omega(k)}h$. We denote by $\mathfrak{h}_0 \subset \mathfrak{h}$ the space $C_0^\infty(\mathbb{R}\backslash\{0\})$.

Theorem 10.1 *i) For all $h \in \mathfrak{h}$ the strong limits*

(10.1) $$W^+(h) := \text{s-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH}$$

exist. They are called the asymptotic Weyl operators. *The asymptotic Weyl operators can be also defined using the norm limit:*

(10.2) $$W^+(h)(H+b)^{-n} = \lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)(H+b)^{-n}\mathrm{e}^{-\mathrm{i}tH}.$$

ii) The map

(10.3) $$\mathfrak{h} \ni h \mapsto W^+(h)$$

is strongly continuous and for $\epsilon > 0$, the map

(10.4) $$\mathfrak{h} \ni h \mapsto W^+(h)(H+b)^{-\epsilon}$$

is norm continuous.
iii) The operators $W^{+}(h)$ satisfy the Weyl commutation relations:

$$W^{+}(h)W^{+}(g) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}Im(h|g)}W^{+}(h+g).$$

iv) The Hamiltonian preserves the asymptotic Weyl operators:

$$\mathrm{e}^{\mathrm{i}tH}W^{+}(h)\mathrm{e}^{-\mathrm{i}tH} = W^{+}(h_{-t}). \tag{10.5}$$

Proof. We have

$$W(h_t) = \mathrm{e}^{-\mathrm{i}tH_0}W(h)\mathrm{e}^{\mathrm{i}tH_0},$$

which implies that, as a quadratic form on $\mathcal{D}(H_0)$, one has

$$\partial_t W(h_t) = -[H_0, \mathrm{i}W(h_t)]. \tag{10.6}$$

Using (10.6) and the fact that $\mathcal{D}(H^n) \subset \mathcal{D}(H_0) \cap \mathcal{D}(V)$, we have, as quadratic forms on $\mathcal{D}(H^n)$,

$$\partial_t \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} = \mathrm{i}\mathrm{e}^{\mathrm{i}tH}[V, W(h_t)]\mathrm{e}^{-\mathrm{i}tH}.$$

Integrating this relation we have as a quadratic form identity on $\mathcal{D}(H^n)$

$$\mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} - W(h) = \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}t'H}[V, W(h_{t'})]\mathrm{e}^{-\mathrm{i}t'H}\mathrm{d}t'. \tag{10.7}$$

Using Prop. 3.13, we obtain that

$$[V, W(h_t)] = W(h_t)\tilde{V}_t,$$

where $\tilde{V}_t$ is the sum of Wick monomials in (3.20) with $s+r \geq 1$. By stationary phase arguments, we obtain that, for $h \in \mathfrak{h}_0$, there exists $\epsilon > 0$ such that

$$h_t = 1\!\!1_{\{|x| \geq \epsilon t\}}h_t + O(t^{-\infty}). \tag{10.8}$$

Using then Lemma 6.3 and the form (3.21) of the symbols of $\tilde{V}_t$, we obtain that

$$\|\tilde{V}_t(N+1)^{-n}\| \in O(t^{-s}).$$

This shows that the identity (10.7) makes sense as an identity between bounded operators from $\mathcal{D}(H^n)$ to $\mathcal{H}$. It also proves that the norm limit (10.2) exists for $h \in \mathfrak{h}_0$.

For $h \in \mathfrak{h}$, let $h_n \in \mathfrak{h}_0$ such that $h = \lim_{n\to\infty} h_n$. Let $0 < \epsilon \leq \frac{1}{2}$. Using the first order estimates and Prop. 3.1 we obtain

$$\begin{aligned}\|\Big(W(h_{n,t}) - W(h_t)\Big)(H+b)^{-\epsilon}\| &\leq \|\Big(W(h_n) - W(h)\Big)(N+1)^{-\epsilon}\|\|(N+1)^{\epsilon}(H+b)^{-\epsilon}\| \\ &\leq C(\|h_n - h\|^{\epsilon}(\|h_n\|^2 + \|h\|)^2 + 1),\end{aligned}$$

which implies

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\|\Big(W(h_{n,t}) - W(h_t)\Big)(H+b)^{-\epsilon}\| = 0.$$

This implies the existence of the norm limit (10.2) for all $h \in \mathfrak{h}$. Now (10.2) implies (10.1). This ends the proof of *i)*. We have

$$\begin{aligned}\| (W^+(h) - W^+(g)) (H+b)^{-\epsilon}\| &\le \lim_{t\to+\infty} \|\mathrm{e}^{\mathrm{i}tH}(W(h_t)-W(g_t))(H+b)^{-\epsilon}\mathrm{e}^{-\mathrm{i}tH}\| \\ &\le C(\|g-h\|^\epsilon(\|g\|^2+\|h\|)^2+1),\end{aligned}$$

by Prop. 3.1, which implies the norm continuity of (10.4). This implies the strong continuity of (10.3) and completes the proof of *ii)*. Finally *iii)* and *iv)* are immediate. □

It follows from the above theorem that $\mathfrak{h} \ni h \mapsto W^+(h)$ is a regular CCR representation. We next follow Sect. 2, introducing field operators, creation/annihilation operators, etc.

Theorem 10.2 *i) For any* $h \in \mathfrak{h}$

$$\phi^+(h) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}W^+(sh)_{|s=0}$$

defines a self-adjoint operator, called the asymptotic field, *such that*

$$W^+(h) = \mathrm{e}^{\mathrm{i}\phi^+(h)}.$$

ii) The operators $\phi^+(h)$ *satisfy in the sense of quadratic forms on* $\mathcal{D}(\phi^+(h_1)) \cap \mathcal{D}(\phi^+(h_2))$ *the canonical commutation relations*

$$[\phi^+(h_2), \phi^+(h_1)] = \mathrm{i}Im(h_2|h_1). \tag{10.9}$$

iii)

$$\mathrm{e}^{\mathrm{i}tH}\phi^+(h)\mathrm{e}^{-\mathrm{i}tH} = \phi^+(h_{-t}).$$

iv) For $h_i \in \mathfrak{h}, 1 \le i \le p$, $\mathcal{D}((H+\mathrm{i})^{p/2}) \subset \mathcal{D}(\Pi_1^p\phi^+(h_i))$, *and*

$$\prod_{i=1}^p \phi^+(h_i)(H+\mathrm{i})^{-p/2} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} \prod_{i=1}^p \phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-p/2}.$$

Proof. Properties *i)* and *ii)* are consequences of the fact that the asymptotic Weyl operators define a regular CCR representation (see Sect. 2). Property *iii)* follows from Thm. 10.1 *iv)*. It remains to prove *iv)*. Let us first establish the existence of the strong limit

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_1^p\phi(h_{i,t})(H+b)^{-p/2}\mathrm{e}^{-\mathrm{i}tH} =: R(h_1,\dots,h_p), \quad \text{for } h_i \in \mathfrak{h}. \tag{10.10}$$

For $u, v \in \mathcal{D}(H^n)$, we have

$$\begin{aligned}&\tfrac{\partial}{\partial t}(v_t, \Pi_1^p\phi(h_{i,t})(H+b)^{-p/2}u_t) \\ &= (v_t, [H, \mathrm{i}\Pi_1^p\phi(h_{i,t})](H+b)^{-p/2}u_t) + (v_t, \partial_t\Pi_1^p\phi(h_{i,t})(H+b)^{-p/2}u_t).\end{aligned}$$

We use again the fact that $H = H_0 + V$ on $\mathcal{D}(H^n)$ and the higher order estimates, which show that

$$\begin{aligned}&[H, \mathrm{i}\Pi_1^p\phi(h_{i,t})](H+b)^{-p/2} + \partial_t\Pi_1^p\phi(h_{i,t})(H+b)^{-p/2} \\ &= [V, \mathrm{i}\Pi_1^p\phi(h_{i,t})](H+b)^{-p/2} + [H_0, \mathrm{i}\Pi_1^p\phi(h_{i,t})](H+b)^{-p/2} + \partial_t\Pi_1^p\phi(h_{i,t})(H+b)^{-p/2} \\ &= [V, \mathrm{i}\Pi_1^p\phi(h_{i,t})](H+b)^{-p/2},\end{aligned}$$

as an identity between quadratic forms on $\mathcal{D}(H^n)$. Using then the fact that $\phi(h)$ maps $\mathcal{D}(N^k)$ into $\mathcal{D}(N^{k-\frac{1}{2}})$, we obtain the identity

$$[V, \mathrm{i}\Pi_1^p \phi(h_{i,t})](H+b)^{-p/2} = \sum_{j=1}^{p} \Pi_1^{j-1}\phi(h_{i,t})[V, i\phi(h_{j,t})]\Pi_{j+1}^p \phi(h_{i,t})(H+b)^{-p/2},$$

as a quadratic form identity on $\mathcal{D}(H^n)$. For $h \in \mathfrak{h}$, the term $[V, \mathrm{i}\phi(h_t)]$ is by Prop. 3.13 a Wick polynomial with kernels of the form $w_p|h_t)$ or $(h_t|w_p$.

By a stationary phase argument, if $h \in \mathfrak{h}_0$, we can find $\epsilon_0 > 0$ such that $\mathbb{1}_{\{|x|\leq\epsilon_0 t\}} h_t \in O(t^{-\infty})$. Using then hypothesis *(Is)* for $s > 1$, Lemma 6.3 and Prop. 3.13, we obtain

$$[V, \mathrm{i}\phi(h_t)] \in O_N(t^{-s})(N+1)^n. \tag{10.11}$$

Using again the higher order estimates, we obtain that if $h_i \in \mathfrak{h}_0$, $1 \leq i \leq p$ then

$$\frac{\partial}{\partial t}(v_t, \Pi_1^p \phi(h_{i,t})(H+b)^{-p/2} u_t) = (v_t, R(t)u_t),\ u, v \in \mathcal{D}(H^n),$$

where $\|R(t)(H+b)^{-n}\| \leq Ct^{-s}$. This proves the existence of the limit (10.10) for $u \in \mathcal{D}(H^n), h_i \in \mathfrak{h}_0$. The estimate

$$\|(N+1)^m(\phi(h_1) - \phi(h_2))(N+1)^{-m-\frac{1}{2}}\| \leq C\|h_1 - h_2\|$$

and a density argument as in the proof of Thm. 10.1 give the existence of (10.10) for $u \in \mathcal{D}(H^n), h_i \in \mathfrak{h}$. Finally it follows again from the higher order estimates that $\Pi_1^p\phi(h_{i,t})(H+b)^{-p/2}$ is bounded uniformly in t, which shows the existence of (10.10) for all $u \in \mathcal{H}$.

We prove now the identity *iv)* by induction on p. We have to show that $\mathcal{D}(H^{p/2}) \subset \mathcal{D}(\Pi_1^p\phi^+(h_i))$ and that $R(h_1, \ldots, h_p) = \Pi_1^p\phi^+(h_i)(H+b)^{-p/2}$. This amounts to show that

$$R(h_1, \ldots, h_p) = \text{s-}\lim_{s\to 0}(\mathrm{i}s)^{-1}(W^+(sh_1) - \mathbb{1})\Pi_2^p\phi^+(h_i)(H+b)^{-p/2}.$$

Note that by the induction assumption $\mathcal{D}(H^{p/2}) \subset \mathcal{D}(\Pi_2^p\phi^+(h_i))$ and

$$\Pi_2^p\phi^+(h_i)(H+b)^{-p/2} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_2^p\phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+b)^{-p/2}. \tag{10.12}$$

Using (10.12) and the fact that $\mathrm{e}^{\mathrm{i}tH}W(h_{1,t})\mathrm{e}^{-\mathrm{i}tH}$ is uniformly bounded in t, we have:

$$\begin{aligned}&(is)^{-1}(W^+(sh_1) - \mathbb{1})\Pi_2^p\phi^+(h_i)(H+b)^{-p/2}\\ &= \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}(\mathrm{i}s)^{-1}(W(sh_{1,t}) - \mathbb{1})\Pi_2^p\phi(h_{i,t})\mathrm{e}^{-\mathrm{i}tH}(H+b)^{-p/2}.\end{aligned}$$

So to prove *iv)*, it suffices to check that

$$\text{s-}\lim_{s\to 0}\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}R(s,t)\mathrm{e}^{-\mathrm{i}tH} = 0, \tag{10.13}$$

for

$$R(s,t) = \Big(\frac{W(sh_{1,t}) - \mathbb{1}}{s} - \mathrm{i}\phi(h_{1,t})\Big)\Pi_2^p\phi(h_{i,t})(H+b)^{-p/2}.$$

We recall that

$$\sup_{|s|\leq 1, \|h\|\leq C} \Big\| \Big(\frac{W(sh) - \mathbb{1}}{s} \Big) (N+1)^{-\frac{1}{2}} \Big\| < \infty, \tag{10.14}$$

and

$$\lim_{s\to 0} \sup_{\|h\|\leq C} \Big\| \Big(\frac{W(sh) - \mathbb{1}}{s} - \mathrm{i}\phi(h) \Big) (N+1)^{-\frac{1}{2}-\epsilon} \Big\| = 0, \ \epsilon > 0. \tag{10.15}$$

Using (10.14) and the higher order estimates, we see that $R(s,t)$ is uniformly bounded for $|s| \leq 1, t \in \mathbb{R}$, and using then (10.15) we see that $\lim_{s\to 0} \sup_{t\in\mathbb{R}} \|R(s,t)u\| = 0$, for $u \in \mathcal{D}(H^{\epsilon})$. This shows (10.13) and completes the proof of the theorem. □

The following theorem follows directly from Thm. 10.1 and from the properties of regular CCR representations.

Theorem 10.3 *1) For any $h \in \mathfrak{h}$, the* asymptotic creation and annihilation operators *defined on $\mathcal{D}(a^{+\sharp}(h)) := \mathcal{D}(\phi^{+}(h)) \cap \mathcal{D}(\phi^{+}(\mathrm{i}h))$ by*

$$\begin{aligned} a^{+*}(h) &:= \tfrac{1}{\sqrt{2}} \left(\phi^{+}(h) - \mathrm{i}\phi^{+}(\mathrm{i}h) \right), \\ a^{+}(h) &:= \tfrac{1}{\sqrt{2}} \left(\phi^{+}(h) + \mathrm{i}\phi^{+}(\mathrm{i}h) \right). \end{aligned}$$

are closed.
ii) The operators $a^{+\sharp}$ satisfy in the sense of forms on $\mathcal{D}(a^{+\#}(h_1)) \cap \mathcal{D}(a^{+\#}(h_2))$ the canonical commutation relations

$$\begin{aligned} &[a^{+}(h_1), a^{+*}(h_2)] = (h_1|h_2)\mathbb{1}, \\ &[a^{+}(h_2), a^{+}(h_1)] = [a^{+*}(h_2), a^{+*}(h_1)] = 0. \end{aligned}$$

iii)

$$\mathrm{e}^{\mathrm{i}tH} a^{+\sharp}(h) \mathrm{e}^{-\mathrm{i}tH} = a^{+\sharp}(h_{-t}). \tag{10.16}$$

iv) For $h_i \in \mathfrak{h}, 1 \leq i \leq p$, $\mathcal{D}((H+\mathrm{i})^{p/2}) \subset \mathcal{D}(\Pi_1^p a^{+\sharp}(h_i))$ and

$$\Pi_1^p a^{+\sharp}(h_i)(H+b)^{-\frac{p}{2}} = \text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} \Pi_1^p a^{\sharp}(h_{i,t})(H+b)^{-\frac{p}{2}} \mathrm{e}^{-\mathrm{i}tH}.$$

10.2 Asymptotic spaces and wave operators

In this subsection, we recall the construction of the asymptotic vacuum spaces and wave operators, due to Høgh-Krohn [HK]. We give a more direct proof of the unitarity of the wave operators based on the existence of a number operator for the CCR representation given by the asymptotic Weyl operators. We define the *asymptotic vacuum space* to be

$$\mathcal{K}^{+} := \{ u \in \mathcal{H} \mid a^{+}(h)u = 0, \ h \in \mathfrak{h} \}.$$

The *asymptotic space* is defined as

$$\mathcal{H}^{+} := \mathcal{K}^{+} \otimes \mathcal{H}.$$

Proposition 10.4 *i) $\mathcal{K}^+$ is a closed $H-$invariant space.*
ii) $\mathcal{K}^+$ is included in the domain of $\Pi_1^p a^{+\sharp}(h_i)$ for $h_i \in \mathfrak{h}$.
iii)

$$\mathcal{H}_{\rm pp}(H) \subset \mathcal{K}^+.$$

Proof. *i)* and *ii)* follow by the properties of CCR relations described in Proposition 4.1. The fact that $\mathcal{K}^+$ is $H-$invariant follows from (10.16). To prove *iii)* we verify that for $u \in \mathcal{D}(H), Hu = \lambda u$, $h \in \mathfrak{h}_0$, $a(h_t){\rm e}^{-{\rm i}tH}u = {\rm e}^{-{\rm i}t\lambda}a(h_t)u \in o(1)$. $\square$

The *asymptotic Hamiltonian* is defined by

$$H^+ := K^+ \otimes 1\!\!1 + 1\!\!1 \otimes {\rm d}\Gamma(\omega),$$

for

$$K^+ := H\Big|_{\mathcal{K}^+}.$$

We also define

$$\begin{array}{ll} & \Omega^+ : \mathcal{H}^+ \to \mathcal{H}, \\ (10.17) & \\ & \Omega^+\psi \otimes a^*(h_1)\cdots a^*(h_p)\Omega := a^{+*}(h_1)\cdots a^{+*}(h_p)\psi, \quad h_1,\dots,h_p \in \mathfrak{h}, \quad \psi \in \mathcal{K}^+. \end{array}$$

The map Ω^+ is called the *wave operator*. It is a particular case of the map Ω_π defined in Prop. 4.2. The following theorem is due to Høgh-Krohn [HK].

Theorem 10.5 *Ω^+ is a unitary map from $\mathcal{H}^+$ to $\mathcal{H}$ such that:*

$$\begin{array}{l} a^{+\sharp}(h)\Omega^+ = \Omega^+ 1\!\!1 \otimes a^\sharp(h), \quad h \in \mathfrak{h}, \\ H\Omega^+ = \Omega^+ H^+. \end{array}$$

Proof. By general properties of regular CCR representations, (see Proposition 4.2) the operator Ω^+ is well defined and isometric. To prove that it is unitary, we will show that the CCR representation $\mathfrak{h} \ni h \mapsto W^+(h)$ admits a densely defined number operator and use Theorem 4.3.

Let n^+ be the quadratic form associated to the CCR representation W^+ as in Subsect. 4.2. Let us show that $\mathcal{D}(n^+)$ is dense in $\mathcal{H}$. For each finite dimensional space $\mathfrak{f} \subset \mathfrak{h}$ if

$$n_\mathfrak{f}^+(u) = \sum_{i=1}^{\dim\mathfrak{f}} \|a^+(h_i)u\|^2,$$

for $\{h_i\}$ an orthonormal base of $\mathfrak{f}$, we have

$$\begin{array}{rl} \|n_\mathfrak{f}^+(u)\|^2 & = \lim\limits_{t\to+\infty} \sum\limits_{i=1}^{\dim\mathfrak{f}} \|a(h_{i,t}){\rm e}^{-{\rm i}tH}u\|^2 \\ & = \lim\limits_{t\to+\infty} ({\rm e}^{-{\rm i}tH}u, {\rm d}\Gamma(P_{\mathfrak{f},t}){\rm e}^{-{\rm i}tH}u), \end{array}$$

if $P_{\mathfrak{f},t}$ is the orthogonal projection on ${\rm e}^{-{\rm i}t\omega}\mathfrak{f}$. But ${\rm d}\Gamma(P_{\mathfrak{f},t}) \leq N$, so

$$n_\mathfrak{f}^+(u) \leq \sup_t \|N^{\frac{1}{2}}{\rm e}^{-{\rm i}tH}u\|^2 \leq C\|(H+b)^{\frac{1}{2}}u\|^2,$$

by the first order estimates (6.4). Therefore

$$\mathcal{D}(H^{\frac{1}{2}}) \subset \mathcal{D}(n^+),$$

which implies that $\mathcal{D}(n^+)$ is dense in $\mathcal{H}$ and hence, by Theorem 4.3, ${\rm Ran}\Omega^+ = \mathcal{H}$. $\square$

10.3 Asymptotic completeness

The definition of the wave operators seems different from the one commonly used in the physics literature, where asymptotic creation operators are only applied to bound states of H, generating the so called *asymptotic states*. In this respect one can ask what property of the wave operators should be called *asymptotic completeness*. A physically important property is the fact that incoming and outgoing asymptotic vacua coincide, that is $\mathcal{K}^+ = \mathcal{K}^-$, where $\mathcal{K}^-$ is defined analogously to $\mathcal{K}^+$, with $t \to -\infty$ replacing $t \to +\infty$ in the definition of the asymptotic Weyl operators.

Since we have seen that $\mathcal{H}_{\rm pp}(H) \subset \mathcal{K}^\pm$, the natural definition of asymptotic completeness is that $\mathcal{H}_{\rm pp}(H) = \mathcal{K}^\pm$.

The following theorem is one of the main results of this paper:

Theorem 10.6 *Assume hypotheses (B1),(Is) for* $s > 1$ *, or if* $\deg P = 4$*, hypotheses (C), (M1), (Is) for* $s > 1$*. Then the* $P(\varphi)_2$ *Hamiltonian* H *has the asymptotic completeness property:*

$$\mathcal{H}_{\rm pp}(H) = \mathcal{K}^\pm.$$

Thm. 10.6 will be proved in Subsect. 12.5, as a consequence of Thm. 12.5 and of the Mourre estimate.

10.4 Extended wave operator

Recall that in Subsect. 7.5 we introduced the extended Hilbert space and the extended Hamiltonian

$$\mathcal{H}^{\rm ext} = \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}), \qquad H^{\rm ext} = H \otimes 1\!\!1 + 1\!\!1 \otimes {\rm d}\Gamma(\omega(k)).$$

Clearly $\mathcal{H}^+$ is a subspace of $\mathcal{H}^{\rm ext}$ and

$$H^+ = H^{\rm ext}\Big|_{\mathcal{H}^+}.$$

Sometimes we will also need the "extended wave operator". Its domain can be chosen to be

$$\mathcal{D}(\Omega^{\rm ext,+}) := \bigoplus_{p=0}^{\infty} \mathcal{D}((H+b)^{\frac{p}{2}}) \otimes \otimes_{\rm s}^{p} \mathfrak{h},$$

which is a dense subset of $\mathcal{H}^{\rm ext}$. Now we set

$$\begin{aligned}&\Omega^{\rm ext,+} : \mathcal{D}(\Omega^{\rm ext,+}) \to \mathcal{H},\\ &\Omega^{\rm ext,+}\psi \otimes a^*(h_1)\cdots a^*(h_p)\Omega := a^{+*}(h_1)\cdots a^{+*}(h_p)\psi, \qquad \psi \in \mathcal{D}((H+b)^{\frac{p}{2}}).\end{aligned} \tag{10.18}$$

Note that $\Omega^{\rm ext,+}$ is an unbounded operator. Clearly,

$$\Omega^{\rm ext,+}\Big|_{\mathcal{H}^+} = \Omega^+. \tag{10.19}$$

We will sometimes treat Ω^+ as a partial isometry equal to zero on the orthogonal complement of $\mathcal{H}^+$ inside $\mathcal{H}^{\rm ext}$. We can then write the following identity:

$$\Omega^+ = \Omega^{\rm ext,+} 1\!\!1_{\mathcal{H}^+}. \tag{10.20}$$

where $1\!\!1_{\mathcal{H}^+}$ denotes the projection onto $\mathcal{H}^+$ inside the space $\mathcal{H}^{\rm ext}$.

10.5 Another construction of the wave operators

Recall that in Subsect. 3.9, we defined the identification operator $I : \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h})$.

Theorem 10.7 *i) Let $u \in \mathcal{D}(\Omega^{\mathrm{ext},+})$. Then the limit*

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} u$$

exists and equals $\Omega^{\mathrm{ext},+}u$.
ii) Let $\chi \in C_0^\infty(\mathbb{R})$. Then $\mathrm{Ran}\chi(H^{\mathrm{ext}}) \subset \mathcal{D}(\Omega^{\mathrm{ext}})$, $I\chi(H^{\mathrm{ext}})$ and $\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})$ are bounded operators and

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} \chi(H^{\mathrm{ext}}) = \Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}}). \quad (10.21)$$

Proof. Let us first show *i)*. Let $u \in \mathcal{D}((H+\mathrm{i})^{k/2}) \otimes \otimes_{\mathrm{s}}^k \mathfrak{h}$. Since by (3.6) $I(H+\mathrm{i})^{-k/2} \otimes 1\!\!1_{\{k\}}(N_\infty)$ is a bounded operator, it suffices to prove *i)* for $u = \psi \otimes \Pi_i^k a^*(h_i)\Omega$, $\psi \in \mathcal{D}((H+\mathrm{i})^{k/2})$, $h_i \in \mathfrak{h}$. It follows from property (3.4) of I that

$$\mathrm{e}^{\mathrm{i}tH} I \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} \psi \otimes \Pi_1^k a^*(h_i)\Omega = \mathrm{e}^{\mathrm{i}tH} \Pi_1^k a^*(h_{i,t}) \mathrm{e}^{-\mathrm{i}tH}\psi.$$

i) follows then from Thm. 10.3 *iv)*.

To prove *ii)*, we observe that since the boson mass is positive, vectors in $\mathcal{H}_{\mathrm{comp}}(H^{\mathrm{ext}})$ are also in $\mathcal{H}_{\mathrm{comp}}(H)$ and in $\mathcal{H}_{\mathrm{comp}}(N_\infty)$. So *ii)* follows from *i)*. □

11 Propagation estimates

In this section we collect various propagation estimates about the evolution $\mathrm{e}^{-\mathrm{i}tH}$, which will be used in the next section. It is essentially similar to [DG1, Sect. 6], the only difference being the control of the interaction term V, which is here much more singular. In all this section we assume hypothesis *(Is)* for $s > 1$.

We will use the following notations for various Heisenberg derivatives:

$$\mathbf{D}_0 = \frac{\partial}{\partial t} + [H_0, \mathrm{i}\cdot], \text{ acting on } B(\Gamma(\mathfrak{h})),$$

$$\mathbf{D} = \frac{\partial}{\partial t} + [H, \mathrm{i}\cdot], \text{ acting on } B(\mathcal{H}).$$

The following easy observation will be used to compute Heisenberg derivatives. It follows from the fact that $H = H_0 + V$ on $\mathcal{D}(H^n)$.

Lemma 11.1 *Let $\mathbb{R} \ni t \mapsto M(t) \in \mathcal{B}(\mathcal{D}(H), \mathcal{H})$ be of class C^1. Then for $\chi \in C_0^\infty(\mathbb{R})$, we have:*

$$\mathbf{D}\chi(H)M(t)\chi(H) = \chi(H)\mathbf{D}_0 M(t)\chi(H) + \chi(H)[V, \mathrm{i}M(t)]\chi(H).$$

We first derive a standard large velocity estimate. It means that no boson can asymptotically propagate in the region $|x| > t$.

Proposition 11.2 *Let $\chi \in C_0^\infty(\mathbb{R})$. For $R' > R > 1$, one has*

$$\int_1^\infty \left\| \mathrm{d}\Gamma\left(1\!\!1_{[R,R']}(\frac{|x|}{t})\right)^{\frac{1}{2}} \chi(H)\mathrm{e}^{-\mathrm{i}tH}u \right\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. Let $F \in C^\infty(\mathbb{R})$ be a cutoff function equal to 1 near ∞, to 0 near the origin, with $F'(s) \geq 1\!\!1_{[R,R']}(s)$. The propagation observable is $\Phi(t) = \chi(H)\mathrm{d}\Gamma\left(F(\frac{|x|}{t})\right)\chi(H)$. The proof is identical to that of [DG1, Prop. 6.1], except for the term $\chi(H)[V, \mathrm{i}\mathrm{d}\Gamma(F(\frac{|x|}{t}))]\chi(H)$, coming from the application of Lemma 11.1. By Prop. 3.13 and Lemma 6.3, $[V, \mathrm{i}\mathrm{d}\Gamma(F(\frac{|x|}{t}))]$ is a sum of Wick monomials with symbols having an L^2 norm $O(t^{-s})$, $s > 1$, by condition *(Is)*. Prop. 3.13 and the higher order estimates give then that $\chi(H)[V, \mathrm{i}\mathrm{d}\Gamma(F(\frac{|x|}{t}))]\chi(H) \in O(t^{-s})$, $s > 1$. Thus this term is integrable in norm. □

The following proposition contains a more subtle propagation estimate. Its intuitive meaning is that along the evolution of an asymptotically free boson the instantaneous velocity $\nabla\omega(k)$ and the average velocity $\frac{x}{t}$ converge to each other as time goes to ∞.

Proposition 11.3 *Let $\chi \in C_0^\infty(\mathbb{R})$, $0 < c_0 < c_1$. Set*

$$\Theta_{[c_0,c_1]}(t) := \mathrm{d}\Gamma\left(\langle \tfrac{x}{t} - \nabla\omega(k), 1\!\!1_{[c_0,c_1]}(\tfrac{x}{t})(\tfrac{x}{t} - \nabla\omega(k))\rangle\right).$$

Then

$$\int_1^\infty \|\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. The propagation observable used to prove the proposition is of the form

$$\Phi(t) = \chi(H)d\Gamma(b(t))\chi(H),$$

for

$$b(t) := R(\frac{x}{t}) - \frac{1}{2}\left(\langle\nabla R(\frac{x}{t}), \frac{x}{t} - \nabla\omega(k)\rangle + \mathrm{hc}\right),$$

with $|\partial_x^\alpha R(x)| \leq C_\alpha$, $\mathrm{supp}R \subset \{|x| \geq \epsilon_0 > 0\}$.

As above it suffices to estimate the term $\chi(H)[V, id\Gamma(b(t))]\chi(H)$, the other terms in the Heisenberg derivative of $\Phi(t)$ being similar to those in [DG1, Prop. 6.2]. By Prop. 3.13, $[V, \mathrm{i}\mathrm{d}\Gamma(b(t))]$ is a sum of Wick momomials with symbols $\mathrm{d}\Gamma(b(t))w_{p,\infty}$, where $w_{p,\infty}$ is the kernel defined in (6.8). We use then pseudodifferential calculus, the fact that $\mathrm{supp}\, R \subset \{|x| \geq \epsilon_0\}$ and Lemma 6.3 to show that $\mathrm{d}\Gamma(b(t))w_\infty \in O(t^{-s})$, $s > 1$. By Prop. 3.13 *i)* this implies that $\chi(H)[V, \mathrm{i}\mathrm{d}\Gamma(b(t))]\chi(H) \in O(t^{-s})$ and hence is integrable in norm. □

The following proposition is an improvement on Prop. 11.3.

Proposition 11.4 *Let $0 < c_0 < c_1$, $J \in C_0^\infty(\{c_0 < |x| < c_1\})$, $\chi \in C_0^\infty(\mathbb{R})$. Then*

$$\int_1^{+\infty} \left\| \mathrm{d}\Gamma\left(\left| J\left(\frac{x}{t}\right)\left(\frac{x}{t} - \partial\omega(k)\right) + \mathrm{hc}\right|\right)^{\frac{1}{2}} \chi(H)\mathrm{e}^{-\mathrm{i}tH}u \right\|^2 \frac{\mathrm{d}t}{t} < C\|u\|^2.$$

Proof. The proof is identical to [DG1, Prop. 6.3], using the argument in the proof of Prop. 11.3 to control the commutators with V. □

Note that Prop. 11.4 is still true if we replace H by $H^{\rm ext}$ and $\mathrm{d}\Gamma(b)$ by $\mathrm{d}\Gamma(b)\otimes 1\!\!1 + 1\!\!1\otimes \mathrm{d}\Gamma(b)$, for $b = \left|J\left(\frac{x}{t}\right)\left(\frac{x}{t} - \partial\omega(k)\right) + \mathrm{hc}\right|$.

The last propagation estimate of this section is the so called *minimal velocity estimate*, based on the Mourre estimate shown in Subsect. 9.2. Since the conjugate operator is different from the one in [DG1], we will give a more detailed proof.

Proposition 11.5 *Assume condition (B1). Let $\chi \in C_0^\infty(\mathbb{R})$ be supported in $\mathbb{R}\backslash(\tau \cup \sigma_{\rm pp}(H))$. Then there exists $\epsilon > 0$ such that*

$$\int_1^{+\infty} \left\| \Gamma\left(1\!\!1_{[0,\epsilon]}\left(\frac{|x|}{t}\right)\right)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\right\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2.$$

Proof. Let us first prove the proposition for χ supported near an energy level $\lambda \in \mathbb{R}\backslash\tau\cup\sigma_{\rm pp}(H)$. By Thm. 9.2, we can find $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 near λ such that for some $c_0 > 0$:

$$\chi(H)[H, \mathrm{i}A]\chi(H) \geq c_0\chi^2(H). \tag{11.1}$$

Let $\epsilon > 0$ be a parameter which will be fixed later. Let $q \in C_0^\infty(|x| \leq 2\epsilon)$, $0 \leq q \leq 1$, $q = 1$ near $\{|x| \leq \epsilon\}$ and let $q^t = q(\frac{x}{t})$.

We use the propagation observable

$$\Phi(t) := \chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H).$$

We fix cutoff functions $\tilde{q} \in C_0^\infty(\mathbb{R})$, $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that

$$\mathrm{supp}\,\tilde{q} \subset \{|x| \leq 4\epsilon\},\ \tilde{q}q = q,\ \tilde{\chi}\chi = \chi.$$

Let us show the following estimate:

$$\|N^k \tfrac{A^m}{t^m}\Gamma(q^t)\chi(H)\| \leq C\epsilon^m + O_\epsilon(t^{-1}),\ m = 1, 2. \tag{11.2}$$

First note that by Lemma 3.2 *iii)*

$$A^{2m} \leq N^{2m-1}\mathrm{d}\Gamma(a^{2m}).$$

Next:

$$\Gamma(q^t)\mathrm{d}\Gamma(a^{2m})\Gamma(q^t) = \mathrm{d}\Gamma((q^t)^2, q^t a^{2m} q^t) \leq \mathrm{d}\Gamma(q^t a^{2m} q^t). \tag{11.3}$$

$$q^t a^{2m} q^t \leq \epsilon^{2m} t^{2m}\omega^{2m}(k) + O(t^{2m-2})\omega(k).$$

Therefore $\Gamma(q^t)\mathrm{d}\Gamma(a^{2m})\Gamma(q^t)$ is less than

$$C\epsilon^{2m}t^{2m}\mathrm{d}\Gamma(\omega^{2m}) + Ct^{2m-2}\mathrm{d}\Gamma(\omega) \leq C\epsilon^{2m}t^{2m}\mathrm{d}\Gamma(\omega)^{2m} + Ct^{2m-2}\mathrm{d}\Gamma(\omega).$$

Therefore

$$\|N^k\frac{A^m}{t^m}\Gamma(q^t)\chi(H)u\|^2 \leq C\epsilon^{2m}\|N^{k+m-\frac{1}{2}}H_0^m\chi(H)u\|^2 + Ct^{-2}\|N^{k+m-\frac{1}{2}}H_0\chi(H)u\|^2.$$

Then we apply the high order estimates.

Now (11.2) implies the uniform boundedness of $\Phi(t)$.

Let us compute the Heisenberg derivative of $\Phi(t)$. Using Lemma 11.1, we have, for $\mathbf{d}_0 q^t = \partial_t q^t + [\omega, \mathrm{i}q^t]$:

$$
\begin{aligned}
\mathbf{D}\Phi(t) &= \chi(H)\mathrm{d}\Gamma(q^t, \mathbf{d}_0 q^t)\tfrac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{hc} \\
&\quad + \chi(H)[V, \mathrm{i}\Gamma(q^t)]\tfrac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{hc} \\
&\quad + t^{-1}\chi(H)\Gamma(q^t)[H, \mathrm{i}A]\Gamma(q^t)\chi(H) \\
&\quad - t^{-1}\chi(H)\Gamma(q^t)\tfrac{A}{t}\Gamma(q^t)\chi(H) \\
&=: R_1(t) + R_2(t) + R_3(t) + R_4(t).
\end{aligned}
\tag{11.4}
$$

We have used the fact that $\Gamma(q^t)$ preserves $\mathcal{D}(H_0)$ and $\mathcal{D}(N^n)$ to expand the commutator $[H, \mathrm{i}\Phi(t)]$ in (11.4).

Let us first estimate $R_2(t)$. By Lemma 3.17 and Lemma 6.3,

$$
[V, \mathrm{i}\Gamma(q^t)] \in (N+1)^{-n} O_N(t^{-s}),\ s > 1,
$$

Therefore by (11.2)

$$
R_2(t) \in O(t^{-s}),\ s > 1. \tag{11.5}
$$

We consider next $R_1(t)$. We have:

$$
\mathbf{d}_0 q^t = -\frac{1}{2t}\langle \frac{x}{t} - \nabla\omega(k), \nabla q(\frac{x}{t})\rangle + \mathrm{hc} + r^t =: \frac{1}{t} g^t + r^t,
$$

where $r^t \in O(t^{-2})$. By the higher order estimates (7.1) $\|\chi(H)d\Gamma(q^t, r^t)\| \in O(t^{-2})$, which using (11.2) yields

$$
\|\chi(H)d\Gamma(q^t, r^t)\frac{A}{t}\Gamma(q^t)\chi(H)\| \in O(t^{-2}).
$$

Then we set

$$
B_1 := \chi(H)\mathrm{d}\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}}, \quad B_2^* := (N+1)^{\frac{1}{2}}\frac{A}{t}\Gamma(q^t)\chi(H),
$$

and use the inequality

$$
\begin{aligned}
\chi(H)d\Gamma(q^t, g^t)\tfrac{A}{t}\Gamma(q^t)\chi(H) + \mathrm{hc} &= t^{-1}B_1 B_2^* + t^{-1}B_2 B_1^* \\
&\geq -t^{-1}B_1 B_1^* - t^{-1}B_2 B_2^*.
\end{aligned}
\tag{11.6}
$$

We have

$$
\begin{aligned}
-B_2 B_2^* &= -\chi(H)\Gamma(q^t)\tfrac{A^2}{t^2}(N+1)\Gamma(q^t)\chi(H) \\
&= \chi(H)\Gamma(q^t)\tilde{\chi}(H)\Gamma(\tilde{q}^t)\tfrac{A^2}{t^2}(N+1)\Gamma(\tilde{q}^t)\tilde{\chi}(H)\Gamma(q^t)\chi(H) + O(t^{-1}) \\
&\geq -\epsilon^2 C_1 \chi(H)\Gamma^2(q^t)\chi(H) + O(t^{-1}),
\end{aligned}
\tag{11.7}
$$

where we used Lemma 7.11 and the boundedness of $\frac{A^2}{t^2}(N+1)\Gamma(q^t)\chi(H)$ in the first step and the estimate analogous to (11.2):

$$
\tilde{\chi}(H)\Gamma(\tilde{q}^t)\frac{A^2}{t^2}(N+1)\Gamma(\tilde{q}^t)\tilde{\chi}(H) \leq C_1\epsilon^2 + O(t^{-2})
$$

in the second step. Next we use that by Lemma 3.4 *v)*:

$$\|(N+1)^{-\frac{1}{2}}d\Gamma(q^t,g^t)u\| \leq \|d\Gamma(g^{t*}g^t)^{\frac{1}{2}}u\|,\ u\in\mathcal{H},$$

to obtain:

$$\begin{aligned}|(u,B_1^*B_1u)| &= \|(N+1)^{-\frac{1}{2}}d\Gamma(q^t,g^t)\chi(H)u\|^2\\ &\leq \|d\Gamma(g^{t*}g^t)^{\frac{1}{2}}\chi(H)u\|^2,\ u\in\mathcal{H}.\end{aligned}$$

Using Prop. 11.3, we obtain

$$\int_1^{+\infty}\|B_1\mathrm{e}^{-\mathrm{i}tH}u\|^2\frac{\mathrm{d}t}{t}\leq C\|u\|^2. \tag{11.8}$$

Next we use Lemma 7.11, to write:

$$\begin{aligned}R_3(t) &= t^{-1}\Gamma(q^t)\chi(H)[H,\mathrm{i}A]\chi(H)\Gamma(q^t)+O(t^{-2})\\ &\geq C_0t^{-1}\Gamma(q^t)\chi^2(H)\Gamma(q^t)-Ct^{-2}\\ &\geq C_0t^{-1}\chi(H)\Gamma^2(q^t)\chi(H)-Ct^{-2}.\end{aligned} \tag{11.9}$$

It remains to estimate $R_4(t)$. We have

$$\begin{aligned}R_4(t) &= -t^{-1}\chi(H)\Gamma(q^t)\tfrac{A}{t}\Gamma(q^t)\chi(H)\\ &= -t^{-1}\chi(H)\Gamma(q^t)\tilde{\chi}(H)\Gamma(\tilde{q}^t)\tfrac{A}{t}\Gamma(\tilde{q}^t)\tilde{\chi}(H)\Gamma(q^t)\chi(H)\\ &\geq -\epsilon C_2t^{-1}\chi(H)\Gamma(q^t)^2\chi(H)+O(t^{-2}).\end{aligned} \tag{11.10}$$

Collecting (11.7), (11.9) and (11.10), we obtain

$$\begin{aligned}&-\epsilon^2t^{-1}B_2^*(t)B_2(t)+R_3(t)+R_4(t)\\ &\geq(-\epsilon^2C_1+C_0-\epsilon C_2)t^{-1}\chi(H)\Gamma(q^t)^2\chi(H)-Ct^{-2}.\end{aligned} \tag{11.11}$$

We pick now ϵ small enough so that $\tilde{C}_0=-\epsilon^2C_1+C_0-\epsilon C_2>0$. Using (11.5), (11.8) and (11.11) we conclude that

$$\mathbf{D}\Phi(t)\geq\frac{\tilde{C}_0}{t}\chi(H)\Gamma^2(q^t)\chi(H)-R(t)-Ct^{-s},\ s>1.$$

where $R(t)$ is integrable along the evolution. By the standard argument, this proves the proposition for χ with support close enough to an energy level $\lambda\subset\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H))$. To prove the proposition for all χ supported in $\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H))$ we argue as in [DG2, Prop. 4.4.7]. □

12 Asymptotic completeness

12.1 Existence of asymptotic localizations

Theorem 12.1 *Assume hypothesis (Is), $s>1$. Let $q\in C_0^\infty(\mathbb{R})$, $0\leq q\leq 1$, $q=1$ on a neighborhood of zero. Set $q^t(x)=q(\frac{x}{t})$. Then there exists*

$$\mathrm{s\text{-}}\lim_{t\to\infty}\mathrm{e}^{\mathrm{i}tH}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH}=:\Gamma^+(q). \tag{12.1}$$

We have

$$\Gamma^+(q\tilde{q}) = \Gamma^+(q)\Gamma^+(\tilde{q}), \tag{12.2}$$

$$0 \le \Gamma^+(q) \le \Gamma^+(\tilde{q}) \le 1\!\mathrm{l},\ \textit{if}\ 0 \le q \le \tilde{q} \le 1, \tag{12.3}$$

$$[H, \Gamma^+(q)] = 0. \tag{12.4}$$

Proof. Let us first prove the existence of (12.1). Using Lemma 7.11 and a density argument, it suffices to prove the existence of

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\chi(H)\Gamma(q^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}$$

for $\chi \in C_0^\infty(\mathbb{R})$. We compute the Heisenberg derivative:

$$\chi(H)\mathbf{D}\Gamma(q^t)\chi(H) = \chi(H)\mathrm{d}\Gamma(q^t, \mathbf{d}_0 q^t)\chi(H) + \chi(H)[V, \mathrm{i}\Gamma(q^t)]\chi(H),$$

by Lemma 11.1. From Lemma 3.17, Lemma 6.3 and hypothesis *(Is)* we obtain

$$\|\chi(H)[V, \mathrm{i}\Gamma(q^t)]\chi(H)\| \in O(t^{-s}). \tag{12.5}$$

Next we compute:

$$\mathbf{d}_0 q^t = \frac{1}{t} g^t + r^t,$$

where

$$g^t = -\tfrac{1}{2}\left((\tfrac{x}{t} - \partial\omega(k))\partial q(\tfrac{x}{t}) + \mathrm{hc}\right)$$

and $r^t \in O(t^{-2})$. Using Lemma 3.4 *v)* and the higher order estimates, we obtain that

$$\|\chi(H)\mathrm{d}\Gamma(q^t, r^t)\chi(H)\| \in O(t^{-2}). \tag{12.6}$$

On the other hand by Lemma 3.4 *v)* we have

$$|(u|\chi(H)\mathrm{d}\Gamma(q^t, g^t)\chi(H)u)| \le \|\mathrm{d}\Gamma(|g^t|)^{\frac{1}{2}}\chi(H)u\|^2. \tag{12.7}$$

Hence the existence of the limit (12.1) follows from (12.5)–(12.7), Proposition 11.4 and Lemma A.1.

(12.4) follows by Lemma 7.11. (12.2) follows from

$$\Gamma(q^t\tilde{q}^t) = \Gamma(q^t)\Gamma(\tilde{q}^t).$$

□

An analogous theorem is true for the free Hamiltonian, but it is much easier. It follows within each n-particle sector by the stationary phase method. Note that in the free case one does not need to assume that the cutoff function q is one at zero.

Proposition 12.2 *Let $q \in C_0^\infty(\mathbb{R})$, $0 \le \le 1$. Then*

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)} = \Gamma(q(\nabla\omega)).$$

12.2 Projection P_0^+

Theorem 12.3 *Let* $\{q_n\} \in C_0^\infty(\mathbb{R})$ *be a decreasing sequence of functions such that* $0 \le q_n \le 1$, $q = 1$ *on a neighborhood of* 0 *and* $\cap_{n=1}^\infty \mathrm{supp}\, q_n = \{0\}$. *Then*

$$P_0^+ := \text{s-}\lim_{n\to\infty} \Gamma^+(q_n) \text{ exists.} \tag{12.8}$$

P_0^+ *does not depend on the choice of the sequence* $\{q_n\}$.
It is an orthogonal projection satisfying

$$[H, P_0^+] = 0.$$

Besides

$$\mathrm{Ran} P_0^+ \subset \mathcal{K}^+. \tag{12.9}$$

The range of P_0^+ can be interpreted as the space of states asymptotically containing no bosons away from the origin.

Proof. The existence of P_0^+ and the fact that it is a projection follow from (12.2), (12.3) and Lemma A.3. To show that P_0^+ does not depend on the choice of $\{q_n\}$, we pick two sequences $\{q_n\}, \{\tilde{q}_n\}$. There exist for each $n \in \mathbb{N}$ an index m_n such that $q_n \ge \tilde{q}_{m_n}, \tilde{q}_n \ge q_{m_n}$. Hence by (12.3) we see that

$$\text{s-}\lim_{n\to\infty} \Gamma^+(q_n) = \text{s-}\lim_{n\to\infty} \Gamma^+(\tilde{q}_n).$$

The fact that $[H, P_0^+] = 0$ follows from (12.4).

Let us now show (12.9). We know that $\mathrm{Ran} P_0^+$ is invariant wrt H, hence $\mathcal{D}(H) \cap \mathrm{Ran} P_0^+$ is dense in $\mathrm{Ran} P_0^+$. Besides, $\mathcal{K}^+$ is closed. Thus it is enough to show that $\mathcal{D}(H) \cap \mathrm{Ran} P_0^+ \subset \mathcal{K}^+$.

Let $u \in \mathcal{D}(H) \cap \mathrm{Ran} P_0^+$. We are going to show that

$$(H + b)^{-\frac{1}{2}} a^+(h) u = 0, \quad h \in \mathfrak{h}, \tag{12.10}$$

which will imply $u \in \mathcal{K}^+$. By the continuity of $\mathfrak{h} \ni h \mapsto (H + b)^{-\frac{1}{2}} a^+(h)$ it is enough to assume that $h \in \mathfrak{h}_0$. By stationary phase arguments we may choose $q \in C_\infty(\mathbb{R})$ with $0 \le q \le 1$, $q(0) = 1$ and $\mathrm{supp}\, q$ contained in a sufficiently small neighborhood of 0 so that $q^t h_t \in o(1)$. Then

$$u = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} \Gamma(q^t) \mathrm{e}^{-\mathrm{i}tH} u,$$

$$(H + b)^{-\frac{1}{2}} a^+(h) = \text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} (H + b)^{-\frac{1}{2}} a(h_t) \mathrm{e}^{-\mathrm{i}tH}.$$

Hence

$$\begin{aligned} (H + b)^{-\frac{1}{2}} a^+(h) u &= \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} (H + b)^{-\frac{1}{2}} a(h_t) \Gamma(q^t) (H + b)^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}tH} (H + b)^{\frac{1}{2}} u \\ &= \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} (H + b)^{-\frac{1}{2}} \Gamma(q^t) a(q^t h_t) (H + b)^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}tH} (H + b)^{\frac{1}{2}} u. \end{aligned} \tag{12.11}$$

But since $q^t h_t \in o(1)$, $a(q^t h_t)(H + b)^{-\frac{1}{2}} \in o(1)$ and therefore (12.11) vanishes. □

12.3 Geometric inverse wave operators

Let $j_0 \in C_0^\infty(\mathbb{R})$, $j_\infty \in C^\infty(\mathbb{R})$, $0 \leq j_0$, $0 \leq j_\infty$, $j_0^2 + j_\infty^2 \leq 1$, $j_0 = 1$ near 0 (and hence $j_\infty = 0$ near 0). Set $j := (j_0, j_\infty)$. Set also $j^t = (j_0^t, j_\infty^t)$, where $j_0^t(x) = j_0(\frac{x}{t})$, $j_\infty^t(x) = j_\infty(\frac{x}{t})$.

As in Subsect. 3.10, we introduce the operator $I(j^t) : \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h})$.

Theorem 12.4 *Assume hypothesis (Is) for $s > 1$.*

i) The following limits exist:

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}} I^*(j^t)\mathrm{e}^{-\mathrm{i}tH}, \tag{12.12}$$

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} I(j^t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}}. \tag{12.13}$$

If we denote (12.12) by $W^+(j)$, then (12.13) equals $W^+(j)^*$.

ii) For a bounded Borel function F one has

$$W^+(j)F(H) = F(H^{\mathrm{ext}})W^+(j).$$

iii) Let $q_0, q_\infty \in C^\infty(\mathbb{R})$, $\nabla q_0, \nabla q_\infty \in C_0^\infty(\mathbb{R})$, $0 \leq q_0, q_\infty \leq 1$, $q_0 = 1$ near 0. Set $\tilde{j} := (\tilde{j}_0, \tilde{j}_\infty) := (q_0 j_0, q_\infty j_\infty)$. Then

$$\Gamma^+(q_0) \otimes \Gamma(q_\infty(\nabla\omega))W^+(j) = W^+(\tilde{j}).$$

iv) Let $q \in C^\infty(\mathbb{R})$, $\nabla q \in C_0^\infty(\mathbb{R})$, $0 \leq q \leq 1$, $q = 1$ near 0. Then

$$W^+(j)\Gamma^+(q) = W^+(qj),$$

where $qj = (qj_0, qj_\infty)$.

v) Let $\tilde{j} = (\tilde{j}_0, \tilde{j}_\infty)$ be another pair satisfying the conditions stated at the beginning of this subsection. (Note that $\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty \leq 1$ and $\tilde{j}_0 j_0 = 1$ near zero). Then

$$W^+(\tilde{j})^* W^+(j) = \Gamma^+(\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty),$$

In particular, if $j_0^2 + j_\infty^2 = 1$, then $W^+(j)$ is isometric.

vi) Let $j_0 + j_\infty = 1$. If $\chi \in C_0^\infty(\mathbb{R})$, then

$$\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})W^+(j) = \chi(H).$$

Proof. Let us first prove the existence of the limit (12.12), the case of (12.13) being similar. Using Lemma 7.12 and a density argument, it suffices to prove the existence of

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}}\chi(H^{\mathrm{ext}})I^*(j^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH},$$

for some $\chi \in C_0^\infty(\mathbb{R})$. We compute the asymmetric Heisenberg derivative

$$\begin{aligned} \chi(H^{\mathrm{ext}})\check{\mathbf{D}}I^*(j^t)\chi(H) &= \chi(H^{\mathrm{ext}})\check{\mathbf{D}}_0 I^*(j^t)\chi(H) \\ &\quad + \mathrm{i}\chi(H^{\mathrm{ext}})(V \otimes 1\!\!1 I^*(j^t) - I^*(j^t)V)\chi(H). \end{aligned}$$

From Lemma 3.17, Lemma 6.3 and hypothesis *(Is)*, we obtain

$$\|\chi(H^{\rm ext})(V\otimes 1\!\!1 I^*(j^t) - I^*(j^t)V)\chi(H)\| \in O(t^{-s}). \tag{12.14}$$

On the other hand by Lemma 3.11, we have $\check{\mathbf{D}}_0 I^*(j^t) = {\rm d}I^*(j^t, \check{\mathbf{d}}_0 j^t)$, and, by pseudodifferential calculus,

$$\check{\mathbf{d}}_0^t j^t = \frac{1}{t}k^t + r^t,$$

where

$$k^t = (k_0^t, k_\infty^t), \quad k_\epsilon^t = -\tfrac{1}{2}((\tfrac{x}{t} - \partial\omega(k))\partial j_\epsilon(\tfrac{x}{t}) + {\rm hc}),\ \epsilon = 0, \infty$$

and $r^t \in O(t^{-2})$. Using Lemma 3.11 *v)* and the higher order estimates we obtain

$$\|\chi(H^{\rm ext}){\rm d}I^*(j^t, r^t)\chi(H)\| \in O(t^{-2}). \tag{12.15}$$

Using then Lemma 3.11 *iv)*, we obtain

$$\begin{aligned} &|(u_2|\chi(H^{\rm ext}){\rm d}I^*(j^t, k^t)\chi(H)u_1)| \\ &\leq \|({\rm d}\Gamma(|k_0^t|)^{\frac{1}{2}}\otimes 1\!\!1)\chi(H^{\rm ext})u_2\|\|{\rm d}\Gamma(|k_0^t|)^{\frac{1}{2}}\chi(H)u_1\| \\ &+\|(1\!\!1\otimes {\rm d}\Gamma(|k_\infty^t|)^{\frac{1}{2}})\chi(H^{\rm ext})u_2\|\|{\rm d}\Gamma(|k_\infty^t|)^{\frac{1}{2}}\chi(H)u_1\|. \end{aligned} \tag{12.16}$$

Hence the existence of the limit (12.12) follows from (12.14)–(12.16), Proposition 11.4 and Lemma A.2.

ii) follows from Lemma 7.12. *iii)* follows from Prop. 12.2 and the fact that

$$\Gamma(q_0^t)\otimes\Gamma(q_\infty^t)I^*(j^t) = I^*(\tilde{j}^t).$$

iv) follows from

$$I^*(j^t)\Gamma(q^t) = I^*((jq)^t).$$

v) follows from

$$I(\tilde{j}^t)I^*(j^t) = \Gamma(\tilde{j}_0^t j_0^t + \tilde{j}_\infty^t j_\infty^t).$$

Up to technical details due to the unboundedness of I, *vi)* can be considered as a special case of *v)* with $\tilde{j} = (1,1)$. To prove *vi)* we note that

$$H^{\rm ext}1\!\!1_{[k,\infty[}(N_\infty) \geq mk + E_0,$$

where $E_0 = \inf\sigma(H)$. Hence for $\chi \in C_0^\infty(\mathbb{R})$ we can find $n \in \mathbb{N}$ such that

$$\chi(H^{\rm ext})1\!\!1_{]n,\infty[}(N_\infty) = 0. \tag{12.17}$$

Therefore

$$\begin{aligned} \Omega^{{\rm ext},+}\chi(H^{\rm ext})W^+(j) &= \Omega^{{\rm ext},+}1\!\!1_{[0,n]}(N_\infty)\chi(H^{\rm ext})W^+(j) && (1)\\ &= \text{s-}\lim_{t\to\infty} {\rm e}^{{\rm i}tH}I1\!\!1_{[0,n]}(N_\infty)\chi(H^{\rm ext})I^*(j^t){\rm e}^{-{\rm i}tH} && (2)\\ &= \text{s-}\lim_{t\to\infty} {\rm e}^{{\rm i}tH}I1\!\!1_{[0,n]}(N_\infty)I^*(j^t){\rm e}^{-{\rm i}tH}\chi(H) && (3), \end{aligned} \tag{12.18}$$

using (12.17) in step (1), Thm. 10.7 *ii)* and Thm. 12.4 *i)* in step (2) and Lemma 7.12 and the boundedness of $I1\!\!1_{[0,n]}(N_\infty)(N_0)^{-n}$ in step (3).

Next we claim that

$$\|I1\!\!1_{]n,\infty[}(N_\infty)I^*(j^t)(N+1)^{-1}\| \leq C(n+1)^{-1}. \tag{12.19}$$

In fact the operator

$$I1\!\!1_{]n,\infty[}(N_\infty)I^*(j^t) = \Gamma(i)1\!\!1_{]n,\infty[}\Big(\mathrm{d}\Gamma\Big(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Big)\Big)\Gamma(j^{t*})$$

commutes with N. On $\otimes_{\mathrm{s}}^n\mathfrak{h}$ it can be written as

$$\sum_{\sharp\{i|\epsilon_i=\infty\}>n} j_{\epsilon_1}^t \otimes \cdots \otimes j_{\epsilon_n}^t,$$

where the indices ϵ_i take the values $0, \infty$. This explicit expression and the fact that $j_0 + j_\infty = 1$ imply

$$\|I1\!\!1_{]n,\infty[}(N_\infty)I^*(j^t)\| \leq 1,$$

$$I1\!\!1_{]n,\infty[}(N_\infty)I^*(j^t)1\!\!1_{[0,n]}(N) = 0,$$

which yields (12.19). Hence

$$\lim_{n\to\infty}\limsup_{t\to\infty}\|\mathrm{e}^{\mathrm{i}tH}I1\!\!1_{]n,\infty[}(N_\infty)I^*(j^t)\mathrm{e}^{-\mathrm{i}tH}\chi(H)\| = 0.$$

Since n can be chosen arbitrarily big and

$$II^*(j^t) = 1\!\!1,$$

(12.18) equals $\chi(H)$. □

12.4 Geometric asymptotic completeness

In this subsection we will show that

$$\mathrm{Ran}P_0^+ = \mathcal{K}^+.$$

We call this property *geometric asymptotic completeness*. It will be convenient to work in the extended space $\mathcal{H}^{\mathrm{ext}}$ and to treat Ω^+ as a partial isometry $\Omega^+ : \mathcal{H}^{\mathrm{ext}} \to \mathcal{H}$, as explained in Subsect. 10.4.

We will give an explicit construction of the inverse wave operator Ω^{+*} in terms of the geometric inverse wave operators $W^+(j)$.

Theorem 12.5 *Assume hypothesis (Is) for $s > 1$. Let $j_n = (j_{0,n}, j_{\infty,n})$ satisfy the conditions of Subsect. 12.3. Additionally, assume that $j_{0,n} + j_{\infty,n} = 1$ and that for any $\epsilon > 0$, there exists m such that, for $n > m$, $\mathrm{supp}\, j_{0,n} \subset [-\epsilon, \epsilon]$ Then*

$$\Omega^{+*} = \mathrm{w} - \lim_{n\to\infty} W^+(j_n).$$

Besides

$$\mathcal{K}^+ = \mathrm{Ran}P_0^+.$$

Proof. Let $q \in C_0^\infty(\mathbb{R})$, $q = 1$ in a neighborhood of 0, $0 \leq q \leq 1$. For sufficiently big n we have $qj_{0,n} = j_{0,n}$. Therefore, for sufficiently big n by Thm. 12.4 *iii)*

$$(\Gamma^+(q) \otimes 1\!\!1)W^+(j_n) - W^+(j_n) = 0.$$

Hence

$$\mathrm{w} - \lim_{n\to\infty} \left(P_0^+ \otimes 1\!\!1 W^+(j_n) - W^+(j_n) \right) = 0. \tag{12.20}$$

Let now $u \in \mathcal{H}$, $\chi \in C_0^\infty(\mathbb{R})$. We have

$$\begin{aligned}
\Omega^{+*}\chi(H) &= \Omega^{+*}\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})W^+(j_n) && (1)\\
&= \mathrm{w} - \lim_{n\to\infty} \Omega^{+*}\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})W^+(j_n) && (2)\\
&= \mathrm{w} - \lim_{n\to\infty} \Omega^{+*}\Omega^{\mathrm{ext},+}\chi(H^{\mathrm{ext}})P_0^+ \otimes 1\!\!1 W^+(j_n) && (3)\\
&= \mathrm{w} - \lim_{n\to\infty} P_0^+ \otimes 1\!\!1 \chi(H^{\mathrm{ext}})W^+(j_n) && (4)\\
&= \mathrm{w} - \lim_{n\to\infty} P_0^+ \otimes 1\!\!1 W^+(j_n)\chi(H) && (5)\\
&= \mathrm{w} - \lim_{n\to\infty} W^+(j_n)\chi(H) && (6).
\end{aligned}$$

We used Theorem 12.4 *vi)* in step (1); step (2) is obvious – we just added $\mathrm{w} - \lim_{n\to\infty}$ to a constant sequence; (12.20) was used in step (3) (note that $P_0^+ \otimes 1\!\!1$ commutes with $\chi(H^{\mathrm{ext}})$); in step (4) we used $\mathcal{K}^+ \supset \mathrm{Ran}P_0^+$, $\Omega^{\mathrm{ext},+}1\!\!1_{\mathcal{K}^+} \otimes 1\!\!1 = \Omega^+$, $\Omega^{+*}\Omega^+ = 1\!\!1_{\mathcal{K}^+} \otimes 1\!\!1$; in step (5) we used Theorem 12.4 *ii)*; finally in step (6) we used again (12.20). The arbitrariness of $\chi \in C_0^\infty(\mathbb{R})$ and a density argument imply

$$\Omega^{+*} = \mathrm{w} - \lim_{n\to\infty} W^+(j_n).$$

Therefore by (12.20), $(P_0^+ \otimes 1\!\!1)\Omega^{+*} = \Omega^{+*}$, ie

$$\mathrm{Ran}\Omega^{+*} \subset \mathrm{Ran}P_0^+ \otimes \Gamma(\mathfrak{h}) \subset \mathcal{K}^+ \otimes \Gamma(\mathfrak{h}).$$

But by construction

$$\mathrm{Ran}\Omega^{+*} = \mathcal{K}^+ \otimes \Gamma(\mathfrak{h}).$$

Hence $\mathcal{K}^+ \otimes \Gamma(\mathfrak{h}) = \mathrm{Ran}P_0^+ \otimes \Gamma(\mathfrak{h})$, and therefore

$$\mathcal{K}^+ = \mathrm{Ran}P_0^+.$$

□

12.5 Asymptotic completeness

In this subsection, we will prove Thm. 10.6.

Proof of Thm. 10.6. By Proposition 10.4 and geometric asymptotic completeness we already know that

$$\mathcal{H}_{\mathrm{pp}}(H) \subset \mathcal{K}^+ = \mathrm{Ran}P_0^+.$$

It remains to prove that $P_0^+ \leq 1\!\!1_{\mathrm{pp}}(H)$. Let $\chi \in C_0^\infty(\mathbb{R}\backslash(\tau \cup \sigma_{\mathrm{pp}}(H)))$. We deduce from Prop. 11.5 in Sect. 11 that there exists $\epsilon > 0$ such that for $q \in C_0^\infty([-\epsilon, \epsilon])$ with $q(x) = 1$ for $|x| < \epsilon/2$ we have

$$\int_1^{+\infty} \|\Gamma(q^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2 \frac{\mathrm{d}t}{t} \leq c\|u\|^2.$$

Since $\|\Gamma(q^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\| \to \|\Gamma^+(q)\chi(H)u\|$, we have $\Gamma^+(q)\chi(H) = 0$. This implies that

$$P_0^+ \leq 1\!\!1_{\tau\cup\sigma_{\mathrm{pp}}}(H).$$

Since τ is a closed countable set and $\sigma_{\mathrm{pp}}(H)$ can accumulate only at τ, we see that $1\!\!1_{\mathrm{pp}}(H) = 1\!\!1_{\tau\cup\sigma_{\mathrm{pp}}}(H)$. This completes the proof of the theorem. □

A Appendix

The following lemma describes an argument commonly used to prove the so called propagation estimates (see [DG1, Sect. 8.4] and references therein).

Lemma A.1 *Let H be a self-adjoint operator and $\mathbf{D}$ the corresponding Heisenberg derivative*

$$\mathbf{D} := \frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}[H, \cdot].$$

Suppose that $\Phi(t)$ is a uniformly bounded family of self-adjoint operators. Suppose that there exist $C_0 > 0$ and operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \ldots, n$, such that

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^{n} B_i^*(t)B_i(t),$$
$$\int_1^\infty \|B_i(t)\mathrm{e}^{-\mathrm{i}tH}\phi\|^2 \mathrm{d}t \leq C\|\phi\|^2, \quad i = 1, \ldots, n.$$

Then there exists C_1 such that

$$\int_1^\infty \|B(t)\mathrm{e}^{-\mathrm{i}tH}\phi\|^2 \mathrm{d}t \leq C_1\|\phi\|^2. \tag{A.1}$$

Next we describe how one uses propagation estimates to prove the existence of asymptotic observables.

Lemma A.2 *Let H_1 and H_2 be two self-adjoint operators. Let ${}_2\mathbf{D}_1$ be the corresponding asymmetric Heisenberg derivative:*

$${}_2\mathbf{D}_1\Phi(t) := \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + \mathrm{i}H_2\Phi(t) - \mathrm{i}\Phi(t)H_1.$$

Suppose that $\Phi(t)$ is a uniformly bounded function with values in self-adjoint operators. Let $\mathcal{D}_1 \subset \mathcal{H}$ be a dense subspace. Assume that

$$|(\psi_2|{}_2\mathbf{D}_1\Phi(t)\psi_1)| \leq \sum_{i=1}^{n} \|B_{2i}(t)\psi_2\|\|B_{1i}(t)\psi_1\|,$$
$$\int_1^\infty \|B_{2i}(t)\mathrm{e}^{-\mathrm{i}tH_2}\phi\|^2 \mathrm{d}t \leq C\|\phi\|^2, \quad \phi \in \mathcal{H}, \quad i = 1, \ldots, n,$$
$$\int_1^\infty \|B_{1i}(t)\mathrm{e}^{-\mathrm{i}tH_1}\phi\|^2 \mathrm{d}t \leq C\|\phi\|^2, \quad \phi \in \mathcal{D}_1, \quad i = 1, \ldots, n.$$

Then the limit

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_2}\Phi(t)\mathrm{e}^{-\mathrm{i}tH_1}$$

exists.

The proof of the following lemma is given in [DG1]:

Lemma A.3 *Let Q_n be a commuting sequence of selfadjoint operators such that:*

$$0 \leq Q_n \leq 1, \qquad Q_{n+1} \leq Q_n, \qquad Q_{n+1}Q_n = Q_{n+1}.$$

Then the limit

$$Q = \text{s-}\lim_{n\to\infty} Q_n.$$

exists and is a projection.

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