Comparaison de deux échantillons iid de même variance et de même taille

$(x_1 \dots x_n) \sim N(\mu_x, \sigma^2)$ $\quad$ $(y_1 \dots y_n) \sim N(\mu_y, \sigma^2)$

On veut tester $\mu_x = \mu_y \ (M_1)$ contre $\mu_y \neq \mu_x \ (M_2)$

- On pose le même prior $\pi_\sigma(\sigma^2) = \frac{1}{\sigma^2}$
- On reparamètre $\mu_x = \mu - \xi$ $\quad \mu_y = \mu + \xi$ ce qui permet de poser le même prior $\pi_\mu(\mu) = 1$ sur les deux modèles et $\xi \sim N(0, \tau^2)$

Rem : prior sur σ^2 et μ sont impropres, mais la vraisemblance intégrée est finie. et les constantes de normalisation se simplifient.

$$B_{21} = \frac{\displaystyle\int \frac{1}{(\cancel{2\pi}\sigma^2)^{\frac{n+m}{2}}} \left[\exp - \frac{\sum_i (x_i-\mu-\xi)^2 + \sum_i (x_i-\mu+\xi)^2}{2\sigma^2}\right] \frac{1}{\sigma^2} \frac{1}{(2\pi\tau^2)^{1/2}} \exp - \frac{(\xi)^2}{2\tau^2} \, d\sigma^2 d\mu \, d\xi}{\displaystyle\int \frac{1}{(\cancel{2\pi}\sigma^2)^{\frac{n+m}{2}}} \left[\exp - \frac{\sum_i (x_i-\mu)^2 + \sum_i (y_i-\mu)^2}{2\sigma^2}\right] \frac{1}{\sigma^2} \, d\sigma^2 d\mu}$$

Calcul du dénominateur :

$$\int \underbrace{\int \exp - \left(\frac{\overbrace{\sum_i (x_i-\mu)^2 + \sum_i (y_i-\mu)^2}^{\beta}}{2\sigma^2}\right) \left((\sigma^2)^{-\frac{2n}{2} - 1} \right) d\sigma^2}_{IG(\alpha,\beta) \times \frac{\Gamma(\alpha)}{\beta^\alpha}} d\mu$$

$\alpha = n$

$$= \int \Gamma(n) \left[\frac{\sum_i (x_i-\mu)^2 + \sum_i (y_i-\mu)^2}{2}\right]^{-n} d\mu.$$

or $\sum_i (x_i-\mu)^2 = \sum_i (x_i - \bar{x} + \bar{x} - \mu)^2 = \sum_i (x_i - \bar{x})^2 + 2\underbrace{\sum_i (x_i - \bar{x})}_{=0}(\bar{x}-\mu) + n(\bar{x}-\mu)^2$

$= s_x^2 + n(\bar{x}-\mu)^2.$

soit $\sum_i (x_i-\mu)^2 + \sum_i (y_i-\mu)^2 = n(\bar{x}-\mu)^2 + s_x^2 + n(\bar{y}-\mu)^2 + s_y^2 = n\left[(\bar{x}-\mu)^2 + (\bar{y}-\mu)^2 + \frac{s_x^2+s_y^2}{n}\right]$

$= n\left[2\mu^2 - 2\mu(\bar{x}+\bar{y}) + \bar{x}^2 + \bar{y}^2 + \frac{s_x^2+s_y^2}{n}\right]$

$= n\left[2\left(\mu^2 - 2\mu\frac{\bar{x}+\bar{y}}{2}\right) + \bar{x}^2 + \bar{y}^2 + \frac{s_x^2+s_y^2}{n}\right]$

$= n\left[2\left(\mu - \frac{\bar{x}+\bar{y}}{2}\right)^2 - \frac{(\bar{x}+\bar{y})^2}{2} + \bar{x}^2 + \bar{y}^2 + \frac{s_x^2+s_y^2}{n}\right]$

$= n\left[2\left(\mu - \frac{\bar{x}+\bar{y}}{2}\right)^2 + \frac{\bar{x}^2 + \bar{y}^2 - 2\bar{x}\bar{y}}{2} + \frac{s_x^2+s_y^2}{n}\right]$

$= n\left[2\left(\mu - \frac{\bar{x}+\bar{y}}{2}\right)^2 + \boxed{\frac{(\bar{x}-\bar{y})^2}{2} + \frac{s_x^2+s_y^2}{n}}\right] \ \tilde{s}^2$

d'où le dénominateur :

$$\int \Gamma(n)\frac{n^{-n}}{2^{-n}}\left[2\left(\mu-\frac{\bar{x}+\bar{y}}{2}\right)^2+\overbrace{\frac{(\bar{x}-\bar{y})^2}{2}+\frac{s_x^2+s_y^2}{n}}^{\tilde{s}^2}\right]^{-n}d\mu.$$

$$=\Gamma(n)\left(\frac{n}{2}\right)^{-n}\left(\frac{1}{\tilde{s}^2}\right)^{-n}\int\underbrace{\left[\frac{2\left(\mu-\frac{\bar{x}+\bar{y}}{2}\right)^2}{\tilde{s}^2}+1\right]^{-\frac{(2n-1)+1}{2}}d\mu.}$$

$\frac{\tilde{s}^2}{2}=(2n-1)\tau^2$

$$\frac{\Gamma((2n-1)/2)}{\Gamma(2n/2)}\sqrt{(2n-1)\pi\tau^2}\times\mathcal{C}\left(2n-1,\frac{\bar{x}+\bar{y}}{2},\frac{\tilde{s}^2}{2(2n-1)}\right).$$

$$=\cancel{\Gamma(n)}\left(\frac{n}{2}\right)^{-n}\left(\frac{1}{\tilde{s}^2}\right)^{-n}\frac{\Gamma(n-1/2)}{\cancel{\Gamma(n)}}\sqrt{\cancel{(2n-1)}\pi\,\tilde{s}^2/2\cancel{(2n-1)}}$$

$$\boxed{\text{Denom}=\left(\frac{n}{2}\right)^{-n}\Gamma\left(n-\frac{1}{2}\right)\sqrt{\frac{\pi}{2}}\ \frac{1}{\tilde{s}^{2n-1}}}$$

Numérateur :

$$\int\frac{1}{(\sigma^2)^{\frac{n+n}{2}}}\exp-\frac{\sum_1^n(x_i-\mu-\xi)^2+\sum_1^n(y_i-\mu+\xi)^2}{2\sigma^2}\ \frac{1}{\sigma^2}\ \frac{1}{(2\pi\tau^2)^{1/2}}\exp-\frac{\xi^2}{2\tau^2}\,d\xi\,d\mu\,d\sigma^2$$

$$=\int\frac{1}{(\sigma^2)^{n+1}}\exp-\frac{\overbrace{\sum_1^n(x_i-\mu-\xi)^2+\sum_1^n(y_i-\mu+\xi)^2}^{\beta}}{2\,\sigma^2}\ \frac{1}{(2\pi\tau^2)^{1/2}}\exp-\frac{\xi^2}{2\tau^2}\,d\sigma^2 d\mu\,d\xi$$

(α : n)

$IG(\alpha,\beta)$

$$=\int\Gamma(n)\left[\frac{\sum_1^n(x_i-\mu-\xi)^2+\sum_1^n(y_i-\mu+\xi)^2}{2}\right]^{-n}\frac{1}{(2\pi\tau^2)^{1/2}}\exp-\frac{\xi^2}{2\tau^2}\,d\mu\,d\xi.$$

$$\sum_1^n(x_i-\mu-\xi)^2+\sum_1^n(x_i-\mu+\xi)^2=n\left[2\mu^2-2\mu(\bar{x}-\cancel{\xi}+\bar{y}+\xi)+(\xi+\bar{x})^2+(\xi-\bar{y})^2+\frac{s_x^2+s_y^2}{n}\right]$$

$$=n\left[2\left(\mu-\frac{\bar{x}+\bar{y}}{2}\right)^2-\frac{(\bar{x}+\bar{y})^2}{2}+(\xi+\bar{x})^2+(\xi-\bar{y})^2+\frac{s_x^2+s_y^2}{n}\right]$$

$$=n\left[2\left(\mu-\frac{\bar{x}+\bar{y}}{2}\right)^2-\frac{1}{2}(\bar{x}^2+\bar{y}^2+2\bar{x}\bar{y})+\bar{x}^2+2\xi\bar{x}+2\xi^2+\bar{y}^2-2\xi\bar{y}+\frac{s_x^2+s_y^2}{n}\right]$$

$$=n\left[2\left(\mu-\frac{\bar{x}+\bar{y}}{2}\right)^2+\frac{1}{2}(\bar{x}^2+\bar{y}^2-2\bar{x}\bar{y})^2+2\xi(\bar{x}-\bar{y})+2\xi^2+\frac{s_x^2+s_y^2}{n}\right]$$

$$=n\left[2\left(\mu-\frac{\bar{x}+\bar{y}}{2}\right)^2+\frac{1}{2}\left[\underbrace{(\bar{x}-\bar{y})^2+4\xi(\bar{x}-\bar{y})+4\xi^2}_{(2\xi+\bar{x}-\bar{y})^2}\right]+\frac{s_x^2+s_y^2}{n}\right]$$

$$=n\left[2\left(\mu-\frac{\bar{x}+\bar{y}}{2}\right)^2+\underbrace{\frac{1}{2}\left((2\xi+\bar{x}-\bar{y})^2+2\frac{s_x^2+s_y^2}{n}\right)}_{\tilde{\tau}^2}\right]\quad\mathcal{C}\left(2n-1,\frac{\bar{x}+\bar{y}}{2},\frac{\tilde{\tau}^2}{2(2n-1)}\right)$$

$$\text{Num} = \cancel{\Gamma(n)} \left(\frac{n}{2}\right)^{-n} \int \sqrt{\frac{\cancel{(2n-1)}\, \pi \tilde{\tau}^2}{2\cancel{(2n-1)}}}\, \frac{\Gamma(n-\frac{1}{2})}{\cancel{\Gamma(n)}} \left(\frac{1}{\tilde{\tau}^2}\right)^{+n} \frac{1}{(2\pi\tau^2)^{1/2}} \exp -\frac{\xi^2}{2\tau^2}\, d\xi \,.$$ 3

avec $\boxed{\tilde{\tau}^2 = \dfrac{(2\xi + \bar{x} - \bar{y})^2 + 2\,\dfrac{s_x^2 + s_y^2}{n}}{2}}$

$$= \left(\frac{n}{2}\right)^{-n} \sqrt{\frac{\pi}{2}}\ \Gamma(n-\tfrac{1}{2}) \int \frac{1}{\tilde{\tau}^{2n-1}_{(\xi)}} \cdot \frac{1}{(2\pi\tau^2)^{1/2}} \exp -\frac{\xi^2}{2\tau^2}\, d\xi \,.$$

$$B_{2,1} = \frac{\int (\tilde{\tau}^2)^{-n+1/2}\, e^{-\xi^2/\tau^2} / (2\pi\tau^2)^{1/2}\, d\xi}{(\tilde{\sigma}^2)^{-n+1/2}}$$

$$\boxed{B_{2,1} = \frac{\int \left[(2\xi + \bar{x} - \bar{y})^2 + 2\,\frac{s_x^2 + s_y^2}{n}\right]^{-n+1/2} e^{-\xi^2/2\tau^2} / (2\pi\tau^2)^{1/2}\, d\xi}{\left[(\bar{x} - \bar{y})^2 + 2\,\frac{s_x^2 + s_1^2}{n}\right]^{-n+1/2}}}$$

$$\frac{s_x^2 + s_y^2}{n} = \frac{\sum_i (x_i - \bar{x})^2 + \sum_i (y_i - \bar{y})^2}{n} = s^2 \,.$$