1 $N-$particle systems

Let us consider a system of N non-relativistic particles in Euclidean space $\mathbb{R}^\nu$. It turns out that for the scattering theory of this system, the statistics, *ie* the bosonic or fermionic character of the particles, and their spin do not play any significant role. Particle spin is usually conserved for the models described in this section, and the inclusion of statistics presents only combinatorial problems.

Therefore one usually assumes that the N particles are spinless and distinguishable, and hence described on the Hilbert space $L^2(\mathbb{R}^{N\nu})$. Its tiime evolution is described by a Hamiltonian of the form

$$H = \sum_{j=1}^{N} \frac{1}{2m_j} D_j^2 + \sum_{1\le i<j\le N} V_{ij}(x_i - x_j), \tag{1.1}$$

m_j being the mass of particle j and V_{ij} the interaction potential between particles i and j.

The case most important for physics is of course the case of *Coulomb interactions*, where $\nu = 3$ and

$$V_{ij}(x) = \frac{q_i q_j}{|x|},$$

q_i being the electric charge of particle i.

A related Hamiltonian arises when certain particle masses (like nuclei) are much larger then others and can henceforth be considered as infinite. The heavy particles are hence treated as fixed classical sources. The Hamiltonian is then:

$$H = \sum_{j=1}^{N} \frac{1}{2m_j} D_j^2 + \sum_{1\le i<j\le N} V_{ij}(x_i - x_j) + \sum_{j=1}^{N}\sum_{k=1}^{P} W_{ik}(x_j - X_k), \tag{1.2}$$

the points X_k, $1 \le k \le P$ being the positions of the heavy particles.

Note that the Hamiltonian in (1.3) is invariant under space translations, while the Hamiltonian in (1.7) is not.

A nice framework, the *Agmon Hamiltonians*, has been introduced by Agmon [**?**]. Besides its mathematical elegance, it allows to treat in a unified way the two cases considered above. It also allows for example to easily incorporate many particle interactions, like for example potentials $v_{i,j,k}(x) = v(x_i - x_j, x_{i-x_k})$, where $v : \mathbb{R}^{2\nu} \to \mathbb{R}$ tends to zero at infinity.

1.1 Agmon Hamiltonians

Let X be an Euclidean space. The scalar product on X will be denoted by $x.x = x^2$. We get also a scalar product on the dual (momentum) space X', denoted by $p.p = p^2$. Using the Lebesgue measure associated to the scalar product, we can consider the Hilbert space $L^2(X)$, on which the Laplacian

$$H_0 = \frac{1}{2}p^2 = \frac{1}{2}D^2$$

for $D = -\mathrm{i}\nabla_x$ is well defined.

Assume that there exists a finite family $\{X_a\}_{a\in\mathcal{A}}$ of subspaces of X, closed under intersections and containing X. The set $\mathcal{A}$ is equipped with an order relation by saying that $a \le b$ if $X_a \supset X_b$.

Since the family $\{\mathcal{X}_a\}$ was supposed to be closed under intersection, we see that $\mathcal{A}$ equipped with $\leq$ becomes a semilattice, in other words for each $a, b \in \mathcal{A}$ there exists a smaller element c such that $a \leq c$, $b \leq c$, denoted by $a \vee b$. One has of course

$$X_{a\vee b} = X_a \cap X_b.$$

let also $a_{\min}$, $a_{\max}$ denote the smallest and largest elements of $\mathcal{A}$, so that $X_{a_{\min}} = X$, $X_{a_{\max}} = \bigcap_{a\in\mathcal{A}} X_a$.

For each $a \in \mathcal{A}$, we choose a real potential $v_a : X \to \mathbb{R}$ *invariant* under translations in X_a:

$$v_a(x) = v_a(x - y),\ \forall y \in X_a.$$

Then an *Agmon Hamiltonian* is an operator on $L^2(X)$ of the form:

$$H = \frac{1}{2}D^2 + \sum_{a\in\mathcal{A}} v_a(x).$$

Note that H commutes with the translations along the space $X_{a_{\max}}$, so if $X_{a_{\max}} \neq \{0\}$, one can separate the motion of the center of mass.

Using the scalar product on X, we can introduce the subspaces:

$$X^a := X_a^{\perp},$$

so that $a \leq b$ iff $X^a \subset X^b$. If we denote by π_a, π^a the orthogonal projections on X_a and X^a, we set

$$x = x^a + x_a,\ \text{for } x^a = \pi^a x,\ x_a = \pi_a x,$$

which allows to write

$$L^2(X) = L^2(X_a) \otimes L^2(X^a).$$

Note that v_a can be considered as a real function on X^a. Clearly to hope for a reasonable scattering theory one has to assume that

$$v_a(x^a) \to 0,\ \text{when } |x^a| \to \infty, \tag{1.3}$$

at least in some averaged sense.

Similarly if we set

$$X'_a := \{p \in X' | \langle p, x\rangle = 0,\ \forall x \in X^a\},\ X^{a'} := \{p \in X' | \langle p, x\rangle = 0,\ \forall x \in X_a\},$$

we can set

$$p = p^a + p_a,\ \text{so that } p^2 = (p^a)^2 = (p_a)^2.$$

For $a \in \mathcal{A}$, one can introduce the *cluster Hamiltonian*

$$H_a := \frac{1}{2}D^2 + V^a(x),$$

for

$$V^a(x) := \sum_{b\leq a} v_b(x).$$

Note that V^a commutes with translation by vectors in X_a, so that V^a can be considered as a real function on X^a. Then we get:

$$H_a = \frac{1}{2}(D_a)^2 + \frac{1}{2}(D^a)^2 + V^a(x^a) =: \frac{1}{2}(D_a)^2 + H^a,$$

where H^a is a Hamiltonian on $L^2(X^a)$. Note that H^a has the same Agmon Hamiltonian structure on X^a, with the set of indices $\mathcal{A}^a = \{b \in \mathcal{A} | b \leq a\}$. This is a first indication of the lattice structure of the $N-$body problem, which plays a fundamental role.

Another notation that will be needed is the *intercluster potentials*:

$$I_a(x) := \sum_{b \not\leq a} v_b(x),$$

so that:

$$H = H_a + I_a.$$

We can now easily discuss the *separation of the center of mass*: if $X_{a_{\max}} \neq \{0\}$, the Hamiltonian H commutes with the translations by vectors in $X_{a_{\max}}$. We write:

$$H = \frac{1}{2}(D_{a_{\max}})^2 + H^{a_{\max}},$$

where $H^{a_{\max}}$ acts on $L^2(X^{a_{\max}})$ and it is clearly sufficient to study $H^{a_{\max}}$. In the sequel we will always assume that $X_{a_{\max}} = \{0\}$, *ie* that the motion of the center of mass has been separated out.

Let us now explain this formalism for the two Hamiltonians introduced above:

In case one, one sets $X = \mathbb{R}^{N\nu}$, equipped with the quadratic form

$$(x, x) := \sum_{i=1}^{N} m_i (x_i)^2. \tag{1.4}$$

Note that with the above notations we have:

$$\sum_{i=1}^{N} \frac{1}{2m_i}(D_i)^2 = \frac{1}{2}D^2.$$

We choose for $\mathcal{A}$ the set of *cluster decompositions*, *ie* the set of partitions $a = \{C_1, \ldots, C_k\}$ of $\{1, \ldots, N\}$. For a pair $\{i, j\}$ one says that $\{i, j\} \leq a$ if $\{i, j\} \subset C_l$ for some $C_l \in a$. To a cluster decomposition $a = \{C_1, \ldots, C_k\} \in \mathcal{A}$, one associates the *collision subspace*

$$X_a := \{x \in \mathbb{R}^{N\nu} | x_i = x_j \text{ if } \{i, jleqa\},$$

corresponding to a particle configuration where all particles in a cluster of a are at the same point of $\mathbb{R}^\nu$, which describes the configuration space for the center of masses of the clusters of a. The space X^a of internal cluster motion is:

$$X^a = \{x \in \mathbb{R}^{N\nu} | \sum_{i \in C_l} m_i x_i = 0,\ 1 \leq l \leq k\}.$$

The order relation introduced above is the familiar order relation on partitions, meaning that $a \leq b$ if a if *finer* than b. We see then that $a_{\min} = \{\{1\}, \ldots, \{N\}\}$, $X_{a_{\min}} = X$, and $a_{\max} = \{\{a, \ldots, N\}\}$, $X_{a_{\max}} = \{x \in \mathbb{R}^{N\nu} | x_i = x_j, \, \forall \, i, j\}$.

In case two, we choose for $\mathcal{A}$ the set of partitions of $\{0, 1, \ldots, N\}$. To a cluster decomposition $a = \{C_0, C_1, \ldots, C_k\}$ where $0 \in C_0$ we associate the subspace:

$$X_a := \{x \in \mathbb{R}^{N\nu} | x_i = 0 \text{ if } i \in C_0, \; x_i = x_j \text{ if } \{i, j\} \subset C_l \text{ for } 1 \leq l \leq k\}.$$

Hence particle indices belonging to C_0 label the particles close to the origin in $\mathbb{R}^\nu$. In fact for the scattering theory, all the heavy particles can be viewed as sitting at the origin of configuration space. The space X^a is again:

$$X^a = \{x \in \mathbb{R}^{N\nu} | \sum_{i \in C_l} m_i x_i = 0, \; 1 \leq l \leq k\}.$$

1.2 Asymptotic velocity

The use of asymptotic observables in scattering theory has a long history (see e.g. [AGJ]). In the time-dependent approach it appeared in the approach of Enss [E3, E4].The introduction of the *asymptotic velocity* observable by Derezinski in [De5, De6], while implicitely present in previous works of Sigal-Soffer[SS1] and Graf [Gr] allows for a very elegant formulation of scattering theory.

Theorem 1.1 *Assume ???. Then there exist a (vector valued) selfadjoint observable P^+, called the* asymptotic velocity*, such that:*

$$\text{s}-\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH} F(\frac{x}{t}) \mathrm{e}^{-\mathrm{i}tH} = F(P^+), \; \forall F \in C_\infty(X).$$

Note for example that the fact that P^+ has a dense domain is reflected physically by the fact that any state propagates with finite speed.

The use of the asymptotic velocity allows for example to split an initial state into parts which belong to various scattering channels. In fact, if we set:

$$Z_a := X_a \setminus \bigcup_{b \not\leq a} X_b, \tag{1.5}$$

we see that $\{Z_a\}_{a \in \mathcal{A}}$ is a partition of X. Hence:

$$1 = \sum_{a \in \mathcal{A}} 1_{\{Z_a\}}(P^+).$$

For $u \in L^2(X)$, $1_{\{Z_a\}}(P^+)u$ is the part of u that asymptotically separate into the clusters of a.

Let us now collect some properties of P^+, which are deduced from a Mourre estimate.

Theorem 1.2 *i) P^+ commutes with the Hamiltonian H,*

ii) $1_{\{0\}}(P^+) = 1^{\text{pp}}(H)$.

iii): joint spectrum of H and P^+: one has:

$$\sigma(P^+, H) = \bigcup_{a \in \mathcal{A}} \{(\xi_a, \tau + \frac{1}{2}\xi_a^2) | \xi_a \in X_a, \; \tau \in \sigma_{\text{pp}}(H^a)\}.$$

One should compare *iii)* with the description of the joint energy-momentum spectrum in relativistic quantum field theory.

1.3 Short-range wave operators

Let us now describe the *short range wave operators* for Agmon Hamiltonians. We consider in this subsection an Agmon Hamiltonian with $X_{a_{\max}} = \{0\}$, it without trivial translation invariance.

We will assume for simplicity the following form of the short-range condition:

$$|v_a(x^a)| \in O(|x^a|^{-\mu}),\ \mu > 1. \tag{1.6}$$

The need to take care of the precise decay rate of the pair potentials at infinity, instead of the very weak assumption (1.3), is well known from the study of the $1-$body problem: for a Schroedinger operator $H = \frac{1}{2}D^2 + V(x)$ for V decaying at infinity like $|x|^{-\mu}$ for $\mu > 1$ one can use as comparison dynamics the free dynamics $\mathrm{e}^{-\mathrm{i}tH_0}$, $H_0 = \frac{1}{2}D^2$, while if V decays like $|x|^{-\mu}$ for $0 < \mu \leq 1$, one has to use a *modified dynamics*. The well known example is *Coulomb scattering*, where one uses the *Dollard dynamics* to define *modified wave operators*

The wave operators should label the various possible *scattering channels*, describing bound clusters of particles, whose center of masses have an asymptotically free motion.

Bound states of a set of clusters a are labelled by states in $\mathcal{H}^{\mathrm{pp}}(H^a)$, the space of bound states of H^a. This leads to the following definition of *wave operators*:

Definition 1.3 *The* short range wave operators *for an Agmon Hamiltonian H are defined by:*

$$\Omega^{\pm}_{\mathrm{sr},a} := \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_a}1^{\mathrm{pp}}(H^a),\ a \in \mathcal{A}.$$

provided the above strong limit exist.

Note that with our definition we have $\Omega^{\pm}_{\mathrm{sr},a_{\max}} = 1^{\mathrm{pp}}(H)$. Clearly the wave operators $\Omega^{\pm}_{\mathrm{sr},a}$ if they exist are isometries and satisfy the *intertwining relations*

$$\Omega^{\pm}_{\mathrm{sr},a}H_a = H\Omega^{\pm}_{\mathrm{sr},a}.$$

This definition leads to the two familiar questions in time-dependent scattering theory:
existence of wave operators: find conditions on the potentials v_a such that the limits:

$$\mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_a}1^{\mathrm{pp}}(H^a),\ a \in \mathcal{A}\text{exist}.$$

asymptotic completeness: find conditions on the potentials v_a such that:

$$\oplus_{a\in\mathcal{A}}\mathrm{Ran}\Omega^{\pm}_{\mathrm{sr},a} = L^2(X),$$

or equivalently:

$$\sum_{a\in\mathcal{A}} \Omega^{\pm}_{\mathrm{sr},a}\Omega^{\pm *}_{\mathrm{sr},a} = 1.$$

Note that in the litterature, one sometimes finds the following weaker definition, called the *weak asymptotic completeness*:

$$\oplus_{a\in\mathcal{A}}\mathrm{Ran}\Omega^{+}_{\mathrm{sr},a} = \oplus_{a\in\mathcal{A}}\mathrm{Ran}\Omega^{-}_{\mathrm{sr},a}.$$

Weak asymptotic completeness implies that the scattering operator (which we will not define) is unitary, and is sufficient to get a complete physical description of a scattering experiment.

However I dont know of examples where weak asymptotic completeness has been shown without proving at the same time (strong) asymptotic completeness.

Let us now explain how the existence of asymptotic velocity easily implies the existence and completeness of short-range wave operators.

Theorem 1.4 *Assume (1.6). Then: i) The short-range wave operators exist.*
ii) the system is asymptotically complete:

$$\bigoplus_{a\in\mathcal{A}} \mathrm{Ran}(\Omega^+_{\mathrm{sr},a}) = L^2(X).$$

To prove *ii)* is clearly suffices to show that:

$$\mathrm{Ran}(\Omega^+_{\mathrm{sr},a}) = 1_{\{Z_a\}}(P^+)\mathcal{H}.$$

The $\subset$ part is easy since it amounts to consider the dynamics generated by H_a. The $\supset$ part is also easy: for a state in $1_{\{Z_a\}}(P^+)\mathcal{H}$, one has $|x^b| \geq ct$ for all $b \not\in a$ and hence the intercluster potential I_a decays like $t^{-\mu}$. Since $\mu > 1$ one can use the Cook argument to show that $u \in \mathrm{Ran}(\Omega^+_{\mathrm{sr},a})$.

1.4 Long-range wave operators

The case of long-range pair potentials, roughly speaking decaying at infinity like $|x|^{-\mu}$ for $0 < \mu \leq 1$ is much more difficult for general N than for $N = 1$. Let us assume for simplicity that the potentials v_a satisfy:

$$\partial^\alpha_{x^a} v_a \in O(|x^a|^{-|\alpha|-\mu}),\ |\alpha| \leq 2,\ \mu > 0. \tag{1.7}$$

The inclusion of a short-range part satisfying the hypotheses (1.6) is easy.

A scattering channel corresponding to a cluster decomposition a is described by a state in $\mathrm{Ran}1_{Z_a}(P^+)$. Because of the long-range nature of the pair potentials, the intercluster potential I_a cannot be entirely neglected on such states. Because the clusters of a move away from each others, it is possible to replace I_a by an *effective time-dependent potential* $I_a(t,x)$, which can for example by defined as:

$$I_a(t,x) = F(\frac{|x|_a}{\ln(t)^{-1}t} \geq 1) I_a(x),$$

where the *intercluster distance* $|x|_a$ is

$$|x|_a := \inf_{b\not\leq a} |x^b|.$$

If the pair potentials v_a satisfy (1.7), then

$$\partial^\alpha_t I_a(t,x) \in O(t^{-\mu'-|\alpha|}),\ |\alpha| \leq 2,\ \text{for each } 0 < \mu' < \mu.$$

Then is follows directly from the very definition of the asymptotic velocity P^+, that the limits

$$\mathrm{s}-\lim_{t\to+\infty} U_a(0,t)\mathrm{e}^{-\mathrm{i}tH} 1_{Z_a}(P^+)$$

exist, where $U_a(t,s)$ is the dynamics generated by the time-dependent Hamiltonian

$$H_a + I_a(t,x).$$

The problem is that the dynamics $U_a(t,0)$ *couples* the internal and external dynamics. In order to prove asymptotic completeness, the essential step is to prove that these two dynamics decouple, ie to prove that one can replace $I_a(t,x)$ by

$$I_a(t,0,x_a).$$

After this step the internal and external dynamics are completely decoupled, and it is a simple matter of $1-$particle long-range scattering to introduce a modified evolution for the time-dependent Hamiltonian $\frac{1}{2}D_a^2 + I_a(t,0,x_a)$.

The simplest way to prove this fact is to show a bound on the *size* of the clusters:

clearly for a state in $1_{Z_a})P^+$, the size of the clusters is $o(t)$. if one can show that the clusters have a size $O(|t|^\beta)$, for some β depending on μ, then one gets:

$$I_a(t,x) - I_a(t,0,x_a) = O(|x^a|)|\nabla_x I_a(t,x)| = O(|t|^{\beta-1-\mu}).$$

The crucial progress was made by Derezinski, who showed that for a state in $1_{Z_a})P^+$, the size of the clusters is bounded by $O(t^{\frac{2}{2+\mu}})$. This exponent is again classical: for the $1-$body problem, there exists zero energy classical trajectories which go to infinity like $t^{\frac{2}{2+\mu}}$. For the Kepler problems these trajectories are the well-known *parabolic trajectories*. Note that parabolic trajectories are known to exist also for the classical $N-$body problem of Celestial Mechanics.

The condition that $\frac{2}{2+\mu} < \mu$, yields the condition

$$\mu > \sqrt{3} - 1 \sim 0.732.$$

One obtain the following result:

Theorem 1.5 *Assume (1.7) for $\mu > \sqrt{3} - 1$. Then: i) The long-range wave operators:*

$$\Omega^+_{\mathrm{lr},a} = \mathrm{s}-\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}S_a(t,D_a)}\mathrm{e}^{-\mathrm{i}tH^a}1^{\mathrm{pp}}(H^a) \text{ exist.}$$

ii) the system is asymptotically complete:

$$\bigoplus_{a\in\mathcal{A}} \mathrm{Ran}(\Omega^+_{\mathrm{lr},a}) = L^2(X).$$

1.5 Further results

The condition $\mu > \sqrt{3} - 1$ in Derezinski's result suffices to cover the physical case of Coulomb interactions. Nevertheless it is interesting mathematically to study slower decay. Sofar, positive results are essentially limited to 3-particle problems. One can focus attention to the so-called *exceptional states*, *ie* possible states not in the range of the wave operators. One is faced with the slightly depressing task to work very hard to study states which ultimately are shown not to

exist!. For 3-body problems, these states consist of a pair, with internal energy asymptotically 0, and a third particle moving away from it. One can concentrate on the motion of the pair, and describe the interaction with the third particle using an effective time-dependent potential. Hence one is faced with the study of the dynamics generated by the time-dependent Hamiltonian

$$H(t) = \frac{1}{2}D^2 + V(x) + W(t,x),$$

where V is a long-range potential and $\partial_x^\alpha W(t,x) \in O(t^{-|\alpha|-\mu})$. Modulo short-range corrections, asymptotic completeness was shown by Wang [**?**] if $V(x) \geq C|x|^{-\mu}$

Again these positive results rely on a detailed study of *classical* one-body problems in an external time-dependent potential.

There exist also *counterexamples* to asymptotic completeness due to Yafaev [Ya2], for a system of 3 one-dimensional particles with pair potentials decaying slower than $|x|^{-\frac{1}{2}}$. In these conuterexamples quantum mechanics comes back in the picture: they are essentially based on *Born-Oppenheimer* approximation. Counterexamples similar in spirit were known for time-dependent [Ya1] and time-independent Schroedinger operators [Y].

2 N-particle systems in constant electromagnetic fields

2.1 Constant electric fields

A system of N non-relativistic particles of masses m_i and charges q_i in a constant external electric field E, is described by the *Stark Hamiltonian*:

$$H = H = \sum_{j=1}^{N} \frac{1}{2m_j} D_j^2 + q_j E.x + \sum_{1 \leq i < j \leq N} V_{ij}(x_i - x_j). \tag{2.1}$$

There are several aspects of Stark Hamiltonians not present for ordinary $N-$particle Hamiltonians.

First of all Stark Hamiltonians are unbounded below. This causes some problems if one wants to include singular potentials. Also the proof of the Mourre estimate, an essential tool in the study of the spectral properties of H, relies on an induction argument. Because of the electric field, the internal energy of subsystems is neither bounded from above not from below if the total energy is bounded, which complicates this inductive proof.

Secondly the notion of a short-range potential is different for Stark Hamiltonian: because the position x_t along the electric field of a particle grows like t^2, potentials decaying as $|x|^{-\mu}$ for $\mu > \frac{1}{2}$ in the direction of the electric field are now short-range potentials.

Finally, because of tunneling, bound states typically dissapear in a constant electric field.

Again it is easy and convenient to pass to the framework of Agmon Hamiltonians: a constant electric field corresponds clearly to a vector $F \in X'$. However to describe trajectories, it is convenient to identify F with a vector E in X, using the scalar product on X. One can then define *Agmon-Stark Hamiltonians*:

$$H = \frac{1}{2}D^2 + E.x + \sum_{a \in \mathcal{A}} v_a(x).$$

For $a \in \mathcal{A}$, one can write $E = E_a + E^a$, where $E_a \in X_a$, $E^a \in X^a$. For the physical case (2.1), we see that $E_a = 0$ iff $\sum_{j \in C_l} q_j = 0$ for each l and $E^a = 0$ iff $\frac{q_i}{m_i} = \frac{q_j}{m_j}$ for each pair $\{i, j\} \subset C_l$. C_l being a cluster of a.

The cluster Hamiltonians are then:

$$H_a = H - I_a = \frac{1}{2}(D_a)^2 + E_a.x_a + H^a,$$

where the internal Hamiltonian $H^a = \frac{1}{2}(D^a)^2 + E^a.x^a + V^a(x^a)$ acting on $L^2(X^a)$ is again an Agmon-Stark Hamiltonian. Clearly if $E^a = 0$ then the electric field only acts on the center of mass motion of the clusters of a, and the Hamiltonian H^a is an usual $N-$particle Hamiltonian.

This can be easily formulated by using the partition $X = \bigcup_{a \in \mathcal{A}} Z_a$, where we recall that:

$$Z_a := X_a \backslash \bigcup_{b \not\leq a} X_b.$$

Clearly there exists a unique $\tilde{a} \in \mathcal{A}$ such that $E \in Z_{\tilde{a}}$, and one has:

$$\tilde{a} = \max\{a \in \mathcal{A} | E^a = 0\}.$$

Let us give a set of assumptions (see [HMS1, HMS2]) under which the results below hold:
one says that a potential v_a is *short-range* if

$$|v_a(x^a)| = O(|x^a|^{-\mu_a}),$$

where $\mu_a > \frac{1}{2}$ if $E^a \neq 0$, $\mu_a > 1$ if $E^a = 0$.

One says that it is *long-range* if for $0 \leq |\alpha| \leq 1$:

$$|\partial^\alpha_{x^a} v_a(x^a)| O(|x^a|^{-\mu_a - |\alpha|}),$$

where $\mu_a > 0$ if $E^a \neq 0$, $\mu_a > \sqrt{3} - 1$ if $E^a = 0$.

Note that with some technical effort, Coulomg type singularities can also be treated (see [HMS1, HMS2]).

An important intermediate result (see [HMS1] and references therein), is:

Theorem 2.1 *Assume $E^a \neq 0$. Then one has:*

$$\sigma_{\rm pp}(H^a) = \emptyset,\ \sigma_{\rm sc}(H^a) = \emptyset.$$

This implies that only the clusters $a \leq \tilde{a}$ can contribute to the scattering channels of H.

The key result, proven in [HMS2] (see also [AT1, AT2]) is that the motion of the system is asymptotically localized in a neighborhood of the field direction:

Theorem 2.2 *i) let $q \in C^{\infty(X)}$ be a function homogeneous of degree 0 near infinity, and supported in a conical neighborhood of $\frac{E}{|E|}$ not intersecting the subspaces X_b for $b \not\leq \tilde{a}$. Then one has:*

$$\mathrm{e}^{-\mathrm{i}tH} u = q(x) \mathrm{e}^{-\mathrm{i}tH} u + o(1),\ t \to +\infty,\ \forall u \in L^2(X),$$

ii) one has:

$$\mathrm{e}^{-\mathrm{i}tH} u = F(\frac{|x|}{t^2} \geq |E| - \epsilon) \mathrm{e}^{-\mathrm{i}tH} u + o(1)\ t \to +\infty,\ \forall u \in L^2(X).$$

In the short-range case this allows to replace the full evolution $\mathrm{e}^{-\mathrm{i}tH}$ by $\mathrm{e}^{-\mathrm{i}tH_{\tilde{a}}}$. Since $H^{\tilde{a}}$ is an usual $N-$particle Hamiltonian, one can complete the proof by invoking the known results in Subsect. 1.3. In the long-range case one has to introduce a modified dynamics for the motion of the center of masses. Due to the fact that in a constant electric field $|x|$ grows like t^2, it suffices to add a pure phase factor

$$S_{\tilde{a}}(t) = \int_0^t I_{\tilde{a}}(E\frac{s^2}{2})\mathrm{d}s.$$

This gives the following results:

Theorem 2.3 *Assume the potentials v_a are short-range. Then the wave operators:*

$$\Omega^{\pm}_{\mathrm{sr},a} := \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_a}1^{\mathrm{pp}}(H^a) \text{ exist for all } a \leq \tilde{a}.$$

The wave operators are complete:

$$\bigoplus_{a\leq\tilde{a}} \mathrm{Ran}\Omega^{\pm}_{\mathrm{sr},a} = L^2(X).$$

Theorem 2.4 *Assume the potentials v_a are long-range. Then the wave operators:*

$$\Omega^{\pm}_{\mathrm{lr},a} := \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}S_{\tilde{a}}(t)}\mathrm{e}^{-\mathrm{i}tH_a}1^{\mathrm{pp}}(H^a) \text{ exist for all } a \leq \tilde{a}.$$

The wave operators are complete:

$$\bigoplus_{a\leq\tilde{a}} \mathrm{Ran}\Omega^{\pm}_{\mathrm{lr},a} = L^2(X).$$

Asymptotic observables do not play such a central role for Stark Hamiltonians, because the motion is essentially localized in the direction of the electric field. Nevertheless it is possible to construct and study asymptotic observables in this case also (see [A]).

2.2 Constant magnetic fields

A system of N non-relativistic particles of masses m_i and charges q_i coupled to a constant magnetic field is described by the Hamiltonian:

$$H = H = \sum_{j=1}^{N} \frac{1}{2m_j}(D - q_i Jx_i)^2 + \sum_{1\leq i<j\leq N} V_{ij}(x_i - x_j). \tag{2.2}$$

Here Jx is the vector potential associated with the magnetic field. It is convenient to work in the *transversal gauge*. In this case, if the space dimension ν is 3, we have:

$$J = \begin{pmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the magnetic field is $\vec{B} = (0, 0, 2b)$, $b > 0$, and if $\nu = 2$:

$$J = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}.$$

If $\nu = 3$, we write the coordinates of the particles as

$$x_i = (y_i, z_i), \text{ where } y_i \in \mathbb{R}^2,\ z_i \in \mathbb{R}.$$

and if $\nu = 2$ we simply omit the z variable in the formulas.

Again it is convenient to pass the the formalism of Agmon Hamiltonians. We set $X = \mathbb{R}^{N\nu}$, equipped with the Euclidean structure $x.x$ defined in (1.4), and write (with obvious notations):

$$X = Y \otimes^{\perp} Z.$$

The $N-$particle vector potential is

$$Ax = (q_1 Jx_1, \ldots, q_N Jx_N),$$

and should be considered as an antisymmetric operator $A : X \to X'$ or equivalently as an antisymmetric bilinear form on X.

If we assume that *all particles are charged* then

$$Z = \mathrm{Ker} A.$$

The main new feature of magnetic $N-$particle Hamiltonians is that while invariant under translations of the center of mass, the generators of these translations do not commute with one another. Therefore care has to be taken with the separation of the center of mass (see [**?**], [GŁ1]).

The generator of translations of the center of mass is the *pseudomomentum of the center of mass*:

$$K_{a_{\max}} := D_{a_{\max}} + (Ax)_{a_{\max}},$$

which satisfies the following commutation relations:

$$[\langle x, K_{a_{\max}}\rangle, \mathrm{i}\langle x', K_{a_{\max}}\rangle] = -2\langle x, A_{a_{\max}, a_{\max}} x'\rangle,$$

where $A_{a_{\max}, a_{\max}}$ is the restriction of A to $X_{a_{\max}} \times X_{a_{\max}}$. The component of $K_{a_{\max}}$ along Z is the usual momentum $D_{z_{a_{\max}}}$. The other two components commute iff the *total charge* :

$$Q = \sum_{i=1}^{N} q_i$$

vanishes. A similar analysis can be applied to the channel Hamiltonian

$$H_a = \frac{1}{2}(D - Ax)^2 + \sum_{b \leq a} v_a,$$

which commute now with

$$K_a := D_a + (Ax)_a.$$

This leads to the distinction between *charged clusters* (whose total charge is non zero) and *neutral clusters*. For the analysis of H_a it is convenient to construct unitary operators U_a where the non commuting components of K_a are transformed into (multiples of) positions and momentum variables. We will not give their definition here but refer to [GŁ1, GŁ4] for details. From these transformations, one sees quite easily that:

a charged pair of two particles has a bounded motion in the y variable, and a free motion in the z variable,

a neutral pair of two particles has a ballistic motion in the y variable and a free motion in the z variable. An example of this fact for classical particles can easily be constructed using the properties of the Lorentz force (see [GŁ4]).

Using these transformations and proving a Mourre estimate for the motion in the z direction, one can show the following result [GŁ1]. Because of the special role played by the z direction in the three dimensional case, one has first to separate out the motion of the center of mass in the z direction. A convenient way to do this is to define a subspace $\mathcal{H}_{\rm bound} \subset L^2(X)$ corresponding to states where the N particles stay together but can move in the y and z directions (if the total charge is 0) or only along z (if the total charge is not 0). The space of *scattering states* is then

$$\mathcal{H}_{\rm scatt} := \mathcal{H}_{\rm bound}^{\perp}.$$

Theorem 2.5 *Assume that the system has no neutral subsystems and that the pair potentials are short-range. Then the wave operators:*

$$\Omega^+_{{\rm sr},a} := {\rm s}-\lim_{t\to+\infty} {\rm e}^{{\rm i}tH}{\rm e}^{-{\rm i}tH_a}\Pi_a \ \textit{exist.}$$

The system is asymptotically complete:

$$\bigoplus_{a\in\mathcal{A},a\neq a_{\max}} {\rm Ran}\Omega^+_{{\rm sr},a} = \mathcal{H}_{\rm scatt}.$$

Here Π_a is a projection on states describing bound clusters of a (its precise defintion uses the reducing transformation U_a). Note that in this theorem the total charge of the system can be 0. The size of the clusters in the y variables is asymptotically bounded.

The analog situation in two dimensions is much simpler: there are no scattering states and if the total charge is non zero, $\sigma(H)$ is a closed countable set of eigenvalues with infinite multiplicity (see [GŁ4]).

One can next consider the case of long-range interactions. As in the standard case, the main problem is to decouple internal and external motion and to this end to prove a bound on the size of the charged clusters. A bound in the z direction can be obtained by the same arguments as in the standard case. The proof of a bound in the y direction is very different and relies on the *center of orbit observable:*

$$C := \frac{1}{2}(y + A^{-1}D_y).$$

Classically the two components of C are the position of the center of the circle followed by the classical particle It is easy to see that a bound on C yields a bound on y, because

$$C - y = \frac{1}{2}A^{-1}(D_y - Ay),$$

and $D_y - Ay$ is controlled by the total energy. Using then a construction inspired by the Graf vector field, but using now the indefinite *charge metric*

$$q(y,y) := \sum_{i=1}^{N} q_i y_i . y_i,$$

it is possible to show that along the evolution C (and hence y) is bounded by $O(t^\alpha)$ for each $\alpha > \frac{1}{2}$. This leads to the following result:

Theorem 2.6 *Assume that the system has no neutral subsystems and that the pair potentials are long-range (with decay rate $\mu > \sqrt{3} - 1$). Then the wave operators:*

$$\Omega^+_{\mathrm{sr},a} := \mathrm{s}- \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}S_a(t,D_{z_a})}\mathrm{e}^{-\mathrm{i}tH_a}\Pi_a \text{ exist.}$$

The system is asymptotically complete:

$$\bigoplus_{a\in\mathcal{A}, a\neq a_{\max}} \mathrm{Ran}\Omega^+_{\mathrm{sr},a} = \mathcal{H}_{\mathrm{scatt}}.$$

Here the modification $S_a(t, D_{z_a})$ is for example a standard Dollard modifier in the z direction.

With some considerable efforts, it is also possible to prove asymptotic completeness for 3−particle systems having one or two neutral pairs. This result holds in two or three space dimensions ([GŁ3, GŁ4]).

2.3 concluding remarks

Time-dependent scattering theory, essentially based on the Mourre estimate and phase space propagation estimates due to Sigal and Soffer, has proven to be an extremely efficient and flexible tool to answer the basic questions of scattering theory. The $N-$body scattering theory in the eigthies, which to the outsider I was at the time looked like a dangerous and mysterious jungle, has turned into a clean and quiet jardin à la francaise, a sure sign of a mature and well understood theory.

However open ground still lies near the boundaries of this garden: for example scattering theory in constant magnetic fields is not completely understood in presence of neutral clusters of particles. Similarly there are very few results about *dispersive* $N-$particle systems, where the non-relativistic kinetic energy

$$\sum_{i=1}^{N} \frac{1}{2m_i} p_i^2$$

is replaced by a general kinetic energy $\omega(p)$, like the relativistic one:

$$\sum_{i=1}^{N} \sqrt{p_i^2 + m_i^2}.$$

Also systems in time-periodic electro-magnetic fields are not completely understood (see eg [**?**]).

When one thinks hard about the time-dependent method, one realizes its defects. The first obvious one is that it is of little use to study scattering amplitudes: here stationary methods seem to rule.

But in my opinion its main defect is the following one:

this method does not allow to understand *mean motion*: since it is based only on differential inequalities, it cannot control a motion which is a superposition of a mean ballistic motion and a bounded periodic one. This is the very reason why dispersive systems or neutral clusters in a magnetic field are not well understood (except for $N = 3$ in the last case). A new method to control mean motion would certainly allow to handle many interesting problems still open. I think this is a challenging and important question.

3 Stationary scattering theory

The stationary approach to scattering theory, relying on the study of the resolvent $(H - z)^{-1}$ when $\mathrm{Im} z \to 0$ is much older than the time-dependent approach. It seems in general less efficient to prove the basic results of scattering theory, like asymptotic completeness, but is the only way to get results on scattering quantitiees, like properties of the scattering matrix.

For $N-$body scattering theory, it started with the approach of Faddeev [**?**].Faddeev used clever resolvent identities (named afterwards Faddeev equations) and the stationary method to study asymptotic completeness for a certain class of short-range potentials in dimension 3 or bigger for 3-body systems [**?**]. Unfortunately, his method required to impose certain implicit assumptions on the potentials.

After Faddeev the stationary method was developed by Ginibre and Moulin [**?**] and Thomas [**?**] for 3-body systems, by Hagedorn [**?**] for 4-body systems and by Sigal [**?**] for N-body systems. All of these papers had the same drawback, namely, implicit assumptions. The only exception was the work of Loss and Sigal [**?**], which contained a stationary proof of the asymptotic completeness of a certain (rather small) class of 3-body systems without implicit assumptions.

One should also mention proofs of asymptotic completeness for some special N-body systems [**?**], [**?**].

Another approach is the one of Kato [**?**], based on the concept of $H-$*moothness* of an operator. For a long time its application was restricted to 2-body problems. However the progress in the time-dependent method, coming from the consideration of propagation estimates in phase space, led to similar progress in Kato's method. Yafaev [**?**] was able to prove asymptotic completeness for short-range $N-$body Hamiltonians by proving that a certain differential operator was $H-$smooth. Note that a certain function used in Yafaev's construction, related to the Graf's function, played a crucial role in the estimate on the size of clusters of Derezinski.

Later Yafaev [**?**] was also able to derive a stationary formula for the scattering matrix $S(\lambda)$, which allowed him to show that $S(\lambda)$ is strongly continuous w.r.t. λ. The scattering matrix is

defined using the *two spaces approach* to scattering theory: clearly if one labels the scattering channels by indices $\alpha = (a, \lambda^a)$, where a is a cluster deoomposition and λ^a an eigenvalue of H^a for an eigenstate ψ^a (repeated as usual with its multiplicity), one can introduce the auxiliary Hilbert space

$$\tilde{\mathcal{H}} := \oplus_{\alpha,\, a \neq a_{\max}} L^2(X_a),$$

auxiliary Hamiltonian

$$\tilde{H} := \oplus_{\alpha,\, a \neq a_{\max}} \lambda^a + \frac{1}{2} D_a^2,$$

and identification operator:

$$J := \sum_{\alpha,\, a \neq a_{\max}} J_\alpha,\ J_\alpha u_a := \psi^a \otimes u_a,$$

in such a way that whole set of channel wave operators can be descirbed with just:

$$\Omega^{\pm} := \mathrm{s}-\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} J \mathrm{e}^{-\mathrm{i}t\tilde{H}}.$$

The scattering operator is then $S = \Omega^{+*}\Omega^{-}$ commutes with $\tilde{H}$ and hence can be decomposed as a direct integral

$$S = \int_{\sigma(\tilde{H})} S(\lambda) \mathrm{d}\lambda,$$

where $S(\lambda)$ defined λ a.e. is the *scattering matrix*. Yafaev's proof relies on two ingredients: first one can replace the identification operator J by a better one $\tilde{J}$ localizing in the region where propagation takes place. Second one has to use resolvent estimates of the type:

$$\|G_a R(\lambda \pm \mathrm{i}\epsilon) G_b\| \in O(1),$$

where G_a is a first order differential operator of the form

$$G_a = \chi(x)|x|^{-\frac{1}{2}} \nabla_a^{\perp},$$

where χ is a cutoff function in a conical neighborhood of Z_a and $\nabla_a^{\perp}$ is the projection of the gradient ∇_{x_a} on the direction orthogonal to x_a.

Related but more detailed results were obtained by Isozaki [**?**, **?**] for 3−particles Hamiltonians. In [**?**] Isozaki considers 3−body Hamiltonians in for particles in $\mathbb{R}^3$ and looks at scattering matrices with 2 to 3 particles scattering processes. The scattering matrix have kernels

$$\hat{S}_{0,\alpha}(\lambda; \theta, \omega),\ \theta \in S^5,\ \omega \in S^2.$$

He shows first that $\hat{S}_{0,\alpha}$ is continuous in all variables when θ is outside the collision planes.

He also derives pointwise asymptotics of $\hat{S}_{0,\alpha}(\lambda; \theta, \omega)$ when θ approaches a collision plane X_a. The singular terms in these asymptotics is related to zero energy eigenfunctions and zero energy resonances for the Hamiltonian H^a.

In [**?**] related results are obtained for the spatial asymptotics of generalized eigenfunctions.

Note that Isozaki's results are related to the results of Vasy that we will briefly describe below. The difference is in the way one desribes the singularity of a given distribution: in Isozaki's approach asymptotics of the distribution near the points where it is not continuous are derived. Vasy uses the microlocal approach where singularities of a distribution are described by studying its *wave front set*.

3.1 Microlocal approach to stationary scattering theory

In this subsection we will try to briefly describe the results of Vasy [**?**] on singularities of generalized eigenfunctions and scattering matrices for $N-$particle Hamiltonians. Vasy's approach is decidedly microlocal: properties of a distribution are described in phase space and PDE methods like pseudodifferential calculus and propagation of singularies theorems are used.

The goal is of course to study asymptotic properties for $|x| \to \infty$ of *generalized eigenfunctions* of H, which can be considered as particular cases of distributions $u \in \mathcal{S}'(X)$ solving:

$$(H - \lambda)u = 0.$$

To discuss asymptotic properties of u when $|x| \to \infty$, it is convenient to introduce the radial compactification of $X = \mathbb{R}^n$ to the upper half-sphere S^n_+. We follows Vasy's notation and denote the points in Euclidean space $\mathbb{R}^n$ by w:

$$\mathrm{RC} : \mathbb{R}^n \ni w \mapsto (1/(1+|w|^2)^{\frac{1}{2}}, \mathrm{x}/(1+|w|^2)^{\frac{1}{2}}) \in S^n_+.$$

The boundary at infinity of S^n_+ is $S^{n-1} = \partial S^{n-1}$. The images of collision planes X_a in S^n_+ are denoted by $\tilde{X}_a$ (extended up to the boundary) and one sets:

$$C_a := \mathrm{cl}\tilde{X}_a \cap S^{n-1}.$$

Clearly $\{C_a\}_{a\in\mathcal{A}}$ inherits the semi-lattice structure from $\{X_a\}$. The following analog of the sets Z_a in (1.5) will also be needed:

$$C'_a = C_a \setminus \cup_{b \not\leq a} C_b. \tag{3.1}$$

Extending $|x|$ in $|x| \leq 1$ to make it smooth, one gets the *boundary defining function*

$$x := |w|^{-1}.$$

Various spaces of functions and distributions have natural definitions in this setup: for example the space of polyhomogeneous symbols of order $m \in \mathbb{R}$ $S^m_{\mathrm{phg}}(\mathbb{R}^n)$ becomes $x^{-m}C^\infty(S^n_+)$ (in particular the space $\mathcal{S}(\mathbb{R}^n)$ becomes $\dot{C}^\infty(S^n_+)$, the space of C^∞ functions on S^n_+) vanishing to infinite order on S^{n-1}.

Similarly the image of the weighted Sobolev space $\langle ww\rangle^{-p}H^m(\mathbb{R}^n)$ is denoted by $H^{m,p}(S^n_+)$. With this notation the *limiting absorption principle* takes the form:

$$R(\lambda \pm \mathrm{i}0) : H^{k,l}(S^n_+) \to H^{k+2,l'}(S^n_+),\ \forall l > /12,\ l' < -\frac{1}{2}.$$

To define a microlocal calculus, one needs essentially three objects:

1) a 'garbage space' $\mathcal{S}$ *ie* a space of functions considered as being 'smooth'. For the $N-$body problem this would be the space $\mathcal{S}(\mathbb{R}^n)$ (note that the notion of smoothness here is decay at spatial infinity).

2) a space X (typically the cotangent space of some manifold).

3) a class of functions on X with a quantization rule $m \mapsto Op(m)$ sending a symbol m onto a (pseudo-differential) operator $Op(m)$ with a good asymptotic calculus. One requires that $Op(m)$

sends $\mathcal{S}$ into itself for all m, and that there is a good notion of 'essential support' ie that one can make sense of the property that a symbol m 'vanishes' at a point $x \in X$.

Using these three ojects, one can define a notion of *wavefront set* of a distribution u by saying that a point $x \in X$ does not belong to the wavefront set of u if there exist m with $m(x) \neq 0$ such that $\mathrm{Op}(m)u \in \mathcal{S}$. The main result one often proves in this setting is the so-called *propagation of singularities theorem*, which says that if u satisfies $\mathrm{Op}(m)u \in \mathcal{S}$, for m of 'real principal type', then the wavefront set of u is a union of Hamiltonian curves of m in X.

It is out of the scope of this review to give a precise definition of these objects, for which we refer to [**?**, **?**, **?**] and we will only indicate the definition of the phase space:

the *scattering vector fields* is the space

$$\mathcal{V}_{\mathrm{sc}}(S^n_+) = x\mathcal{V}_{[\mathrm{b}]}(S^n_+),$$

where $\mathcal{V}_b(S^n_+)$ is the space of vector fields on S^n_+ tangent to the boundary S^{n-1}. If one introduces local coordinates $(x, y_1, \ldots, y_{n-1})$ on S^n_+, then a basis of $\mathcal{V}_{\mathrm{sc}}(S^n_+)$ is given by the vector fields

$$x^2\partial_x,\ x\partial_{y_j},\ 1 \leq j \leq n-1,$$

which correspond to the usual vector fields p_{w_j} under RC. The space $\mathcal{V}_{\mathrm{sc}}(S^n_+)$ is the space of sections of a vector bundle ${}^{\mathrm{sc}}TS^n_+$, whose dual is denoted by ${}^{\mathrm{sc}}T^*(S^n_+)$. This space would be sufficient to handle the two-body scattering [**?**].

In the $N-$body case, one has to take care of the collision planes. Over one of the sets C'_a defined in (3.1) one replaces the fiber of ${}^{\mathrm{sc}}T^*(S^n_+)$ by $\tilde{X}_a$. This reflects the fact that near a collision plane, only the motion in the external variables can be described semiclassically.

In this way one finally obtains the phase space:

$${}^{\mathrm{sc}}\dot{T}^*S^n_+ = \cup_a {}^{\mathrm{sc}}T^*_{C'_a}\tilde{X}_a.$$

The *scattering wave from set* ${}^{\mathrm{sc}}\mathrm{WF}u$ of a distribution u is now a closed subset of ${}^{\mathrm{sc}}\dot{T}^*S^n_+$.

On this phase space Vasy introduces a rather intricate notion of *broken Hamiltonian curves*, roughly defined as follows: first of all Hamiltonian vector fields are scaled by a factor of $|w|$. In this way for example a free trajectory which takes an infinite time to go to one point on S^{n-1} to its antipodal now does this journey in a time equal to π.

In the free region, a broken Hamiltonian curve is simply a Hamilton trjactory for this rescaled Hamiltonian. When it hits a collision plane C_a, it is either reflected or can continue along C_a respecting the energy conservation (at that point of course eigenvalues of subsystems must be taken into account)

The main result of [**?**] is now that if $u \in \mathcal{S}'(\mathbb{R}^n)$ is a solution of $(H-\lambda)u = 0$, then ${}^{\mathrm{sc}}\mathrm{WF}u$ is a union of broken Hamiltonian curves.

Vasy also proved a similar result about the wavefront set of the *scattering amplitudes*. These being now distributions on products of spheres, the wavefront set is now the usual notion of wavefront set for distributions on a manifold.

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