On the relativistic KMS condition for the $P(\phi)_2$ model

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Summary. The relativistic KMS condition introduced by Bros and Buchholz provides a link between quantum statistical mechanics and quantum field theory. We show that for the $P(\phi)_2$ model at positive temperature, the two point function for fields satisfies the relativistic KMS condition.

1 Introduction

The operator algebraic framework allows to characterize the equilibrium states of a quantum system by first principles: when the dynamical law is changed by a local perturbation, which is slowly switched on and slowly switched off again, an equilibrium state returns to its original form at the end of this procedure. In a pioniering work Haag, Kastler and Trych-Pohlmeyer [15] showed that this characterization of an equilibrium state leads to a sharp mathematical criterion, the so-called *KMS condition*, first encountered by Haag, Hugenholtz and Winnink [14] and more implicitly by Kubo [23], Martin and Schwinger [24].

On the other hand, the vacuum states of a relativistic QFT are characterized by Poincaré invariance and the spectrum condition [27]. Both the KMS and the spectrum condition can be formulated in terms of analyticity properties of the correlation functions, but for almost 40 years the connection between these two conditions was not investigated in depth, and algebraic quantum statistical mechanics and algebraic quantum field theory were treated in this respect as disjoint subjects. But finally increased interest in cosmology and heavy-ion collisions led to a need to combine QFT and quantum statistical mechanics.

It was first recognized by Bros and Buchholz [6] that the thermal equilibrium states of a relativistic QFT should have stronger analyticity properties in configuration space than those imposed by the traditional KMS condition. The result of their careful analysis is a *relativistic KMS condition* which can be understood as a remnant of (and, in applications, as a substitute for) the relativistic spectrum condition in the vacuum sector.

1.1 Content of the Paper

The content of this short paper is as follows. In Section 2 we recall the construction of the $P(\phi)_2$ model at positive temperature presented in [12]. In Section 3 we review the main ingredients of the Euclidean approach to thermal field theory. In Section 4 we show that the Wightman two-point function for fields satisfies the relativistic KMS condition. Section 5 is devoted to a brief discussion of the implications of the relativistic KMS condition.

2 The spatially-cutoff $P(\phi)_2$ model at positive temperature

The construction of interacting thermal quantum fields in [12] includes several of the original ideas of Høegh-Krohn [17], but instead of starting from the interacting system in a box the authors started from the Araki-Woods representation for the free thermal system in infinite volume. We briefly recall the main steps of this construction.

2.1 Preliminary Material

The KMS condition

A state ω_β over a C^*-algebra $\mathfrak{A}$ equipped with a C^*-dynamics τ_t is called a (τ, β)-KMS state for some $\beta \in \mathbb{R} \cup \{\pm\infty\}$, if for all $A, B \in \mathfrak{A}$ there exists a function $F_{A,B}$ which is continuous in the strip $\{z \in \mathbb{C} \mid 0 \le \Im z \le \beta\}$ and analytic and bounded in the open strip $\{z \in \mathbb{C} \mid 0 < \Im z < \beta\}$, with boundary values given by

$$F_{A,B}(t) = \omega_\beta\big(A\tau_t(B)\big),\ F_{A,B}(t+i\beta) = \omega_\beta\big(\tau_t(B)A\big),\ \forall\, t \in \mathbb{R}.$$

Thus KMS states are characterized by a real parameter β, which has the meaning of inverse temperature.

The relativistic KMS condition

Lorentz invariance is always broken by a KMS state. A KMS state might also break spatial translation or rotation invariance, but the maximal propagation velocity of signals is not affected by such a lack of symmetry.

Definition 1. *A state ω_β over a C^*-algebra $\mathfrak{A}$ satisfies the* relativistic KMS condition *at inverse temperature $\beta > 0$ if there exists some positive timelike vector $e \in V_+$, $e^2 = 1$, such that for every pair of elements A, B of $\mathfrak{A}$ there exists a function $F_{A,B}$ which is analytic in the tube domain*

$$\mathcal{T}_\beta := \{z \in \mathbb{C}^2 \mid \Im z \in V_+ \cap (\beta e + V_-)\},$$

and continuous at the boundary sets $\mathbb{R}^2$ and $\mathbb{R}^2 + i\beta e$ with boundary values given by

$$F_{A,B}(t,x) = \omega_\beta\big(A\alpha_{t,x}(B)\big) \quad \text{and} \quad F_{A,B}(t+i\beta e, x) = \omega_\beta\big(\alpha_{t,x}(B)\big)$$

for all $(t,x) \in \mathbb{R}^2$. Here $V_\pm = \{(t,x) \in \mathbb{R}^2 \mid |t| < |x|,\ \pm t \ge 0\}$ denote the forward/backward light-cones.

Remark 1. Note that in the limit $\beta \to \infty$ the tube $\mathcal{T}_\beta$ tends toward the forward tube $\mathbb{R}^2 + \mathrm{i}V_+$.

As we will discuss in Section 5, the relativistic KMS condition has numerous applications. For instance, the famous Reeh-Schlieder property (in the thermal representation) is a direct consequence of the relativistic KMS condition (and weak additivity), no matter if the spatial translation or rotation invariance is broken by the KMS state or not [19].

2.2 The free neutral scalar field at temperature β^{-1}

Let $\mathfrak{h} = H^{-\frac{1}{2}}(\mathbb{R})$ be the complex Sobolev space of order $-\frac{1}{2}$ equipped with the norm

$$\|h\|^2 = (h, (2\nu)^{-1}h)_{L^2(\mathbb{R},\mathrm{d}x)}$$

for $\nu := (D_x^2 + m^2)^{\frac{1}{2}}$.

Let $\mathfrak{W}(\mathfrak{h})$ be the abstract Weyl C^*-algebra over $\mathfrak{h}$. On $\mathfrak{W}(\mathfrak{h})$ we define the free time evolution $\{\tau_t^\circ\}_{t\in\mathbb{R}}$ by

$$\tau_t^\circ\big(W(h)\big) = W(\mathrm{e}^{\mathrm{i}t\nu}h), \quad h \in \mathfrak{h},\ t \in \mathbb{R}.$$

For $m > 0$ the unique (τ°, β)-KMS state on the Weyl algebra $\mathfrak{W}(\mathfrak{h})$ is given by

$$\omega_\beta^\circ\big(W(h)\big) := \mathrm{e}^{-\frac{1}{4}(h,(1+2\rho)h)}, \quad h \in \mathfrak{h}, \tag{1}$$

where $\rho := (\mathrm{e}^{\beta\nu} - 1)^{-1}$, $\beta > 0$.

The Araki-Woods representation

A realization of the GNS representation associated to the pair $\big(\mathfrak{W}(\mathfrak{h}), \omega_\beta^\circ\big)$ is provided by the Araki-Woods representation. We set:

$$\mathcal{H}_{AW} := \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}}),$$

$$\Omega_{AW} := \Omega_F,$$

$$\pi_{AW}(W(h)) = W_{AW}(h) := W_F\Big((1+\rho)^{\frac{1}{2}}h \oplus \overline{\rho}^{\frac{1}{2}}\overline{h}\Big), \quad h \in \mathfrak{h}.$$

Here $\overline{\mathfrak{h}}$ is the conjugate Hilbert space to $\mathfrak{h}$, $W_F(.)$ denotes the Fock Weyl operator on the bosonic Fock space $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ and $\Omega_F \in \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ is the Fock vacuum.

The generator of the time evolution

Since ω_β° is τ°-invariant, the time evolution can be unitarily implemented in the representation π_{AW}:

$$\mathrm{e}^{itL_{AW}}\pi_{AW}(A)\Omega_{AW} := \pi_{AW}\big(\tau_t^\circ(A)\big)\Omega_{AW}, \quad A \in \mathfrak{W}(\mathfrak{h}). \tag{2}$$

The generator L_{AW} of the free time evolution, uniquely fixed by (2) and the request $L_{AW}\Omega_{AW} = 0$, is called the (free) Liouvillean. It is equal to $\mathrm{d}\Gamma(\nu \oplus -\overline{\nu})$.

Thermal fields

The Araki-Woods representation π_{AW} is a regular representation, i.e., the map $\mathbb{R} \ni \lambda \mapsto W_\pi(\lambda h)$ is strongly continuous for any $h \in \mathfrak{h}$. Thus one can define *field operators*

$$\phi_{AW}(h) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda} W_{AW}(\lambda h)\Big|_{\lambda=0}, \quad h \in \mathfrak{h},$$

which satisfy in the sense of quadratic forms on $\mathcal{D}(\phi_{AW}(h_1)) \cap \mathcal{D}(\phi_{AW}(h_2))$ the commutation relations

$$[\phi_{AW}(h_1), \phi_{AW}(h_2)] = \mathrm{i}\Im(h_1, h_2), \quad h_1,\ h_2 \in \mathfrak{h}.$$

The net of local von Neumann algebras

The von Neumann algebra generated by $\{\pi_{AW}(W(h)) \mid h \in \mathfrak{h}\}$ is denoted by $\mathfrak{R}_{AW}$.

To define the net of local von Neumann algebras, we introduce the $\mathbb{R}-$linear map

$$\begin{array}{rccc} U\colon & \mathfrak{h} = H^{-\frac{1}{2}}(\mathbb{R}) & \rightarrow & \mathcal{S}'(\mathbb{R}) \\ & h & \mapsto & \Re h + \mathrm{i}\nu^{-1}\Im h. \end{array}$$

For $I \subset \mathbb{R}$ a bounded open interval, we define the following real vector subspace of $\mathfrak{h}$

$$\mathfrak{h}_I := \{h \in \mathfrak{h} \mid \operatorname{supp} Uh \subset I\}. \tag{3}$$

We denote by $\mathfrak{R}_{AW}(I)$ the von Neumann algebra generated by $\{W_{AW}(h) \mid h \in \mathfrak{h}_I\}$. The algebra

$$\mathfrak{A} := \overline{\bigcup_{I \subset \mathbb{R}} \mathfrak{R}_{AW}(I)}^{(C^*)}$$

is called the *algebra of local observables*. The union is over all open bounded intervals $I \subset \mathbb{R}$ and the symbol $\overline{\bigcup_{I \subset \mathbb{R}} \mathfrak{R}_{AW}(I)}^{(C^*)}$ denotes the C^*-inductive limit (see e.g. [20, Proposition 11.4.1.]).

Remark 2. We note that the Araki-Woods representation is locally normal w.r.t. the vacuum representation of the free field. Consequently, the C^*-algebra $\mathfrak{A}$ is identical (up to $*$-isomorphisms) to the algebra

$$\mathfrak{A}_{\mathrm{F}} := \overline{\bigcup_{I \subset \mathbb{R}} \pi_{\mathrm{F}}\big(\mathfrak{W}(\mathfrak{h}_I)\big)''}^{(C^*)},$$

where π_{F} is the Fock representation.

2.3 The spatially-cutoff $P(\phi)_2$ model at temperature β^{-1}

Let $P(\lambda) = \sum_{j=0}^{2n} a_j \lambda^j$ be a real valued polynomial, which is bounded from below, and let $l \in \mathbb{R}^+$ be a spatial cutoff parameter. The spatially cutoff $P(\phi)_2$ model on the real line $\mathbb{R}$ at temperature β^{-1} is then defined by the formal interaction term

$$V_l = \int_{-l}^{l} :P(\phi_{AW}(x)):\mathrm{d}x, \quad l > 0.$$

Here : : denotes the Wick ordering (see e.g. [13]) with respect to the *temperature* β^{-1} *covariance on* $\mathbb{R}$:

$$C_0(h_1,h_2) := \Big(h_1, \frac{(1+\mathrm{e}^{-\beta\nu})}{2\nu(1-\mathrm{e}^{-\beta\nu})}h_2\Big)_{L^2(\mathbb{R})}, \quad h_1,h_2 \in \mathcal{S}(\mathbb{R}). \tag{4}$$

Using a sequence of functions $\{\delta_k\}$ approximating the delta-function, the limit

$$V_l := \lim_{k\to\infty} \int_{-l}^{l} :P(\phi_{AW}(\delta_k(.-x))): \mathrm{d}x$$

exists on a dense set of vectors in $\Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$. An approximation of the Dirac δ function can be fixed by setting $\delta_k(x) := k\chi(kx)$, where χ is a function in $C^\infty_{0\,\mathbb{R}}(\mathbb{R}^d)$ with $\int \chi(x)\mathrm{d}x = 1$.

The perturbed KMS system

It can be shown (see [12]) that the operator sum $L_{AW} + V_l$ is essentially selfadjoint on $\mathcal{D}(L_{AW})\cap\mathcal{D}(V_l)$ and if we set $H_l := \overline{L_{AW} + V_l}$, then the perturbed time-evolution on $\mathfrak{A}$ is given by

$$\tau_t^l(A) := \mathrm{e}^{\mathrm{i}tH_l} A \mathrm{e}^{-\mathrm{i}tH_l}, \quad A \in \mathfrak{A}.$$

The perturbed KMS state ω_l on $\mathfrak{A}$ is normal w.r.t. the Araki-Woods representation π_{AW}. In fact, the GNS vector $\Omega_{AW} \in \Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ belongs to $\mathcal{D}\big(\mathrm{e}^{-\frac{\beta}{2}H_l}\big)$ and the perturbed KMS state ω_l is the vector state induced by the state vector

$$\Omega_l := \frac{\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}}{\|\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}\|}.$$

These results are in complete analogy to the analytic perturbation theory for bounded perturbations due to Araki (see e.g. [5]). Identical formulas, valid for a certain class of unbounded perturbations, have recently been derived in [8].

2.4 The translation invariant $P(\phi)_2$ model at temperature β^{-1}

Existence of the dynamics

Let $I \subset \mathbb{R}$ a bounded open interval. For $t \in \mathbb{R}$ fixed, the norm limit

$$\lim_{l\to\infty} \tau_t^l(A) =: \tau_t(A)$$

exists for all $A \in \mathfrak{R}_{AW}(I)$. In fact, for A and t fixed, $\tau_t^l(A)$ is independent of l for l sufficiently large.

By construction the elements of the local von Neumann algebras $\mathfrak{R}_{AW}(I)$, I open and bounded, are norm dense in $\mathfrak{A}$. Thus the map $\tau\colon t \mapsto \tau_t$ extends to a group of $*$-automorphisms of $\mathfrak{A}$. Moreover, if $\{\alpha_x\}_{x\in\mathbb{R}}$ denotes the group of space translations over $\mathfrak{A}$, defined by

$$\alpha_x\big(W_{AW}(h)\big) := W_{AW}(\mathrm{e}^{\mathrm{i}x.k}h), \; x \in \mathbb{R},$$

where k is the momentum operator acting on $\mathfrak{h}$, then

$$\tau_t \circ \alpha_x = \alpha_x \circ \tau_t$$

for all $t, x \in \mathbb{R}$. Consequently the time evolution is translation invariant.

Existence of the thermodynamic limit

Let $\{\omega_l\}_{l>0}$ be the family of (τ^l, β)-KMS states for the spatially cutoff $P(\phi)_2$ models constructed in Subsection 2.3. Then it has been shown in [12] that

$$\mathrm{w}-\lim_{l\to+\infty}\omega_l =: \omega_\beta \text{ exists over } \mathfrak{A}. \tag{5}$$

Moreover, the state ω_β has the following properties:

(i) ω_β is a (τ, β)-KMS state over $\mathfrak{A}$;
(ii) ω_β is *locally normal*, i.e., if $I \subset \mathbb{R}$ is an open and bounded interval, then $\omega_{\beta|\mathfrak{R}_{AW}(I)}$ is normal w.r.t. the Araki-Woods representation;
(iii) ω_β is invariant under spatial translations, i.e.,

$$\omega_\beta(\alpha_x(A)) = \omega_\beta(A), \quad x \in \mathbb{R},\ A \in \mathfrak{A};$$

(iv) ω_β has the *spatial clustering property*, i.e.,

$$\lim_{x\to\infty}\omega_\beta(A\alpha_x(B)) = \omega_\beta(A)\omega_\beta(B) \quad \forall A, B \in \mathfrak{A}.$$

The rate of the spatial clustering is related to the infrared properties of the Liouvillean [18].

3 Euclidean approach

As far as the formulation of the spatially cut-off thermal $P(\phi)_2$ model is concerned, the Euclidean approach is only used to show that the sum of the operators L_{AW} and V_l (which are both unbounded from below) is essentially selfadjoint on the intersection of their domains.

However, for the existence of the thermodynamic limit (5) the Euclidean approach is used in a more sophisticated manner. The key argument in the proof of (5), *Nelson symmetry*, will be crucial for the proof of the relativistic KMS condition too. In order to formulate it, we briefly recall the Euclidean approach, in a framework adapted to the thermal $P(\phi)_2$ model (see [11], [21] for a more general abstract framework).

3.1 Euclidean reconstruction theorem

Euclidean measures on the cylinder

Let $S_\beta = [-\beta/2, \beta/2]$ be the circle of length β. The points in the cylinder $S_\beta \times \mathbb{R}$ are denoted by (t, x). Let $Q := \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ be the space of real valued, $\beta-$periodic Schwartz distributions on $S_\beta \times \mathbb{R}$ and let Σ be the Borel σ-algebra on Q. Let μ be a Borel probability measure on (Q, Σ).

For $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$, we denote by $\phi(f)$ the function

$$\begin{array}{rccc} \phi(f)\colon & Q & \to & \mathbb{R} \\ & q & \mapsto & \langle q, f\rangle. \end{array}$$

For $T \geq 0$, we denote by $\Sigma_{[0,T]}$, the sub σ-algebra of Σ generated by the functions $\mathrm{e}^{\mathrm{i}\phi(f)}$ for $\mathrm{supp}\, f \subset [0, T[\times \mathbb{R}$. Let $r: Q \to Q$ be the time-reflection around $t = 0$ defined by $r\phi(t, x) = \phi(-t, x)$ and let $\tau_t^E: Q \to Q$ be the group of euclidean time translations defined by $\tau_t^E \phi(s, x) = \phi(s-t, x)$. The map r lifts to a $*-$automorphism R of $L^\infty(Q, \Sigma)$ defined by $RF(q) := F(rq)$, and the group τ_t^E lifts to a group $U(t)$ of $*$-automorphisms of $L^\infty(Q, \Sigma)$. This group is unitary on $L^2(Q, \Sigma, \mu)$ if μ is invariant under τ_t^E.

Reconstruction theorem

Let $\mathcal{H}_{\mathrm{OS}} := L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)$. We assume that the measure μ satisfies the *Osterwalder-Schrader positivity*

$$(F, F) := \int_Q R(\overline{F}) F \mathrm{d}\mu \geq 0 \quad \forall F \in \mathcal{H}_{\mathrm{OS}}.$$

Let $\mathcal{N} \subset \mathcal{H}_{\mathrm{OS}}$ be the kernel of $(.,.)$. We set

$$\mathcal{H}_{\mathrm{phys}} := \overline{\mathcal{H}_{\mathrm{OS}}/\mathcal{N}},$$

where the completion of $\mathcal{H}_{\mathrm{OS}}/\mathcal{N}$ is done w.r.t. the norm $(.,.)^{\frac{1}{2}}$. The canonical projection $\mathcal{H}_{\mathrm{OS}}$ to $\mathcal{H}_{\mathrm{phys}}$ is denoted by $\mathcal{V}$.

In $\mathcal{H}_{\mathrm{phys}}$ we have the distinguished vector

$$\Omega_{\mathrm{phys}} := \mathcal{V}(1),$$

where 1 is the constant function equal to 1 on Q.

The unitary group $U(t)$ for $t \geq 0$ does *not* preserve $\mathcal{H}_{\mathrm{OS}}$ (contrary to 0-temperature theories), because distributions supported in the 'future' $[0, \beta/2[\times \mathbb{R}$ come back in the 'past' $]-\beta/2, 0] \times \mathbb{R}$ by time translations. Nevertheless $U(s)$ for $0 \leq s \leq t$ sends $L^2(Q, \Sigma_{[0,\beta/2-t[}, \mathrm{d}\mu)$ into $\mathcal{H}_{\mathrm{OS}}$.

Using the theory of *local symmetric semigroups* (see [10], [22]), it is possible to define a selfadjoint operator L_{phys} on $\mathcal{H}_{\mathrm{phys}}$ such that for $F \in L^2(Q, \Sigma_{[0,\beta/2-t[}, \mathrm{d}\mu)$ and $0 \leq s \leq t$ one has

$$\mathcal{V}(U(s)F) = \mathrm{e}^{-sL_{\mathrm{phys}}} \mathcal{V}(F).$$

Finally if $A \in L^\infty(Q, \Sigma_{\{0\}})$, then multiplication by A preserves $\mathcal{H}_{\mathrm{OS}}$ and $\mathcal{N}$, and one obtains a representation π_{phys} of the *algebra of time-zero fields* $L^\infty(Q, \Sigma_{\{0\}})$:

$$\pi_{\mathrm{phys}}(A)\mathcal{V}(F) := \mathcal{V}(AF), \quad F \in \mathcal{H}_{\mathrm{OS}}.$$

From this reconstruction procedure one obtains a β-KMS system defined as follows:

(i) the C^*-algebra $\mathfrak{A}_{\mathrm{phys}}$ is the von Neumann algebra generated by the operators $\mathrm{e}^{\mathrm{i}tL_{\mathrm{phys}}} \pi_{\mathrm{phys}}(A) \mathrm{e}^{-\mathrm{i}tL_{\mathrm{phys}}}$, $t \in \mathbb{R}$, $A \in L^\infty(Q, \Sigma_{\{0\}})$;

(ii) the dynamics τ_t on $\mathfrak{A}_{\mathrm{phys}}$ is the dynamics generated by the unitary group $\mathrm{e}^{-\mathrm{i}tL_{\mathrm{phys}}}$;

(iii) the β-KMS state on $\mathfrak{A}_{\mathrm{phys}}$ is the state generated by the vector Ω_{phys}.

3.2 Euclidean measure for the translation invariant $P(\phi)_2$ model

The spatially-cutoff $P(\phi)_2$ model at positive temperature allows an Euclidean formulation: the measure $\mathrm{d}\mu_l$ for the spatially cutoff $P(\phi)_2$ model is given by

$$\mathrm{d}\mu_l := \frac{1}{Z_l}\,\mathrm{e}^{-\int_{-\beta/2}^{\beta/2}\int_{-l}^{l} :P(\phi(t,x)):_C \mathrm{d}t\mathrm{d}x}\mathrm{d}\phi_C, \tag{6}$$

where $\mathrm{d}\phi_C$ denotes the Gaussian measure on (Q, Σ) with covariance

$$C(u,u) = (u, (D_t^2 + D_x^2 + m^2)^{-1}u)$$

(with β-periodic b.c.) defined by

$$\int_Q \mathrm{e}^{\mathrm{i}\phi(f)}\mathrm{d}\phi_C = \mathrm{e}^{-C(f,f)/2}, \quad f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}). \tag{7}$$

The partition function Z_l is chosen such that $\int_Q d\mu_l = 1$.

Existence of limiting measure

In order to show that one can remove the spatial cutoff one has to show that

$$\lim_{l\to+\infty}\int_Q F(q)\mathrm{d}\mu_l =: \int_Q F(q)\mathrm{d}\mu_\infty \tag{8}$$

exists and defines a Borel probability measure on $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$.

Nelson Symmetry

Formally exchanging the role of t and x in (6) one notices that $\mathrm{d}\mu_\infty$ is the Euclidean measure of the $P(\phi)_2$ model on the circle at temperature zero. This formal argument can be made rigorous (see [12], [17]). In particular one has:

$$\mathrm{e}^{-\int_{-\beta/2}^{\beta/2}(\int_{-l}^{l} :P(\phi(t,x)):_{C_0}\mathrm{d}x)\mathrm{d}t} = \mathrm{e}^{-\int_{-l}^{l}(\int_{-\beta/2}^{\beta/2} :P(\phi(t,x)):_{C_\beta}\mathrm{d}t)\mathrm{d}x}. \tag{9}$$

Note that on the r.h.s. Wick ordering is done w.r.t. the covariance

$$C_\beta(g_1,g_2) := \left(g_1, \frac{1}{2\nu}g_2\right)_{L^2(S_\beta,\mathrm{d}t)}, \; g_1, g_2 \in \mathcal{S}(S_\beta).$$

The analog of (9) in the zero temperature case is called *Nelson symmetry* (see e.g. [26]).

It was first noticed by Høegh-Krohn [17] that the existence of the limit (8) is equivalent to the uniqueness of the vacuum state for the $P(\phi)_2$ model on the circle.

Using Nelson symmetry, the existence of the limit (8) is proved in [12]. Moreover it is shown in [12] that μ_∞ is OS positive, invariant under space-time translations, and that *sharp-time fields* are well defined: this means that if $\delta_k \in C_0^\infty(S_\beta)$ is a sequence of functions tending to the Dirac distribution δ_0, then the limits

$$\phi(t,h) := \lim_{k\to\infty}\phi(\delta_k(.-t)\otimes h)$$

exist in $\bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\mu_\infty)$ for any $h \in C_0^\infty(\mathbb{R})$.

4 The relativistic KMS condition

In this section we show that two point functions for fields in the thermal $P(\phi)_2$ model satisfy the relativistic KMS condition.

Identification of physical objects

Let $(\mathcal{H}_\beta, \pi_\beta, \Omega_\beta)$ be the GNS triple associated to the KMS state ω_β over the C^*-algebra $\mathfrak{A}$. Let also P_β, L_β be the unique generators of space-time translations such that $L_\beta \Omega_\beta = P_\beta \Omega_\beta = 0$.

As we saw in Subsect. 3.1, the Euclidean reconstruction theorem provides a Hilbert space $\mathcal{H}_{\mathrm{phys}}$, a unit vector Ω_{phys}, a representation π_{phys} of the abelian von Neumann algebra $\mathcal{U}_{\mathrm{AW}}$ generated by $\{W_{\mathrm{AW}}(h) \mid h \in \mathfrak{h}, h \text{ real valued }\}$, and a selfadjoint operator L_{phys}. Since the Euclidean measure μ_∞ is invariant under space translations, we obtain also a selfadjoint operator P_{phys} implementing the space translations.

Let us briefly check that these two families of objects are identical (up to unitary equivalence). In the sequel we will freely identify them.

Let $\mathfrak{U}(I)$ be the abelian von Neumann algebra generated by $\{W_{\mathrm{AW}}(h) \mid h \in \mathfrak{h}, h \text{ real valued}, \operatorname{supp} h \subset I\}$. It was shown in [12] that for $A_j \in \mathfrak{U}(I)$, $1 \leq j \leq n$,

$$\begin{aligned}
&(\Omega_{\mathrm{phys}}, \textstyle\prod_{j=1}^n \mathrm{e}^{\mathrm{i}t_j L_{\mathrm{phys}}} \pi_{\mathrm{phys}}(A_j) \mathrm{e}^{-\mathrm{i}t_j L_{\mathrm{phys}}} \Omega_{\mathrm{phys}}) \\
&= \omega_\beta(\textstyle\prod_{j=1}^n \tau_{t_j}(A_j)) \\
&= (\Omega_\beta, \textstyle\prod_{j=1}^n \mathrm{e}^{\mathrm{i}t_j L_\beta} \pi_\beta(A_j) \mathrm{e}^{-\mathrm{i}t_j L_\beta} \Omega_\beta).
\end{aligned} \tag{10}$$

Thus we can define a map $U : \mathcal{H}_{\mathrm{phys}} \to \mathcal{H}_\beta$ by

$$U \mathrm{e}^{\mathrm{i}tL_{\mathrm{phys}}} \pi_{\mathrm{phys}}(A) \Omega_{\mathrm{phys}} := \mathrm{e}^{\mathrm{i}tL_\beta} \pi_\beta(A) \Omega_\beta, \quad t \in \mathbb{R},\ A \in \mathfrak{U}(I),\ I \subset \mathbb{R}. \tag{11}$$

From (10) we see that U preserves the scalar product, hence it can be uniquely extended by linearity to

$$\mathcal{E} = \mathrm{Vect}\{\mathrm{e}^{\mathrm{i}tL_{\mathrm{phys}}} \pi_{\mathrm{phys}}(A) \Omega_{\mathrm{phys}} \mid t \in \mathbb{R}, A \in \mathfrak{U}(I),\ I \subset \mathbb{R}\}$$

as an isometry. It follows from the Euclidean reconstruction theorem, that $\mathcal{E}$ is dense in $\mathcal{H}_{\mathrm{phys}}$. Moreover it is clear from (11) that U intertwines π_{phys} and π_β restricted to $\mathfrak{U}$ and also intertwines L_{phys} and L_β. Finally U also intertwines P_{phys} and P_β (note that the algebra of time-zero fields is clearly invariant under space translations).

To check that U is unitary, we use a result in [12]: let $\mathcal{B}_\alpha(I)$ be the von Neumann algebra generated by $\{\tau_t(A) \mid A \in \mathfrak{U}(I), |t| < \alpha\}$. Then it was shown in [12, Prop. 6.5] that

$$\mathfrak{R}_{AW}(I) = \bigcap_{\alpha > 0} \mathcal{B}_\alpha(I).$$

Since by construction Ω_β is cyclic for $\pi_\beta(\mathfrak{A})$, this implies that the range of U is dense in $\mathcal{H}_\beta$. Therefore U is unitary.

4.1 Wightman two point function for the thermal $P(\phi)_2$ model

Let $I \subset \mathbb{R}$ be a bounded open interval and $\mathfrak{h}_I$ the real subspace of $\mathfrak{h}$ defined in (3).

By restriction to the local algebra $\mathcal{R}_{\rm AW}(I)$, π_β defines a CCR representation of the real symplectic space $\mathfrak{h}_I$.

Lemma 1. *i) The representation π_β restricted to $\mathcal{R}_{\rm AW}(I)$ is quasi-equivalent to the concrete representation of $\mathcal{R}_{\rm AW}(I)$;*

ii) the CCR representation $\mathfrak{h}_I \ni h \mapsto \pi_\beta(W_{\rm AW}(h)) \in \mathcal{B}(\mathcal{H}_\beta)$ is regular.

Proof. It is well known (see e.g. [12, Lemma 6.2]) that $\mathcal{R}_{\rm AW}(I)$ is a factor. Now it is shown in [20, Prop. 10.3.14] that if $\mathcal{R}$ is a C^*-algebra and π is a factor representation of $\mathcal{R}$, then π is quasi-equivalent to the GNS representation of any π-normal state ω. Applying this fact to $\mathcal{R}_{\rm AW}(I)$ (with its concrete representation) and to ω_β, we obtain that π_β is quasi-equivalent to the concrete representation of $\mathcal{R}_{\rm AW}(I)$. This proves *i)*. We know then that there exists a $*$-isomorphism γ from $\mathcal{R}_{\rm AW}(I)$ into $\pi_\beta(\mathcal{R}_{\rm AW}(I))''$ extending π_β. This isomorphism is automatically weakly continuous. Since the Araki-Woods representation is regular, the same holds true for the GNS representation π_β restricted to $\mathfrak{h}_I$. □

Since π_β is a regular CCR representation, we can define for $h \in \mathfrak{h}_I$ the *Segal field operators*

$$\phi_\beta(h) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\pi_\beta(W_{\rm AW}(sh))\Big|_{s=0}.$$

In the sequel we will consider only the *time-zero fields* φ_β:

$$\varphi_\beta(h) := \phi_\beta(h), \ \text{for } h \in \mathfrak{h}_I, \ h \text{ real}.$$

Remark 3. If we restrict ourselves to time-zero fields, it is possible to give a direct proof that the CCR representation is locally regular, avoiding the use of the local normality of the state ω_β. In fact, let us show that

$$\text{the map } \mathbb{R} \ni s \mapsto \pi_\beta\big(W_{\rm AW}(sh)\big) \text{ is strongly continuous for } h \in \mathfrak{h}_I, \ h \text{ real}, \quad (12)$$

using the Euclidean approach.

It suffices to prove the weak continuity on a dense subspace of $\mathcal{H}_\beta$. From the reconstruction theorem, we see that we may take as a dense subspace of $\mathcal{H}_\beta$ the linear span of the vectors $\mathcal{V}(\prod_1^k \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)})$ for $h_j \in C_{0\mathbb{R}}^\infty(\mathbb{R})$, $0 \le t_j < \beta/2$.

We see that it suffices to prove the continuity of the map

$$\mathbb{R} \ni s \mapsto \int_Q \big(\prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)}\big)\mathrm{e}^{\mathrm{i}s\phi(0,h)}\mathrm{d}\mu_\infty$$

for $h_j \in C_{0\mathbb{R}}^\infty(\mathbb{R})$, $t_j \in S_\beta$. But this follows from the fact that $\phi(0,h) \in L^1(Q, \mathrm{d}\mu_\infty)$, shown in [12].

Lemma 2. *$\Omega_\beta \in \mathcal{D}(\varphi_\beta(h))$, $\forall h \in \mathfrak{h}_I$, h real valued.*

Proof. Clearly it suffices to prove that

$$2 - (\Omega_\beta, \mathrm{e}^{\mathrm{i}s\varphi_\beta(h)}\Omega_\beta) - (\Omega_\beta, \mathrm{e}^{-\mathrm{i}s\varphi_\beta(h)}\Omega_\beta) \leq C|s|^2, \quad 0 \leq s \leq 1. \tag{13}$$

By the reconstruction theorem, the r.h.s. is equal to

$$\int_Q \big(2 - \mathrm{e}^{\mathrm{i}s\phi(0,h)} - \mathrm{e}^{-\mathrm{i}s\phi(0,h)}\big)\mathrm{d}\mu_\infty.$$

Since $\phi(0,h) \in L^2(Q, \mathrm{d}\mu_\infty)$, we obtain (13). □

We now define Wightman two point functions. For a function $h \in C_0^\infty(\mathbb{R})$ we denote by h^- the function $h^-(x) = h(-x)$.

Proposition 1. *There exists a unique $\mathcal{W}_\beta(t,.) \in C^0(\mathbb{R}_t, \mathcal{D}'(\mathbb{R}))$ such that for $h_1, h_2 \in C^\infty_{0\mathbb{R}}(\mathbb{R})$:*

$$\big(\varphi_\beta(h_1)\Omega_\beta, \mathrm{e}^{\mathrm{i}tL_\beta}\varphi_\beta(\alpha_x h_2)\Omega_\beta\big) = h_1 \star h_2^- \star \mathcal{W}_\beta(t,x), \quad (t,x) \in \mathbb{R}^2. \tag{14}$$

Proof. For fixed t the l.h.s. is a bilinear form Q_t w.r.t. h_1 and h_2. Moreover it is shown in [12] that $\|\varphi_\beta(h)\Omega_\beta\| \leq C\|h\|_S$, where $\|.\|_S$ is a Schwartz seminorm. Therefore Q_t is continuous for the topology of $C_0^\infty(\mathbb{R})$, which by translation invariance, implies the existence of $\mathcal{W}_\beta(t,.)$. The continuity w.r.t. the variable t of $\mathcal{W}(t,.)$ follows from the obvious continuity in t of the l.h.s. of (14). □

4.2 Relativistic KMS condition

The rest of this section is devoted to the proof of the following theorem:

Theorem 1. *The distribution $W_\beta(t,x)$ extends holomorphically to $\mathbb{R}^2 + \mathrm{i}V_\beta$, where $V_\beta := \{(t,y) \mid \ |y| < \inf(t, \beta - t)\}$. Therefore for $A_i = \varphi_\beta(h_i)$, $h_i \in C^\infty_{0\mathbb{R}}(\mathbb{R})$ the two-point function $F_{A_1,A_2}(t,x)$ is holomorphic in $\mathbb{R}^2 + \mathrm{i}V_\beta$.*

4.3 Proof of Thm. 1

Let us briefly recall a few facts concerning the 0−temperature $P(\phi)_2$ model on the circle S_β. The Hilbert space is the Fock space $\Gamma(H^{-\frac{1}{2}}(S_\beta))$, where $H^{-\frac{1}{2}}(S_\beta)$ is the Sobolev space of order $-\frac{1}{2}$ on S_β with norm

$$\|g\|^2 = (g, (2b)^{-1}g)_{L^2(S_\beta, \mathrm{d}t)}$$

with $b = (D_t^2 + m^2)^{\frac{1}{2}}$. For $g \in H^{-\frac{1}{2}}(S_\beta)$, we denote by $\phi_C(g)$ the (Fock) field operator acting on $\Gamma(H^{-\frac{1}{2}}(S_\beta))$.

The operator sum

$$\mathrm{d}\Gamma(b) + \int_{S_\beta} :P(\phi_C(t)):_{C_\beta} \mathrm{d}t$$

is essentially selfadjoint and bounded below, and the Hamiltonian of the $P(\phi)_2$ model on S_β is

$$H_C := \mathrm{d}\Gamma(b) + \int_{S_\beta} : P(\phi(t)) :_{C_\beta} \mathrm{d}t - E_C,$$

where E_C is an additive constant such that $\inf \sigma(H_C) = 0$. The Hamiltonian H_C has a unique (up to a phase) ground state (i.e., vacuum state) induced by the state vector Ω_C. Another fact we shall need is the following: if g_1, g_2 are *real* elements of $H^{-\frac{1}{2}}(S_\beta)$, then $(\phi_C(g_1)\Omega_C, \mathrm{e}^{-yH_C}\phi_C(g_2)\Omega_C)$ is real for $y \in \mathbb{R}^+$. This follows from the representation of e^{-yH_C} using the Feynman-Kac-Nelson (FKN) formula.

Finally we note that

$$P_C := \mathrm{d}\Gamma(D_t)$$

is the momentum operator on the circle S_β.

The two-point function for the $P(\phi)_2$ model on the circle

Now consider the two-point function $\mathcal{W}_C$ for the $P(\phi)_2$ model on the circle

$$\mathcal{W}_C(t,y) = \big(\Omega_C, \phi_C(\delta_0)\mathrm{e}^{\mathrm{i}yH_C}\mathrm{e}^{\mathrm{i}tP_C}\phi_C(\delta_0)\Omega_C\big), \quad t \in S_\beta,\ y \in \mathbb{R}.$$

$\mathcal{W}_C$ is a tempered distribution on $S_\beta \times \mathbb{R}$ rigorously defined by

$$\langle \mathcal{W}_C, f \otimes g \rangle := (2\pi)^{\frac{1}{2}} \big(\phi_C(\delta_0)\Omega_C, \tilde{g}(-H_C)\phi_C(f)\Omega_C\big),$$

for $f \in C^\infty(S_\beta)$, $g \in \mathcal{S}(\mathbb{R})$, and $\tilde{g}$ the Fourier transform of g.

To check that $\mathcal{W}_C$ is well defined as a tempered distribution, we use the bound

$$\|(H_C+1)^{-\frac{1}{2}}\phi_C(h)(H_C+1)^{-\frac{1}{2}}\| \le C\|h\|_{H^{-1}(S_\beta)}, \tag{15}$$

which using that $\delta_0 \in H^{-1}(S_\beta)$ yields

$$|\langle \mathcal{W}_C, f \otimes g \rangle| \le C\|(H_C+1)\tilde{g}(-H_C)\|\|f\|_{H^{-1}(S_\beta)}\|(H_C+1)^{\frac{1}{2}}\Omega_C\|^2.$$

The r.h.s. can clearly be estimated in terms of Schwartz seminorms of f and g.

Analytic continuation

We first recall the *spectrum condition on the circle* [14]:

$$|P_C| \le H_C.$$

Set

$$V_\pm := \{(t,y) \in \mathbb{R}^2 \mid |t| < |y|,\ \pm y \ge 0\}.$$

Using $\|\mathrm{e}^{-\epsilon H_C}(H_C+1)\| \le C\epsilon^{-1}$ and the bound (15) we conclude that

$$F(\tau, z) := \big(\Omega_C, \phi_C(\delta_0)\mathrm{e}^{\mathrm{i}zH_C}\mathrm{e}^{\mathrm{i}\tau P_C}\phi_C(\delta_0)\Omega_C\big)$$

is holomorphic in $S_\beta \times \mathbb{R} + \mathrm{i}V_+$, has a moderate growth when $\Im(\tau, z) \to 0$ and

$$\begin{aligned}
&\int F(t+\mathrm{i}\epsilon, y+\mathrm{i}\epsilon)f(t)g(y)\mathrm{d}t\mathrm{d}y \\
&\quad = (2\pi)^{\frac{1}{2}}\big(\phi_C(\delta_0)\Omega_C, \tilde{g}(-H_C)\mathrm{e}^{-\epsilon(H_C+P_C)}\phi_C(f)\Omega_C\big) \\
&\quad \to \langle \mathcal{W}_C, f \otimes g \rangle,
\end{aligned}$$

when $\epsilon \to 0$, i.e.,

$$\lim_{\epsilon \to 0} F(.+\mathrm{i}\epsilon, .+\mathrm{i}\epsilon) = \mathcal{W}_C(.,.) \text{ in } \mathcal{S}'(S_\beta \times \mathbb{R}).$$

Locality on the circle

Clearly

$$\mathcal{W}_C = \lim_{k\to+\infty} G_k \text{ in } \mathcal{S}'(S_\beta \times \mathbb{R}),$$

where

$$G_k(t,y) := \big(\Omega_C, \phi_C(\delta_k)\mathrm{e}^{\mathrm{i}yH_C}\mathrm{e}^{\mathrm{i}tP_C}\phi_C(\delta_k)\Omega_C\big),$$

and δ_k is a sequence in $C^\infty(S_\beta)$ with support in $\{t \in \mathbb{R} \mid |t| \le k^{-1}\}$ and tending to δ_0 when $k \to \infty$.

Now using locality (i.e., finite speed of light) on the circle, we see that if $(t,y) \in V_{\beta,k}$,

$$V_{\beta,k} = \{(t,y) \mid |y| < \inf(t, \beta - t) - 2k^{-1}\},$$

then

$$[\phi_C(\delta_k), \mathrm{e}^{\mathrm{i}yH_C}\mathrm{e}^{\mathrm{i}tP_C}\phi_C(\delta_k)\mathrm{e}^{-\mathrm{i}tP_C}\mathrm{e}^{-\mathrm{i}yH_C}] = 0,$$

because no signal can go from supp δ_k to supp $(\delta_k + t)$ in time y if $(t,y) \in V_{\beta,k}$. This fact can be shown by exactly the same arguments as those used for the $P(\phi)_2$ model on $\mathbb{R}$. Thus for $(t,y) \in V_{\beta,k}$, the function G_k is real valued.

Edge of the wedge

According to the Schwarz reflection principle, $\mathcal{W}_C$ can now be view as the boundary value of a function holomorphic in $V_\beta - iV_+$:

$$\mathcal{W}_C(t,x) = H(t,x)$$

where

$$H(\tau,z) := \big(\phi_C(\delta_0)\Omega_C, \mathrm{e}^{-\mathrm{i}zH_C}\mathrm{e}^{-\mathrm{i}\tau P_C}\phi_C(\delta_0)\Omega_C\big).$$

Thus

$$\overline{\mathcal{W}_C(t,y)} = \mathcal{W}_C(\bar{t},\bar{y}), \quad (t,y) \in V_\beta + i(V_+ \cup V_-). \tag{16}$$

We can now apply the edge of the wedge theorem [27]. It implies that there exists an open ball $B(0,d) := \{z \in \mathbb{C}^2 \mid |z| < d\}$ such that $\mathcal{W}_C(t,y)$ is holomorphic in $V_\beta + i\Gamma$, where

$$\Gamma := V_+ \cup V_- \cup B(0,d).$$

Moreover, $\mathcal{W}_C(t,iy)$ is real for $t \in S_\beta$, $y > 0$ (by using the representation of e^{-yH_C} as a Feynman-Kac-Nelson (FKN) kernel). Thus $\mathcal{W}_C(t,iy)$ is real for $t \in S_\beta$, $y \in \mathbb{R}$, by (16). Applying the Schwarz reflection principle one more time, we conclude that

$$\mathcal{W}_C(t,y) = \mathcal{W}_C(t,-y) \quad \forall (t,y) \in V_\beta + i\Gamma.$$

Schwinger two-point function for the thermal $P(\phi)_2$ model

Let $h \in C^\infty_{0\mathbb{R}}(\mathbb{R})$ and set

$$I(t,x) := \int_{\mathcal{S}'(S_\beta\times\mathbb{R})} \phi(0,h)\phi(t,\alpha_x h)\mathrm{d}\mu_\infty, \quad t \in S_\beta, x \in \mathbb{R},$$

where $\alpha_x h(y) = h(y-x)$. By [13, Thm. 7.2], we get:

$$
\begin{aligned}
I &= 2\int_{-\infty<x_1\leq x_2<\infty} h(x_1)\big(\Omega_C,\phi_C(\delta_0)\mathrm{e}^{-(x_2-x_1)H_C}\phi_C(\delta_t)\Omega_C\big)h(x_2-x)\mathrm{d}x_1\mathrm{d}x_2\\
&= 2\int_{-\infty<x_1\leq x_2<\infty} h(x_1-x)\big(\Omega_C,\phi_C(\delta_t)\mathrm{e}^{-(x_2-x_1)H_C}\phi_C(\delta_0)\Omega_C\big)h(x_2)\mathrm{d}x_1\mathrm{d}x_2.
\end{aligned}
$$

But $\big(\Omega_C,\phi_C(\delta_0)\mathrm{e}^{-(x_2-x_1)H_C}\phi_C(\delta_t)\Omega_C\big)$ is real by the FKN formula and hence

$$
\big(\Omega_C,\phi_C(\delta_0)\mathrm{e}^{-(x_2-x_1)H_C}\phi_C(\delta_t)\Omega_C\big)=\big(\Omega_C,\phi_C(\delta_t)\mathrm{e}^{-(x_2-x_1)H_C}\phi_C(\delta_0)\Omega_C\big),
$$

which yields:

$$
\begin{aligned}
I(t,x) = 2\int_{-\infty<x_1\leq x_2<\infty} &\big(h(x_1)r(t,x_2-x_1)h(x_2-x)\\
&+h(x_1-x)r(t,x_2-x_1)h(x_2)\big)\mathrm{d}x_1\mathrm{d}x_2,
\end{aligned}
\tag{17}
$$

for

$$
r(t,x)=\big(\Omega_C,\phi_C(\delta_0)\mathrm{e}^{-xH_C}\phi_C(\delta_t)\Omega_C\big)=\mathcal{W}_C(t,\mathrm{i}x),\quad x\geq 0.
$$

We have seen that $\mathcal{W}_C(t,y)=\mathcal{W}_C(t,-y)$ for $(t,y)\in V_\beta+\mathrm{i}\Gamma$, which when restricted to $t\in S_\beta$, $\Re y=0$, yields $\mathcal{W}_C(t,\mathrm{i}x)=\mathcal{W}_C(t,-\mathrm{i}x)$.

Therefore exchanging the variables x_1 and x_2 in the r.h.s. of (17), we obtain:

$$
2I(t,x)=\int_{\mathbb{R}^2} h(x_1)\mathcal{W}_C(t,x_2-x_1)h(x_2-x)\mathrm{d}x_1\mathrm{d}x_2 = h\star h^-\star\mathcal{W}_C(t,x), \tag{18}
$$

where $h^-(x):=h(-x)$.

Wightman two-point function for the thermal $P(\phi)_2$ model

Let Ω_β be the GNS vector for the thermal state ω_β, and let P_β and L_β be the generators of space-time translations such that $P_\beta\Omega_\beta=L_\beta\Omega_\beta=0$. We recall that $\varphi_\beta(h)$ for $h\in C^\infty_{0\mathbb{R}}(\mathbb{R})$ is the field operator in the GNS representation.

Let $\mathcal{W}_\beta(t,x)$ be the two-point function for the thermal $P(\phi)_2$ model, defined by

$$
\big(\varphi_\beta(h_1)\Omega_{\beta,}\mathrm{e}^{\mathrm{i}tL_\beta}\varphi_\beta(\alpha_x h_2)\Omega_\beta\big)=h_1\star h_2^-\star\mathcal{W}_\beta(t,x), \tag{19}
$$

for $h_i\in C^\infty_{0\mathbb{R}}(\mathbb{R})$. Since $\mathrm{d}\mu_\infty$ is the Euclidean measure for the thermal $P(\phi)_2$ model on the line, we have

$$
\mathcal{W}_\beta(\mathrm{i}t,x)=\mathcal{W}_C(t,\mathrm{i}x),\quad 0<t<\beta/2,\ x\in\mathbb{R}. \tag{20}
$$

As we have seen, $\mathcal{W}_C(t,y)$ is holomorphic in $V_\beta+\mathrm{i}\Gamma$. Thus we deduce from (20) that $\mathcal{W}_\beta(t,x)$ is holomorphic in $\Gamma+\mathrm{i}V_\beta$. Applying (19) with $h_1=h_2=:h$, we deduce from this fact that

$$
\varphi_\beta(h)\Omega_\beta\in\mathcal{D}(\mathrm{e}^{-(sL+yP)/2})\quad\forall(s,y)\in V_\beta.
$$

But by the spectral theorem this clearly implies that $\mathcal{W}_\beta(t,x)$ is holomorphic in $\mathbb{R}^2+\mathrm{i}V_\beta$. $\square$

5 Outlook

The condition of locality leads to strong constraints on the general form of the thermal two-point functions which allow one to apply the techniques of the Jost–Lehmann–Dyson representation. As has been shown by Bros and Buchholz [6][7], the interacting two-point function $\mathcal{W}_\beta$ can be represented in the form

$$\mathcal{W}_\beta(t,x) = \int_0^\infty \mathrm{d}m\, \mathcal{D}_\beta(x,m)\, \mathcal{W}_\beta^{(0)}(t,x,m).$$

Here $\mathcal{D}_\beta(x,m)$ is a distribution in x, m which is symmetric in x, and

$$\mathcal{W}_\beta^{(0)}(x,m) = (2\pi)^{-1} \int \mathrm{d}\nu \mathrm{d}p\, \varepsilon(\nu)\delta(\nu^2 - p^2 - m^2)(1 - \mathrm{e}^{-\beta\nu})^{-1}\mathrm{e}^{i(\nu t - px)}$$

is the two-point correlation function of the free field of mass m in a thermal equilibrium state at inverse temperature β. In contrast to the vacuum case, the damping factors $\mathcal{D}_\beta(x,m)$ depend in general in a non-trivial way on the spatial variables x. They describe the dissipative effects of the thermal background on the propagation of sharply localized excitations.

If the underlying equilibrium state satisfies the relativistic KMS condition, the function $D_\beta(x,m)$ is regular in x and admits an analytic continuation into the domain $\{z \in \mathbb{C} \mid |\Im z| < \beta/2\}$.

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