# Thermal Quantum Fields without Cut-offs in 1+1 Space-time Dimensions

Christian Gérard[*] and Christian D. Jäkel[†]

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**Abstract**

We construct interacting quantum fields in 1+1 dimensional Minkowski space, representing neutral scalar bosons at positive temperature. Our work is based on prior work by Klein and Landau and Høegh-Krohn.

# Contents

# 1  Introduction

Constructive thermal field theory allows one to circumvent (at least in 1+1 space-time dimensions) the severe infrared problems (see e.g. [St]) of thermal perturbation theory. A class of models representing scalar neutral bosons with polynomial interactions was constructed by Høegh-Krohn [H-K] more than twenty years ago. Shortly afterwards, several related results on the construction of self-interacting thermal fields were announced by Fröhlich [Fr2].

Our first paper was devoted to the construction of neutral and charged thermal fields with *spatially cutoff* interactions in 1+1 space–time dimensions, using the notion of *stochastically positive* KMS systems due to Klein and Landau [KL1].

The construction of interacting thermal quantum fields without cutoffs presented here includes several of the original ideas of Høegh-Krohn [H-K], but instead of starting from the interacting system in a box we start from the *Araki-Woods representation* for the free thermal system in infinite volume. This 'algebraic' approach eliminates some cumbersome limiting procedures present in Høegh-Krohn's work due to the introduction of boxes. We provide complete proofs for a number of statements which where only touched upon in Høegh-Krohn's work. The list of 'new' contributions contains the Wick (re-)ordering with respect to different covariance functions, the existence of interacting sharp-time fields, the identification of local algebras, the existence and uniqueness of the solution of Høegh-Krohn's time dependent heat-equation, local normality of the interacting KMS state, uniqueness of the weak* accumulation point of the sequence of approximating KMS states, and a number of inequalities that enter into a rigorous construction at several points. Although some of our results were probably already known by the experts (most of our work is based on results by Glimm and Jaffe, Høegh-Krohn, Fröhlich, Klein and Landau, and Simon) more than twenty years ago, we feel that it is worth while to present the arguments in full detail.

We will provide a detailed description of the content of this paper in the next subsection. But before we do so, we give a rough outline of the main ideas.

Let $\mathfrak{h}$ and $\epsilon$ denote the one-particle Hilbert space and the one-particle energy for a single neutral scalar boson. On the Weyl algebra $\mathcal{W}(\mathfrak{h})$ we define a quasi-free $(\tau^\circ, \beta)$-KMS state $\omega_\beta^\circ$ for the time evolution $\{\tau_t^\circ\}_{t \in \mathbb{R}}$ by

$$\omega_\beta^\circ\big(W(h)\big) := \mathrm{e}^{-\frac{1}{4}(h,(1+2\rho)h)}, \quad \tau_t^\circ\big(W(h)\big) = W(\mathrm{e}^{\mathrm{i}t\epsilon}h),\ h \in \mathfrak{h},\ t \in \mathbb{R},$$

where $\rho := (\mathrm{e}^{\beta\epsilon} - 1)^{-1}$, $\beta > 0$.

2

A convenient realization of the GNS representation associated to the pair $\big(\mathcal{W}(\mathfrak{h}), \omega_\beta^\circ\big)$ is the *Araki-Woods representation* defined by:

$$\mathcal{H}_{AW} := \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}}),$$

$$\Omega_{AW} := \Omega,$$

$$\pi_{AW}(W(h)) = W_{AW}(h) := W_{\mathrm{F}}\big((1+\rho)^{\frac{1}{2}} h \oplus \overline{\rho}^{\frac{1}{2}} \overline{h}\big), \quad h \in \mathfrak{h}.$$

Here $\overline{\mathfrak{h}}$ is the conjugate Hilbert space to $\mathfrak{h}$, $W_F(.)$ denotes the Fock Weyl operator on $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ and $\Omega \in \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ is the Fock vacuum. The von Neumann algebra generated by $\{\pi_{AW}(W(h)) \mid h \in \mathfrak{h}\}$ is denoted by $\mathcal{R}_{AW}$. The *local von Neumann algebra* generated by $\{\pi_{AW}(W(h)) \mid h \in \mathfrak{h}_I\}$ is denoted by $\mathcal{R}_{AW}(I)$. Here $I \subset \mathbb{R}$ is an open and bounded interval and $\mathfrak{h}_I$ will be defined in (6.4).

Since $\omega_\beta^\circ$ is $\tau^\circ$-invariant, there exists a standard implementation (see [DJP]) of the time evolution in the representation $\pi_{AW}$:

$$\mathrm{e}^{iL_{AW}t} \pi_{AW}(A)\Omega_{AW} := \pi_{AW}\big(\tau_t^\circ(A)\big)\Omega_{AW} \quad \text{and} \quad L_{AW}\Omega_{AW} = 0.$$

The generator $L_{AW}$ of the free time evolution is called the (free) Liouvillean.

*Euclidean techniques* were used in our first paper to define the operator sum

$$H_l := L_{AW} + \int_{-l}^{l} {:}P(\phi(0,x)){:}_{C_0} \, \mathrm{d}x$$

and to show that $H_l$ is essentially selfadjoint.

Using Trotter's product formula as in [GJ2], a finite propagation speed argument shows that

$$\tau_t^l(A) = \mathrm{e}^{iH_l t} A \mathrm{e}^{-iH_l t}$$

is independent of $l$ for $t \in \mathbb{R}$ and $A \in \mathcal{R}_{AW}(I)$ fixed, if $I$ is bounded and $l$ is sufficiently large. Thus there exists a limiting dynamics $\tau$ such that

$$(1.1) \qquad \lim_{l \to \infty} \big\| \tau_t^l(A) - \tau_t(A) \big\| = 0$$

for all $A \in \mathcal{R}_{AW}(I)$, $I$ bounded. This norm convergence extends to the *norm closure*

$$\mathcal{A} := \overline{\bigcup_{I \subset \mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)}$$

of the *local von Neumann algebras*. The $C^*$-algebra $\mathcal{A}$ is called the *algebra of local observables*.

It follows from general results of [KL1] that the vector $\Omega_l \in \mathcal{H}_{AW}$,

$$(1.2) \qquad \Omega_l := \frac{\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}}{\|\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}\|},$$

induces a $(\tau^l, \beta)$-KMS state $\omega_l$ for the $W^*$-dynamical system $(\mathcal{A}, \tau^l)$. Equation (1.2) should be compared with similar expressions which are well-known (see e.g. [BR, Theorem 5.4.4]) for bounded perturbations and which have recently been derived for a class of unbounded perturbations in [DJP, Theorem 5.6].

The existence of weak limit points (which are states) of the net $\{\omega_l\}_{l>0}$ is a consequence of the Banach-Alaoglu theorem (see [BR, Theorem 2.3.15]).

That fact that all limit states satisfy the *KMS condition* w.r.t. the pair $(\mathcal{A}, \tau)$ follows from (1.1), which itself is a consequence of finite propagation speed.

3

Since $\mathcal{A}$ is the norm closure of the *weakly closed* local algebras, all limit points are *locally normal* KMS states w.r.t. the Araki-Woods representation [TW].

To prove that there is only one accumulation point is more delicate. Following Høegh-Krohn [H-K] we will use *Nelson symmetry* to relate the interacting vacuum theory on the circle to the interacting thermal model on the real line.

## 1.1  Content of this paper

In Section 2 we recall the notions of *stochastically positive KMS systems* and associated *generalized path spaces*, due to Klein and Landau [KL1]. The property corresponding to stochastic positivity in the 0-temperature case is called *Nelson-Symanzik positivity*.

In Subsection 2.1 we recall the characterization of the thermal equilibrium states of a dynamical system $(\mathcal{B}, \tau)$ by the *KMS condition* and the definition of *Euclidean Green's functions*. The notion of a *stochastically positive KMS systems* $(\mathcal{B}, \mathcal{U}, \tau, \omega)$ rests on the introduction of a distinguished abelian sub-algebra $\mathcal{U}$ of the observable algebra $\mathcal{B}$. In our case this algebra will be the algebra generated by the time-zero fields.

In Subsection 2.2 we recall the notion of a *generalized path space* $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$. It consists of a probability space $(Q, \Sigma, \mu)$, a distinguished sub $\sigma$-algebra $\Sigma_0$, a one-parameter group $t \mapsto U(t)$ of automorphisms of $L^\infty(Q, \Sigma, \mu)$ such that $\Sigma = \bigvee_{t \in \mathbb{R}} U(t)\Sigma_0$ and a reflection $R$, acting as an automorphism on $L^\infty(Q, \Sigma, \mu)$ such that $R^2 = \mathbb{1}$, $RU(t) = U(-t)R$.

Klein and Landau (see [KL1]) have shown that for $\beta > 0$ there is a one to one correspondence between stochastically positive $\beta$-KMS systems and $\beta$-periodic OS-positive path spaces (for $\beta = \infty$ the object associated to an OS-positive path space is called a *positive semigroup structure*, see [K]). The role of OS-positivity is to ensure the positivity of the inner product in the Hilbert space $\mathcal{H}$ on which the real time quantum fields act. A similar reconstruction theorem allowing to go from Euclidean Green's functions to a KMS system was also shown in a slightly different framework in [GJO1, GJO2]. The case of Euclidean structures corresponding to quasi-free KMS states (which give rise to Gaussian path spaces) was considered in [KL2, GJO3].

The *reconstruction theorem* provides a concrete realization of the *GNS triple* $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated to the pair $(\mathcal{B}, \omega)$. The *Liouvillean $L$* implements the time evolution in the GNS representation $\pi_\omega$.

In Subsection 2.3 we recall some results from [KL1] (with some improvements in [GeJ]) concerning perturbations of generalized path spaces obtained from Feynman-Kac-Nelson kernels. The main examples of FKN kernels are those obtained from a selfadjoint operator $V$ on the physical Hilbert space $\mathcal{H}_\omega$, which is affiliated to $\mathcal{U} \cong L^\infty(K, \nu_\omega)$. If $\mathrm{e}^{-\beta V} \in L^1(K, \nu_\omega)$ and

$$V \in L^p(K, \nu_\omega), \quad \mathrm{e}^{-\frac{\beta}{2}V} \in L^q(K, \nu_\omega) \quad \text{for} \quad p^{-1} + q^{-1} = \frac{1}{2}, \quad 2 \le p, q \le \infty,$$

then the operator sum $L + V$ is essentially selfadjoint on $\mathcal{D}(L) \cap \mathcal{D}(V)$ and the perturbed time-evolution $\tau_V$ on $\mathcal{B}$ is given by

$$\tau_{V,t}(B) = \mathrm{e}^{\mathrm{i}t\overline{L+V}} B \mathrm{e}^{-\mathrm{i}t\overline{L+V}}.$$

The KMS state $\omega_V$ for the pair $(\mathcal{B}, \tau_V)$ is the vector state induced by

$$\Omega_V := \frac{\mathrm{e}^{-\frac{\beta}{2}\overline{L+V}}\Omega_\omega}{\|\mathrm{e}^{-\frac{\beta}{2}\overline{L+V}}\Omega_\omega\|}.$$

4

The *Liouvillean* $L_V$ for the perturbed $\beta$-KMS system $(\mathcal{B}_V, \tau_V, \omega_V)$ equals $\overline{L + V - JVJ}$. ($J$ denotes the modular conjugation associated to the pair $(\mathcal{B}, \Omega)$). It satisfies

$$\mathrm{e}^{itL_V} A \Omega_V = \tau_{V,t}(A) \Omega_V \text{ and } L_V \Omega_V = 0.$$

In Section 3 we recall some standard facts about Gaussian measures on distribution spaces and fix some notation. *Gaussian measures* are reviewed in Subsection 3.2. *Sharp-time free fields* are introduced in Subsection 3.3. If the space dimension $d$ is 1, then it is possible to define similarly *sharp-space free fields*. This is done in Subsection 3.4.

In Section 4 we recall two well known path spaces supported by $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \mathrm{d}\phi_C)$, where $S_\beta$ is the circle of length $\beta$. In Subsection 4.1 we identify the generalized path space on $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \mathrm{d}\phi_C)$ corresponding to the free massive scalar field on the circle $S_\beta$ at temperature 0.

In Subsection 4.2 we identify the generalized path space on $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \mathrm{d}\phi_C)$ corresponding to the free massive scalar field on the real line $\mathbb{R}$ at temperature $\beta^{-1}$. The physical Hilbert space associated to this path space can be unitarily identified with the Fock space $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The KMS vector $\Omega_{AW}$ is identified with the Fock vacuum vector $\Omega$ in $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The dynamics $\tau^\circ$ can be unitarily implemented in $\pi_{AW}$: The (free) Liouvillean $L_{AW}$ is identified with $\mathrm{d}\Gamma(\epsilon \oplus -\overline{\epsilon})$.

In Section 5 we describe perturbations of the two path spaces defined in Subsects. 4.1 and 4.2. The perturbed path spaces are obtained from FKN kernels corresponding to $P(\phi)_2$ interactions.

In Subsection 5.1 we recall some well known facts concerning the Wick ordering of Gaussian random variables. In 1+1 space-time dimensions Wick ordering is sufficient to eliminate the UV divergences of polynomial interactions. As it turns out, the leading order in the UV divergences is independent of the temperature. Thus it is a matter of convenience whether one uses the thermal covariance function $C_0$ or the vacuum covariance function $C_{\mathrm{vac}}$ to define the Wick ordering.

In Subsection 5.2 the $P(\phi)_2$ model on the circle $S_\beta$ at temperature 0 is discussed. It is specified by the formal interaction

$$V_{\mathsf{c}} = \int_{S_\beta} {:}P(\phi(t,0)){:}_{C_\beta} \, \mathrm{d}t.$$

Here $P(\lambda)$ is a real valued polynomial, which is bounded from below. The time-evolution $x \mapsto \mathrm{e}^{ixH_{\mathsf{c}}^{\mathrm{ren}}}$ is generated by $H_{\mathsf{c}}^{\mathrm{ren}} := H_{\mathsf{c}} - E_{\mathsf{c}}$, where $E_{\mathsf{c}} := \inf(\sigma(H_{\mathsf{c}}))$ and

$$H_{\mathsf{c}} = \overline{\mathrm{d}\Gamma\big((D_t^2 + m^2)^{\frac{1}{2}}\big) + V_{\mathsf{c}}}.$$

The operator $H_{\mathsf{c}}$ is bounded from below and has a unique vacuum state $\omega_{\mathsf{c}}(\,.\,) = (\Omega_{\mathsf{c}}, \,.\, \Omega_{\mathsf{c}})$ such that $(\Omega_{\mathsf{c}}, \Omega) > 0$ and $H_{\mathsf{c}}^{\mathrm{ren}} \Omega_{\mathsf{c}} = 0$. The renormalized energy operator $H_{\mathsf{c}}^{\mathrm{ren}}$ is called the $P(\phi)_2$ *Hamiltonian on the circle* $S_\beta$.

Some bounds are provided in Proposition 5.4, which are used in the sequel to prove the existence of interacting sharp-time fields.

The *spatially cutoff* $P(\phi)_2$ *model on the real line* $\mathbb{R}$ *at temperature* $\beta^{-1}$ is discussed in Subsection 5.3. It is specified by the formal interaction

$$V_l = \int_{-l}^{l} {:}P(\phi(0,x)){:}_{C_0} \, \mathrm{d}x.$$

Here $P(\lambda)$ is once again a real valued polynomial, which is bounded from below, and $l \in \mathbb{R}^+$ is a spatial cutoff parameter. The perturbed KMS state $\omega_l$ turns out to be normal w.r.t. the Araki-Woods representation $\pi_{AW}$. In fact, it is the vector state induced by

$$\Omega_l := \frac{\mathrm{e}^{-\frac{\beta}{2}H_l} \Omega_{AW}}{\|\mathrm{e}^{-\frac{\beta}{2}H_l} \Omega_{AW}\|},$$

5

where $H_l$ is the selfadjoint operator $H_l := \overline{L_{AW} + V_l}$. The perturbed time-evolution on $\mathcal{B}$ is given by

$$\tau_t^l(B) := \mathrm{e}^{\mathrm{i}tH_l} B \mathrm{e}^{-\mathrm{i}tH_l}, \quad B \in \mathcal{B}.$$

The following consequence of Lemma 5.3 will be important in Section 7:

$$(1.3) \qquad \mathrm{e}^{-\int_{-\beta/2}^{\beta/2} U(t) \int_{-l}^{l} :P(\phi(0,x)):_{C_0} \mathrm{d}x \mathrm{d}t} = \mathrm{e}^{-\int_{-l}^{l} U_\mathrm{c}(x) \int_{S_\beta} :P(\phi(t,0)):_{C_\beta} \mathrm{d}t \mathrm{d}x}.$$

The analog of (1.3) in the zero temperature case is called *Nelson symmetry* (see e.g. [Si]).

The thermodynamic limit is discussed in Section 6. We prove that the limits

$$\lim_{l \to +\infty} \tau_t^l(A) =: \tau_t(A) \quad \text{and} \quad \lim_{l \to +\infty} \omega_l(A) =: \omega_\beta(A)$$

exist for $A$ in the $C^*$-algebra of local observables $\mathcal{A}$ and that $(\mathcal{A}, \tau, \omega_\beta)$ is a $\beta$-KMS system, describing the *translation invariant $P(\phi)_2$ model at temperature $\beta^{-1}$*.

In Subsection 6.1 we recall localization properties of the classical solutions of the Klein-Gordon equation.

In Subsection 6.2 we introduce the net of local algebras $I \to \mathcal{R}_{AW}(I)$ for the free thermal field: for a bounded open interval $I \subset \mathbb{R}$, the symbol $\mathcal{R}_{AW}(I)$ denotes the von Neumann algebra generated by $\{W_{AW}(h) \mid h \in \mathfrak{h}_I\}$. By a result of Araki [Ar1], the local von Neumann algebras for the free thermal scalar field are regular from the inside and from the outside:

$$\bigcap_{J \supset \overline{I}} \mathcal{R}_{AW}(J) = \mathcal{R}_{AW}(I) = \bigvee_{\overline{J} \subset I} \mathcal{R}_{AW}(J).$$

Moreover, if $I$ is bounded, then the local algebra $\mathcal{R}_{AW}(I)$ is $*$-isomorphic to the unique hyper-finite factor of type $\mathrm{III}_1$.

In Subsection 6.3 the existence of the limiting dynamics is discussed. For $t \in \mathbb{R}$ fixed, the norm limit

$$\lim_{l \to \infty} \tau_t^l(B) =: \tau_t(B)$$

exists for all $B$ in

$$\mathcal{A} := \overline{\bigcup_{I \subset \mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)},$$

where the $I$'s are open and bounded. *Finite propagation speed* is used to show that $\tau_t^l(B)$, for $B \in \mathcal{R}_{AW}(I)$ and $|t| \le T$, is independent of $l$ for $l > |I| + T$. The proof uses *Trotter's product formula*, which requires that $L_{AW} + V_l$ is essentially self-adjoint on $\mathcal{D}(L_{AW}) \cap \mathcal{D}(V_l)$.

In order to apply the results of Section 7 to the $C^*$-algebra $\mathcal{A}$, it is necessary to identify the local von Neumann algebra $\mathcal{R}_{AW}(I)$ with the von Neumann algebra obtained by applying the interacting dynamics $\tau$ to the local abelian algebra of time-zero fields. This is done in Subsection 6.4: for $I \subset \mathbb{R}$ a bounded open interval, we denote by $\mathcal{U}_{AW}(I)$ the abelian von Neumann algebra generated by $\{W_{AW}(h) \mid h \in \mathfrak{h}_I, h \text{ real valued}\}$. We denote by $\mathcal{B}_\alpha(I)$ the von Neumann algebra generated by

$$\{\tau_t(A) \mid A \in \mathcal{U}_{AW}(I), |t| < \alpha\}.$$

We set $\mathcal{B}(I) := \bigcap_{\alpha > 0} \mathcal{B}_\alpha(I)$ and show that $\mathcal{B}(I) = \mathcal{R}_{AW}(I)$.

Taking the existence of the interacting path space (which we will construct in Section 7) for granted, we show that the net $\{\omega_l\}_{l > 0}$ has a unique accumulation point. This is done in Subsection 6.5, using the identification of algebras established in the previous subsection. Thus

$$\mathrm{w} - \lim_{l \to +\infty} \omega_l =: \omega_\beta \text{ exists on } \mathcal{A}.$$

6

The state $\omega_\beta$ is a $(\tau, \beta)$-KMS state on $\mathcal{A}$. It follows from a result of Takesaki and Winnink [TW] that $\omega_\beta$ is *locally normal*, i.e., if $I$ is an open and bounded interval, then $\omega_{\beta | \mathcal{R}_{AW}(I)}$ is normal w.r.t. the Araki-Woods representation; thus $\omega_{\beta | \mathcal{R}_{AW}(I)}$ is also normal with respect to the Fock representation. Moreover, $\omega_\beta$ is invariant under spatial translations and satisfies the *space-clustering property*:

$$\lim_{x \to \infty} \omega_\beta(A\alpha_x(B)) = \omega_\beta(A)\omega_\beta(B), \ A, B \in \mathcal{A}.$$

Finally, the main results of this paper, namely the explicit construction of the translation invariant $P(\phi)_2$ model at positive temperature is given in Section 7. Following ideas of Høegh-Krohn [H-K], *Nelson symmetry* is used to establish the existence of the model in the thermodynamic limit.

The first step is to construct the *interacting path space* supported by $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ describing the translation invariant $P(\phi)_2$ model at temperature $\beta^{-1}$.

Following Høegh-Krohn [H-K] we consider the operator $W_{[-\infty,\infty]}(f)$ solving the time-dependent heat equation

$$\frac{\mathrm{d}}{\mathrm{d}b} W_{[a,b]}(f) = W_{[a,b]}(f)\big(-H_{\mathsf{C}}^{\mathrm{ren}} + \mathrm{i}\phi(f_b)\big), \quad a \leq b,$$

where $f_b(\cdot) := f(\cdot, b) \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ for $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$. We show that for $f \in C^\infty_{0\,\mathbb{R}}(S_\beta \times \mathbb{R})$

$$\lim_{l \to +\infty} \int e^{\mathrm{i}\phi(f)} \mathrm{d}\mu_l = (\Omega_{\mathsf{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathsf{C}})$$

exists and that the map

$$\begin{array}{ccc} \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}) & \to & \mathbb{R}^+ \\ f & \mapsto & (\Omega_{\mathsf{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathsf{C}}) \end{array}$$

is the generating functional of a *Borel probability measure* $\mu$ on $(Q, \Sigma)$. The measure $\mu$ is invariant under space translations, time translations and time-reflection.

In Subsection 7.2 we prove the *existence of interacting sharp-time fields*. (Note that the necessary bounds (5.9) depend on the dimension of space-time.) This result allows us to equip the probability space $(Q, \Sigma, \mu)$ with an *OS-positive $\beta$-periodic path space structure*:

- $U(t)$ is the group of transformations generated by the time translations $\mathfrak{T}_s$ induced on $Q$ by the map $(t, x) \mapsto (t+s, x)$;

- $R$ is the transformation generated by the (euclidean) time reflection at $t = 0$;

- $\Sigma_0$ is the sub$-\sigma$-algebra of $\Sigma$ generated by the functions $\{\phi(0, h) \mid h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\}$.

In Subsection 7.3 some properties of the *associated interacting $\beta$-KMS system* $(\mathcal{B}, \mathcal{U}, \tilde{\tau}, \tilde{\omega})$ are discussed. We prove the *convergence of sharp-time Schwinger functions* and show that

$$\tilde{\omega}\big(\alpha_x(W_{AW}(h))\big) = \tilde{\omega}\big(W_{AW}(h)\big)$$

for all $x \in \mathbb{R}$ and

$$\lim_{x \to \infty} \tilde{\omega}\big(W_{AW}(h)\alpha_x(W_{AW}(g))\big) = \tilde{\omega}\big(W_{AW}(h)\big)\tilde{\omega}\big(W_{AW}(g)\big)$$

for $h, g \in C^\infty_{0\,\mathbb{R}}(\mathbb{R})$.

In Appendix A we discuss the abstract time-dependent heat equation

(1.4)
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} U(t, s) = -\big(H + \mathrm{i}R(t)\big)U(t, s), \ s \leq t, \\ U(s, s) = \mathbb{1}. \end{cases}$$

Here $H \geq 0$ is a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $R(t)$, $t \in \mathbb{R}$, is a family of closed operators with $\mathcal{D}(H^\gamma) \subset \mathcal{D}(R(t))$ for some $0 \leq \gamma < 1$. We show that there exists a unique solution $U(t,s)$ such that $U(s,s) = \mathbb{1}$ and

$$U(t,r)U(r,s) = U(t,s) \text{ for } s \leq r \leq t.$$

In Subsection A.2 we consider the dissipative case when $R(t)$ is selfadjoint for $t \in \mathbb{R}$. We establish an approximation of $U(t,s)$ by time-ordered products and prove some bounds on $U(t,s)$, which are used in the main text to show the existence of interacting sharp-time fields and the convergence of sharp-time Schwinger functions.

Finally we establish a lemma which is used in the main text to prove spatial clustering for the translation invariant $P(\phi)_2$ model at temperature $\beta^{-1}$.

# 2 Stochastically positive KMS systems and generalized path spaces

In this section we briefly recall the notions of *stochastically positive KMS systems* and associated *generalized path spaces*, due to Klein and Landau [KL1]. We will also need the corresponding notions at 0-temperature, which can be found in [K].

## 2.1 Stochastically positive KMS systems

Let $\mathcal{B}$ be a $C^*$-algebra and let $\{\tau_t\}_{t \in \mathbb{R}}$ be a one parameter group of $*$-automorphisms of $\mathcal{B}$. We recall that a state $\omega$ on $\mathcal{B}$ is a $(\tau, \beta)$-*KMS state* or $(\mathcal{B}, \tau, \omega)$ is a $\beta - KMS$ *system*, if for each pair $A, B \in \mathcal{B}$ there exists a function $F_{A,B}(z)$ holomorphic in the strip $I_\beta^+ = \{z \in \mathbb{C} \mid 0 < \mathrm{Im} z < \beta\}$ and continuous on $\overline{I_\beta^+}$ such that

$$F_{A,B}(t) = \omega(A\tau_t(B)) \quad \text{and} \quad F_{A,B}(t + \mathrm{i}\beta) = \omega(\tau_t(B)A) \quad \forall t \in \mathbb{R}.$$

For $A_i \in \mathcal{B}$ and $t_i \in \mathbb{R}$, $1 \leq i \leq n$, the *Green's functions* are defined as follows:

$$G(t_1, \ldots, t_n; A_1, \ldots, A_n) := \omega\big(\prod_{i=1}^n \tau_{t_i}(A_i)\big).$$

It is well known (see [Ar2, Ar3]) that the Green's functions are holomorphic in

$$I_\beta^{n+} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \mathrm{Im} z_i < \mathrm{Im} z_{i+1}, \ \mathrm{Im} z_n - \mathrm{Im} z_1 < \beta\},$$

continuous on $\overline{I_\beta^{n+}}$ and bounded there by $\prod_1^n \|A_i\|$. Therefore one can define the *Euclidean Green's functions*:

$${}^E G(s_1, \ldots, s_n; A_1, \ldots, A_n) := G(\mathrm{i}s_1, \ldots, \mathrm{i}s_n; A_1, \ldots, A_n) \text{ for } s_1 \leq \cdots \leq s_n, \ s_n - s_1 \leq \beta.$$

The following class of $\beta$-KMS systems has been introduced by Klein and Landau [KL1].

**Definition 2.1** *Let $(\mathcal{B}, \tau, \omega)$ be a $\beta$-KMS system and let $\mathcal{U} \subset \mathcal{B}$ be an abelian $^*$-sub-algebra. The KMS system $(\mathcal{B}, \mathcal{U}, \tau, \omega)$ is stochastically positive if*

  (i) *the $C^*$-algebra generated by $\bigcup_{t \in \mathbb{R}} \tau_t(\mathcal{U})$ is equal to $\mathcal{B}$;*
  (ii) *the Euclidean Green's functions $^E G(s_1, \ldots, s_n; A_1, \cdots, A_n)$ are positive for all $A_1, \ldots, A_n$ in $\mathcal{U}^+ = \{A \in \mathcal{U} \mid A \geq 0\}$.*

In applications it is more convenient to use a version of stochastic positivity, which is adapted to von Neumann algebras.

**Definition 2.2** *Let $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ be a weakly closed abelian sub-algebra of $\mathcal{B}$. Assume that the dynamics $\tau \colon \mathcal{B} \to \mathcal{B}$ is given by*

$$\tau_t(B) := \mathrm{e}^{\mathrm{i}tL} B \mathrm{e}^{-\mathrm{i}tL}, \ \ B \in \mathcal{B},$$

*where $L$ is a selfadjoint operator on $\mathcal{H}$. Moreover, assume that $\omega$ is a $\beta$-KMS state for the $W^*$-dynamical system $(\mathcal{B}, \tau)$. Then the KMS system $(\mathcal{B}, \mathcal{U}, \tau, \omega)$ is stochastically positive if*

  (i) *the von Neumann algebra generated by $\bigcup_{t \in \mathbb{R}} \tau_t(\mathcal{U})$ is equal to $\mathcal{B}$;*
  (ii) *the Euclidean Green's functions $^E G(s_1, \ldots, s_n; A_1, \cdots, A_n)$ are positive for all $A_1, \ldots, A_n$ in $\mathcal{U}^+ = \{A \in \mathcal{U} \mid A \geq 0\}$.*

## 2.2 Generalized path spaces

Stochastically positive $\beta$-KMS systems can be associated to *generalized path spaces* (see [KL1], [K]). Let us first recall some terminology.

If $\Xi_i$, for $i$ in an index set $I$, is a family of subsets of a set $Q$, we denote by $\bigvee_{i \in I} \Xi_i$ is the $\sigma$-algebra generated by all the $\Xi_i$, $i \in I$.

**Definition 2.3** *A generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ consists of*

  (i) *a probability space $(Q, \Sigma, \mu)$;*
  (ii) *a distinguished sub $\sigma$-algebra $\Sigma_0 \subset \Sigma$;*
  (iii) *a one-parameter group $\mathbb{R} \ni t \mapsto U(t)$ of measure preserving automorphisms of $L^\infty(Q, \Sigma, \mu)$, strongly continuous in measure, such that $\Sigma = \bigvee_{t \in \mathbb{R}} U(t)\Sigma_0$;*
  (iv) *a measure preserving automorphism $R$ of $L^\infty(Q, \Sigma, \mu)$ such that $R^2 = \mathbb{1}$, $RU(t) = U(-t)R$ and $RE_0 = E_0 R$, where $E_0$ is the conditional expectation with respect to $\Sigma_0$.*

A path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is said to be *supported* by the probability space $(Q, \Sigma, \mu)$.

It follows from (iii) and (iv) that $U(t)$ extends to a strongly continuous group of isometries of $L^p(Q, \Sigma, \mu)$ and $R$ extends to an isometry of $L^p(Q, \Sigma, \mu)$ for $1 \leq p < \infty$.

We say that the path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is *$\beta$-periodic* for $\beta > 0$ if $U(\beta) = \mathbb{1}$. On a $\beta$-periodic path space one can consider the one-parameter group $U(t)$ as being indexed by the circle $S_\beta = [-\beta/2, \beta/2]$.

For $I \subset \mathbb{R}$, we denote by $E_I$ the conditional expectation with respect to the $\sigma$-algebra $\Sigma_I := \bigvee_{t \in I} \Sigma_t$.

**Definition 2.4** (0-temperature case): *A generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is OS-positive, if $E_{[0,+\infty[} R E_{[0,+\infty[} \geq 0$ as an operator on $L^2(Q, \Sigma, \mu)$.*

(Positive temperature case): *A $\beta$-periodic path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is OS-positive, if $E_{[0,\beta/2]} R E_{[0,\beta/2]} \geq 0$ as an operator on $L^2(Q, \Sigma, \mu)$.*

For simplicity of notation we will consider $\beta$ as a parameter in $]0, +\infty]$, the case $\beta = +\infty$ corresponding to the 0-temperature case.

It is shown in [KL1] that for $\beta > 0$ there is a one to one correspondence between stochastically positive $\beta$-KMS systems and $\beta$-periodic OS-positive path spaces. For $\beta = \infty$ the object associated to an OS-positive path space is called a *positive semigroup structure* (see [K]).

Let us describe in more details one part of this correspondence, which is an example of a *reconstruction theorem*. Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be an OS-positive path space, $\beta$-periodic if $\beta < \infty$. We set
$$\mathcal{H}_{\mathrm{OS}} := L^2(Q, \Sigma_{[0,\beta/2]}, \mu).$$
Let $\mathcal{N} \subset \mathcal{H}_{\mathrm{OS}}$ be the kernel of the positive quadratic form
$$(\psi, \psi) := \int_Q \overline{\psi} R \psi \mathrm{d}\mu.$$

Then the *physical Hilbert space* is
$$\mathcal{H} := \text{ completion of } \mathcal{H}_{\mathrm{OS}}/\mathcal{N},$$

where the completion is done with respect to the positive definite scalar product $(.,.)$. Let us denote by $\mathcal{V}$ the canonical map $\mathcal{V}\colon \mathcal{H}_{\mathrm{OS}} \to \mathcal{H}_{\mathrm{OS}}/\mathcal{N}$. Then in $\mathcal{H}$ there is the *distinguished unit vector*
$$\Omega := \mathcal{V}1,$$

where $1 \in \mathcal{H}_{\mathrm{OS}}$ is the constant function equal to 1 on $Q$.

For $A \in L^\infty(Q, \Sigma_0, \mu)$ one defines $\tilde{A} \in \mathcal{B}(\mathcal{H})$ by

(2.1)
$$\tilde{A}\mathcal{V}\psi := \mathcal{V}A\psi.$$

(Note that multiplication by $A$ preserves $\mathcal{N}$, since $A$ is by assumption $\Sigma_0$ measurable). One denotes by $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ the abelian von Neumann algebra $\mathcal{U} := \{\tilde{A} \mid A \in L^\infty(Q, \Sigma_0, \mu)\}$. It is shown in [KL1, K] that the map $A \mapsto \tilde{A}$ is a weakly continuous *-isomorphism between $L^\infty(Q, \Sigma_0, \mu)$ and $\mathcal{U}$.

Finally, setting $\mathcal{M}_t = L^2(Q, \Sigma_{[0,\beta/2-t]}, \mu)$ for $0 \le t \le \beta/2$ and $\mathcal{D}_t = \mathcal{V}\mathcal{M}_t$, one can define $P(s)\colon \mathcal{D}_t \to \mathcal{H}$ for $0 \le s \le t$ by
$$P(s)\mathcal{V}\psi := \mathcal{V}U(s)\psi, \ \psi \in \mathcal{M}_t.$$

The triple $(P(t), \mathcal{D}_t, \beta/2)$ forms a *local symmetric semigroup* (see [Fr1, KL3]), and there exists a unique selfadjoint operator $L$ on $\mathcal{H}$ such that $P(s)u = \mathrm{e}^{-sL}u$ for $u \in \mathcal{D}_t$ and $0 \le s \le t$. The selfadjoint operator constructed in this way is said to be *associated* to the local symmetric semigroup $(P(t), \mathcal{D}_t, \beta/2)$.

Next one defines:

- $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ as the von Neumann algebra generated by $\{\mathrm{e}^{\mathrm{i}tL}A\mathrm{e}^{-\mathrm{i}tL} \mid t \in \mathbb{R}, A \in \mathcal{U}\}$;

- $\tau\colon t \mapsto \tau_t$ as the weakly continuous group of *-automorphisms of $\mathcal{B}$, which is given by
$$\tau_t(B) = \mathrm{e}^{\mathrm{i}tL}B\mathrm{e}^{-\mathrm{i}tL}$$

  for $t \in \mathbb{R}$ and $B \in \mathcal{B}$;

- $\omega$ as the vector state on $\mathcal{B}$ given by $\omega(B) = (\Omega, B\Omega)$ for $B \in \mathcal{B}$.

- the modular conjugation $J$ associated to the KMS system $(\mathcal{B}, \tau, \omega_\Omega)$ as the unique extension of
$$J\mathcal{V}\psi = \mathcal{V}(R_{\beta/4}\overline{\psi}), \quad \psi \in L^2(Q, \Sigma, \mu),$$

  where
$$R_{\beta/4} := U(\beta/4)RU(-\beta/4) = RU(-\beta/2) = U(\beta/2)R$$

  is the reflection at $t = \beta/4$ in $\mathcal{H}_{OS}$.

10

It is shown in [KL1] that $(\mathcal{B}, \mathcal{U}, \tau, \omega)$ is a stochastically positive $\beta$-KMS system. The relationship between the two objects is fixed by the following identity:

$$(2.2) \qquad {}^E G(s_1, \ldots, s_n; \tilde{A}_1, \ldots, \tilde{A}_n) = \int_Q \left(\prod_{i=1}^n U(s_i) A_i\right) d\mu$$

for $A_i \in L^\infty(Q, \Sigma_0, \mu)$, $1 \le i \le n$, and $s_1 \le \cdots \le s_n$, $s_n - s_1 \le \beta$.

## 2.3 Perturbations of generalized path spaces

We now describe perturbations of generalized path spaces obtained from a Feynman-Kac-Nelson kernel. Unless stated otherwise, we will consider the case $\beta < \infty$.

Let $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ be an OS-positive $\beta$-periodic path space. Let $V$ be a selfadjoint operator on $\mathcal{H}$, which is affiliated to $\mathcal{U}$. Using the isomorphism between $\mathcal{U}$ and $L^\infty(Q, \Sigma_0, \mu)$ we can view $V$ as a real $\Sigma_0$-measurable function on $Q$, which we still denote by $V$.

Assume that $V \in L^1(Q, \Sigma_0, \mu)$ and $\exp(-\beta V) \in L^1(Q, \Sigma_0, \mu)$. Then (see [KL1] or [GeJ, Proposition 6.2]) the function $F := \exp\left(-\int_{-\beta/2}^{\beta/2} U(t) V dt\right)$ belongs to $L^1(Q, \Sigma, \mu)$. One can hence define the perturbed measure $d\mu_V := \left(\int_Q F d\mu\right)^{-1} F d\mu$. The perturbed path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_V)$ is OS-positive and $\beta$-periodic (see [KL1]). Hence we can associate to this perturbed path space a stochastically positive $\beta$-KMS system $(\mathcal{B}_V, \mathcal{U}_V, \tau_V, \omega_V)$.

The following concrete realization of the perturbed $\beta$-KMS system $(\mathcal{B}_V, \mathcal{U}_V, \tau_V, \omega_V)$ has been obtained in [KL1] (with some improvements in [GeJ]):

- the physical Hilbert space $\mathcal{H}_V$ obtained from the reconstruction theorem outlined in the previous subsection is equal to the physical Hilbert space $\mathcal{H}$ of the unperturbed $\beta$-KMS system $(\mathcal{B}, \mathcal{U}, \tau, \omega)$;

- the von Neumann algebra $\mathcal{B}_V$ and the abelian algebra $\mathcal{U}_V$ are equal to $\mathcal{B}$ and $\mathcal{U}$, respectively;

- the operator sum $L + V$ is essentially selfadjoint on $\mathcal{D}(L) \cap \mathcal{D}(V)$ and if $H_V := \overline{L + V}$, then the perturbed time-evolution $\tau_V$ on $\mathcal{B}$ is given by

$$\tau_{V,t}(B) = e^{itH_V} B e^{-itH_V}, \quad B \in \mathcal{B};$$

- the distinguished vector $\Omega$ of the unperturbed KMS system belongs to $\mathcal{D}\left(e^{-\frac{\beta}{2} H_V}\right)$ and the perturbed KMS state $\omega_V$ is given by $\omega_V(B) = (\Omega_V, B\Omega_V)$, where

$$\Omega_V := \frac{e^{-\frac{\beta}{2} H_V} \Omega}{\|e^{-\frac{\beta}{2} H_V} \Omega\|}.$$

The following result is shown in [GeJ, Theorem 6.12]: If $e^{-\beta V} \in L^1(Q, \Sigma_0, \mu)$ and

$$V \in L^p(Q, \Sigma_0, \mu), \quad e^{-\frac{\beta}{2} V} \in L^q(Q, \Sigma_0, \mu) \quad \text{for} \quad p^{-1} + q^{-1} = \frac{1}{2}, \quad 2 \le p, q \le \infty,$$

then the operator sum $H_V - JVJ$ is essentially selfadjoint and the *Liouvillean* $L_V$ (for a general definition of Liouvilleans see, e.g., [DJP]) for the perturbed $\beta$-KMS system $(\mathcal{B}_V, \tau_V, \omega_V)$ is equal to $\overline{H_V - JVJ}$. Here $J$ denotes the modular conjugation associated to the pair $(\mathcal{B}, \Omega)$.

## 2.4 Perturbed dynamics associated to FKN kernels

Let us describe in more details the construction of $H_V = \overline{L + V}$ given in [KL1] which is based on the Feynman-Kac-Nelson formula. Note that the results of this subsection are also

valid in the 0-temperature case $\beta = +\infty$. Let $V$ be a real $\Sigma_0$-measurable function such that $V \in L^1(Q, \Sigma_0, \mu)$ and $\mathrm{e}^{-TV} \in L^1(Q, \Sigma_0, \mu)$ for some $T > 0$ if $\beta = \infty$ and for $T = \beta$ if $\beta < \infty$. Set

$$F_{[0,s]} := \mathrm{e}^{-\int_0^s U(t)V \mathrm{d}t}, \ \ 0 \leq s \leq \inf(T, \beta)/2,$$

which belongs to $L^2(Q, \Sigma_{[0,\inf(T,\beta)/2]}, \mu)$. The family $\{F_{[0,s]}\}_{0 \leq s \leq \inf(T,\beta)/2}$ is called a *Feynman-Kac-Nelson kernel*.

For $0 \leq t \leq \inf(T, \beta)/2$ we set

$$\mathcal{M}_t := \text{ linear span of } \bigcup_{0 \leq s \leq \inf(T,\beta)/2 - t} F_{[0,s]} L^\infty(Q, \Sigma_{[0,\inf(T,\beta)/2-t]}, \mu)$$

and

$$U_V(s): \ \ \begin{matrix} \mathcal{M}_t & \to & L^2(Q, \Sigma_{[0,\inf(T,\beta)/2]}, \mu) \\ \psi & \mapsto & F_{[0,s]} U(s) \psi, \end{matrix} \ \ 0 \leq s \leq t.$$

Setting finally
(2.3)
$$\mathcal{D}_t = \mathcal{V}(\mathcal{M}_t),$$

one can show that

$$P_V(s): \ \ \begin{matrix} \mathcal{D}_t & \to & \mathcal{H} \\ \mathcal{V}(\psi) & \mapsto & \mathcal{V}(F_{[0,s]} U(s) \psi) \end{matrix}$$

is a well defined linear operator, and that $\big(P_V(t), \mathcal{D}_t, \inf(T, \beta)/2\big)$ is a local symmetric semigroup on $\mathcal{H}$. Now let $H_V$ be the unique selfadjoint operator associated to the local symmetric semigroup $(\mathcal{D}_t, P_V(t), \inf(T, \beta)/2)$. It follows (see [KL1]) that $H_V = \overline{L + V}$.

In the sequel we will need the following result.

**Proposition 2.5** *Let $V \in L^2(Q, \Sigma_0, \mu)$ be a real function such that $\mathrm{e}^{-TV} \in L^1(Q, \Sigma_0, \mu)$ for some $T > 0$ and $V_n := V \mathbb{1}_{\{|V| \leq n\}}$ for $n \in \mathbb{N}$. Denote by $L$ the selfadjoint operator on $\mathcal{H}$ associated to the OS-positive path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$. Let $H_n$ be the closure of $L + V - V_n$. Then*

$$\mathrm{e}^{-\mathrm{i}tL} = \text{s-} \lim_{n \to \infty} \mathrm{e}^{-\mathrm{i}tH_n}, \ \ t \in \mathbb{R}.$$

Note that the selfadjoint operators $H_n$ are associated to local symmetric semigroups $\big(P_n(t), \mathcal{D}_t^{(n)}, T/2\big)$ obtained from the FKN kernels

$$F_{[0,s]}^{(n)} := \mathrm{e}^{-\int_0^s U(t)(V-V_n)\mathrm{d}t},$$

and the operator $L$ is associated to the local symmetric semigroup $\big(P_\infty(t), \mathcal{D}_t, T/2\big)$ obtained from the FKN kernels $F_{[0,s]}^{(\infty)} = 1$.
**Proof.** We first claim that

(2.4)
$$\sup_{0 \leq s \leq T/2} \|F_{[0,s]}^{(n)} - 1\|_{L^1(Q, \Sigma, \mu)} \to 0 \text{ for } n \to \infty.$$

In order to prove (2.4), we recall the following bound from [KL4, Theorem 6.2 (i)]:

(2.5)
$$\|\mathrm{e}^{-\int_a^b U(t)V \mathrm{d}t}\|_{L^p(Q, \Sigma, \mu)} \leq \|\mathrm{e}^{-(b-a)V}\|_{L^p(Q, \Sigma, \mu)}, \ \ 1 \leq p < \infty.$$

Now let $W$ be a real measurable function on $Q$. Using $1 - \mathrm{e}^{-a} = a \int_0^1 \mathrm{e}^{-\theta a} \mathrm{d}\theta$ we find

$$1 - \mathrm{e}^{-\int_0^s U(t)W \mathrm{d}t} = \int_0^s U(t)W \mathrm{d}t \int_0^1 \mathrm{e}^{-\theta \int_0^s U(t)W \mathrm{d}t} \mathrm{d}\theta.$$

12

This yields

$$
\begin{aligned}
\big\|1 - \mathrm{e}^{-\int_0^s U(t)W\mathrm{d}t}\big\|_{L^1} &\leq |s|\,\|W\|_{L^2} \int_0^1 \big\|\mathrm{e}^{-\theta \int_0^s U(t)W\mathrm{d}t}\big\|_{L^2}\mathrm{d}\theta \\
&\leq |s|\,\|W\|_{L^2} \int_0^1 \big\|\mathrm{e}^{-\theta s W}\big\|_{L^2}\mathrm{d}\theta \\
&\leq |s|\,\|W\|_{L^2}\big(1 + \int_0^1 \big\|\mathrm{e}^{\theta s W_-}\big\|_{L^2}\mathrm{d}\theta\big) \\
&\leq \tfrac{T}{2}\|W\|_{L^2}\big(1 + \|\mathrm{e}^{TW_-}\|_{L^1}\big),
\end{aligned}
$$

where $W_- = \sup(0, -W)$ denotes the negative part of $W$. In the first line we have used the Cauchy-Schwarz inequality and the fact that $U(t)$ is unitary on $L^2(Q, \Sigma, \mu)$, in the second line the estimate (2.5).

By assumption $V \in L^2(Q, \Sigma, \mu)$ and $\mathrm{e}^{-TV} \in L^1(Q, \Sigma, \mu)$. Thus $V - V_n \to 0$ in $L^2(Q, \Sigma, \mu)$ and $\mathrm{e}^{T(V-V_n)_-} \to 0$ in $L^1(Q, \Sigma, \mu)$. Applying the above bound for $W = V - V_n$, we obtain (2.4).

Before we finish the proof, we extract a Lemma.

**Lemma 2.6** *Let $\big(P_n(t), \mathcal{D}_t^{(n)}, T\big)$ for $n \in \mathbb{N} \cup \{\infty\}$ be a family of local symmetric semigroups on a Hilbert space $\mathcal{H}$. Let $H_n$, $n \in \mathbb{N} \cup \{\infty\}$, denote the associated selfadjoint operators.*

*Assume that there exists a family $\{\mathcal{L}_t\}$ for $0 < t \leq T' \leq T$ of subspaces of $\mathcal{H}$ with*

$$
(2.6) \qquad \mathcal{L}_t \subset \mathcal{D}_t^{(n)}, \quad \bigcup_{0 < t \leq T'} \mathcal{L}_t \ \text{dense in } \mathcal{H}.
$$

*Assume moreover that*

$$
(2.7) \qquad \lim_{n \to \infty} (\Psi, P_n(s)\Psi) = (\Psi, P_\infty(s)\Psi), \ \Psi \in \mathcal{L}_t, \ 0 \leq s \leq t \leq T',
$$

$$
(2.8) \qquad \sup_n \sup_{0 \leq s \leq t} (\Psi, P_n(s)\Psi) < \infty, \ \Psi \in \mathcal{L}_t, \ 0 \leq t \leq T'.
$$

*Then* $\ \mathrm{s\text{-}}\lim_{n \to \infty} \mathrm{e}^{-\mathrm{i}t H_n} = \mathrm{e}^{-\mathrm{i}t H_\infty} \ $ *for all $t \in \mathbb{R}$.*

**Proof.** Let us fix $0 < t \leq T'$ and $\Psi \in \mathcal{L}_t$. From [KL3, Lemma 1], we know that there exist positive measures $\{\nu_n\}$ on $\mathbb{R}$ such that

$$
(\Psi, P_n(s)\Psi) = \Big\|P_n\Big(\tfrac{s}{2}\Big)\Psi\Big\|^2 = \int_{\mathbb{R}} \mathrm{e}^{-sa}\mathrm{d}\nu_n(a), \ \ 0 \leq s \leq t.
$$

Moreover, one has (see [KL3, Lemma 1])

$$
(\Psi, \mathrm{e}^{-\mathrm{i}y H_n}\Psi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ya}\mathrm{d}\nu_n(a).
$$

Set

$$
f_n(z) := \int_{\mathbb{R}} \mathrm{e}^{-za}\mathrm{d}\nu_n(a), \ \ z \in\, ]0, t[ + \mathrm{i}\mathbb{R}.
$$

The family $\{f_n\}$ is uniformly bounded on $]0, t[ + \mathrm{i}\mathbb{R}$ by (2.8) and converges pointwise to $f_\infty$ on $]0, T[$ by (2.7). Applying Lemma B.3 we conclude that $f_n(z)$ converges to $f_\infty(z)$ for all $z \in \mathrm{i}\mathbb{R}$. This implies that on $\mathcal{L}_t$

$$
\mathrm{w} - \lim_{n \to \infty} \mathrm{e}^{-\mathrm{i}y H_n} = \mathrm{e}^{-\mathrm{i}y H_\infty} \ \ \forall y \in \mathbb{R}.
$$

Since by hypothesis $\bigcup_{0 < t \leq T} \mathcal{L}_t$ is dense in $\mathcal{H}$ and for unitary operators weak convergence implies strong convergence, this completes the proof of the lemma $\square$.

**Proof of Proposition 2.5 (second part).** Let us now fix a convenient family of sub-spaces $\mathcal{L}_t$. For $0 < t < T/4$ we set $\mathcal{L}_t = \mathcal{V}\mathcal{R}_t$, where $\mathcal{R}_t$ equals $L^\infty(Q, \Sigma_{[0,\beta/2-t]}, \mu)$ if $\beta < \infty$ and $\mathcal{R}_t$ equals $L^\infty(Q, \Sigma_{[0,+\infty[}, \mu)$ if $\beta = +\infty$. Clearly $\mathcal{L}_t$ is included in the spaces $\mathcal{D}_t^{(n)}$ defined in (2.3) (with $V$ replaced by $V - V_n$). Moreover, $\bigcup_{0<t\leq T/4} \mathcal{L}_t$ is dense in $\mathcal{H}$, hence hypothesis (2.6) of Lemma 2.6 is satisfied. Let us now fix some $\Psi \in \mathcal{L}_t$, i.e., $\Psi = \mathcal{V}\psi$ for some $\psi \in \mathcal{R}_t$. Using (2.4) we obtain that

$$\lim_{n\to\infty} (\Psi, P_n(s)\Psi) = (\Psi, P_\infty(s)\Psi) \text{ for } 0 \leq s \leq t$$

and $\sup_n \sup_{0\leq s\leq t}(\Psi, P_n(s)\Psi) < \infty$. Hence hypotheses (2.7) and (2.8) of Lemma 2.6 are satisfied. Thus we can apply Lemma 2.6 and this completes the proof of Proposition 2.5 $\square$.

# 3 Gaussian measures

In this Section we recall some standard facts about Gaussian measures on distribution spaces.

## 3.1 Distribution spaces

Let $S_\beta = [-\beta/2, \beta/2]$ (with end points identified) be the circle of length $\beta > 0$. Points in $S_\beta \times \mathbb{R}^d$, $d \geq 1$, will be denoted by $(t, x)$.

The Fréchet space of Schwartz functions on $\mathbb{R}^d$ will be denoted by $\mathcal{S}(\mathbb{R}^d)$. For coherence of notation, the Fréchet space $\mathcal{D}(S_\beta)$ of smooth periodic functions on $S_\beta$ will also be denoted by $\mathcal{S}(S_\beta)$.

In addition, we denote by $\mathcal{S}(S_\beta \times \mathbb{R}^d)$ the Fréchet space of Schwartz functions on $S_\beta \times \mathbb{R}$, i.e., the space of smooth functions on $S_\beta \times \mathbb{R}^d$, which are $\beta$-periodic in $t$ and such that for all $p \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$

$$\left|(1+|x|)^{|\alpha|}\partial_t^p \partial_x^\alpha f(t,x)\right| \leq C_{p,\alpha}.$$

We will denote by $\mathcal{S}'(\mathbb{R}^d)$, $\mathcal{S}'(S_\beta)$ and $\mathcal{S}'(S_\beta \times \mathbb{R}^d)$ the duals of $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}(S_\beta)$ and $\mathcal{S}(S_\beta \times \mathbb{R}^d)$. The spaces of real elements in these spaces will be denoted by $\mathcal{S}'_{\mathbb{R}}(\mathbb{R}^d)$, $\mathcal{S}'_{\mathbb{R}}(S_\beta)$ and $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}^d)$.

We set $D_t = \mathrm{i}^{-1}\partial_t$ and $D_x = \mathrm{i}^{-1}\partial_x$, and we will denote by $D_t^2$ the selfadjoint operator on $L^2(S_\beta)$ defined by

$$D_t^2 := -\partial_t^2, \quad \mathcal{D}(D_t^2) := \left\{u \in L^2(S_\beta) \mid \partial_t^2 u \in L^2(S_\beta),\, u(0) = u(\beta)\right\}.$$

We denote by $D_t^2 + D_x^2$ the selfadjoint operator on $L^2(S_\beta \times \mathbb{R}^d)$ with domain

$$\mathcal{D}(D_t^2 + D_x^2) := \left\{u \in L^2(S_\beta \times \mathbb{R}^d) \mid (D_t^2 + D_x^2)u \in L^2(S_\beta \times \mathbb{R}^d),\, u \text{ is } \beta\text{-periodic in } t\right\}.$$

We denote by $\mathcal{S}(\mathbb{Z} \times \mathbb{R}^d)$ the Fréchet space of sequences $\{u_n\}_{n\in\mathbb{N}}$ with values in $\mathcal{S}(\mathbb{R}^d)$ such that

$$\sum_{n\in\mathbb{Z}} |n|^p \|(D_x^2 + x^2)^{p/2}u_n\|_{L^2(\mathbb{R}^d)} < \infty \quad \forall p \in \mathbb{N}.$$

We now fix the notation concerning partial Fourier transforms. We first define the (unitary) partial Fourier transform with respect to $t$:

$$\mathcal{F}_t: \begin{array}{ccc} \mathcal{S}(S_\beta \times \mathbb{R}^d) & \to & \mathcal{S}(\mathbb{Z} \times \mathbb{R}^d) \\ u & \mapsto & \{\hat{u}_n\} \end{array},$$

where $\hat{u}_n(x) = \beta^{-\frac{1}{2}} \int_{S_\beta} e^{-i\nu_n t} u(t,x) dt$. (The coefficients $\nu_n = 2\pi n/\beta$, $n \in \mathbb{N}$, are called in physics *Matsubara frequencies*). Its inverse is

$$u(t,x) = \beta^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} e^{i\nu_n t} \hat{u}_n(x).$$

The (unitary) partial Fourier transform with respect to $x$ is

$$\mathcal{F}_x: \quad \begin{matrix} \mathcal{S}(S_\beta \times \mathbb{R}^d) & \to & \mathcal{S}(S_\beta \times \mathbb{R}^d) \\ u & \mapsto & \hat{u} \end{matrix} \quad,$$

where $\hat{u}(t,p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot p} u(t,x) dx$. Its inverse is

$$u(t,x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot p} \hat{u}(t,p) dp.$$

For later use we fix two approximations of the Dirac $\delta$ functions in $t$ and $x$. We set, for $k \in \mathbb{N}$,

$$\delta_k(s) := \beta^{-1} \sum_{|n| \leq k} e^{i\nu_n s} \quad \text{and} \quad \delta_k(x) := k\chi(kx),$$

where $\chi$ is a function in $C_{0\,\mathbb{R}}^\infty(\mathbb{R}^d)$ with $\int \chi(x) dx = 1$.

## 3.2   Gaussian measures

We set
(3.1)  $$C(f,g) = \big(f, (D_t^2 + D_x^2 + m^2)^{-1} g\big), \quad f, g \in \mathcal{S}(S_\beta \times \mathbb{R}^d),$$

where $(.,.)$ is the scalar product on $L^2(S_\beta \times \mathbb{R}^d)$.

Let $Q := \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}^d)$ and let $\Sigma$ be the Borel $\sigma$-algebra on $Q$. If $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}^d)$, then $\phi(f)$ denotes the coordinate function

$$\phi(f): \quad \begin{matrix} Q & \to & \mathbb{C} \\ q & \mapsto & \langle q, f \rangle \end{matrix}.$$

Let $F$ be a Borel function on $\mathbb{R}$. Then $F(\phi(f))$ denotes the function

$$F(\phi(f)): \quad \begin{matrix} Q & \to & \mathbb{C} \\ q & \mapsto & F(\langle q, f \rangle) \end{matrix}.$$

We denote by $d\phi_C$ the Gaussian measure on $(Q, \Sigma)$ with covariance $C$ defined by

(3.2)  $$\int_Q e^{i\phi(f)} d\phi_C = e^{-C(f,f)/2}, \quad f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}^d).$$

We have
(3.3)  $$\int_Q \phi(f)^p d\phi_C = \begin{cases} 0, & p \text{ odd}, \\ (p-1)!! C(f,f)^{p/2}, & p \text{ even}, \end{cases}$$

where $n!! = n(n-2)(n-4)\cdots 1$. One easily deduces from (3.3) that $e^{\phi(f)} \in L^1(Q, \Sigma, d\phi_C)$ if $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}^d)$.

The cylindrical functions $F\big(\phi(f_1), \ldots, \phi(f_n)\big)$, $f_i \in \mathcal{S}_{\mathbb{R}}(\mathbb{R} \times S_\beta)$, $F$ a Borel function on $\mathbb{R}^n$ and $n \in \mathbb{N}$, are dense in $L^p(Q, \Sigma, d\phi_C)$ for $1 \leq p < \infty$.

15

## 3.3 Sharp-time fields

We now recall some standard results about the existence of *sharp-time fields*. We will make use of the following well known identity (see [KL2]):

$$(3.4) \qquad \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \frac{e^{i\nu_n t}}{\nu_n^2 + \epsilon^2} = \frac{e^{-|t|\epsilon} + e^{-(\beta - |t|)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})} \quad \text{for } \epsilon > 0, \ \nu_n = \frac{2\pi n}{\beta}, \ 0 \le |t| \le \beta.$$

For $h_1, h_2 \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^d)$, $0 \le t_1, t_2 \le \beta$, and $k \in \mathbb{N}$

$$C\big(\delta_k(. - t_1) \otimes h_1, \delta_k(. - t_2) \otimes h_2\big)$$
$$= \quad \beta^{-1} \sum_{|n| \le k} e^{i\nu_n(t_1 - t_2)} \big(\hat{h}_{1n}, (\nu_n^2 + D_x^2 + m^2)^{-1}\hat{h}_{2n}\big)_{L^2(\mathbb{R}^d)}.$$

Using (3.4) we see that

$$\lim_{k \to \infty} C\big(\delta_k(. - t_1) \otimes h_1, \delta_k(. - t_2) \otimes h_2\big) = \Big(h_1, \frac{e^{-|t_2 - t_1|\epsilon} + e^{-(\beta - |t_2 - t_1|)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})} h_2\Big)_{L^2(\mathbb{R}^d)},$$

where $\epsilon := (D_x^2 + m^2)^{\frac{1}{2}}$.

Using (3.3) this implies that, for $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^d)$ and $t \in S_\beta$ fixed, the sequence of functions $\{\phi(\delta_k(. - t) \otimes h)\}_{k \in \mathbb{N}}$ is Cauchy in $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, d\phi_C)$.

We set

$$(3.5) \qquad \phi(t, h) := \lim_{k \to \infty} \phi(\delta_k(. - t) \otimes h)$$

and

$$(3.6) \qquad C_0(t_1, h_1, t_2, h_2) := \Big(h_1, \frac{e^{-|t_2 - t_1|\epsilon} + e^{-(\beta - |t_2 - t_1|)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})} h_2\Big)_{L^2(\mathbb{R}^d)}.$$

We note that $\phi(t, h)$ belongs to $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, d\phi_C)$. For later use we define the *temperature $\beta^{-1}$ covariance on $\mathbb{R}^d$*:

$$(3.7) \qquad C_0(h_1, h_2) := \Big(h_1, \frac{(1 + e^{-\beta\epsilon})}{2\epsilon(1 - e^{-\beta\epsilon})} h_2\Big)_{L^2(\mathbb{R}^d)}, \quad h_1, h_2 \in \mathcal{S}(\mathbb{R}^d).$$

## 3.4 Sharp-space fields

If $d = 1$, then it is possible to define similarly *sharp-space fields*. We first recall another well-known identity, which is analogous to (3.4):

$$(3.8) \qquad (2\pi)^{-1} \int_{\mathbb{R}} \frac{e^{ipx}}{p^2 + b^2} dp = \frac{e^{-b|x|}}{2b} \quad \text{for } b > 0, \ x \in \mathbb{R}.$$

For $g_1, g_2 \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ and $x_1, x_2 \in \mathbb{R}$ one has

$$(3.9) \qquad \begin{aligned} & C\big(g_1 \otimes \delta_k(. - x_1), g_2 \otimes \delta_k(. - x_2)\big) \\ & = \int_{\mathbb{R}} \hat{\chi}^2(\tfrac{p}{k}) e^{ip(x_1 - x_2)} \big(g_1, (D_t^2 + p^2 + m^2)^{-1} g_2\big)_{L^2(S_\beta)} dp. \end{aligned}$$

Using (3.8) and $\hat{\chi}(0) = (2\pi)^{-\frac{1}{2}}$ we find

$$(3.10) \qquad \lim_{k \to \infty} C\big(g_1 \otimes \delta_k(. - x_1), g_2 \otimes \delta_k(. - x_2)\big) = \Big(g_1, \frac{e^{-|x_1 - x_2|b}}{2b} g_2\Big)_{L^2(S_\beta)},$$

where $b := (D_t^2 + m^2)^{\frac{1}{2}}$. Now we can use (3.3) again: for $g \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ and $x \in \mathbb{R}$ fixed, the sequence of functions $\{\phi\big(g \otimes \delta_k(. - x)\big)\}_{k \in \mathbb{N}}$ is Cauchy in $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, d\phi_C)$.

We set

$$\phi(g, x) := \lim_{k \to \infty} \phi\big(g \otimes \delta_k(. - x)\big)$$

and

(3.11) $$C_\beta(g_1, x_1, g_2, x_2) := \Big(g_1, \frac{\mathrm{e}^{-|x_1 - x_2|b}}{2b} g_2\Big)_{L^2(S_\beta)}.$$

We note that $\phi(g, x)$ belongs to $\bigcap_{1 \le k < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$. For later use we define the 0-temperature covariance on $S_\beta$:

(3.12) $$C_\beta(g_1, g_2) := \Big(g_1, \frac{1}{2b} g_2\Big)_{L^2(S_\beta)}, \quad g_1, g_2 \in \mathcal{S}(S_\beta).$$

## 3.5 Some elementary properties

From (3.3), (3.6) and (3.11) we deduce that the maps

(3.13) $$\begin{array}{rcl} H_{\mathrm{IR}}^{-1}(S_\beta \times \mathrm{IR}^d) & \to & \bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C) \\ f & \mapsto & \phi(f) \end{array},$$

(3.14) $$\begin{array}{rcl} S_\beta \times H_{\mathrm{IR}}^{-\frac{1}{2}}(\mathrm{IR}^d) & \to & \bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C) \\ (t, h) & \mapsto & \phi(t, h) \end{array}$$

and

(3.15) $$\begin{array}{rcl} H_{\mathrm{IR}}^{-\frac{1}{2}}(S_\beta) \times \mathrm{IR} & \to & \bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C) \\ (g, x) & \mapsto & \phi(g, x) \end{array}$$

are continuous.

For $f \in \mathcal{S}_{\mathrm{IR}}(S_\beta \times \mathrm{IR})$, $t \in S_\beta$ and $x \in \mathrm{IR}$ we set

$$\begin{array}{rccc} f_t: & \mathrm{IR} & \to & \mathbb{C} \\ & x & \mapsto & f(t, x) \end{array}, \qquad \begin{array}{rccc} f_x: & S_\beta & \to & \mathbb{C} \\ & t & \mapsto & f(t, x) \end{array}.$$

We note that $f_t \in \mathcal{S}_{\mathrm{IR}}(\mathrm{IR})$ and $f_x \in \mathcal{S}_{\mathrm{IR}}(S_\beta)$.

**Lemma 3.1** *If $f \in \mathcal{S}_{\mathrm{IR}}(S_\beta \times \mathrm{IR})$, then the following identity holds on $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\int_{\mathrm{IR}} \phi(f_x, x)\mathrm{d}x = \int_{S_\beta} \phi(t, f_t)\mathrm{d}t = \phi(f).$$

**Proof.** Let $f \in \mathcal{S}_{\mathrm{IR}}(S_\beta \times \mathrm{IR})$ and $k \in \mathrm{IN}$. The map

$$\begin{array}{rcl} \mathrm{IR} & \to & H^{-1}(S_\beta \times \mathrm{IR}) \\ x & \mapsto & f_x \otimes \delta_k(. - x) \end{array}$$

is continuous. Since $f \in \mathcal{S}_{\mathrm{IR}}(S_\beta \times \mathrm{IR})$, the bound $\|f_x \otimes \delta_k(. - x)\|_{H^{-1}(S_\beta \times \mathrm{IR})} \in O\big(|x|^{-\infty}\big)$ holds true. Hence by (3.15) the map

$$\begin{array}{rcl} \mathrm{IR} & \to & \bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C) \\ x & \mapsto & \phi\big(f_x \otimes \delta_k(. - x)\big) \end{array}$$

is continuous and $\big\|\phi\big(f_x \otimes \delta_k(.-x)\big)\big\|_{L^p(Q, \Sigma, \mathrm{d}\phi_C)} \in O\big(|x|^{-\infty}\big)$. Therefore $\int_{\mathrm{IR}} \phi\big(f_x \otimes \delta_k(.-x)\big)\mathrm{d}x$ is well defined as an element of $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$. Moreover,

$$\int_{\mathrm{IR}} \phi\big(f_x \otimes \delta_k(. - x)\big)\mathrm{d}x = \phi\Big(\int_{\mathrm{IR}} f_x \otimes \delta_k(. - x)\mathrm{d}x\Big) = \phi(f * \delta_k),$$

17

where the convolution product $*$ acts only in the space variable $x$. Since $\lim_{k\to\infty} f * \delta_k = f$ holds in $H^{-1}(S_\beta \times \mathbb{R})$, we obtain from (3.13)

$$\lim_{k\to\infty} \int_{\mathbb{R}} \phi\big(f_x \otimes \delta_k(.-x)\big)\mathrm{d}x = \phi(f) \quad \text{in} \quad \bigcap_{1\le p<\infty} L^p(Q,\Sigma,\mathrm{d}\phi_C).$$

It follows from (3.9) and (3.10) that

$$\lim_{k\to\infty} \sup_{x\in\mathbb{R}} |x|^N \big\|\phi\big(f_x \otimes \delta_k(.-x)\big) - \phi(f_x,x)\big\|_{L^p(Q,\Sigma,\mathrm{d}\phi_C)} = 0$$

for $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and $N \in \mathbb{N}$. Hence

$$\lim_{k\to\infty} \int_{\mathbb{R}} \phi\big(f_x \otimes \delta_k(.-x)\big)\mathrm{d}x = \int_{\mathbb{R}} \phi(f_x,x)\mathrm{d}x.$$

This proves the first identity of the lemma. The second one can be shown by similar arguments $\square$.

# 4   Path spaces supported by $(\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R}), \Sigma, \mathrm{d}\phi_C)$

In this section we recall two well known path spaces supported by $(Q,\Sigma,\mathrm{d}\phi_C)$. The first is associated to the free neutral scalar field of mass $m$ on $S_\beta$ at temperature $0$; the second is associated to the free neutral scalar field of mass $m$ on $\mathbb{R}$ at temperature $\beta^{-1}$.

We recall that $(t,x)$ denotes a point in $S_\beta \times \mathbb{R}$, and refer to $t$ as the (euclidean) time and to $x$ as space variable. The time translation induced on $Q$ by the map $(t,x) \mapsto (t+s,x)$ will be denoted by $\mathfrak{T}_s\colon Q \to Q$ and the spatial translations induced on $Q$ by the map $(t,x) \mapsto (t,x+y)$ will be denoted by $\mathfrak{a}_y\colon Q \to Q$. Note that for the thermal model on the real line, $t$ is the (euclidean) time and $x$ is the space variable, while for the model on the circle $t$ has to be interpreted as the 'position variable on the circle' and $x$ has to be identified with the 'euclidean time variable'.

## 4.1   The free massive euclidean field on the circle at $0$-temperature

In this subsection we identify the generalized path space on $(Q,\Sigma,\mathrm{d}\phi_C)$ corresponding to the free massive scalar field on the circle $S_\beta$ at temperature $0$.

Let $\Sigma_0^{\mathsf{c}}$ be the sub $\sigma$-algebra of $\Sigma$ generated by the functions $\{\phi(g,0) \mid g \in \mathcal{S}_{\mathbb{R}}(S_\beta)\}$.

We denote by $\{U_{\mathsf{c}}(x)\}_{x\in\mathbb{R}}$ the 1-parameter group generated by the spatial translations $\{\mathfrak{a}_x\}_{x\in\mathbb{R}}$. More precisely, if $F\colon Q \to \mathbb{C}$ is a function on $Q$, then $U_{\mathsf{c}}(x)F(q) := F\big(\mathfrak{a}_{-x}(q)\big)$ for $q \in Q$. Applying (3.2) we see that $x \mapsto U_{\mathsf{c}}(x)$ is a strongly continuous unitary group on $L^2(Q,\Sigma,\mathrm{d}\phi_C)$, and hence extends to a group of measure-preserving automorphisms of $L^\infty(Q,\Sigma,\mathrm{d}\phi_C)$ which is continuous in measure.

Let $r_{\mathsf{c}}\colon Q \to Q$ be the space reflection around $x = 0$. We denote by $R_{\mathsf{c}}$ the measure preserving transformation of $(Q,\Sigma,\mathrm{d}\phi_C)$ generated by $r_{\mathsf{c}}$.

For $g \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ we have
(4.1)                                $U_{\mathsf{c}}(x)\phi(g,0) = \phi(g,x).$

Using then Lemma 3.1, we see that $\Sigma = \bigvee_{x\in\mathbb{R}} U_{\mathsf{c}}(x)\Sigma_0^{\mathsf{c}}$.

Hence $(Q,\Sigma,\Sigma_0^{\mathsf{c}}, U_{\mathsf{c}}(x), R_{\mathsf{c}}, \mathrm{d}\phi_C)$ is a generalized path space. Moreover, it is OS-positive (see e.g. [KL2]).

It describes the *free neutral scalar euclidean field of mass $m$ on the circle $S_\beta$ at temperature* $0$.

Let us now briefly describe a well-known concrete form of the physical objects associated to this path space by the reconstruction theorem. Let $H^{-\frac{1}{2}}(S_\beta)$ be the Sobolev

space of order $-\frac{1}{2}$ equipped with its canonical complex structure i and scalar product $(h_1, (2b)^{-1}h_2)_{L^2(S_\beta)}$, where $b = (D_t^2 + m^2)^{\frac{1}{2}}$. Then the physical Hilbert space can be unitarily identified with the bosonic Fock space $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$ over $H^{-\frac{1}{2}}(S_\beta)$. The distinguished unit vector $\Omega_{\mathsf{C}}^{\circ} := \mathcal{V}1$ is identified with the Fock vacuum $\Omega$ in $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$. The (free) Hamiltonian is

$$H_{\mathsf{C}}^{\circ} = \mathrm{d}\Gamma(b).$$

The abelian von Neumann algebra $\mathcal{U}_{\mathsf{C}}$ obtained from the reconstruction theorem can be identified with the von Neumann algebra generated by $\{W_F(g) \mid g \in H_{\mathbb{R}}^{-\frac{1}{2}}(S_\beta)\}$. In fact, if $A = \mathrm{e}^{\mathrm{i}\phi(g,0)}$ for $g \in \mathcal{S}_{\mathbb{R}}(S_\beta)$, then the operator $\tilde{A}$ defined in (2.1) is identified with the Fock Weyl operator $W_F(g) = \mathrm{e}^{\mathrm{i}\phi_F(g)}$ on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$.

## 4.2 The free massive euclidean field on $\mathbb{R}$ at temperature $\beta^{-1}$

We now identify the generalized path space on $(\mathcal{S}_{\mathbb{R}}'(S_\beta \times \mathbb{R}), \Sigma, \mathrm{d}\phi_C)$ corresponding to the free massive scalar euclidean field on $\mathbb{R}$ at temperature $\beta^{-1}$.

Let $\Sigma_0$ be the sub $\sigma$-algebra of $\Sigma$ generated by the functions $\{\phi(0,h) \mid h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\}$. We denote by $\{U(t)\}_{t\in S_\beta}$ the one parameter group generated by $\{\mathfrak{T}_t\}_{t\in S_\beta}$. If $F: Q \to \mathbb{C}$ is a function on $Q$, then $U(t)F(q) := F\big(\mathfrak{T}_{-t}(q)\big)$ for $q \in Q$. Using (3.2) we see that $t \mapsto U(t)$ is a strongly continuous $\beta$-periodic unitary group on $L^2(Q, \Sigma, \mathrm{d}\phi_C)$. Hence it extends to a group of measure-preserving automorphisms of $L^\infty(Q, \Sigma, \mathrm{d}\phi_C)$ which is continuous in measure.

Let $r$ be the (euclidean) time reflection around $t = 0$. We denote by $R$ the measure preserving transformation of $(Q, \Sigma, \mathrm{d}\phi_C)$ generated by $r$.

For $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ we have

(4.2) $$U(t)\phi(0,h) = \phi(t,h).$$

Again by Lemma 3.1, we see that $\Sigma = \bigvee_{t\in S_\beta} U(t)\Sigma_0$. Hence $(Q, \Sigma, \Sigma_0, U(t), R, \mathrm{d}\phi_C)$ is a generalized path space. Moreover, it is $\beta$-periodic and OS-positive (see e.g. [KL2]). It describes the *free neutral scalar field of mass $m$ on $\mathbb{R}$ at temperature $\beta^{-1}$*.

We now describe a well known concrete form of the $\beta$-KMS system associated to the generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mathrm{d}\phi_C)$. Let $\mathfrak{h} := H^{-\frac{1}{2}}(\mathbb{R})$ be the Sobolev space of order $-\frac{1}{2}$, equipped with its canonical complex structure i and scalar product $(h_1, h_2) = (h_1, (2\epsilon)^{-1}h_2)_{L^2(\mathbb{R})}$, where $\epsilon = (D_x^2 + m^2)^{\frac{1}{2}}$. On $\mathfrak{h}$ we consider the unitary dynamics $\mathrm{e}^{-\mathrm{i}t\epsilon}$.

On the Weyl algebra $\mathcal{W}(\mathfrak{h})$ we define a state $\omega_\beta^{\circ}$ and a one-parameter group of automorphisms $\{\tau_t^{\circ}\}_{t\in\mathbb{R}}$ by

(4.3) $$\omega_\beta^{\circ}\big(W(h)\big) := \mathrm{e}^{-\frac{1}{4}(h,(1+2\rho)h)}, \quad \tau_t^{\circ}\big(W(h)\big) := W(\mathrm{e}^{\mathrm{i}t\epsilon}h), \; h \in \mathfrak{h}, \; t \in \mathbb{R},$$

where $\rho := (\mathrm{e}^{\beta\epsilon} - 1)^{-1}$, $\beta > 0$. It can be easily seen that $\omega_\beta^{\circ}$ is a quasi-free $(\tau^{\circ}, \beta)$-KMS state on $\mathcal{W}(\mathfrak{h})$.

Let us now recall some terminology. If $\mathfrak{h}$ is a complex vector space, then the *conjugate vector space* $\overline{\mathfrak{h}}$ is the real vector space $\mathfrak{h}$ equipped with the complex structure $-\mathrm{i}$. We will denote by $\mathfrak{h} \ni h \mapsto \overline{h} \in \overline{\mathfrak{h}}$ the (anti-linear) identity operator. If $a \in \mathcal{L}(\mathfrak{h})$, then we denote by $\overline{a} \in \mathcal{L}(\overline{\mathfrak{h}})$ the operator $\overline{a}\overline{h} := \overline{ah}$. If $\mathfrak{h}$ is a Hilbert space, then $\overline{\mathfrak{h}}$ is equipped with the Hilbert space structure $(\overline{h_1}, \overline{h_2}) := (h_2, h_1)$.

We recall a convenient realization of the GNS representation associated to $\big(\mathcal{W}(\mathfrak{h}), \omega_\beta^{\circ}\big)$, which is called the *right Araki-Woods representation*. It is specified by setting

$$\mathcal{H}_{AW} := \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}}),$$

$$\Omega_{AW} := \Omega,$$

$$\pi_{AW}(W(h)) = W_{AW}(h) := W_{\mathrm{F}}\big((1+\rho)^{\frac{1}{2}}h \oplus \overline{\rho}^{\frac{1}{2}}\overline{h}\big), \quad h \in \mathfrak{h}.$$

Here $W_F(.)$ denotes the Fock Weyl operator on $\Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ and $\Omega \in \Gamma(\mathfrak{h}\oplus\overline{\mathfrak{h}})$ is the Fock vacuum.

The physical Hilbert space associated to the path space $(Q, \Sigma, \Sigma_0, U(t), R, \mathrm{d}\phi_C)$ can be unitarily identified with $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The distinguished vector $\mathcal{V}1$ is identified with the Fock vacuum vector $\Omega$ in $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. The Liouvillean $L_{AW}$ satisfies

$$\mathrm{e}^{iL_{AW}t}\pi_{AW}(A)\Omega_{AW} = \pi_{AW}\big(\tau_t^\circ(A)\big)\Omega_{AW} \quad \text{and} \quad L_{AW}\Omega_{AW} = 0,$$

and can be identified with $\mathrm{d}\Gamma(\epsilon \oplus -\overline{\epsilon})$.

The abelian von Neumann algebra $\mathcal{U}_{AW}$ obtained by the reconstruction theorem can be identified with the abelian von Neumann algebra generated by $\{W_{AW}(h) \mid h \in H_{\mathbb{R}}^{-\frac{1}{2}}(\mathbb{R})\}$. In fact, if $A = \mathrm{e}^{i\phi(0,h)}$ for $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, then the operator $\tilde{A}$ defined in (2.1) is identified with the Weyl operator $W_{AW}(h) = \mathrm{e}^{i\phi_{AW}(h)}$ on $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$.

The von Neumann algebra $\mathcal{B}_{AW}$ generated by $\bigcup_{t\in\mathbb{R}} \tau_t^\circ(\mathcal{U}_{AW})$ can be identified with the von Neumann algebra $\mathcal{R}_{AW}$ generated by $\{W_{AW}(h) \mid h \in H^{-\frac{1}{2}}(\mathbb{R})\}$.

# 5 Perturbations of path spaces

In this section we describe perturbations of the two path spaces defined in Subsects. 4.1 and 4.2, obtained from FKN kernels corresponding to $P(\phi)_2$ interactions.

## 5.1 Interaction terms

We recall some well known facts concerning the Wick ordering of Gaussian random variables. Let $(K, \nu)$ be a probability space and $X$ a real vector space equipped with a positive quadratic form $f \mapsto c(f, f)$ called a *covariance*. Let $f \mapsto \phi(f)$ be a $\mathbb{R}$-linear map from $X$ into the space of real measurable functions on $K$.

The *Wick ordering* $:\phi(f)^n:_c$ with respect to the covariance $c$ is defined by the following generating series:

$$(5.1) \qquad :\mathrm{e}^{\alpha\phi(f)}:_c := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :\phi(f)^n:_c = \mathrm{e}^{\alpha\phi(f)}\mathrm{e}^{-\frac{\alpha^2}{2}c(f,f)}.$$

Thus

$$(5.2) \qquad :\phi(f)^n:_c = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m!)}\phi(f)^{n-2m}\left(-\frac{1}{2}c(f,f)\right)^m,$$

where $[.]$ denotes the integer part.

**Lemma 5.1**

(i) *For $f \in L^1(S_\beta\times\mathbb{R})\cap L^2(S_\beta\times\mathbb{R})$ the following limit exists in $\bigcap_{1\leq p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\lim_{(k,k')\to\infty} \int_{S_\beta\times\mathbb{R}} f(t,x) :\phi\big(\delta_k(.-t)\otimes\delta_{k'}(.-x)\big)^n:_C \mathrm{d}t\mathrm{d}x.$$

*It will be denoted by $\int_{S_\beta\times\mathbb{R}} f(t,x) :\phi(t,x)^n:_C \mathrm{d}t\mathrm{d}x$.*

(ii) *For $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the following limit exists in $\bigcap_{1\leq p<\infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\lim_{k\to\infty} \int_{\mathbb{R}} h(x) :\phi(0, \delta_k(.-x))^n:_{C_0} \mathrm{d}x.$$

*It will be denoted by $\int_{\mathbb{R}} h(x) :\phi(0, x)^n:_{C_0} \mathrm{d}x$.*

(iii)   *For $g \in L^1(S_\beta) \cap L^2(S_\beta)$ the following limit exists in $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$:*

$$\lim_{k' \to \infty} \int_{S_\beta} g(t) : \phi(\delta_k(. - t), 0)^n :_{C_\beta} \mathrm{d}t.$$

*It will be denoted by $\int_{S_\beta} g(t) : \phi(t, 0)^n :_{C_\beta} \mathrm{d}t$.*

We recall that the covariances $C$, $C_0$ and $C_\beta$ have been defined in (3.1), (3.7) and (3.12), respectively. In Lemma 5.1 the probability space is $(Q, \Sigma, \mathrm{d}\phi_C)$ and the real vector spaces are equal to $\mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$, $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $\mathcal{S}_{\mathbb{R}}(S_\beta)$, respectively.

   **Proof.**  The proof is straightforward, adapting standard arguments (see e.g. [Si], [GeJ, Section 9]) used for the spatially cutoff $P(\phi)_2$ model at 0-temperature $\square$.

**Remark 5.2**  *If $P = P(\lambda)$ is a polynomial, then the functions*

$$\int_{S_\beta \times \mathbb{R}} f(t, x) : P(\phi(t, x)) :_C \mathrm{d}t\mathrm{d}x, \quad \int_{\mathbb{R}} h(x) : P(\phi(0, x)) :_{C_0} \mathrm{d}x \quad and \quad \int_{S_\beta} g(t) : P(\phi(t, 0)) :_{C_\beta} \mathrm{d}t$$

*are well defined, by linearity. It can be easily shown (see [GeJ, Proposition 8.4]) using the so-called* Wick reordering identities *that there exists a linear invertible map between polynomials*

$$P \mapsto \tilde{P}$$

*with $\deg P = \deg \tilde{P}$, $\deg(P - \tilde{P}) \le \deg(P) - 1$ such that*

$$\int_{\mathbb{R}} h(x) : P(\phi(0, x)) :_{C_0} \mathrm{d}x = \int_{\mathbb{R}} h(x) : \tilde{P}(\phi(0, x)) :_{\mathrm{vac}} \mathrm{d}x.$$

*Here $: :_{\mathrm{vac}}$ denotes Wick ordering with respect to the 0-temperature covariance $(h, \frac{1}{2\epsilon}h)_{L^2(\mathbb{R})}$.*

**Lemma 5.3**  *Let $P$ be a polynomial, $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g \in L^1(S_\beta) \cap L^2(S_\beta)$. Set*

(5.3)
$$V_0(h) := \int_{\mathbb{R}} h(x) : P(\phi(0, x)) :_{C_0} \mathrm{d}x,$$
$$V_\beta(g) := \int_{S_\beta} g(t) : P(\phi(t, 0)) :_{C_\beta} \mathrm{d}t,$$

*as functions on $Q$.*
   *Then*

(5.4) $$\int_{S_\beta} g(t) U(t) V_0(h) \mathrm{d}t = \int_{S_\beta \times \mathbb{R}} \big(g(t) \otimes h(x)\big) : P(\phi(t, x)) :_C \mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}} h(x) U_{\mathsf{c}}(x) V_\beta(g) \mathrm{d}x$$

*as functions on $Q$.*

**Proof.**  Let $W$ be a function in $L^p(Q, \Sigma, \mathrm{d}\phi_C)$ for some $1 \le p < \infty$. The one parameter groups $\{U(t)\}_{t \in S_\beta}$ and $\{U_{\mathsf{c}}(x)\}_{x \in \mathbb{R}}$, defined in (4.2) and (4.1), are strongly continuous groups of isometries of $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$. Therefore the functions $\int_{\mathbb{R}} h(x) U_{\mathsf{c}}(x) W \mathrm{d}x$ and $\int_{S_\beta} g(t) U(t) W \mathrm{d}t$ belong to $L^p(Q, \Sigma, \mathrm{d}\phi_C)$.
   Together with Lemma 5.1 this implies that all three functions given in (5.4) belong to $L^p(Q, \Sigma, \mathrm{d}\phi_C)$. Let us now prove that they are identical. By linearity, we may assume that $P(\lambda) = \lambda^n$. Using Lemma 5.1 and the Wick identity (5.2), it follows that

$$\int_{S_\beta \times \mathbb{R}} \big(g(t) \otimes h(x)\big) : P(\phi(t, x)) :_C \mathrm{d}t\mathrm{d}x = \lim_{(k, k') \to \infty} F(k, k') \text{ in } L^p(Q, \Sigma, \mathrm{d}\phi_C),$$

where

$$F(k,k') = \sum_{m=0}^{[n/2]} \frac{n!\left(-\frac{1}{2}C(\delta_{k,k'},\delta_{k,k'})\right)^m}{m!(n-2m)!} \int_{S_\beta \times \mathbb{R}} \left(g(t) \otimes h(x)\right)\phi\left(\delta_k(.-t) \otimes \delta_{k'}(.-x)\right)^m \mathrm{d}t\mathrm{d}x$$

and $\delta_{k,k'}(t,x) := \delta_k(t) \otimes \delta_{k'}(x)$. Since

$$\lim_{k\to\infty} C(\delta_{k,k'},\delta_{k,k'}) = C_0(\delta_{k'},\delta_{k'}),$$

the definition given in (3.5) of sharp-time fields implies that

$$\lim_{k\to\infty} F(k,k') = \int_{S_\beta} g(t)V_{k'}(t,h)\mathrm{d}t \ \text{ in } \ L^p(Q,\Sigma,\mathrm{d}\phi_C),$$

where

$$V_{k'}(t,h) = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m)!}\left(-\frac{1}{2}C_0(\delta_{k'},\delta_{k'})\right)^m \int_{\mathbb{R}} h(x)\phi\left(t,\delta_{k'}(.-x)\right)^m \mathrm{d}x.$$

Note that (4.2) implies $V_{k'}(t,h) = U(t)V_{k'}(0,h)$. By Lemma 5.1 (ii) we know that

$$\lim_{k'\to\infty} V_{k'}(0,h) = \int_{\mathbb{R}} h(x) :P(\phi(0,x)):_{C_0} \mathrm{d}x \ \text{ in } \ L^p(Q,\Sigma,\mathrm{d}\phi_C)$$

and hence

$$\lim_{k\to\infty} \int_{S_\beta} g(t)V_{k'}(t,h)\mathrm{d}t = \int_{S_\beta} g(t)U(t)V_0(h)\mathrm{d}t \ \text{ in } \ L^p(Q,\Sigma,\mathrm{d}\phi_C).$$

Applying Lemma B.1 with $E = L^p(Q,\Sigma,\mathrm{d}\phi_C)$ we obtain the first identity in (5.4). The second identity follows by the same argument, taking first the limit $k' \to \infty$ and using then that

$$\lim_{k'\to\infty} C(\delta_{k,k'},\delta_{k,k'}) = C_\beta(\delta_k,\delta_k) \ \square.$$

## 5.2 The $P(\phi)_2$ model on the circle $S_\beta$ at temperature $0$

Let $P(\lambda)$ be a real valued polynomial, which is bounded from below. The $P(\phi)_2$ *model on the circle* $S_\beta$ is specified by the formal interaction term

$$V_{\mathsf{C}} := V_\beta\left(1_{[-\beta/2,\beta/2]}\right) = \int_{S_\beta} :P(\phi(t,0)):_{C_\beta} \mathrm{d}t.$$

This expression can be given two equivalent meanings: first of all, as recalled in Lemma 5.1, it can be viewed as a $\Sigma_0^{\mathsf{C}}$ measurable function $V_{\mathsf{C}} \in \bigcap_{1\le p<\infty} L^p(Q,\Sigma_0^{\mathsf{C}},\mathrm{d}\phi_C)$. Secondly, $V_{\mathsf{C}}$ can be considered as a selfadjoint operator on $\Gamma\left(H^{-\frac{1}{2}}(S_\beta)\right)$ affiliated to the abelian algebra $\mathcal{U}_{\mathsf{C}}$. More precisely, for $t \in S_\beta$ and $\Lambda \gg 1$ an UV cutoff parameter, we define an approximation $h_{\Lambda,t} \in H^{-\frac{1}{2}}(S_\beta)$ of the Dirac delta-function $\delta(.-t) \in H^{-\frac{1}{2}}(S_\beta)$ by

$$h_{\Lambda,t} := 1\!\!1_{[0,\Lambda]}(b)\delta(.-t) \in H^{-\frac{1}{2}}(S_\beta),$$

where $b = (D_t^2 + m^2)^{\frac{1}{2}}$. Setting $\phi_\Lambda(t,0) := \phi_F(h_{\Lambda,t})$ one obtains by well-known arguments that

$$V_{\mathsf{C}} = \lim_{\Lambda\to\infty} \int_{S_\beta} :P(\phi_\Lambda(t,0)):_{C_\beta} \mathrm{d}t$$

22

on a dense set of vectors in $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$. Since $h_{\Lambda,t} \in H_{\mathbb{R}}^{-\frac{1}{2}}(S_\beta)$ is a real valued function, it is easy to see that $V_{\mathsf{c}}$ is a selfadjoint operator affiliated to $\mathcal{U}_{\mathsf{c}}$.

It is then easy to verify, by adapting well-known results for the spatially cutoff $P(\phi)_2$ model on the real line $\mathbb{R}$ at 0-temperature (see [S-H.K]) that $V_{\mathsf{c}} \in \bigcap_{1 \le p < \infty} L^p(Q, \Sigma_0^{\mathsf{c}}, \mathrm{d}\phi_C)$ and $\mathrm{e}^{-TV_{\mathsf{c}}} \in L^1(Q, \Sigma_0^{\mathsf{c}}, \mathrm{d}\phi_C)$ for all $T > 0$. Now consider, for $0 \le b - a < \infty$,

$$(5.5) \qquad G_{[a,b]} := \mathrm{e}^{-\int_a^b U_{\mathsf{c}}(x) V_{\mathsf{c}}\, \mathrm{d}x}$$

as a function on $Q$. It follows from Jensen's inequality (see [KL4, Theorem 6.2]) that

$$(5.6) \qquad \|G_{[a,b]}\|_{L^p(Q,\Sigma,\mathrm{d}\phi_C)} \le \|\mathrm{e}^{-(b-a)V_{\mathsf{c}}}\|_{L^p(Q,\Sigma,\mathrm{d}\phi_C)}$$

and hence $G_{[a,b]} \in \bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$. From the results recalled in Subsection 2.3, we obtain a selfadjoint operator

$$H_{\mathsf{c}} = \overline{\mathrm{d}\Gamma(b) + V_{\mathsf{c}}}$$

on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$ associated to the FKN kernel $\{G_{[0,s]}\}$. The Hamiltonian $H_{\mathsf{c}}$ is called the $P(\phi)_2$ *Hamiltonian on the circle $S_\beta$.*

**Proposition 5.4** *The Hamiltonian $H_{\mathsf{c}}$ is bounded from below and has a unique normalized ground state such that $(\Omega_{\mathsf{c}}, \Omega) > 0$. We set*

$$\omega_{\mathsf{c}}(\,.\,) = (\Omega_{\mathsf{c}},\,.\,\Omega_{\mathsf{c}}).$$

*Moreover, for $c \gg 1$,*
$$(5.7) \qquad \|\phi_F(g)(H_{\mathsf{c}} + c)^{-\frac{1}{2}}\| \le C\|g\|_{H^{-\frac{1}{2}}(S_\beta)},$$

$$(5.8) \qquad \pm\phi_F(g) \le C\|g\|_{H^{-\frac{1}{2}}(S_\beta)}(H_{\mathsf{c}} + c)^{\frac{1}{2}}$$

*and*
$$(5.9) \qquad \pm\phi_F(g) \le C\|g\|_{H^{-1}(S_\beta)}(H_{\mathsf{c}} + c)$$

*for all $g \in H^{-\frac{1}{2}}(S_\beta)$. As before, $W_F(g) = \mathrm{e}^{\mathrm{i}\phi_F(g)}$ is the Fock Weyl operator on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$.*

**Proof.** The existence and uniqueness of the vacuum state can be shown by following the proofs of the corresponding results for spatially cutoff $P(\phi)_2$ models. For example, one easily obtains (see e.g. [Si, Theorem V.20] or [DG, Theorem 6.4 (ii)]) that

$$(5.10) \qquad (\mathrm{d}\Gamma(b) + 1) \le C(H_{\mathsf{c}} + c) \text{ for } c \gg 1.$$

Since $\mathrm{d}\Gamma(b)$ has compact resolvent on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$, it follows that $H_{\mathsf{c}}$ is bounded from below with a compact resolvent and hence has a ground state. The uniqueness of the vacuum (i.e., the ground state of $H_{\mathsf{c}}$) follows from a Perron-Frobenius argument (see e.g. [Si, Theorem V.17]). Since $b \ge m > 0$, we see that it suffices to check (5.7) and (5.8), with $H_{\mathsf{c}}$ replaced by the number operator $N$, which is immediate. To prove (5.9) we use (5.10) and the well known bound (see e.g. [Ge, Appendix])

$$\pm\phi_F(g) \le \|b^{-\frac{1}{2}}g\|_{H^{-\frac{1}{2}}(S_\beta)}(\mathrm{d}\Gamma(b) + 1) \;\square.$$

23

Without proof we quote the following result (see [HO]).

**Theorem 5.5** *Let $H_{\mathsf{C}}^{\mathrm{ren}} := H_{\mathsf{C}} - E_{\mathsf{C}}$, where $E_{\mathsf{C}} := \inf(\sigma(H_{\mathsf{C}}))$ and let $P_{\mathsf{C}}$ denote the generator of the translations along the circle $S_\beta$. The joint spectrum of $H_{\mathsf{C}}^{\mathrm{ren}}$ and $P_{\mathsf{C}}$ is purely discrete and is contained in the forward light cone.*

Consequently the correlation function

$$(t,x) \mapsto \big(\Omega_{\mathsf{C}}, A\mathrm{e}^{ixH_{\mathsf{C}}^{\mathrm{ren}}+itP_{\mathsf{C}}}B\Omega_{\mathsf{C}}\big), \quad A, B \in \mathcal{B}\big(\Gamma(H^{-\frac{1}{2}}(S_\beta))\big),$$

allows an analytic continuation to the tube $\mathbb{R}^2 + iV_+$, where $V_+ := \{(t,x) \mid |t| < x;\ x > 0\}$ denotes the forward light cone (with $t$ and $x$ reversed, due to our conventions).

## 5.3 The spatially cutoff $P(\phi)_2$ model on $\mathbb{R}$ at temperature $\beta^{-1}$

Let $P(\lambda)$ be a real valued polynomial, which is bounded from below (as in Subsection 5.2), and let $l \in \mathbb{R}^+$ be a spatial cutoff parameter. The *spatially cutoff $P(\phi)_2$ model on $\mathbb{R}$* is specified by the formal interaction term (see (5.3))

$$V_l := V_0\big(\mathbb{1}_{[-l,l]}\big) = \int_{-l}^{l} :P(\phi(0,x)): _{C_0}\ \mathrm{d}x.$$

Again this formal expression can be given two equivalent meanings: first of all, as recalled in Lemma 5.1, it can be viewed as a $\Sigma_0$-measurable function $V_l \in \bigcap_{1 \le p < \infty} L^p(Q, \Sigma_0, \mathrm{d}\phi_C)$. Secondly, $V_l$ can be considered as a selfadjoint operator on $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ affiliated to the abelian von Neumann algebra $\mathcal{U}_{AW}$. As in Subsection 5.2 we define an approximation $h_{\Lambda,x} \in H^{-\frac{1}{2}}(\mathbb{R})$ of the Dirac delta-function $\delta(.-x) \in H^{-\frac{1}{2}}(\mathbb{R})$. For $x \in \mathbb{R}$ and $\Lambda \gg 1$ we set

$$h_{\Lambda,x} := \mathbb{1}_{[0,\Lambda]}(\epsilon)\delta(.-x) \in H^{-\frac{1}{2}}(\mathbb{R})$$

and introduce cutoff fields $\phi_\Lambda(0,x) := \phi_{AW}(h_{\Lambda,x})$, where $\phi_{AW}(h)$ is the selfadjoint field operator associated to $W_{AW}(h)$, $h \in \mathfrak{h}$.

As before, the limit

(5.11) $$V_l = \lim_{\Lambda \to \infty} \int_{-l}^{l} :P(\phi_\Lambda(0,x)): _{C_0}\ \mathrm{d}x$$

exists on a dense set of vectors in $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$. Since $h_{\Lambda,x} \in H_{\mathbb{R}}^{-\frac{1}{2}}(\mathbb{R})$, one obtains that $V_l$ is a selfadjoint operator affiliated to $\mathcal{U}_{AW}$.

Adapting well known arguments (see [GeJ, Section 8.2]) it can be shown that $\mathrm{e}^{-TV_l} \in L^1(Q, \Sigma_0, \mu)$ for all $T > 0$. Consequently, we can associate to $V_l$ the FKN kernel

$$F_{[a,b]}^l := \mathrm{e}^{-\int_a^b U(t)V_l\,\mathrm{d}t}, \quad 0 \le b - a \le \beta,$$

and the measure

$$\mathrm{d}\mu_l := \frac{F_{[-\beta/2,\beta/2]}^l \mathrm{d}\phi_C}{\int_Q F_{[-\beta/2,\beta/2]}^l \mathrm{d}\phi_C}.$$

The generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu_l)$ is $\beta$-periodic and OS-positive. The associated $\beta$-KMS system is called the *spatially cutoff $P(\phi)_2$ model on $\mathbb{R}$ at temperature $\beta^{-1}$*. Applying the abstract results recalled in Subsection 2.3, we obtain the following facts:

- the physical Hilbert space $\mathcal{H}_{V_l}$ is equal to $\mathcal{H}_{AW} = \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$;
- the $W^*$-algebra $\mathcal{B}_{V_l}$ and the abelian algebra $\mathcal{U}_{V_l}$ are equal to $\mathcal{R}_{AW}$ and $\mathcal{U}_{AW}$, respectively;

- the operator sum $L_{AW} + V_l$ is essentially selfadjoint on $\mathcal{D}(L_{AW}) \cap \mathcal{D}(V_l)$ and if $H_l := \overline{L_{AW} + V_l}$, then the perturbed time-evolution on $\mathcal{B}$ is given by $\tau_t^l(B) := \mathrm{e}^{\mathrm{i}tH_l} B \mathrm{e}^{-\mathrm{i}tH_l}$, $B \in \mathcal{B}$;

- the GNS vector $\Omega_{AW} \in \Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ belongs to $\mathcal{D}\big(\mathrm{e}^{-\frac{\beta}{2}H_l}\big)$ and the perturbed KMS state $\omega_l$ is given by $\omega_l(B) = (\Omega_l, B\Omega_l)$, where $\Omega_l := \|\mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}\|^{-1} \mathrm{e}^{-\frac{\beta}{2}H_l}\Omega_{AW}$.

The following consequence of Lemma 5.3 will be important in Section 7:

$$(5.12) \qquad F_{[-\beta/2,\beta/2]}^l = G_{[-l,l]},$$

where $G_{[a,b]}$ was defined in (5.5). The analog identity in the temperature zero case is called *Nelson symmetry*.

# 6   The thermodynamic limit

In this section we prove that the limits

$$\lim_{l\to+\infty} \tau_t^l(A) =: \tau_t(A) \quad \text{and} \quad \lim_{l\to+\infty} \omega_l(A) =: \omega_\beta(A)$$

exist for $A$ in the $C^*$-algebra of local observables $\mathcal{A}$ and that $(\mathcal{A}, \tau, \omega_\beta)$ is a $\beta$-KMS system, describing the *translation invariant $P(\phi)_2$ model at temperature $\beta^{-1}$*.

## 6.1   Preparations

We first recall a well known relationship between $\mathrm{e}^{-\mathrm{i}t\epsilon}$ and the Klein-Gordon equation: let

$$(6.1) \qquad \begin{aligned} U: \quad H^{-\frac{1}{2}}(\mathbb{R}) \quad &\to \quad H_{\mathbb{R}}^{-\frac{1}{2}}(\mathbb{R}) + \mathrm{i} H_{\mathbb{R}}^{\frac{1}{2}}(\mathbb{R}). \\ h \quad &\mapsto \quad f := \mathrm{Re}\,h + \mathrm{i}\epsilon^{-1}\mathrm{Im}\,h. \end{aligned}$$

(Note that $U$ is $\mathbb{R}$-linear but not $\mathbb{C}$-linear). Then

$$(6.2) \qquad U\mathrm{e}^{-\mathrm{i}t\epsilon} = T(t)U, \text{ where } T(t)f = f_t,$$

and $f_t$ is the solution of the Klein-Gordon equation

$$\begin{cases} \big(\partial_t^2 - \partial_x^2 + m^2\big)f_t = 0, \\ f_{t=0} = f, \quad \big(\partial_t f\big)_{t=0} = -\epsilon^2\mathrm{Im}\,f + \mathrm{i}\mathrm{Re}\,f. \end{cases}$$

Moreover if $h_i \in H^{-\frac{1}{2}}(\mathbb{R})$ and $Uh_i = f_i$ for $i = 1, 2$, then

$$(6.3) \qquad \sigma(h_1, h_2) := \mathrm{Im}(h_1, h_2)_{H^{-\frac{1}{2}}(\mathbb{R})} = \mathrm{Im}\int_{\mathbb{R}} \overline{f}_1(x)f_2(x)\mathrm{d}x.$$

For $I \subset \mathbb{R}$ a bounded open interval we define the real vector subspace $\mathfrak{h}_I$ of $\mathfrak{h}$

$$(6.4) \qquad \mathfrak{h}_I := \{h \in \mathfrak{h} \mid \mathrm{supp}\,Uh \subset I\}.$$

It follows from (6.2) that $\mathrm{i}\epsilon: \mathcal{D}(\epsilon) \cap \mathfrak{h}_I \to \mathfrak{h}_I$, and hence $(1 + \alpha\epsilon^2)^{-1}: \mathfrak{h}_I \to \mathfrak{h}_I$ for $\alpha > 0$. In particular $\mathcal{D}(\epsilon) \cap \mathfrak{h}_I$ is dense in $\mathfrak{h}_I$. Moreover (6.3) shows that $\mathfrak{h}_I$ and $\mathfrak{h}_J$ are orthogonal for the symplectic form $\sigma$ if $I \cap J = \emptyset$.

## 6.2 The net of local algebras

We start by recalling a result of Araki [Ar1, Thm. 1] which will be useful later on. Let us recall a standard notation: If $\mathcal{H}_1$, $\mathcal{H}_2$ are two vector subspaces of a Hilbert space $\mathcal{H}$, then $\mathcal{H}_1 \vee \mathcal{H}_2$ denotes $\overline{\mathcal{H}_1 + \mathcal{H}_2}$. If $\mathcal{R}_1$, $\mathcal{R}_2$ are two *-sub-algebras of $\mathcal{B}(\mathcal{H})$, then $\mathcal{R}_1 \vee \mathcal{R}_2$ denotes the von Neumann algebra generated by $\mathcal{R}_1 \cup \mathcal{R}_2$.

**Proposition 6.1** *Let $X$ be a Hilbert space and let $Z$ be a real vector subspace of $X$. Let $\mathcal{W}(Z) \subset \mathcal{W}(X)$ denote the $C^*$-algebra generated by $\{W(x) \mid x \in Z\}$ and let $\pi_F : \mathcal{W}(X) \to \mathcal{B}(\Gamma(X))$ be the Fock representation. Then*

$$(6.5) \qquad \bigcap_\alpha \pi_F\big(\mathcal{W}(Z_\alpha)\big)'' = \pi_F\big(\mathcal{W}\,(\cap_\alpha Z_\alpha)\big)'', \quad \bigvee_\alpha \pi_F\big(\mathcal{W}(Z_\alpha)\big)'' = \pi_F\big(\mathcal{W}\,(\vee_\alpha Z_\alpha)\big)''$$

*and*

$$(6.6) \qquad\qquad\qquad \pi_F\big(\mathcal{W}(Z)\big)' = \pi_F\big(\mathcal{W}(Z^\perp)\big)'',$$

*where $Z_\alpha$ is a family of real vector subspaces of $X$ and $Z^\perp$ is the vector space orthogonal to $Z$ for the symplectic form $\sigma(x_1, x_2) = \mathrm{Im}(x_1, x_2)$.*

We now define the *net of local von Neumann algebras* $I \to \mathcal{R}_{AW}(I)$ describing free thermal scalar bosons. Let $I \subset \mathbb{R}$ be a bounded open interval. We denote by $\mathcal{R}_{AW}(I)$ the von Neumann algebra generated by

$$\{W_{AW}(h) \mid h \in \mathfrak{h}_I\}.$$

**Lemma 6.2**

   (i)    *The local von Neumann algebras for the free thermal field are regular from the inside and regular from the outside:*

$$\bigcap_{J \supset \overline{I}} \mathcal{R}_{AW}(J) = \mathcal{R}_{AW}(I) = \bigvee_{\overline{J} \subset I} \mathcal{R}_{AW}(J);$$

   (ii)    *The net of local von Neumann algebras for the free thermal field is* additive*:*

$$\mathcal{R}_{AW}(I) = \bigvee_{J_i} \mathcal{R}_{AW}(J_i) \; \text{if } I = \cup_i J_i;$$

   (iii)    *For each open and bounded interval $I$, the local observable algebra $\mathcal{R}_{AW}(I)$ is $*$-isomorphic to the unique hyper-finite factor of type $III_1$.*

**Proof.** Recalling the definition of the Araki-Woods representation we see that, with the notation introduced above,

$$\mathcal{R}_{AW}(I) = \pi_F\big(\mathcal{W}(Z_I)\big)'',$$

where $Z_I \subset \mathfrak{h} \oplus \overline{\mathfrak{h}}$ is the vector subspace

$$Z_I = \{(1+\rho)^{\frac{1}{2}} h \oplus \overline{\rho}^{\frac{1}{2}} \overline{h} \mid h \in \mathfrak{h}_I\}.$$

Clearly $\bigcap_{J \supset \overline{I}} Z_J = \bigvee_{\overline{J} \subset I} Z_J = Z_I$, which using (6.5) implies (i). Part (ii) is a direct consequence of (6.5). To prove (iii) we use (6.6) and (6.5) which implies that

$$\mathcal{R}_{AW}(I) \cap \mathcal{R}_{AW}(I)' = \pi_F\big(\mathcal{W}(Z_I \cap Z_I^\perp)\big)'',$$

where $Z^\perp$ is the orthogonal space to $Z$ in $\mathfrak{h} \oplus \overline{\mathfrak{h}}$ for the symplectic form $\sigma(f, g) = \mathrm{Im}(f, g)$ on $\mathfrak{h} \oplus \overline{\mathfrak{h}}$. We claim that

$$(6.7) \qquad\qquad\qquad Z_I \cap Z_I^\perp = \{0\},$$

which will imply that $\mathcal{R}_{AW}(I)$ is a factor. To prove our claim we pick $h \in \mathfrak{h}_I$ such that $(1+\rho)^{\frac{1}{2}} h \oplus \overline{\rho}^{\frac{1}{2}} \overline{h} \in Z_I^\perp$. This implies that $\mathrm{Im}(h, g) = 0$ for all $g \in \mathfrak{h}_I$. Hence to prove (6.7) it suffices to check that

$$(6.8) \qquad \mathfrak{h}_I \cap \mathfrak{h}_I^\perp = \{0\}.$$

But if $h \in \mathfrak{h}_I \cap \mathfrak{h}_I^\perp$, we have $\mathrm{Im}(h, i\epsilon(1 + \alpha\epsilon^2)^{-1}h) = 0$ for $\alpha > 0$, since $i\epsilon(1 + \alpha\epsilon^2)^{-1}h \in \mathfrak{h}_I$ for $h \in \mathfrak{h}_I$. Letting $\alpha \to 0$ this yields $\mathrm{Re}(h, \epsilon h) = (h, \epsilon h) = 0$, since $\epsilon$ is selfadjoint. Using that $\epsilon \geq m > 0$ this implies that $h = 0$, which proves (6.8) and hence (6.7). Thus $\mathcal{R}_{AW}(I)$ is a factor, if $I$ is bounded. Note that (6.8) shows that $\pi_F(\mathcal{W}(\mathfrak{h}_I))''$ is a factor, and it is well known (see e.g. [BD'AF][L] and lit. cit.) that $\pi_F(\mathcal{W}(\mathfrak{h}_I))''$ is $*$-isomorphic to the unique hyper-finite factor of type III$_1$. Thus Lemma 6.3 below completes the proof of the lemma $\square$.

We now recall an easy fact about the restriction of the free KMS state $\omega_\beta^\circ$ to the local algebras $\mathcal{W}(\mathfrak{h}_I)$.

**Lemma 6.3** *Let $I \subset \mathbb{R}$ be a bounded open interval. Then the representations $\pi_{AW}$ and $\pi_F$ of $\mathcal{W}(\mathfrak{h}_I)$ are quasi-equivalent.*

**Proof.** Let $\mathfrak{h}$ be a Hilbert space and let $\epsilon \geq m > 0$ be a positive selfadjoint operator on $\mathfrak{h}$. Let $\omega_\beta^\circ$ be the quasi free state on $\mathcal{W}(\mathfrak{h})$ defined by $\omega_\beta^\circ(W(h)) = \mathrm{e}^{-\frac{1}{4}(h, (1+2\rho)h)}$, where $\rho = (\mathrm{e}^{\beta\epsilon} - 1)^{-1}$. Then it is well known that $\omega_\beta^\circ$ is normal with respect to the Fock representation $\pi_F$ of $\mathcal{W}(X)$ iff $\mathrm{Tr}\, \mathrm{e}^{-\beta\epsilon} < \infty$ (see e.g. [BR, Prop. 5.2.27]).

This fact implies that if $\mathfrak{h}_1 \subset \mathfrak{h}$ is a complex vector subspace, then the restriction of $\omega_\beta^\circ$ to $\mathcal{W}(\mathfrak{h}_1)$ is $\pi_F$-normal iff $\mathrm{Tr}(E\mathrm{e}^{-\beta\epsilon}E) < \infty$, where $E$ is the orthogonal projection onto $\mathfrak{h}_1$.

We will apply this remark to $\mathfrak{h} = H^{-\frac{1}{2}}(\mathbb{R})$, $\rho = (\mathrm{e}^{\beta\epsilon} - 1)^{-1}$ and $\mathfrak{h}_1 = \mathbb{C}\mathfrak{h}_I$. Let $E_I$ denote the orthogonal projection on $\mathbb{C}\mathfrak{h}_I$. Let $\chi \in C_{0\,\mathbb{R}}^\infty$ such that $\chi \equiv 1$ near $I$ and $x = i\partial_k$. If $h \in \mathfrak{h}_I$, then $\mathrm{Re}h = \chi(x)\mathrm{Re}h$ and $\mathrm{Im}h = \epsilon\chi(x)\epsilon^{-1}\mathrm{Im}h$. Using pseudodifferential calculus, we see that the operators $(1 + |x|)^N \chi(x)$ and $(1 + |x|)^N \epsilon\chi(x)\epsilon^{-1}$ are bounded on $H^{-\frac{1}{2}}(\mathbb{R})$ for all $N \in \mathbb{N}$. This implies that

$$(6.9) \qquad \left\|(1 + |x|)^N h\right\|_{H^{-\frac{1}{2}}(\mathbb{R})} \leq C \|h\|_{H^{-\frac{1}{2}}(\mathbb{R})}, \quad h \in \mathfrak{h}_I.$$

Clearly (6.9) extends to $\mathbb{C}\mathfrak{h}_I$, which implies that $(1 + |x|)^N E_I$ is bounded for all $N \in \mathbb{N}$. Since $\mathrm{e}^{-\beta\epsilon}(1 + |x|)^{-N}$ is trace class for $N$ large enough we see that $E_I\mathrm{e}^{-\beta\epsilon}E_I$ is trace class. Using the arguments given above we obtain that $\omega_\beta^\circ$ restricted to $\mathcal{W}(\mathbb{C}\mathfrak{h}_I)$ (and hence also to $\mathcal{W}(\mathfrak{h}_I)$) is $\pi_F$-normal.

Finally we have seen in the proof of Lemma 6.2 that $\pi_F(\mathcal{W}(\mathfrak{h}_I))''$ is a factor, hence $\pi_F$ is a factor representation of $\mathcal{W}(\mathfrak{h}_I)$. It is shown in [KR, Prop. 10.3.14] that if $\mathcal{R}$ is a $C^*$-algebra and $\pi$ is a factor representation of $\mathcal{R}$, then $\pi$ is quasi-equivalent to the GNS representation of any $\pi$-normal state $\omega$. Since the restriction of $\pi_{AW}$ to $\mathcal{W}(\mathfrak{h}_I)$ is the GNS representation for the quasi-free state $\omega_\beta^\circ$, this completes the proof of the lemma $\square$.

## 6.3 Existence of the limiting dynamics

The $C^*$-algebra of local observables $\mathcal{A}$ is defined as follows:

$$\mathcal{A} := \overline{\bigcup_{I \subset \mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)},$$

where the union is over all open bounded intervals $I \subset \mathbb{R}$ and the symbol $\overline{\bigcup_{I \subset \mathbb{R}} \mathcal{R}_{AW}(I)}^{(*)}$ denotes the $C^*$-inductive limit (see e.g. [KR, Proposition 11.4.1.]).

We denote by $\{\alpha_x\}_{x \in \mathbb{R}}$ the group of space translations on $\mathcal{A}$, defined by

$$\alpha_x(W_{AW}(h)) := W_{AW}(\mathrm{e}^{ix.k}h), \quad x \in \mathbb{R},$$

where $k$ is the momentum operator acting on $\mathfrak{h} = H^{-\frac{1}{2}}(\mathbb{R})$.

**Theorem 6.4** (Existence of limiting dynamics). *Let $I \subset \mathbb{R}$ be a bounded open interval. For $t \in \mathbb{R}$ fixed, the norm limit*

$$\lim_{l \to \infty} \tau_t^l(B) =: \tau_t(B)$$

*exists for all $B \in \mathcal{R}_{AW}(I)$. The map $\tau : t \mapsto \tau_t$ defines a group of $*$-automorphisms of $\mathcal{A}$ such that $\tau_t \circ \alpha_x = \alpha_x \circ \tau_t$ for all $t, x \in \mathbb{R}$. Moreover,*

$$(6.10) \qquad \tau_t : \mathcal{R}_{AW}(I) \to \mathcal{R}_{AW}\big(I + ] - t, t[\big).$$

**Proof.** The proof follows the well-known proof in the 0-temperature case, which is based on finite propagation speed (see [GJ2, Theorem 4.1.2]). To prove the existence of the limit and the group property, it suffices to show that $\tau_t^l(B)$, for $B \in \mathcal{R}_{AW}(I)$ and $|t| \leq T$, is independent of $l$ for $l > |I| + T$.

It follows from (6.2) and Huygens principle that

$$(6.11) \qquad \tau_t^\circ : \mathcal{R}_{AW}(I) \to \mathcal{R}_{AW}\big(I + ] - t, t[\big).$$

Moreover (6.3) implies that $\mathcal{R}_{AW}(I_1) \subset \mathcal{R}_{AW}(I_2)'$, if $I_1 \cap I_2 = \emptyset$.

The dynamics $\tau_t^l$ is unitarily implemented by $e^{itH_l}$, where $H_l = \overline{L_{AW} + V_l}$ for

$$V_l = \int_{]-l,l[} :P(\phi(0, x)):_{C_0} \, dx.$$

Trotter's formula yields $e^{itH_l} = \text{s-}\lim_{n \to \infty} (e^{itL_{AW}/n} e^{itV_l/n})^n$ and hence

$$(6.12) \qquad \tau_t^l(A) = \text{s-}\lim_{n \to \infty} \big(\tau_{t/n}^\circ \circ \gamma_{t/n}^l\big)^n(A), \quad A \in \mathcal{B}(\mathcal{H}_{AW}),$$

where $\gamma_t^l(A) := e^{itV_l} A e^{-itV_l}$. Note that for $l' > l$

$$V_{l'} = V_l + \int_{]-l',l'[ \, \setminus \, ]-l,l[} :P(\phi(0, x)):_{C_0} \, dx.$$

Since $V_{l'} - V_l$ is affiliated to $\mathcal{R}_{AW}\big(] - l', l'[ \, \setminus \, [-l, l]\big)$, we see that $\gamma_t^l = \gamma_t^{l'}$ on $\mathcal{R}_{AW}(I)$ for $l, l' > |I|$. Using (6.11) and (6.12), this implies that $\tau_t^l = \tau_t^{l'}$ on $\mathcal{R}_{AW}(I)$ for $|t| \leq T$ and $l, l' > |I| + T$. This proves our claim. The same argument using again (6.11) proves (6.10).

It remains to check that $\tau$ and $\alpha$ commute. Let $T > 0$ and $I$ a bounded interval. For $|t| \leq T$ the time evolution is locally (i.e., applied to elements in $\mathcal{R}_{AW}(I)$) generated by $H_l$ if $l > |I| + t$. Now $\alpha_x$ is implemented by $e^{ixP}$ with $P = d\Gamma(k \oplus \overline{k})$. It follows that $\alpha_x \circ \tau_t \circ \alpha_x^{-1}$ is implemented by $e^{itH_{l,x}}$ with $H_{l,x} = e^{ixP} H_l e^{-ixP}$. It is easy to see that

$$H_{l,x} = L_{AW} + \int_{]-l+x,l+x[} :P(\phi(0, x)):_{C_0} \, dx.$$

By the same argument as above, $\tau_t$ is implemented by $e^{itH_{l,x}}$ for $|t| \leq T$ if $l > |I| + |T| + |x|$, which implies that $\alpha_x \circ \tau_t \circ \alpha_x^{-1} = \tau_t$ $\square$.

## 6.4 An identification of local algebras

In order to apply the results of Section 7 to the algebra of local observables $\mathcal{A}$, it is necessary to identify the local Weyl algebra $\mathcal{R}_{AW}(I)$ with the von Neumann algebra $\mathcal{B}(I)$ obtained by applying the interacting dynamics $\tau$ to the local abelian algebra of time-zero fields $\mathcal{U}_{AW}(I)$. This is done in Proposition 6.5 below. Note that by similar arguments the corresponding result holds also in the 0-temperature case.

For $I \subset \mathbb{R}$ a bounded open interval, we denote by $\mathcal{U}_{AW}(I)$ the von Neumann algebra generated by $\{W_{AW}(h) \mid h \in \mathfrak{h}_I,\ h$ real valued$\}$. Note that $\mathcal{U}_{AW}(I) \subset \mathcal{R}_{AW}(I)$ is abelian. We denote by $\mathcal{B}_\alpha(I)$ the von Neumann algebra generated by

$$(6.13) \qquad \{\tau_t(A) \mid A \in \mathcal{U}_{AW}(I),\ |t| < \alpha\}.$$

**Proposition 6.5** *Set $\mathcal{B}(I) := \bigcap_{\alpha>0} \mathcal{B}_\alpha(I)$. Then*

$$\mathcal{B}(I) = \mathcal{R}_{AW}(I).$$

**Proof.** Let us first prove that $\mathcal{B}(I) \subset \mathcal{R}_{AW}(I)$. Using (6.10) and $\mathcal{U}_{AW}(I) \subset \mathcal{R}_{AW}(I)$, we see that $\mathcal{B}_\alpha(I) \subset \mathcal{R}_{AW}(I+] - \alpha, \alpha[)$ for all $\alpha > 0$. According to Lemma 6.2 (i) this implies $\mathcal{B}(I) \subset \mathcal{R}_{AW}(I)$.

Let us now prove that $\mathcal{R}_{AW}(I) \subset \mathcal{B}(I)$. Using Lemma 6.2 (i) it suffices to show that for all $\overline{J} \subset I$ and $\alpha \ll 1$ one has

$$(6.14) \qquad \mathcal{R}_{AW}(J) \subset \mathcal{B}_\alpha(I).$$

To this end we fix $I$ and $J$ with $\overline{J} \subset I$ and set $\delta = \frac{1}{2}\mathrm{dist}(J, I^c)$. We will first prove that

$$(6.15) \qquad \mathrm{e}^{\mathrm{i}tL_{AW}} A \mathrm{e}^{-\mathrm{i}tL_{AW}} \in \mathcal{B}_\alpha(I),\ A \in \mathcal{U}_{AW}(J),\ |t| < \alpha,$$

if $\alpha < \delta$. The proof of Theorem 6.4 shows that for $|t| \leq \delta$ the unitary group $\mathrm{e}^{\mathrm{i}tH_I}$, with $H_I := \overline{L_{AW} + V_I}$ and

$$V_I := \int_I :P(\phi(0,x)): \mathrm{d}x,$$

induces the correct dynamics $\tau$ on $\mathcal{R}_{AW}(J)$. Applying then Proposition 2.5, we obtain

$$\mathrm{e}^{\mathrm{i}tL_{AW}} = \text{s-}\lim_{n\to\infty} \mathrm{e}^{\mathrm{i}tH_I^{(n)}},\ t \in \mathbb{R},$$

for $H_I^{(n)} = \overline{L_{AW} + V_I - V_I^{(n)}}$, where $V_I^{(n)} = V_I \mathbb{1}_{\{|V_I| \leq n\}}$. Since $V_I^{(n)}$ is bounded,

$$H_I^{(n)} = \overline{L_{AW} + V_I} - V_I^{(n)} = H_I - V_I^{(n)},$$

and hence by Trotter's formula

$$\mathrm{e}^{\mathrm{i}tH_I^{(n)}} = \text{s-}\lim_{p\to\infty} \big(\mathrm{e}^{\mathrm{i}tH_I/p} \mathrm{e}^{-\mathrm{i}tV_I^{(n)}/p}\big)^p.$$

This yields, for $A \in \mathcal{R}_{AW}(J)$,

$$\mathrm{e}^{\mathrm{i}tL_{AW}} A \mathrm{e}^{-\mathrm{i}tL_{AW}} = \text{s-}\lim_{n\to\infty} \text{s-}\lim_{p\to\infty} \big(\mathrm{e}^{\mathrm{i}tH_I/p} \mathrm{e}^{-\mathrm{i}tV_I^{(n)}/p}\big)^p A \big(\mathrm{e}^{\mathrm{i}tV_I^{(n)}/p} \mathrm{e}^{-\mathrm{i}tH_I/p}\big)^p.$$

Using again Theorem 6.4 we obtain, for $|t| < \alpha$,

$$\big(\mathrm{e}^{\mathrm{i}tH_I/p} \mathrm{e}^{-\mathrm{i}tV_I^{(n)}/p}\big)^p A \big(\mathrm{e}^{\mathrm{i}tV_I^{(n)}/p} \mathrm{e}^{-\mathrm{i}tH_I/p}\big)^p = \big(\tau_{t/p} \circ \gamma_{t/p}^{(n)}\big)^p(A),$$

where $\gamma^{(n)}$ is the dynamics implemented by the unitary group $t \mapsto \mathrm{e}^{-\mathrm{i}tV_I^{(n)}}$. Since $V_I$ is affiliated to $\mathcal{U}_{AW}(I)$, $\mathrm{e}^{-\mathrm{i}tV_I^{(n)}} \in \mathcal{U}_{AW}(I)$ and hence $\big(\tau_{t/p} \circ \gamma_{t/p}^{(n)}\big)^p(A) \in \mathcal{B}_\alpha(I)$ for $|t| < \alpha$. Since $\mathcal{B}_\alpha(I)$ is weakly closed, we obtain (6.15).

Let us now prove (6.14). Clearly the operators $W_{AW}(h)$ for $h \in \mathfrak{h}_J$ and $h$ real valued belong to $\mathcal{U}_{AW}(J)$ and hence to $\mathcal{B}_\alpha(I)$. Let us now pick $h \in \mathfrak{h}_J \cap \mathcal{D}(\epsilon)$ and $h$ real valued. (This is possible; see the discussion presented at the end of Subsection 6.1). Applying (6.15) to $A = W_{AW}(h)$, we obtain that $W_{AW}(\mathrm{e}^{\mathrm{i}t\epsilon}h) \in \mathcal{B}_\alpha(I)$ for $|t| < \alpha$. Hence $W_{AW}\big(t^{-1}(\mathrm{e}^{\mathrm{i}t\epsilon}h - h)\big) \in \mathcal{B}_\alpha(I)$ for $|t| < \alpha$. Letting $t \to 0$ and using the fact that the map $\mathfrak{h} \ni h \mapsto W_{AW}(h)$ is continuous for the strong operator topology, we obtain that $W_{AW}(\mathrm{i}\epsilon h) \in \mathcal{B}_\alpha(I)$. But any vector $h \in \mathfrak{h}_J$ can be approximated in norm by vectors of the form $h_1 + \mathrm{i}\epsilon h_2$, with $h_i \in \mathfrak{h}_J$ real and $h_2 \in \mathcal{D}(\epsilon)$. This implies that for all $h \in \mathfrak{h}_J$ the operators $W_{AW}(h)$ belong to $\mathcal{B}_\alpha(I)$ and hence $\mathcal{R}_{AW}(J) \subset \mathcal{B}_\alpha(I)$. This completes the proof of the proposition $\square$.

## 6.5 Existence of the limiting state

**Theorem 6.6** (Existence of limiting state). *Let $\{\omega_l\}_{l>0}$ be the family of $(\tau^l, \beta)$-KMS states for the spatially cutoff $P(\phi)_2$ models constructed in Subsection 5.3.*
    *Then*

$$\mathrm{w} - \lim_{l \to +\infty} \omega_l =: \omega_\beta \text{ exists on } \mathcal{A}.$$

*The state $\omega_\beta$ on $\mathcal{A}$ has the following properties:*

   (i)   *$\omega_\beta$ is a $(\tau, \beta)$-KMS state on $\mathcal{A}$;*

   (ii)  *$\omega_\beta$ is locally normal, i.e., if $I$ is an open and bounded interval, then $\omega_{\beta|\mathcal{R}_{AW}(I)}$ is normal w.r.t. the Araki-Woods representation;*

   (iii) *$\omega_\beta$ is invariant under spatial translations, i.e.,*

$$\omega_\beta(\alpha_x(A)) = \omega_\beta(A), \quad x \in \mathbb{R}, \ A \in \mathcal{A};$$

   (iv) *$\omega_\beta$ has the spatial clustering property, i.e.,*

$$\lim_{x \to \infty} \omega_\beta(A\alpha_x(B)) = \omega_\beta(A)\omega_\beta(B) \quad \forall A, B \in \mathcal{A}.$$

**Remark 6.7** *Let $\mathcal{R}$ be a $C^*$-algebra, $\pi_i \colon \mathcal{R} \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, two quasi-equivalent representations of $\mathcal{R}$. Then there exists a $*$-isomorphism $\tau$ between $\pi_1(\mathcal{R})''$ and $\pi_2(\mathcal{R})''$ intertwining the two representations. This isomorphism is automatically weakly continuous. Therefore the representation $\pi_2$ extends uniquely from $\mathcal{R}$ to $\pi_1(\mathcal{R})''$ and is quasi-equivalent to the concrete representation of $\pi_1(\mathcal{R})''$ in $\mathcal{B}(\mathcal{H}_1)$.*

    *Applying this easy observation to the representations $\pi_{AW}$ and $\pi_F$ of $\mathcal{W}(\mathfrak{h}_I)$, which are quasi-equivalent by Lemma 6.3, we see that the Fock representation $\pi_F$ extends by weak continuity from $\pi_{AW}(\mathcal{W}(\mathfrak{h}_I))$ to $\mathcal{R}_{AW}(I)$ and is quasi-equivalent to the Araki-Woods representation. Since two quasi-equivalent representations have the same set of normal states, we obtain that $\omega_{\beta|\mathcal{R}_{AW}(I)}$ is also normal with respect to the Fock representation.*

    **Proof.** The family $\{\omega_l\}_{l>0}$ of states on $\mathcal{A}$ is weak$^*$ compact by the Banach-Alaoglu theorem. Let $\omega_1$ be one of the limit points of $\{\omega_l\}_{l>0}$. Then we can find a subnet[1] $\{\omega^r\}_{r \in R}$ such that $\omega_1 = \mathrm{w} - \lim_{r \in R} \omega^r$.

    We claim that $\omega_1$ is a $(\tau, \beta)$-KMS state. Let $A, B \in \mathcal{A}$. Writing

$$\omega_1(A\tau_t(B)) - \omega^r(A\tau_t^{l_r}(B)) = (\omega_1 - \omega^r)(A\tau_t(B)) + \omega^r\big(A\tau_t(B) - A\tau_t^{l_r}(B)\big)$$

and using that $\lim_{l \to \infty} \|\tau^l(A) - \tau_t(A)\| = 0$ for $A \in \mathcal{A}$ and $t \in \mathbb{R}$ fixed, we find

$$(6.16) \qquad \omega_1(A\tau_t(B)) = \lim_{r \in R} \omega^r\big(A\tau_t^{l_r}(B)\big), \ t \in \mathbb{R}.$$

The same argument shows

$$(6.17) \qquad \omega_1(\tau_t(B)A) = \lim_{r \in R} \omega^r\big(\tau_t^{l_r}(B)A\big), \ t \in \mathbb{R}.$$

Since the $\omega^r$'s are $(\tau^{l_r}, \beta)$-KMS states there exist functions $F^r(z)$, which are holomorphic in $I_\beta^+ = \{0 < \mathrm{Im}z < \beta\}$ and continuous in $\overline{I_\beta^+}$, such that $F^r(t) = \omega^r\big(A\tau_t^{l_r}B\big)$ and $F^r(t + \mathrm{i}\beta) = \omega^r\big(\tau_t^{l_r}(B)A\big)$. Moreover, one has $\sup_{z \in I_\beta} |F^r(z)| \leq \|A\|\|B\|$. Applying Vitali's theorem and

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[1]A net $\{y_\beta\}_{\beta \in B}$ is a subnet of a net $\{x_\alpha\}_{\alpha \in A}$ if there exists a map $B \ni \beta \mapsto \alpha(\beta) \in A$ such that: i.) $y_\beta = x_{\alpha(\beta)}$ for all $\beta \in B$; ii.) for all $\alpha_0 \in A$ there exists some $\beta_0$ such that $\alpha(\beta) \geq \alpha_0$ whenever $\beta \geq \beta_0$.

possibly extracting a subnet, we know that $\lim_{r\to\infty} F^r(z) = F(z)$ exists and is holomorphic and bounded in $I_\beta^+$. By Lemma B.3, we obtain that $F$ is continuous on $\overline{I_\beta^+}$ and

$$F(t) = \lim_{r\to\infty} F^r(t), \quad F(t+\mathrm{i}\beta) = \lim_{r\to\infty} F^r(t+\mathrm{i}\beta).$$

Using (6.16) and (6.17) this implies that $\omega_1$ is a $(\tau,\beta)$-KMS state.

We now apply a result of Takesaki and Winnink [TW]: clearly $I \to \{\mathcal{R}_{AW}(I)\}$ is a net of von Neumann algebras (see [TW, Section 2]). The algebras $\mathcal{R}_{AW}(I)$ are $\sigma$-finite, since the Hilbert space $\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})$ on which they act is separable. The algebras $\mathcal{R}_{AW}(I)$ have separable preduals since there exists a faithful normal representation (namely the defining Araki-Woods representation $\pi_{AW}$) of $\mathcal{R}_{AW}(I)$ on a separable Hilbert space (namely the Araki-Woods Hilbert space $\mathcal{H}_{AW}$). Moreover, as factors on a separable infinite dimensional Hilbert space they are properly infinite. Applying the following theorem [TW, Theorem 1], we obtain that the KMS state $\omega_1$ is *normal* on $\mathcal{R}_{AW}(I)$

**Theorem 6.8** *Let $\mathcal{A}$ be a $C^*$-algebra, $\mathbb{R} \ni t \mapsto \tau_t$ a one-parameter group of $*-$automorphisms of $\mathcal{A}$ and $\omega$ a $(\tau,\beta)-$KMS state on $\mathcal{A}$. If there exists a net of $\sigma$-finite, properly infinite von Neumann algebras $\mathcal{M}_\alpha$ with separable preduals such that*

(i)    *to all pairs $\mathcal{M}_\alpha$, $\mathcal{M}_\beta$ in $\{\mathcal{M}_\alpha\}_{\alpha\in\Gamma}$ there exists $\mathcal{M}_\gamma$ with the property*

$$\mathcal{M}_\alpha \cup \mathcal{M}_\beta \subset \mathcal{M}_\gamma;$$

(ii)   *every $\mathcal{M}_\alpha$ contains the unit of $\mathcal{A}$;*

(iii)  *$\mathcal{A}$ is the norm closure of the von Neumann algebras $\mathcal{M}_\alpha$, i.e.,*

$$\mathcal{A} := \overline{\bigcup_{\alpha\in\Gamma} \mathcal{M}_\alpha}^{(C^*)};$$

*then $\omega$ is locally normal, i.e. the restriction of $\omega$ to each von Neumann algebra $\mathcal{M}_\alpha$ is a normal state.*

Let us now show that all limit states are identical. Let us denote by $\mathcal{U}_0(I)$ the abelian $C^*$-algebra generated by

$$\left\{ F(\phi_{AW}(h)) \mid h \in C_{0\,\mathbb{R}}^\infty(I),\ F \in C_0^\infty(\mathbb{R}) \right\}$$

and by $\mathcal{T}_\alpha(I)$ the $*$-algebra generated by $\left\{ \tau_t(A) \mid A \in \mathcal{U}_0(I),\ |t| < \alpha \right\}$.

From Theorem 6.4 and Proposition 7.6 we deduce that

$$\lim_{l\to\infty} \omega_l \Big(\prod_1^n \tau_{t_i}(A_i)\Big) = \tilde{\omega}\Big(\prod_1^n \tilde{\tau}_{t_i}(A_i)\Big) = \omega_1\Big(\prod_1^n \tau_{t_i}(A_i)\Big),\ \ A_i \in \mathcal{U}_0(I),\ t_i \in \mathbb{R},$$

where $\tilde{\omega}$ and $\tilde{\tau}$ are defined in Subsection 7.2. Therefore all weak accumulation points of $\{\omega_l\}_{l>0}$ coincide on the algebras $\mathcal{T}_\alpha(I) \subset \mathcal{R}_{AW}(I+] - \alpha,\alpha[)$. We note that $\mathcal{T}_\alpha(I)$ is weakly dense in the von Neumann algebra $\mathcal{B}_\alpha(I)$ defined in (6.13). Moreover, we have seen that all limit states are normal on the local algebras $\mathcal{R}_{AW}(I)$, $I$ open and bounded. Therefore they coincide on the von Neumann algebras $\mathcal{B}_\alpha(I)$, and hence by Proposition 6.5 on $\mathcal{R}_{AW}(I)$. Consequently, they also coincide on the norm closure $\mathcal{A}$. Thus the weak$^*$ compact family $\{\omega_l\}_{l>0}$ has a unique accumulation point, which implies that

$$\omega_\beta := \mathrm{w} - \lim_{l\to\infty} \omega_l \text{ exists on } \mathcal{A}.$$

We have already seen that $\omega_\beta$ is a locally normal $(\tau,\beta)$-KMS state on $\mathcal{A}$, which completes the proof of (i) and (ii). Property (iii) follows from the invariance of the state $\tilde{\omega}$ under

31

space translations shown in Lemma 7.7 and the same density argument as above. It remains to prove (iv). Let $(\mathcal{H}_\beta, \pi_\beta, \Omega_\beta)$ denote the GNS objects associated to $(\mathcal{A}, \omega_\beta)$. The group $\{\alpha_x\}_{x \in \mathbb{R}}$ is implemented in $\mathcal{H}_\beta$ by a strongly continuous group of unitary operators $\{e^{\mathrm{i}x P_\beta}\}_{x \in \mathbb{R}}$ with $P_\beta \Omega_\beta = 0$. Lemma 7.7 (ii) implies that, for $A, B \in \mathcal{T}_\alpha(I)$,

(6.18) $$\lim_{x \to \infty} \big(\pi_\beta(A)\Omega_\beta, e^{\mathrm{i}x P_\beta}\pi_\beta(B)\Omega_\beta\big) = \big(\pi_\beta(A)\Omega_\beta, \Omega_\beta\big)(\Omega_\beta, \pi_\beta(B)\Omega_\beta).$$

Since $\mathcal{R}_{AW}(I)$ is a factor, the representation $\pi_\beta$ provides a weakly continuous *-isomorphism between $\mathcal{R}_{AW}(I)$ and $\mathcal{R}_\beta(I) = \pi_\beta(\mathcal{R}_{AW}(I)) = \pi_\beta(\mathcal{R}_{AW}(I))''$. Hence, by the same weak density argument as above, (6.18) extends to all $A, B \in \mathcal{R}_{AW}(I)$. Thus the space clustering property holds on $\mathcal{R}_{AW}(I)$ for all $I$, $I$ open and bounded, and extends to $\mathcal{A}$ by norm density.

# 7 Construction of the interacting path space

In this section we construct the interacting path space supported by $\mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ describing the translation invariant $P(\phi)_2$ model at temperature $\beta^{-1}$ and study some of its properties.

## 7.1 Construction of the interacting measure

Let $H^{\mathrm{ren}}_{\mathsf{C}} = H^{\mathrm{ren}}_{\mathsf{C}} - E_{\mathsf{C}}$ be the renormalized $P(\phi)_2$ Hamiltonian on $S_\beta$ defined in Subsection 5.2. Let $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$. For $x \in \mathbb{R}$ the function $f_x$ defined in Subsection 3.5 belongs to $\mathcal{S}_{\mathbb{R}}(S_\beta)$. We will apply the results of Appendix A to the selfadjoint operator $H = H^{\mathrm{ren}}_{\mathsf{C}}$, $R(x) = \phi_F(f_x)$ (replacing the variable $t$ in Appendix A by the variable $x$).

It follows from the bound (5.7) in Proposition 5.4 and the fact that the map

$$\begin{aligned} \mathbb{R} &\to \mathcal{B}\big(\Gamma(H^{-\frac{1}{2}}(S_\beta))\big) \\ x &\mapsto \phi(f_x)(H^{\mathrm{ren}}_{\mathsf{C}} + 1)^{-\frac{1}{2}} \end{aligned}$$

is infinitely differentiable that the hypothesis (A.3) in Subsection A.1 is satisfied. Similarly, using the bound (5.8) and the fact that the map $x \mapsto \|f_x\|_{H^{-\frac{1}{2}}(S_\beta)}$ is in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we see that hypotheses (A.7) and (A.8) in Subsection A.2 is satisfied. Therefore we can apply all the abstract results from Subsections A.1 and A.2. In particular there exists a solution $U(b, a)$ of the time-dependent heat equation:

$$\frac{\mathrm{d}}{\mathrm{d}b}U(b, a) = (-H^{\mathrm{ren}}_{\mathsf{C}} + \mathrm{i}\phi_F(f_b))U(b, a),\ U(a, a) = \mathbb{1}.$$

We will set for $-\infty \leq a \leq b \leq +\infty$:

$$W_{[a,b]}(f) := U(b, a)^*.$$

**Proposition 7.1** *Let $f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and assume that* $\operatorname{supp} f \subset S_\beta \times [-a, a]$. *Then*

$$\int_Q e^{\mathrm{i}\phi(f)}G_{[-l,l]}\mathrm{d}\phi_C = e^{-2lE_{\mathsf{C}}}\big(e^{-(l-a)H^{\mathrm{ren}}_{\mathsf{C}}}\Omega^\circ_{\mathsf{C}}, W_{[-a,a]}(f)e^{-(l-a)H^{\mathrm{ren}}_{\mathsf{C}}}\Omega^\circ_{\mathsf{C}}\big),$$

*where $\Omega^\circ_{\mathsf{C}}$ is the free vacuum on $\Gamma\big(H^{-\frac{1}{2}}(S_\beta)\big)$.*

**Proof.** Let us first introduce a notation which we will use throughout the proof. If $A$ is a $\Sigma$-measurable function on $Q$, the image of $A$ under $U_{\mathsf{C}}(x)$ for $x \in \mathbb{R}$ will be denoted by $U_{\mathsf{C}}(x)(A)$. On the other hand, the expression $U_{\mathsf{C}}(x)A$ will denote the operator product of the operator $U_{\mathsf{C}}(x)$ and the operator of multiplication by $A$, acting on $L^2(Q, \Sigma, \mathrm{d}\phi_C)$.

Using Lemma 3.1 we find

$$e^{\mathrm{i}\phi(f)} = e^{\mathrm{i}\int_{-a}^a \phi(f_x, x)\mathrm{d}x} = e^{\mathrm{i}\int_{-a}^a U_{\mathsf{C}}(x)(\phi(f_x, 0))\mathrm{d}x}.$$

We will approximate the above integral using Riemann sums. Let $n, p \in \mathbb{N}$ and $0 \leq j \leq 2np$. We set $x_j = -a + j\frac{a}{np}$ and $z_j = -a + [j/p]\frac{a}{n}$, where $[.]$ denotes the integer part.

It follows from (3.15) that the map $x \mapsto \phi(f_x, x) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\phi_c)$ is continuous. Therefore

$$\int_{-a}^{a} U_{\mathsf{C}}(x)\big(\phi(f_x, 0)\big)dx = \lim_{n,p \to \infty} \sum_{j=0}^{2np-1} (x_{j+1} - x_j)U_{\mathsf{C}}(x_j)\big(\phi(f_{z_j}, 0)\big)$$

in $\bigcap_{1 \leq p \leq \infty} L^p(Q, \Sigma, d\phi_C)$ and hence

(7.1)
$$\begin{aligned}
e^{i\int_{-a}^{a} U_{\mathsf{C}}(x)(\phi(f_x,0))dx} &= \lim_{n,p \to \infty} \prod_{j=0}^{2np-1} e^{i(x_{j+1}-x_j)U_{\mathsf{C}}(x_j)(\phi(f_{z_j},0))} \\
&= \lim_{n,p \to \infty} \prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}\big)
\end{aligned}$$

in $\bigcap_{1 \leq p \leq \infty} L^p(Q, \Sigma, d\phi_C)$, where in the last line we use the fact that $U_{\mathsf{C}}(x_j)$ is an automorphism of $L^\infty(Q, \Sigma, d\phi_C)$. Since $G_{[a,b]}$ is a FKN kernel,

$$\begin{aligned}
G_{[-l,l]} &= G_{[-l,-a]}\Big[\prod_{j=0}^{2np-1} G_{[x_j,x_{j+1}]}\Big]G_{[a,l]} \\
&= G_{[-l,-a]}\Big[\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(G_{[0,x_{j+1}-x_j]}\big)\Big]G_{[a,l]}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&G_{[-l,l]} \prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}\big) \\
=\ &G_{[-l,-a]}\Big[\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}G_{[0,x_{j+1}-x_j]}\big)\Big]G_{[a,l]}.
\end{aligned}$$

Next let $A_j$, $0 \leq j < 2np-1$, be the multiplication operators by $\Sigma$-measurable functions. Using the identity $U_{\mathsf{C}}(x_j)(A_j) = U_{\mathsf{C}}(x_j)A_jU_{\mathsf{C}}(-x_j)$ and the fact that $U_{\mathsf{C}}(x)$ is an automorphism of $L^\infty(Q, \Sigma, d\phi_C)$, we obtain as an operator identity on $L^2(Q, \Sigma, d\phi_C)$:

$$\prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)(A_j) = U_{\mathsf{C}}(x_0) \prod_{j=0}^{2np-1} A_jU_{\mathsf{C}}(x_{j+1} - x_j)U_{\mathsf{C}}(-x_{2np}).$$

In the above identity the product on the l.h.s. is the operator of multiplication by the product of the functions $U_{\mathsf{C}}(x_j)(A_j)$ and the product on the r.h.s. is an operator product. Using that $x_0 = -x_{2np} = -a$ and that $U_{\mathsf{C}}(-a)^* = U_{\mathsf{C}}(a)$ we get

$$\begin{aligned}
&\int_Q G_{[-l,l]} \prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}\big)d\phi_C \\
=\ &\int_Q G_{[-l,-a]}U_{\mathsf{C}}(-a)\Big[\prod_{j=0}^{2np-1} e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}U_{\mathsf{C}}(x_{j+1} - x_j)G_{[0,x_{j+1}-x_j]}\Big]U_{\mathsf{C}}(-a)G_{[a,l]}d\phi_C \\
=\ &\int_Q G_{[-l+a,0]}\Big[\prod_{j=0}^{2np-1} e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}U_V(x_{j+1} - x_j)\Big]G_{[0,l-a]}d\phi_C
\end{aligned}$$

for $U_V(s) = G_{[0,s]}U_{\mathsf{C}}(s)$.

Let us now set for $0 \leq k \leq 2n$, $y_k = -a + k\frac{a}{n}$. We note that $z_j = y_{[j/p]}$ and that $(x_{j+1} - x_j) = \frac{a}{np} = (y_{k+1} - y_k)/p$. We obtain that

$$\begin{aligned}
&\int_Q G_{[-l,l]} \prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\big(e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}\big)d\phi_C \\
=\ &\int_Q G_{[-l+a,0]} \prod_{k=0}^{2n-1} \big(e^{i(y_{k+1}-y_k)\phi(f_{y_k},0)/p}U_V\big(\tfrac{y_{k+1}-y_k}{p}\big)\big)^P G_{[0,l-a]}d\phi_C \\
=\ &\int_Q R_{\mathsf{C}}(G_{[0,l-a]}) \prod_{k=0}^{2n-1} \big(e^{i(y_{k+1}-y_k)\phi(f_{y_k},0)/p}U_V\big(\tfrac{y_{k+1}-y_k}{p}\big)\big)^P G_{[0,l-a]}d\phi_C.
\end{aligned}$$

33

Taking into account the construction of $H_{\mathsf{C}}$ recalled in Subsection 5.2 we find

$$\int_Q G_{[-l,l]} \prod_{j=0}^{2np-1} U_{\mathsf{C}}(x_j)\left(e^{i(x_{j+1}-x_j)\phi(f_{z_j},0)}\right)d\phi_C$$

$$= \left(e^{-(l-a)H_{\mathsf{C}}}\Omega_{\mathsf{C}}^{\circ}, \prod_{k=0}^{2n-1}\left(e^{i(y_{k+1}-y_k)\phi(f_{y_k})/p}e^{-(y_{k+1}-y_k)H_{\mathsf{C}}/p}\right)^P e^{-(l-a)H_{\mathsf{C}}}\Omega_{\mathsf{C}}^{\circ}\right)$$

$$= e^{-2lE_{\mathsf{C}}}\left(e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}, \prod_{k=0}^{2n-1}\left(e^{i(y_{k+1}-y_k)\phi(f_{y_k})/p}e^{-(y_{k+1}-y_k)H_{\mathsf{C}}^{\mathrm{ren}}/p}\right)^P e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}\right).$$

Letting now $n$ and $p$ tend to $\infty$ and using Proposition A.5, we obtain the proposition $\square$.

**Theorem 7.2**

(i) Let $f \in C_{0\,\mathrm{IR}}^{\infty}(S_\beta \times \mathrm{IR})$. Then

$$\lim_{l\to+\infty} \int e^{i\phi(f)}d\mu_l = (\Omega_{\mathsf{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathsf{C}}),$$

where $\Omega_{\mathsf{C}}$ is the unique vacuum state of $H_{\mathsf{C}}$.

(ii) The map

$$\mathcal{S}_{\mathrm{IR}}(S_\beta \times \mathrm{IR}) \ni f \mapsto (\Omega_{\mathsf{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathsf{C}})$$

is the generating functional of a Borel probability measure $\mu$ on $(Q, \Sigma)$.

(iii) The measure $\mu$ is invariant under space translations $\{\mathfrak{a}_x\}_{x\in\mathrm{IR}}$, time translations $\{\mathfrak{T}_t\}_{t\in S_\beta}$ and the time reflection $r$.

(iv) The functions $\phi(f)$ belong to $\bigcap_{1\leq p<\infty} L^p(Q, \Sigma, \mu)$ for $f \in \mathcal{S}_{\mathrm{IR}}(S_\beta \times \mathrm{IR})$. Moreover,

$$\int_Q \phi(f)^n d\mu = n! \int_{-\infty<x_1\leq\cdots\leq x_n<\infty} \left(\Omega_{\mathsf{C}}, \left[\prod_1^{n-1} \phi(f_{x_k})e^{-(x_{k+1}-x_k)H_{\mathsf{C}}^{\mathrm{ren}}}\right]\phi(f_{x_n})\Omega_{\mathsf{C}}\right)dx_1\ldots dx_n.$$

(v) Let $f_i \in C_{0\,\mathrm{IR}}^{\infty}(S_\beta \times \mathrm{IR})$ for $1 \leq i \leq n$. Then

$$\lim_{l\to+\infty} \int_Q \prod_{i=1}^{n} \phi(f_i)d\mu_l = \int_Q \prod_{i=1}^{n} \phi(f_i)d\mu.$$

**Proof.** Note first that applying Proposition 7.1 for $f = 0$, we obtain $W_{[-a,a]}(0) = e^{-2aH_{\mathsf{C}}^{\mathrm{ren}}}$:

$$\int_Q G_{[-l,l]}d\phi_C = e^{-2lE_{\mathsf{C}}}\left(e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}, e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}\right)..$$

Let $f \in C_{0\,\mathrm{IR}}^{\infty}(S_\beta \times \mathrm{IR})$ with $\mathrm{supp}\,f \subset S_\beta \times [-a,a]$ for some $a \in \mathrm{IR}$. Using Proposition 7.1 we find

$$\int e^{i\phi(f)}d\mu_l = \frac{\left(e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}, W_{[-a,a]}(f)e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}\right)}{\left(e^{-lH_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}, e^{-lH_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ}\right)}.$$

Now $\lim_{l\to+\infty} e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^{\circ} = (\Omega_{\mathsf{C}}, \Omega_{\mathsf{C}}^{\circ})\Omega_{\mathsf{C}}$, where $\Omega_{\mathsf{C}}$ is the eigenvector for the simple eigenvalue $\{0\}$ of $H_{\mathsf{C}}^{\mathrm{ren}}$. Thus

$$\lim_{l\to+\infty} \int e^{i\phi(f)}d\mu_l = (\Omega_{\mathsf{C}}, W_{[-a,a]}(f)\Omega_{\mathsf{C}}),$$

34

Because supp $f \subset S_\beta \times [-a, a]$, we see that $\left(\Omega_{\mathsf{C}}, W_{[s,t]}(f)\Omega_{\mathsf{C}}\right)$ is constant for $s \leq -a$, $t \geq a$, which proves (i).

To prove (ii) we apply Minlos theorem (see e.g. [GV]). As a limit of functionals of Borel probability measures on $(Q, \Sigma)$ the functional $f \mapsto \left(\Omega_{\mathsf{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathsf{C}}\right)$ is of positive type. It remains to show that the map

$$
\begin{aligned}
\mathcal{S}(S_\beta \times \mathbb{R}) &\rightarrow & \mathbb{C} \\
f &\mapsto & \left(\Omega_{\mathsf{C}}, W_{[-\infty,\infty]}(f)\Omega_{\mathsf{C}}\right)
\end{aligned}
$$

is continuous. Using the bound (5.8) we obtain

$$
\pm\left(\phi_F(f_{2,x}) - \phi_F(f_{1,x})\right) \leq Cr(x)(H_{\mathsf{C}}^{\mathrm{ren}} + 1)^{\frac{1}{2}} \text{ for } f_1,\ f_2 \in \mathcal{S}(S_\beta \times \mathbb{R}),
$$

where $C > 0$ is some constant and

$$
r(x) := \|(f_{2,x} - f_{1,x})\|_{H^{-\frac{1}{2}}(S_\beta)}.
$$

Clearly $\|r\|_{L^2(\mathbb{R})} \leq C\|f_1 - f_2\|_p$, where $\|.\|_p$ is a Schwartz semi-norm on $\mathcal{S}(S_\beta \times \mathbb{R})$. Applying Lemma A.7 for $\delta = \frac{1}{2}$, we obtain

$$
\left\|W_{[-\infty,+\infty]}(f_2) - W_{[-\infty,+\infty]}(f_1)\right\| \leq C\|f_2 - f_1\|_p,
$$

which proves the desired continuity result.

Let us now verify (iii). The measure $\mu$ is invariant under time translations and time reflection as the weak limit of the time translation and time reflection invariant measures $\mu_l$. The fact that $\mu$ is invariant under space translations follows directly from (i) and Remark A.9.

To prove (iv) we apply Lemma B.2, using the estimates in Proposition A.6 (ii). We obtain that $\phi(f) \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, \mu)$. The formula in (iv) follows from Proposition A.6 (iii).

It remains to prove (v). Let $f \in C_{0\,\mathrm{IR}}^\infty(S_\beta \times \mathbb{R})$ with supp $f \subset S_\beta \times [-a, a]$. We consider the family of functions

$$
u_l(\lambda) = \int e^{\mathrm{i}\lambda\phi(f)} \mathrm{d}\mu_l \text{ for } \lambda \in \mathbb{C}.
$$

Since $e^{\phi(f)} \in \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, \mathrm{d}\phi_C)$ and $F_{[-\beta/2,\beta/2]}^l \in L^{1+\epsilon}(Q, \Sigma, \mathrm{d}\phi_C)$, the functions $u_l(\lambda)$ are entire and

$$
(7.2) \qquad\qquad \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} u_l(0) = \mathrm{i}^n \int \phi(f)^n \mathrm{d}\mu_l.
$$

Using Proposition 7.1 and $\lambda\phi(f) = \phi(\lambda f)$ for $\lambda \in \mathbb{R}$ we find

$$
u_l(\lambda) = \frac{\left(W_{[-a,a]}(\lambda f)e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^\circ, e^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^\circ\right)}{\|e^{-lH_{\mathsf{C}}^{\mathrm{ren}}}\Omega_{\mathsf{C}}^\circ\|} \quad \text{for } \lambda \in \mathbb{R}.
$$

The r.h.s. is an entire function by Lemma A.3. Therefore this identity extends to $\lambda \in \mathbb{C}$. Applying (5.8) and Proposition A.6 (i) with $\delta = 1/2$ we obtain

$$
(7.3) \qquad\qquad |u_l(\lambda)| \leq e^{C|\mathrm{Im}\lambda|^2},\ l \in \mathbb{R}^+,\ \lambda \in \mathbb{C}.
$$

We have seen above that

$$
\lim_{l\to\infty} u_l(\lambda) = \int_Q e^{\mathrm{i}\lambda\phi(f)} \mathrm{d}\mu \text{ for } \lambda \in \mathbb{R}.
$$

By Vitali's theorem we obtain

$$
\lim_{l\to\infty} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} u_l(0) = \mathrm{i}^n \int_Q \phi(f)^n \mathrm{d}\mu.
$$

Using (7.2) and multi-linearity, this proves (v) □.

## 7.2 Existence and properties of sharp-time fields

**Proposition 7.3** *Let $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $t \in S_\beta$. Then the sequence $\phi\big(\delta_k(.-t) \otimes h\big)$ is Cauchy in $\bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mu)$ and hence*

$$\phi(t, h) := \lim_{k \to \infty} \phi\big(\delta_k(.-t) \otimes h\big) \in \bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mu).$$

*Moreover, the map*

$$\begin{array}{ccc} S_\beta & \to & \bigcap_{1 \le p < \infty} L^p(Q, \Sigma, \mu) \\ t & \mapsto & \phi(t, h) \end{array}$$

*is continuous for each $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$.*

**Proof.** For $p \ge 1$ we have

$$\int_Q \big(\phi(\delta_k(.-t) \otimes h) - \phi(\delta_{k'}(.-t) \otimes h)\big)^{2p} \mathrm{d}\mu$$

$$= (-\mathrm{i})^{2p} \frac{\mathrm{d}^{2p}}{\mathrm{d}\lambda^{2p}} \Big(\Omega_{\mathsf{C}}, W_{[-\infty,+\infty]}\big(\lambda(\delta_k(.-t) \otimes h - \delta_{k'}(.-t) \otimes h)\big)\Omega_{\mathsf{C}}\Big)_{|\lambda=0}.$$

If $f = \delta_k(.-t) \otimes h$, then for $x \in \mathbb{R}$ the function $f_x \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ is equal to $\delta_k(.-t)h(x)$. It follows then from the estimate (5.9) in Proposition 5.4 that

$$\pm\big(\phi_F(\delta_k(.-t)h(x)) - \phi_F(\delta_{k'}(.-t)h(x))\big) \le c \, \|\delta_k(.-t) - \delta_{k'}(.-t)\|_{H^{-1}(S_\beta)} |h(x)| \big(H_{\mathsf{C}}^{\mathrm{ren}} + 1\big).$$

Applying now Lemma A.8 we obtain that

(7.4)
$$\big\| \tfrac{\mathrm{d}^{2p}}{\mathrm{d}\lambda^{2p}} W_{[-\infty,+\infty]}\big(\lambda(\delta_k(.-t) \otimes h - \delta_{k'}(.-t) \otimes h)\big)\big\|$$

$$\le c_p \|\delta_k(.-t) - \delta_{k'}(.-t)\|_{H^{-1}(S_\beta)}^{2p} \|h\|_\infty^{2p} \, \mathrm{e}^{\|h\|_1 \|h\|_\infty^{-1}}.$$

Since $\delta_k(.-t)$ converges to $\delta(.-t)$ in $H^{-1}(S_\beta)$, we see that $\phi\big(\delta_k(.-t) \otimes h\big)$ is Cauchy in $L^{2p}(Q, \Sigma, \mu)$. A similar argument shows that $t \mapsto \phi(t, h) \in L^{2p}(Q, \Sigma, \mu)$ is continuous, using the fact that $t \mapsto \delta(.-t) \in H^{-1}(S_\beta)$ is continuous $\square$.

Using the existence of sharp-time fields, we can equip the probability space $(Q, \Sigma, \mu)$ with an OS-positive $\beta$-periodic path space structure: We recall that $U(t)$ is the group of transformations generated by the (euclidean) time translations $\mathfrak{T}_t$ and $R$ is the transformation generated by time reflection, and $\Sigma_0$ is the sub$-\sigma$-algebra of $\Sigma$ generated by the functions $\{\phi(0, h) \mid h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})\}$.

**Theorem 7.4** *$(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is an OS-positive $\beta$-periodic generalized path space.*

**Proof.** Since the measure $\mu$ is invariant under time translations and time reflection, we see that $U(t)$ and $R$ are measure preserving automorphisms of $L^\infty(Q, \Sigma, \mu)$. Proposition 7.3 implies that the map $S_\beta \ni t \mapsto \mathrm{e}^{\mathrm{i}\phi(t,h)} \in L^2(Q, \Sigma, \mu)$ is continuous. Hence $U(t)$ is a strongly continuous group on $L^2(Q, \Sigma, \mu)$. This implies that $U(t)$ is strongly continuous in measure on $L^\infty(Q, \Sigma, \mu)$. Clearly it is $\beta$-periodic.

The generalized path space $(Q, \Sigma, \Sigma_0, U(t), R, \mu)$ is OS-positive, since $\mu$ is the weak limit of the measures $\mu_l$, which are associated to OS-positive path spaces. Finally we have already seen that $\Sigma = \bigvee_{t \in S_\beta} \Sigma_t$. This completes the proof of the theorem $\square$.

By the reconstruction theorem, we obtain a stochastically positive $\beta$-KMS system

$$(\tilde{\mathcal{B}}, \tilde{\mathcal{U}}, \tilde{\tau}, \tilde{\omega})$$

which describes the *translation invariant $P(\phi)_2$ model at temperature $\beta^{-1}$*.

## 7.3 Properties of the interacting $\beta$-KMS system

We first prove the convergence of sharp-time Schwinger functions.

**Proposition 7.5** *Let $h_i \in C_{0\,\mathrm{I\!R}}^\infty(\mathrm{I\!R})$ and $t_i \in S_\beta$ for $1 \le i \le n$. Then*

$$\lim_{l \to \infty} \int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j, h_j)} \mathrm{d}\mu_l = \int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j, h_j)} \mathrm{d}\mu.$$

**Proof.** Let $a > 0$ such that $\operatorname{supp} h_j \subset [-a, a]$. By Proposition 7.3, we know that

$$\phi(t_j, h_j) = \lim_{k \to \infty} \phi\big(\delta_k(. - t_j) \otimes h_j\big) \text{ in } L^1(Q, \Sigma, \mu).$$

After extracting a subsequence, this implies that

$$\phi(t_j, h_j) = \lim_{k \to \infty} \phi\big(\delta_k(. - t_j) \otimes h_j\big) \text{ pointwise } \mu \text{ a.e. on } Q,$$

and hence

(7.5)
$$\begin{aligned}
\int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j,h_j)} \mathrm{d}\mu &= \lim_{k \to \infty} \int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-t_j)\otimes h_j)} \mathrm{d}\mu \\
&= \lim_{k \to \infty} \big(\Omega_{\mathsf{C}}, W_{[-a,a]}\big(\textstyle\sum_1^n \delta_k(. - t_j) \otimes h_j\big) \Omega_{\mathsf{C}}\big),
\end{aligned}$$

by Theorem 7.2 (i). Note that for all $l > 0$

$$\phi(t_j, h_j) = \lim_{k \to \infty} \phi\big(\delta_k(. - t_j) \otimes h_j\big) \text{ in } L^1(Q, \Sigma, \mu_l),$$

because this convergence holds in $L^2(Q, \Sigma, \mathrm{d}\phi_C)$ and

$$\mathrm{d}\mu_l := \frac{G_{[-l,l]} \mathrm{d}\phi_C}{\int_Q G_{[-l,l]} \mathrm{d}\phi_C},$$

where $G_{[-l,l]} \in L^2(Q, \Sigma, \mathrm{d}\phi_C)$ as a consequence of (5.6). By the same arguments as above, we obtain

$$\int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j, h_j)} \mathrm{d}\mu_l = \lim_{k \to \infty} \frac{\big(\mathrm{e}^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}} \Omega_{\mathsf{C}}^{\mathrm{o}}, W_{[-a,a]}\big(\sum_1^n \delta_k(. - t_j) \otimes h_j\big) \mathrm{e}^{-(l-a)H_{\mathsf{C}}^{\mathrm{ren}}} \Omega_{\mathsf{C}}^{\mathrm{o}}\big)}{\|\mathrm{e}^{-lH_{\mathsf{C}}^{\mathrm{ren}}} \Omega_{\mathsf{C}}^{\mathrm{o}}\|^2}.$$

Let us denote by $F(k, l)$ the quantity on the r.h.s.. Applying (7.4) we obtain

$$\lim_{k \to \infty} F(k, l) = \int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(t_j, h_j)} \mathrm{d}\mu_l \text{ uniformly w.r.t. } l.$$

As we have seen, Theorem 7.2 (i) implies

$$\lim_{l \to \infty} F(k, l) = \int_Q \prod_1^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(.-t_j)\otimes h_j)} \mathrm{d}\mu.$$

Applying now Lemma B.1 (ii) and using (7.5) we obtain the proposition $\square$.

Let us denote by $\mathcal{U}_0 \subset \mathcal{B}\big(\Gamma(\mathfrak{h} \oplus \overline{\mathfrak{h}})\big)$ the $C^*$-algebra generated by

$$\big\{F\big(\phi_{AW}(h_1), \ldots, \phi_{AW}(h_n)\big) \mid h_i \in C_{0\,\mathrm{I\!R}}^\infty(\mathrm{I\!R}), \ F \in C_0^\infty(\mathrm{I\!R}^n), \ n \in \mathrm{I\!N}\big\}.$$

The isomorphism between $L^\infty(Q, \Sigma_0, \mathrm{d}\phi_C)$ and $\mathcal{U}_{AW}$, which we recalled in Subsection 2.3, maps the operator $F\big(\phi_{AW}(h_1), \ldots, \phi_{AW}(h_n)\big)$ onto the function $F\big(\phi(0, h_1), \ldots, \phi(0, h_n)\big)$. This function is $\Sigma_0$-measurable. We will still denote by $A$ the image of such a function $A$ in the abelian algebra $\tilde{\mathcal{U}}$ provided by the reconstruction theorem for the translation invariant $P(\phi)_2$ model.

37

**Proposition 7.6** *Let* $A_i \in \mathcal{U}_0$ *and* $t_i \in \mathbb{R}$, $1 \leq i \leq n$. *Then*

$$\lim_{l \to +\infty} \omega_l \Big( \prod_1^n \tau_{t_i}^l(A_i) \Big) = \tilde{\omega} \Big( \prod_1^n \tilde{\tau}_{t_i}(A_i) \Big).$$

**Proof.** Let us fix $A_i \in \mathcal{U}_0$ and set

$$G^l(t_1, \ldots, t_n) := \omega_l \Big( \prod_1^n \tau_{t_i}^l(A_i) \Big) \quad \text{and} \quad G(t_1, \ldots, t_n) = \tilde{\omega} \Big( \prod_1^n \tilde{\tau}_{t_i}(A_i) \Big).$$

Due to the KMS condition, the functions $G^l$ and $G$ are holomorphic in

$$I_\beta^{n+} = \big\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \mathrm{Im} z_i < \mathrm{Im} z_{i+1}, \ \mathrm{Im} z_n - \mathrm{Im} z_1 < \beta \big\},$$

continuous on $\overline{I_\beta^{n+}}$ and bounded by $\prod_1^n \|A_i\|$.

We first claim that

(7.6) $$\lim_{l \to \infty} G^l(\mathrm{i}s_1, \ldots, \mathrm{i}s_n) = G(\mathrm{i}s_1, \ldots, \mathrm{i}s_n) \text{ for } s_1 \leq \cdots \leq s_n, \ s_n - s_1 \leq \beta.$$

Using Proposition 7.5 and the identity (2.2) we see that (7.6) holds for $A_j = \mathrm{e}^{\mathrm{i}\phi_{AW}(h_j)}$, $h_j \in C_{0\,\mathbb{R}}^\infty(\mathbb{R})$. Using functional calculus we can extend (7.6) to arbitrary $A_j \in \mathcal{U}_0$.

Let us now consider, for $s_2 \leq \cdots \leq s_n$ and $s_n - s_2 \leq \beta$, the functions

$$u_l(z) := G^l(z, \mathrm{i}s_2, \ldots, \mathrm{i}s_n),$$

which are holomorphic in $\{0 < \mathrm{Im} z < s_2\}$ and continuous on $\{0 \leq \mathrm{Im} z \leq s_2\}$. Since the family $\{u_l\}$ is uniformly bounded, we can apply Lemma B.3. It follows that

$$\lim_{l \to +\infty} G^l(t_1, \mathrm{i}s_2, \ldots, \mathrm{i}s_n) = G(t_1, \mathrm{i}s_2, \ldots, \mathrm{i}s_n)$$

for $s_2 \leq \cdots \leq s_n$, $s_n - s_2 \leq \beta$ and $t_1 \in \mathbb{R}$. Iterating this argument, we obtain

$$\lim_{l \to +\infty} G^l(t_1, \ldots, t_n) = G(t_1, \ldots, t_n).$$

This completes the proof of the proposition $\square$.

Let us denote by $\{\alpha_x\}_{x \in \mathbb{R}}$ the group of space translations on $\mathcal{U}_0$ defined by $\alpha_x\big(W_{AW}(h)\big) = W_{AW}\big(h(\,.\,-x)\big)$ for $h \in C_{0\,\mathbb{R}}^\infty(\mathbb{R})$.

**Lemma 7.7** *Let* $A_j \in \mathcal{U}_0$ *and* $t_j \in \mathbb{R}$, $1 \leq j \leq n$. *Set* $A = \prod_{j=1}^k \tau_{t_j}(A_j)$ *and* $B = \prod_{j=k+1}^n \tau_{t_j}(A_j)$. *It follows that*

  (i) $\tilde{\omega}(\alpha_x(A)) = \tilde{\omega}(A)$ *for all* $x \in \mathbb{R}$;
  (ii) $\lim_{x \to \infty} \tilde{\omega}(A\alpha_x(B)) = \tilde{\omega}(A)\tilde{\omega}(B)$.

**Proof.** Property (i) follows directly from the invariance of the measure $\mu$ under the space translations $\{\alpha_x\}_{x \in \mathbb{R}}$ shown in Theorem 7.2 (iii). It remains to prove (ii). We set

$$G_x(t_1, \ldots, t_n) := \tilde{\omega}\big( \textstyle\prod_{j=1}^l \tau_{t_j}(A_j) \prod_{j=l+1}^n \alpha_x \circ \tau_{t_j}(A_j) \big),$$
$$G_\infty(t_1, \ldots, t_n) := \tilde{\omega}\big( \textstyle\prod_{j=1}^l \tau_{t_j}(A_j) \big) \cdot \tilde{\omega}\big( \textstyle\prod_{j=l+1}^n \tau_{t_j}(A_j) \big).$$

Due to the KMS condition, the functions $G_x$ and $G_\infty$ are holomorphic in $I_\beta^{n+}$ and bounded by $\prod_{j=1}^n \|A_j\|$. We claim that, for $s_1 \leq \cdots \leq s_n$ and $s_n - s_1 \leq \beta$,

(7.7) $$\lim_{x \to \infty} G_x(\mathrm{i}s_1, \ldots, \mathrm{i}s_n) = G_\infty(\mathrm{i}s_1, \ldots, \mathrm{i}s_n).$$

38

Let us prove (7.7). Let us first assume that $A_j = \mathrm{e}^{\mathrm{i}\phi(0,h_j)}$ for $h_j \in C_{0\,\mathbb{R}}^\infty(\mathbb{R})$. Then

$$
\begin{aligned}
G_x(\mathrm{i}s_1,\ldots,\mathrm{i}s_n) &= \int_Q \prod_{j=1}^l \mathrm{e}^{\mathrm{i}\phi(\delta(\,.\,-s_j)\otimes h_j)} \prod_{j=l+1}^n \mathrm{e}^{\mathrm{i}\phi(\delta(\,.\,-s_j)\otimes h_j(\,.\,-x))}\mathrm{d}\mu, \\
G_\infty(\mathrm{i}s_1,\ldots,\mathrm{i}s_n) &= \int_Q \prod_{j=1}^l \mathrm{e}^{\mathrm{i}\phi(\delta(\,.\,-s_j)\otimes h_j)}\mathrm{d}\mu \times \int_Q \prod_{j=l+1}^n \mathrm{e}^{\mathrm{i}\phi(\delta(\,.\,-s_j)\otimes h_j)}\mathrm{d}\mu.
\end{aligned}
$$

By Proposition 7.3 we have

$$
\begin{aligned}
G_x(\mathrm{i}s_1,\ldots,\mathrm{i}s_n) &= \lim_{k\to+\infty} \int_Q \prod_{j=1}^l \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j)} \prod_{j=l+1}^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j(\,.\,-x))}\mathrm{d}\mu, \\
G_\infty(\mathrm{i}s_1,\ldots,\mathrm{i}s_n) &= \lim_{k\to\infty} \int_Q \prod_{j=1}^l \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j)}\mathrm{d}\mu \times \int_Q \prod_{j=l+1}^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j)}\mathrm{d}\mu.
\end{aligned}
$$

From Theorem 7.2 we get

$$
\begin{aligned}
&\int_Q \prod_{j=1}^l \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j)} \prod_{j=l+1}^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j(\,.\,-x))}\mathrm{d}\mu \\
&= \big(\Omega_\mathsf{C}, W_{[-\infty,+\infty]}(R_{1,k} + \mathfrak{a}_x(R_{2,k}))\Omega_\mathsf{C}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_Q \prod_{j=1}^l \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j)}\mathrm{d}\mu \times \int_Q \prod_{j=l+1}^n \mathrm{e}^{\mathrm{i}\phi(\delta_k(\,.\,-s_j)\otimes h_j)}\mathrm{d}\mu \\
&= \big(\Omega_\mathsf{C}, W_{[-\infty,+\infty]}(R_{1,k})\Omega_\mathsf{C}\big) \cdot \big(\Omega_\mathsf{C}, W_{[-\infty,+\infty]}(R_{2,k})\Omega_\mathsf{C}\big),
\end{aligned}
$$

where

$$
R_{1,k} = \phi\Big(\sum_{j=1}^l \delta_k(\,.\,-s_j) \otimes h_j\Big) \ \text{and} \ R_{2,k} = \phi\Big(\sum_{j=l+1}^n \delta_k(\,.\,-s_j) \otimes h_j\Big).
$$

As before (see Section 4.1), the group of spatial translations induced on $Q$ by the map $(t,y) \mapsto (t, y+x)$ has been denoted by $\{\mathfrak{a}_x\}_{x\in\mathbb{R}}$. Applying Lemma A.10 we find

$$
\Big|\big(\Omega_\mathsf{C}, W(R_{1,k} + \mathfrak{a}_x(R_{2,k}))\Omega_\mathsf{C}\big) - \big(\Omega_\mathsf{C}, W(R_{1,k})\Omega_\mathsf{C}\big)\big(\Omega_\mathsf{C}, W(R_{2,k})\Omega_\mathsf{C}\big)\Big| \le \mathrm{e}^{-(|x|-C)a},
$$

where $a > 0$ is the spectral gap of $H_\mathsf{C}^{\mathrm{ren}}$ and $W(.) := W_{[-\infty,+\infty]}(.)$. Letting $k \to \infty$ and using Proposition 7.3 we obtain

$$
\big|G_x(\mathrm{i}s_1,\ldots,\mathrm{i}s_n) - G_\infty(\mathrm{i}s_1,\ldots,\mathrm{i}s_n)\big| \le \mathrm{e}^{-(|x|-C)a}.
$$

Using functional calculus, we conclude that (7.7) holds for all $A_j \in \mathcal{U}_0$. To complete the proof of the lemma, we can now argue as in the proof of Proposition 7.6, using Lemma B.3 $\square$.

# A  A time-dependent heat equation

Let $H \ge 0$ be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and let $R(t)$, $t \in \mathbb{R}$, be a family of closed operators with $\mathcal{D}(H^\gamma) \subset \mathcal{D}(R(t))$ for some $0 \le \gamma < 1$. We consider the following time-dependent heat equation:

(A.1)
$$
\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}U(t,s) = -\big(H + \mathrm{i}\lambda R(t)\big)U(t,s), \ s \le t, \\ U(s,s) = \mathbb{1}. \end{cases}
$$

This equation is (formally) equivalent to the following integral equation:

(A.2)
$$
U(t,s) = \mathrm{e}^{-(t-s)H} - \mathrm{i}\lambda \int_s^t \mathrm{e}^{-(t-\tau)H}R(\tau)U(\tau,s)\mathrm{d}\tau.
$$

In the main text we only use the results of this section in the *dissipative* case, i.e., when $R(t)$ is selfadjoint for all $t \in \mathbb{R}$. However, part of the results are valid and will be proved in the general case.

The solution of (A.1) will be denoted by $U(t,s)$ or $U_\lambda(t,s)$. If we want to display its dependence on the family $R(t)$, then the solution of (A.1) will be denoted by $U(t,s;R)$.

## A.1 Existence of solutions

We assume that the maps

(A.3)
$$\begin{array}{ccc} \mathbb{R} & \to & \mathcal{B}(\mathcal{H}) \\ t & \mapsto & R(t)(H+1)^{-\gamma} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{R} & \to & \mathcal{B}(\mathcal{H}) \\ t & \mapsto & R^*(t)(H+1)^{-\gamma} \end{array}$$

are Hölder continuous of some order $\epsilon > 0$.

In the sequel we will use the following result.

**Lemma A.1** *Assume (A.3). Then*

(A.4)
$$\|e^{-(t-\tau)H}R(\tau)\| \le c_\gamma \|R(\tau)^*(H+1)^{-\gamma}\| \left(|t-\tau|^{-\gamma} + 1\right) \quad \forall \tau \le t.$$

**Proof.** We have $\|e^{-(t-\tau)H}R(\tau)\| \le \|R(\tau)^*(H+1)^{-\gamma}\| \|(H+1)^\gamma e^{-(t-\tau)H}\|$. This proves the lemma, using

(A.5)
$$|(\lambda+1)^\gamma e^{-s\lambda}| \le c_\gamma(s^{-\gamma} + 1) \quad \text{for} \quad s, \lambda \ge 0 \,\square.$$

The following result is shown in [H, Theorem 7.1.3].

**Proposition A.2** *There exists a unique solution $U(t,s)$ of (A.1) such that*

(i) $U(s,s) = \mathbb{1}$ *and* $U(t,r)U(r,s) = U(t,s)$ *for* $s \le r \le t$;

(ii) $t \mapsto U(t,s) \in \mathcal{B}(\mathcal{H})$ *is strongly continuous in* $[s, +\infty[$ *and strongly differentiable in* $]s, +\infty[$.

**Lemma A.3** *The map $\lambda \mapsto U_\lambda(.,s)\Psi \in C\big([s,+\infty[, \mathcal{H}\big)$ is entire analytic for each $\Psi \in \mathcal{H}$.*

**Proof.** Let $\Psi \in \mathcal{H}$. For $s \le T < \infty$, set $V(t)\Psi = e^{-(t-s)H}\Psi$ and

$$\|U(.,s)\Psi\| := \sup_{t \in [s,T]} \|U(t,s)\Psi\|_{\mathcal{H}}.$$

If we define a map

$$\begin{array}{cccc} K: & C([s,T], \mathcal{H}) & \to & C([s,T], \mathcal{H}) \\ & W(.) & \mapsto & -i \int_s^{(.)} e^{-(.-\tau)H}R(\tau)W(\tau)d\tau, \end{array}$$

then the integral equation (A.2) can be rewritten as $(1-K)(U(.,s)\Psi) = V(.)\Psi$. Now (A.4) implies

$$\|KU(t,s)\Psi\| \le \|U(.,s)\Psi\| \sup_{\tau \in [s,T]} \|R^*(\tau)(H+1)^{-\gamma}\| \int_s^t \big(|t-\tau|^{-\gamma} + 1\big)d\tau,$$

and hence

$$\|KU(.,s)\Psi\| \le c \sup_{\tau \in [s,T]} \|R^*(\tau)(H+1)^{-\gamma}\| \, |T-s|^{1-\gamma} \|U(.,s)\Psi\|,$$

which shows that $K \in \mathcal{B}\big(C([s,T], \mathcal{H})\big)$. Then $U_\lambda(.,s)\Psi$ solves $(1-\lambda K)\big(U_\lambda(.,s)\Psi\big) = V(.)\Psi$, which implies that $\lambda \mapsto U_\lambda(.,s)\Psi$ is entire analytic $\square$.

## A.2    The dissipative case

We now consider the dissipative case when $R(t)$ is selfadjoint for $t \in \mathbb{R}$. We first prove a result about approximation by time-ordered products. We will make use of an extension of Gronwall's inequality to integral equations, shown in [H, Lemma 7.1.1].

**Lemma A.4** *Let $b \geq 0$ and $\gamma > 0$. Let $a(t)$ and $u(t)$ be non negative locally integrable functions on $s \leq t \leq T < \infty$ such that*

$$u(t) \leq a(t) + b \int_s^t (t - \tau)^{-\gamma} u(\tau) d\tau \text{ for } s \leq t \leq T.$$

*Then*

$$u(t) \leq a(t) + cb^{(1-\gamma)^{-1}} \int_s^t E(t - \tau) a(\tau) d\tau \text{ for } s \leq t \leq T,$$

*where $|E(r)| \leq c_T(|r|^{-\gamma})$ on $[0, T - s]$.*

**Proposition A.5** *Assume that $R(t)$ is selfadjoint and that (A.3) holds.*
   *Then for $s \leq t$ there exists a sequence $\{p_n\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} p_n = +\infty$ such that*

$$U(t, s) = \text{s-} \lim_{n \to \infty} \prod_{n-1}^{0} \left( e^{-(t_{j+1} - t_j) H / p_n} e^{-i(t_{j+1} - t_j) R(t_j) / p_n} \right)^{p_n},$$

*where $t_j = s + \frac{(t-s)j}{n}$ for $0 \leq j \leq n$.*

**Proof.** Let $R_i$, $i = 1, 2$, be two families of closed operators satisfying (A.3) and let $U^{(i)}(t, s)$ be the associated propagators. Then

$$
\begin{aligned}
U^{(1)}(t, s) - U^{(2)}(t, s) = \quad &-i \int_s^t e^{-(t-\tau)H} R_1(\tau) \big( U^{(1)}(\tau, s) - U^{(2)}(\tau, s) \big) d\tau \\
&-i \int_s^t e^{-(t-\tau)H} \big( R_1(\tau) - R_2(\tau) \big) U^{(2)}(\tau, s) d\tau.
\end{aligned}
$$

This implies, using Lemma A.1, that

$$
\begin{aligned}
&\| U^{(1)}(t, s) - U^{(2)}(t, s) \| \\
\leq \quad & c_\gamma \int_s^t (|t - \tau|^{-\gamma} + 1) \, \| (H + 1)^{-\gamma} R_1(\tau) \| \, \| U^{(1)}(\tau, s) - U^{(2)}(\tau, s) \| d\tau \\
& + c_\gamma \int_s^t (|t - \tau|^{-\gamma} + 1) \, \| (H + 1)^{-\gamma} \big( R_1(\tau) - R_2(\tau) \big) \| \, \| U^{(2)}(\tau, s) \| d\tau.
\end{aligned}
$$

We now apply Gronwall's inequality, as given in Lemma A.4, with

$$b = \sup_{s \leq t \leq T} \| (H + 1)^{-\gamma} R_1(t) \|,$$
$$a(t) \equiv a = c_T \sup_{s \leq t \leq T} \| (H + 1)^{-\gamma} \big( R_1(t) - R_2(t) \big) \| \times \sup_{s \leq t \leq T} \| U^{(2)}(t, s) \|.$$

We obtain

$$
\begin{aligned}
& \sup_{s \leq t \leq T} \| U^{(1)}(t, s) - U^{(2)}(t, s) \| \\
\text{(A.6)} \quad \leq \quad & c_T \sup_{s \leq t \leq T} \| (H + 1)^{-\gamma} R_1(t) \|^{(1-\gamma)^{-1}} \\
& \times \sup_{s \leq t \leq T} \| (H + 1)^{-\gamma} \big( R_1(t) - R_2(t) \big) \| \times \sup_{s \leq t \leq T} \| U^{(2)}(t, s) \|.
\end{aligned}
$$

Let us now prove the proposition. For $s < t$ fixed, $n \in \mathbb{N}$, we set

$$t_j := s + \frac{(t - s)j}{n}, \ \ 0 \leq j \leq n, \quad R_n(\tau) = \sum_{n=0}^{n-1} \mathbb{1}_{[t_j, t_{j+1}[}(\tau) R(t_j).$$

41

Note that $H + iR(t_j)$ with domain $\mathcal{D}(H)$ is the generator of a $C_0$-semigroup of contractions, since it is closed and maximal accretive, using (A.3).

If $U^{(n)}(t, s)$ is the solution of (A.1) for the piecewise constant family of operators $\{R_n(t)\}$, then one can easily verify that:

$$U^{(n)}(t, s) = \prod_{n-1}^{0} e^{-(t_{j+1}-t_j)(H+iR_n(t_j))}.$$

Since $\mathbb{R} \ni t \mapsto (H + 1)^{-\gamma} R(t) \in \mathcal{B}(\mathcal{H})$ is continuous, we conclude that

$$\lim_{n \to \infty} \sup_{s \le t \le T} \left\| (H + 1)^{-\gamma} \big( R(t) - R_n(t) \big) \right\| = 0.$$

Using (A.6) we get

$$\lim_{n \to \infty} \sup_{s \le t \le T} \| U(t, s) - U^{(n)}(t, s) \| = 0.$$

Applying next [Ch] we obtain

$$e^{-(t_{j+1}-t_j)(H+iR(t_j))} = \text{s-} \lim_{p \to \infty} \left( e^{-(t_{j+1}-t_j)H/p} e^{-i(t_{j+1}-t_j)R(t_j)/p} \right)^p.$$

Using the fact that $e^{-\tau(H+iR(t_j))}$, $e^{-\tau H}$ and $e^{-i\tau R(t_j)}$ are all contractions, we conclude that there exists a sequence $p_n \to \infty$ such that

$$U(t, s) = \text{s-} \lim_{n \to \infty} \prod_{n-1}^{0} \left( e^{-(t_{j+1}-t_j)H/p_n} e^{-i(t_{j+1}-t_j)R(t_j)/p_n} \right)^{p_n}.$$

This completes the proof of the proposition $\square$.

**Proposition A.6** *Assume that $R(t)$ is selfadjoint and satisfies (A.3). Assume moreover that the function*
(A.7) $\qquad\qquad t \mapsto \|(H + 1)^{-\gamma} R(t)\|$ *is in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$*

*and*
(A.8) $\qquad\qquad \pm R(t) \le r(t)(H + 1)^\delta, \quad 0 \le \delta < 1,$

*for some $r \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the limit $U_\lambda(+\infty, -\infty) := \text{w}-\lim_{(t,s)\to(+\infty,-\infty)} U_\lambda(t, s)$ exists for all $\lambda \in \mathbb{C}$.*

*(i) the function $\mathbb{C} \ni \lambda \mapsto U_\lambda(+\infty, -\infty)$ is entire and satisfies*

$$\|U_\lambda(+\infty, -\infty)\| \le e^{c|\text{Im}\lambda|^{(1-\delta)^{-1}}} \quad \forall \lambda \in \mathbb{C};$$

*(ii) the derivatives w.r.t. $\lambda$ are uniformly bounded:*

$$\sup_{\lambda \in \mathbb{R}} |\partial_\lambda^n U_\lambda(+\infty, -\infty)| < \infty \quad \forall n \in \mathbb{N};$$

*(iii) for $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ the derivatives at $\lambda = 0$ are given by the following formula:*

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} U_\lambda(+\infty, -\infty)_{|\lambda=0}$$
$$= \; n!(-\mathrm{i})^n \int_{-\infty < t_1 \le \cdots \le t_n < \infty} \mathbb{1}_{\{0\}}(H) \left( \prod_n^2 R(t_k) e^{-(t_k - t_{k-1})H} \right) R(t_1) \mathbb{1}_{\{0\}}(H) \mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

**Proof.** Let $\Psi \in \mathcal{H}$. Using Proposition A.2 and the fact that $R(t)$ is selfadjoint we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|U_\lambda(t,s)\Psi\|^2 \;=\; -2\mathrm{Re}\big(U_\lambda(t,s)\Psi,(H+\mathrm{i}\lambda R(t))U_\lambda(t,s)\Psi\big)
$$
$$
= -2\big(U_\lambda(t,s)\Psi,(H-\mathrm{Im}\lambda R(t))U_\lambda(t,s)\Psi\big).
$$

Now

$$
H - \mathrm{Im}\lambda R(t) \geq H - |\mathrm{Im}\lambda|\, r(t)(H+1)^\delta \geq c\big(|\mathrm{Im}\lambda|\, r(t)\big)^{(1-\delta)^{-1}},
$$

since $\inf_{s\geq 0}\; s - t(s+1)^\delta = -c\,t^{(1-\delta)^{-1}}$ for $t\geq 0$. This yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|U_\lambda(t,s)\Psi\|^2 \leq c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}} r(t)^{(1-\delta)^{-1}}\|U_\lambda(t,s)\Psi\|^2,
$$

and hence

$$
\|U_\lambda(t,s)\Psi\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}} \int_s^t r(\tau)^{(1-\delta)^{-1}}\,\mathrm{d}\tau}\|\Psi\|.
$$

Since $r \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we have $r^{(1-\delta)^{-1}} \in L^1(\mathbb{R})$, which yields

(A.9)
$$
\sup_{s\leq t}\|U_\lambda(t,s)\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}}} \qquad \forall \lambda \in \mathbb{C}.
$$

Let us now prove that

(A.10)
$$
\mathrm{w} - \lim_{(t,s)\to(+\infty,-\infty)} U_\lambda(t,s) \text{ exists for all } \lambda \in \mathbb{R}.
$$

For $\Psi \in \mathcal{H}$, $\Phi \in \mathcal{D}(H)$ and $0 \leq \gamma < 1$ we find

$$
\big(\Phi,(U_\lambda(t,s)-\mathrm{e}^{-(t-s)H})\Psi\big) = -\mathrm{i}\lambda \int_s^t \Big(\mathrm{e}^{-(t-\tau)H}(H+1)^\gamma\Phi,(H+1)^{-\gamma}R(\tau)U(\tau,s)\Psi\Big)\mathrm{d}\tau.
$$

Using dominated convergence and hypothesis (A.7) we obtain the existence of

$$
\lim_{(t,s)\to(+\infty,-\infty)}\big(\Phi,U_\lambda(t,s)\Psi\big) \text{ for } \Psi \in \mathcal{H} \text{ and } \Phi \in \mathcal{D}(H^\gamma).
$$

Applying a density argument and the uniform bound (A.9) this proves (A.10).

Now $\{\lambda \mapsto U_\lambda(t,s) \mid s \leq t\}$ is a locally uniformly bounded family of entire functions. Applying Lemma B.3 and (A.10) we obtain that

$$
U_\lambda(+\infty,-\infty) = \mathrm{w} - \lim_{(t,s)\to(+\infty,-\infty)} U_\lambda(t,s)
$$

exists for all $\lambda \in \mathbb{C}$. Moreover, the map $\lambda \mapsto U_\lambda(+\infty,-\infty)$ is entire and

$$
\|U_\lambda(+\infty,-\infty)\| \leq \mathrm{e}^{c|\mathrm{Im}\lambda|^{(1-\delta)^{-1}}} \qquad \forall \lambda \in \mathbb{C}.
$$

If $f(z)$ is a bounded holomorphic function in a strip $\{|\mathrm{Im}z| < a\}$, then it follows easily from Cauchy's formula that $\sup_{x\in\mathbb{R}} |\partial_x^n f(x)| < \infty$ for all $n \in \mathbb{N}$. This completes the proof of (ii).

Let us now prove (iii). Set, as in Subsection A.1,

$$
\begin{aligned}
K:\quad C([s,T],\mathcal{H}) &\rightarrow C([s,T],\mathcal{H}) \\
W(.) &\mapsto -\mathrm{i}\int_s^{(.)} \mathrm{e}^{-(.-\tau)H}R(\tau)W(\tau)\mathrm{d}\tau,
\end{aligned}
$$

and $V(t)\Psi = \mathrm{e}^{-(t-s)H}\Psi$. From the integral equation $(\mathbb{1}-\lambda K)U_\lambda(.,s)\Psi = V(.)\Psi$ we deduce that

$$
\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}U_\lambda(t,t_0)_{|\lambda=0}\Psi &= n!K^n V(t)\Psi \\
&= n!(-\mathrm{i})^n \int_{t_0\leq t_1\leq\cdots\leq t_n\leq t} \mathrm{e}^{-(t-t_n)H}\Big[\prod_n^1 R(t_k)\mathrm{e}^{-(t_k-t_{k-1})H}\Big]\Psi\mathrm{d}t_1\ldots\mathrm{d}t_n.
\end{aligned}
$$

43

The function $\mathbb{C} \ni \lambda \mapsto U_\lambda(t,s)\Psi$ is entire and uniformly bounded in $\{|\text{Im}\lambda| \leq a\}$ for $-\infty < s \leq t < +\infty$. Therefore

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}U_\lambda(+\infty,-\infty)\Psi = \lim_{(t,s)\to(+\infty,-\infty)} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}U_\lambda(t,s)\Psi.$$

Setting $t_{n+1} = t$ we find

$$\left\| \mathrm{e}^{-(t-t_n)H} \prod_n^1 R(t_k)\mathrm{e}^{-(t_k-t_{k-1})H} \right\| \leq c \prod_n^1 \big( |t_{k+1} - t_k|^{-\gamma} + 1 \big) \| (H+1)^{-\gamma} R(t_k) \|.$$

From Lebesgue dominated convergence we deduce that

$$\lim_{s\to-\infty} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}U_\lambda(t,s)_{|\lambda=0} = n!(-\mathrm{i})^n \int_{-\infty < t_1 \leq \cdots \leq t_n \leq t} \Big[ \prod_n^1 \mathrm{e}^{-(t_{k+1}-t_k)H} R(t_k) \Big] \mathbb{1}_{\{0\}}(H)\mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

A similar argument yields

$$\lim_{t\to+\infty} \lim_{s\to-\infty} \tfrac{\mathrm{d}^n}{\mathrm{d}\lambda^n}U_\lambda(t,s)_{|\lambda=0}$$

$$= \; n!(-\mathrm{i})^n \int_{-\infty < t_1 \leq \cdots \leq t_n \leq +\infty} \mathbb{1}_{\{0\}}(H) \Big[ \prod_n^2 R(t_k)\mathrm{e}^{-(t_k-t_{k-1})H} \Big] R(t_1)\mathbb{1}_{\{0\}}(H)\mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

Applying Lemma B.1 we obtain (iii) $\square$.

We will use the following lemma to show that the limiting functional obtained in Theorem 7.2 defines a Borel measure on $\mathcal{S}'(S_\beta \times \mathbb{R})$.

**Lemma A.7** *Let $R_i(t)$, $i = 1,2$, be selfadjoint families satisfying (A.3) and (A.7). Assume that*

$$\pm\big(R_1(t) - R_2(t)\big) \leq r(t)(H+1)^\delta, \; 0 \leq \delta < 1,$$

*for $r \in L^{(1-\delta)^{-1}}(\mathbb{R})$. Then*

$$\|U(+\infty,-\infty;R_2) - U(+\infty,-\infty;R_1)\| \leq c\|r\|_{(1-\delta)^{-1}}.$$

**Proof.** Let us denote by $Z_\lambda(t,s)$ the operator $U(t,s;R_1 + \lambda(R_2 - R_1))$. By the same arguments as used in the proof of Proposition A.6, we see that $\lambda \mapsto Z_\lambda(t,s)$ is an entire analytic function, which satisfies the bound

$$\|Z_\lambda(t,s)\| \leq \mathrm{e}^{c|\text{Im}\lambda|^\gamma\|r\|_\gamma^\gamma} \quad \text{for } \lambda \in \mathbb{C} \text{ and } \gamma = (1-\delta)^{-1}.$$

As in Proposition A.6, the limit of $Z_\lambda(t,s)$ when $(t,s) \to (+\infty,-\infty)$ exists for $\lambda \in \mathbb{R}$ fixed. Applying again Vitali's theorem, we obtain the existence of $Z_\lambda(+\infty,-\infty)$ for all $\lambda \in \mathbb{C}$, and the bound

$$\|Z_\lambda(+\infty,-\infty)\| \leq \mathrm{e}^{c|\text{Im}\lambda|^\gamma\|r\|_\gamma^\gamma} \quad \forall\lambda \in \mathbb{C}.$$

Applying Cauchy's formula on the circle of radius $R$ centered around $\lambda \in \mathbb{R}$ yields

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}Z_\lambda(+\infty,-\infty) \right| \leq R^{-1}\mathrm{e}^{cR^\gamma\|r\|_\gamma^\gamma}.$$

Optimizing this bound w.r.t. $R$ we get

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}Z_\lambda(+\infty,-\infty) \right| \leq c\|r\|_\gamma.$$

Integrating in $\lambda$ from 0 to 1 we obtain the lemma $\square$.

## A.3 Some additional results

We now prove some bounds on $U(t,s)$, which we use in the main text to show the existence of sharp-time fields and the convergence of sharp-time Schwinger functions.

**Lemma A.8** *Let $R_i(t)$, $i = 1,2$, be two families of selfadjoint operators satisfying (A.3) and (A.7). Assume that*

(A.11) $$\pm\bigl(R_2(t) - R_1(t)\bigr) \le r(t)(H+1) \ \text{for } r \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}).$$

*Set $Z_\lambda(t,s) := U\bigl(t,s; R_1 + \lambda(R_2 - R_1)\bigr)$ for $-\infty \le s \le t \le +\infty$. Then*

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} Z_\lambda(t,s) \right\| \le n!\, \|r\|_\infty^n\, \mathrm{e}^{\|r\|_1 \|r\|_\infty^{-1}}.$$

**Proof.** Since $R_i(t)$ satisfy (A.3) and (A.7), the function $\lambda \mapsto Z_\lambda(t,s)$ is entire. We still denote by $Z_\lambda(t,s)$ its extension to $\lambda \in \mathbb{C}$. As in the proof of Proposition A.6, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \|Z_\lambda(t,s)\Psi\|^2 = -2\Bigl( Z_\lambda(t,s)\Psi, \bigl(H - \mathrm{Im}\lambda(R_2(t) - R_1(t))\bigr) Z_\lambda(t,s)\Psi \Bigr).$$

Now

$$H - \mathrm{Im}\lambda\bigl(R_2(t) - R_1(t)\bigr) \ge H - r(t)\,|\mathrm{Im}\lambda|\,(H+1) \ge -r(t)\,|\mathrm{Im}\lambda|$$

for $|\mathrm{Im}\lambda| \le \|r\|_\infty^{-1}$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|Z_\lambda(t,s)\Psi\|^2 \le 2\,|\mathrm{Im}\lambda|\,r(t)\|\Psi\|^2 \ \text{ for } \ |\mathrm{Im}\lambda| \le \|r\|_\infty^{-1}.$$

Hence

$$\|Z_\lambda(t,s)\| \le \mathrm{e}^{|\mathrm{Im}\lambda|\|r\|_1} \ \text{ for } \ |\mathrm{Im}\lambda| \le \|r\|_\infty^{-1}.$$

We apply Cauchy's formula on a circle of radius $\|r\|_\infty^{-1}$ and obtain

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} Z_\lambda(t,s) \right\| \le n!\, \|r\|_\infty^n\, \mathrm{e}^{\|r\|_1 \|r\|_\infty^{-1}} \ \text{ for } \ \lambda \in \mathbb{R}.$$

This completes the proof of the lemma $\square$.

Finally we prove a lemma which is used in the main text to prove spatial clustering.

**Remark A.9** *Let $t_0 \in \mathbb{R}$ and define the time-translated family by $\xi_{t_0}\bigl(R(t)\bigr) := R(t - t_0)$. Then clearly*

$$U\bigl(t,s; \xi_{t_0}(R)\bigr) = U(t - t_0, s - t_0; R) \ \text{for} \ -\infty < s \le t < +\infty.$$

*Letting $(s,t) \to (-\infty, +\infty)$ we obtain that $U\bigl(+\infty, -\infty; \xi_{t_0}(R)\bigr) = U(+\infty, -\infty; R)$.*

**Lemma A.10** *Assume that $0$ is a simple eigenvalue of $H$ and that $H$ has a spectral gap, i.e.,*

$$]0, a] \cap \sigma(H) = \emptyset \ \text{for some } a > 0.$$

*Let $\{R_1(t)\}$, $\{R_2(t)\}$ be two selfadjoint families of operators satisfying (A.3) and (A.7) with $R_i(t) \equiv 0$ for $|t| \ge T$. If $\Omega$ is a normalized ground state of $H$, then*

$$\left| \bigl(\Omega, U^\infty(R_1 + \xi_t(R_2))\Omega\bigr) - \bigl(\Omega, U^\infty(R_1)\Omega\bigr)\bigl(\Omega, U^\infty(R_2)\Omega\bigr) \right| \le \mathrm{e}^{-(|t| - 2T)a}$$

*for $|t| > 2T$, where $U^\infty(R) := U(+\infty, -\infty; R)$.*

**Proof.** It suffices to consider the case $t > 0$. Using the group property and considering the supports of $R_i(.)$, we find

(A.12)
$$
\begin{aligned}
U\big(t, s; R_1 + \xi_{t_0}(R_2)\big) &= \ U\big(t, t_0 - T; \xi_{t_0}(R_2)\big)\mathrm{e}^{-(t_0-2T)H}U(T, s; R_1) \\
&= \ U(t - t_0, -T; R_2)\mathrm{e}^{-(t_0-2T)H}U(T, s; R_1)
\end{aligned}
$$

for $s \leq -T$, $t_0 > 2T$ and $t > t_0 + T$. Since $H$ has a spectral gap of length $a$,

(A.13)
$$
\big\|\mathrm{e}^{-(t_0-2T)H} - |\Omega\rangle\langle\Omega|\,\big\| \leq \mathrm{e}^{-(t_0-2T)a}.
$$

Moreover, since $H\Omega = 0$ and $\operatorname{supp} R_i(.) \subset [-T, T]$,

(A.14)
$$
\begin{aligned}
\big(\Omega, U(t - t_0, -T; R_2)\Omega\big) &= \big(\Omega, U(t - t_0, s; R_2)\Omega\big), \\
\big(\Omega, U(T, s; R_2)\Omega\big) &= \big(\Omega, U(t, s; R_2)\Omega\big).
\end{aligned}
$$

Combining (A.12), (A.13), (A.14) and letting $(t, s) \to (+\infty, -\infty)$ we obtain the lemma $\square$.

# B  Miscellaneous results

**Lemma B.1** *Let $F \colon \mathbb{R}^2 \to E$ be a map with value in a metric space $E$.*

(i) *Assume that*
$$
\lim_{k,k'\to\infty} F(k, k') = F_\infty \ \text{exists},
$$
$$
\lim_{k'\to\infty} F(k, k') = G(k) \ \text{exists} \ \forall k \in \mathbb{N},
$$
$$
\lim_{k\to\infty} G(k) = G_\infty \ \text{exists}.
$$

*Then $F_\infty = G_\infty$.*

(ii) *Assume that*
$$
\lim_{k'\to\infty} F(k, k') = G(k) \ \text{exists},
$$
$$
\lim_{k\to\infty} F(k, k') = F(k') \ \text{exists and the convergence is uniform w.r.t. } k',
$$
$$
\lim_{k\to\infty} G(k) = G_\infty \ \text{exists}.
$$

*Then $\lim_{k'\to\infty} F(k') = G_\infty$.*

The proof is easy and left to the reader.

**Lemma B.2** *Let $(Q, \Sigma, \mu)$ be a probability space. Let $f$ be a real measurable function on $Q$ and set*
$$
C(t) := \int_Q \mathrm{e}^{\mathrm{i}tf}\,\mathrm{d}\mu.
$$

*Then $f \in \bigcap_{1\leq p<\infty} L^p(Q, \Sigma, \mu)$ if and only if $\sup_{t\in\mathbb{R}} |\partial_t^n C(t)| < \infty$ for all $n \in \mathbb{N}$. If this is the case, then*
$$
\partial_t^n C(t) = \mathrm{i}^n \int_Q f^n \mathrm{e}^{\mathrm{i}tf}\,\mathrm{d}\mu.
$$

**Proof.** The $\Rightarrow$ part and the formula for $\partial_t^n C(t)$ is obvious by differentiating under the integral sign. It remains to prove the $\Leftarrow$ part. Let $\chi(\tau) = \mathrm{e}^{-\tau^2/2}$ and let $p \geq 1$. By monotone convergence it suffices to prove that
$$
\sup_{n\in\mathbb{N}} \int_Q f^{2p}\chi\left(\frac{f}{n}\right)\mathrm{d}\mu < \infty
$$

46

in order to show that $f \in L^{2p}(Q, \Sigma, \mu)$. We have

$$\tau^{2p}\chi\left(\frac{\tau}{n}\right) = \frac{n^{2p+1}}{2\pi}\int e^{it\tau}\left(\partial_t^{2p}\hat{\chi}(nt)\right)dt.$$

Hence

$$\int_Q f^{2p}\chi\left(\frac{f}{n}\right)d\mu = \frac{n^{2p+1}}{2\pi}\int_Q\int_{\mathbb{R}} e^{itf}\left(\partial_t^{2p}\hat{\chi}(nt)\right)dtd\mu$$

$$= \frac{n^{2p+1}}{2\pi}\int_{\mathbb{R}} C(t)\left(\partial_t^{2p}\hat{\chi}(nt)\right)dt$$

$$= \frac{(-1)^{2p}}{2\pi}n\int_{\mathbb{R}}\left(\partial_t^{2p}C(t)\right)\hat{\chi}(nt)dt,$$

using Fubini's theorem and integrating by parts $2p$ times. Since $\hat{\chi} \in L^1(\mathbb{R})$ and $\partial_t^{2p}C$ is uniformly bounded, we obtain that $\sup_{n\in\mathbb{N}}\int_Q f^{2p}\chi(n^{-1}f)d\mu < \infty$, which completes the proof of the lemma $\square$.

**Lemma B.3** *Let $I$ be a directed set and $\{u_\alpha\}_{\alpha\in I}$ a net of functions which are holomorphic in an open set $\Omega \subset \mathbb{C}$.*

(i) *Assume that the family $\{u_\alpha\}$ is locally uniformly bounded in $\Omega$ and that there exists a set $\Gamma \subset \Omega$ having an accumulation point in $\Omega$ such that*

$$\lim_{\alpha\in I} u_\alpha(z) \text{ exists for } z \in \Gamma.$$

*Then $\lim_{\alpha\in I} u_\alpha = u$ exists in the compact-open topology on $\Omega$ and $u$ is a holomorphic function in $\Omega$." (We recall that the compact-open topology on $C(\Omega)$ is the topology of uniform convergence on all compact subsets of $\Omega$.)*

(ii) *Assume moreover that $\Omega$ is bounded with a smooth boundary and that*

$$\sup_{\alpha\in I}\sup_{z\in\Omega}|u_\alpha(z)| < \infty.$$

*Then $u$ is continuous on $\overline{\Omega}$ and $\lim_{\alpha\in I}\sup_{z\in\partial\Omega}|u_\alpha(z) - u(z)| = 0$.*

**Proof.** Let us first prove (i). By Vitali's theorem the family $\{u_\alpha\}$ is compact for the compact-open topology. Let $\{u_\beta\}_{\beta\in J}$ be a subnet converging to a continuous function $u$. Assume that the net $\{u_\alpha\}_{\alpha\in I}$ does not converge to $u$. Then there exists a bounded open set $\Omega_1 \subset \Omega$ and a subnet $\{u_\gamma\}_{\gamma\in J_1}$ such that $\sup_{z\in\Omega_1}|u_\gamma(z) - u(z)| \geq \epsilon_0 > 0$ for $\gamma \in J_1$. Applying again Vitali's theorem to the net $\{u_\gamma\}_{\gamma\in J_1}$, we obtain another subnet $\{u_\delta\}_{\delta\in J_2}$ such that $\lim_{\delta\in J_2} u_\delta = v$, with $v \neq u$. But $u$ and $v$ are holomorphic in $\Omega$, as limits of holomorphic functions for the compact-open topology and coincide on $\Gamma$ by hypothesis. Since $\Gamma$ has an accumulation point in $\Omega$, we have $u = v$ which gives a contradiction.

Let us now prove (ii). Assume the contrary and let $\{u_\beta\}_{\beta\in J}$ be a subnet such that

$$\inf_{\beta\in J}\sup_{z\in\partial\Omega}|u_\beta(z) - u(z)| \geq \epsilon > 0.$$

Since $\Delta u = 0$ in $\Omega$, we see that $u$ belongs to the Sobolev space $H^2(\Omega)$. Using that $\Delta u_\beta = 0$ in $\Omega$ and the fact that the family $\{u_\beta\}_{\beta\in J}$ is uniformly bounded in $\Omega$, we obtain similarly that $\{u_\beta\}_{\beta\in J}$ is a bounded family in $H^2(\Omega)$. Hence (i) implies $\lim_{\beta\in J} u_\beta = u$ in $\mathcal{D}'(\Omega)$. Finally we note that the injection $H^2(\Omega) \to H^{3/2}(\Omega)$ is compact. Extracting again a subnet, we obtain $\lim_{\gamma\in J_1} u_\gamma = u$ in $H^{3/2}(\Omega)$. Together with the trace theorem this implies that $\lim_{\gamma\in J_1} u_\gamma = u$ in $H^1(\partial\Omega)$ and hence in $C(\partial\Omega)$. This gives a contradiction $\square$.

# References

[Ar1]  H. Araki: A lattice of von Neumann algebras associated with the quantum theory of a free Bose field, J. Math. Phys. 4 (1963) 1343–1362.

[Ar2]  H. Araki: Relative Hamiltonian for faithful normal states of a von Neumann algebra, Publ. Res. Int. Math. Soc. 9 (1973) 165–209.

[Ar3]  H. Araki: Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of Tomita-Takesaki theory, in $C^*$-algebras and their applications to Statistical Mechanics and Quantum Field Theory, D. Kastler Ed., North Holland (1976).

[BR]  O. Bratteli and D.W. Robinson: Operator Algebras and Quantum Statistical Mechanics Vol. I,II, Springer-Verlag, New York-Heidelberg-Berlin(1981)

[BD'AF]  D. Buchholz, C. D'Antoni and K. Fredenhagen: The universal structure of local algebras, Comm. Math. Phys. 111 (1987) 123–135

[Ch]  P.R. Chernoff: Note on product formulas for operator semigroups, J. Funct. Anal. 2 (1968) 238–242

[DG]  J. Derezinski, C. Gérard: Spectral scattering theory of spatially cut-off $P(\varphi)_2$ Hamiltonians, Comm. Math. Phys. 213 (2000) 39–125.

[DJP]  J. Derezinski, V. Jaksic, C.A. Pillet: Perturbations of $W^*$-dynamics, Liouvilleans and KMS states, preprint mp-arc 03–94 (2003).

[Fr1]  J. Fröhlich: Unbounded, symmetric semigroups on a separable Hilbert space are essentially selfadjoint. Adv. in Appl. Math. 1 (1980) 237–256.

[Fr2]  J. Fröhlich: The reconstruction of quantum fields from Euclidean Green's functions at arbitrary temperatures, Helv. Phys. Acta 48 (1975) 355–363.

[Ge]  C. Gérard: On the existence of ground states for massless Pauli-Fierz Hamiltonians. Ann. Henri Poincaré 1 (2000) 443–458.

[GeJ]  C. Gérard, C. Jäkel: Thermal quantum fields with spatially cut-off interactions in 1+1 space-time dimensions, preprint math-ph/0307053 (2003).

[GV]  I.M. Gelfand, N.J. Vilenkin: Generalized functions. Vol. 4: Applications of harmonic analysis, Academic Press (1964).

[GJ1]  J. Glimm, A. Jaffe: The $\lambda\varphi_2^4$ quantum field theory without cutoffs. II. The field operators and the approximate vacuum, Ann. of Math. 91 (1970) 362–401.

[GJ2]  J. Glimm, A. Jaffe: Collected Papers, Volume 1: Quantum Field Theory and Statistical Mechanics, Birkhäuser, (1985).

[GJO1]  Gielerak, R., Jakóbczyk, L., and Olkiewicz, R., Reconstruction of KuboMartin-Schwinger structure from Euclidean Green functions J. Math. Phys. 35 (1994) 3726–3744

[GJO2]  Gielerak, R., Jakóbczyk, L., and Olkiewicz, R., W*-KMS structure from multi-time Euclidean Green functions J. Math. Phys. 35 (1994) 6291–6303

[GJO3]  Gielerak, R., Jakóbczyk, L., and Olkiewicz, R., Stochastically positive structures on Weyl algebras. The case of quasi-free states J. Math. Phys. 39 (1998) 6291–6328

[HO]  E.P. Heifets, E.P. Osipov: The energy momentum spectrum in the $P(\varphi)_2$ quantum field theory, Comm. Math. Phys. 56 (1977) 161–172.

[H]  D. Henry: Geometric theory of semilinear parabolic equations, Springer Lect. Notes in Math. 840 (1981).

[H-K]  R. Høegh-Krohn:  Relativistic quantum statistical mechanics in two-dimensional space-time, Comm. Math. Phys. 38 (1974) 195–224.

[K]  A. Klein: The semigroup characterization of Osterwalder-Schrader path spaces and the construction of Euclidean fields, J. Funct. Anal. 27 (1978) 277–291.

[KL1]  A. Klein, L. Landau:  Stochastic processes associated with KMS states, J. Funct. Anal. 42 (1981) 368–428.

[KL2]  A. Klein, L. Landau: Periodic Gaussian Osterwalder-Schrader positive processes and the two-sided Markov property on the circle, Pacific J. Math. 94 (1981) 341–367.

[KL3]  A. Klein, L. Landau: Construction of a unique selfadjoint generator for a symmetric local semigroup, J. Funct. Anal. 44 (1981) 121–137.

[KL4]  A. Klein, L. Landau: Singular perturbations of positivity preserving semigroups via path space techniques, J. Funct. Anal. 20 (1975) 44–82.

[KR]  R.V. Kadison, J.R. Ringrose: *Fundamentals of the Theory of Operator Algebras II*, Academic Press, New York (1986)

[L]  R. Longo: Algebraic and modular structure of von Neumann algebras of physics, Proc. Symposia in Pure Mathematics, Vol. 38 (1982) 551–566.

[Si]  B. Simon: *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory*, Princeton University Press (1974).

[S-H.K]  B. Simon, R. Høegh-Krohn: Hypercontractive semigroups and two dimensional self-coupled Bose fields, J. Funct. Anal. 9 (1972) 121–180.

[St]  O. Steinmann: Perturbative quantum field theory at positive temperatures: an axiomatic approach, Comm. Math. Phys. 170 (1995) 405–415.

[TW]  M. Takesaki, M. Winnink: Local normality in quantum statistical mechanics, Comm. Math. Phys. 30 (1973) 129–152.