p-adic Hodge theory, deformations and local Langlands

Christophe Breuil

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WARNING: These notes are informal and are not intended to be published. I apologize for the inaccuracies, flaws and English mistakes that they surely contain.

Contents

1 Lecture 1: Introduction, motivation and contents

1.1

I've been asked by the organizers to lecture on p-adic Hodge theory with a view to modular forms. The result is these notes, which have three aims.

The first aim is to provide the basics of classical p-adic Hodge theory (together with the complementary talks) and to give many examples coming from modular forms. So the listener who is only interested in learning about this subject will find, I hope, in the contents of this course and of the complementary talks the main results he might need some day.

The second aim is to provide the audience with several open questions which, for some of them, are open essentially because nobody made a real effort to solve them. To give a flavour, here are three examples of such presumably accessible problems which naturally arise from this course:
1) The problem of determining the filtered modules of p-adic Hodge theory associated to cuspidal newforms on $\Gamma_1(N)$ when p^2 divides N (the other cases being known): see Lecture 3.
2) The problem of extending Henniart-Langlands correspondence from GL_2 to GL_n. This correspondence arose from the following question: is there a Langlands correspondence for smooth representations of the inertia group (instead of classically the Weil-Deligne group)? See Lecture 7.
3) The problem of determining the reduction modulo p of the p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ associated to *all* newforms of low weight k on $\Gamma_1(pN)$ ($p \nmid N$) (the $\Gamma_1(N)$ case is known; for $\Gamma_1(pN)$, the case k even and $p \nmid \mathrm{cond}(\mathrm{char}(f))$ is in these notes, some cases where $p \,|\, \mathrm{cond}(\mathrm{char}(f))$ are in [40]): see Lecture 9.
Other open questions will be mentionned during the course (in particular in the last lecture).

The third aim is to go further than p-adic Hodge theory and to suggest a possible link between p-adic Hodge theory for $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ (or the Weil group of $\mathbf{Q}_p$) on the one side and the theory of p-adic and modulo p representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ on the other side (not to mention GL_n of a p-adic local field!). In other terms, to suggest hypothetic continuous p-adic and modulo p Langlands correspondences. In fact, this question is the underlying motivation of this course. It is also quite natural since there are already archimedean, ℓ-adic and modulo ℓ Langlands correspondences for GL_n, and

I do not know any instances of a mathematical theory having archimedean and ℓ-adic shapes and no p-adic one.

I will now describe the results of this course (with their origin) and try to explain why they can suggest the idea of such correspondences.

1.2

The starting point is the calculations that led to the complete proof of the full Shimura-Taniyama-Weil conjecture ([9], [17]) following Wiles'method. In [9], the problem was to find which local residual representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ (actually $\mathrm{Gal}(\overline{\mathbf{Q}}_3/\mathbf{Q}_3)$ led to Barsotti-Tate deformation rings isomorphic to $\mathbf{Z}_p[[X]]$ (actually $\mathbf{Z}_3[[X]]$). All the previous computations (e.g. [14] and [15]) suggested the deformation ring was $\mathbf{Z}_p[[X]]$ precisely when a certain "multiplicity 1" phenomena occured on a certain modulo p representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ (related to the deformation problem) appearing in spaces of weight 2 modular forms. Using these modular representations of $\mathrm{GL}_2(\mathbf{Z}_p)$, it was possible to guess which of the Barsotti-Tate deformation rings could be isomorphic to $\mathbf{Z}_p[[X]]$ and to check in [9] this was indeed true for some of them when $p = 3$ (actually it was only proven that they were non zero and "smaller" than $\mathbf{Z}_3[[X]]$ since this was enough to get the remaining cases of modularity). Then a conjecture was made ([9], Conjecture 1.3.1) putting this mysterious prediction into a precise mathematical assertion and in the same time generalizing and unifying earlier conjectures of [15] (Conjectures 1.2.2 and 1.2.3). The new feature of [9] in the cases where this conjecture was (almost) checked was the total disproportion between the delicate computations of the deformation rings and the straightforward computations of the modular representations of $\mathrm{GL}_2(\mathbf{Z}_3)$. However both computations were in perfect accord. This suggested something was going on...

In this course, essentially based on a joint work with A. Mézard ([10]), I will state a far reaching generalization of Conjecture 1.3.1 of [9] (although the experienced reader will notice we only consider types and not "extended types") and I will explain non trivial cases (the "semi-stable cases of even weight") where this generalized conjecture holds. Basically the conjecture says that the Hilbert-Samuel multiplicities of the reductions modulo p of the various deformation rings arising from p-adic Hodge theory (in arbitrary weight smaller than p, not just weight 2) are *equal* to certain multiplicities computed on some modular representations of $\mathrm{GL}_2(\mathbf{Z}_p)$ whatever their value is (not just 1). When the multiplicity is 1, this implies the deformation ring must be $\mathbf{Z}_p[[X]]$. To check the conjecture for semi-stable deformations, we use integral p-adic Hodge theory, which will also be surveyed in this course. These semi-stable cases have their own interest (independantly of the conjecture): for instance they give a very precise description of the local residual representations at p associated to newforms of low weight on $\Gamma_0(pN)$.

Now I try to explain how this conjecture could be related to some p-adic or modulo p Langlands correspondences.
First, it is explained in this course (following Fontaine) how to associate an n-dimensional Weil-Deligne representation, hence a classical smooth irreducible representation of GL_n by the local Langlands correspondence ([27], [26]), to an n-dimensional potentially semi-stable p-adic Galois representation. This already gives a step towards the representation theory of GL_n. But there are two problems: the first is that the representations you get are smooth, that is algebraic, and not at all p-adic, the second is that you have lost a crucial part of the initial p-adic Galois representation, namely the Hodge-Tate weights and the Hodge filtration. So the very rough idea is: "Try to incorporate in some way the Hodge-Tate weights and the Hodge filtration to these algebraic representations of GL_n and you might get the sought-after p-adic representations of GL_n." At the moment, of course, noone has the slightest idea how to do such a thing (assuming it is the right thing to do). In the last lecture however, I will describe non trivial examples where it seems one can really do something.
Secondly, our computations of the semi-stable deformation rings show that they *heavily depend* on the Hodge filtration associated to 2-dimensional semi-stable p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$. And so does their Hilbert multiplicity. However, one can *still* predict these various multiplicities by taking certain smooth irreducible representations of $\mathrm{GL}_2(\mathbf{Z}_p)$, reducing them modulo p and looking at their decomposition. It suggests there should be something like the Hodge filtration also somewhere on the GL_2 side, or, in other words, a possible "correspondence" between 2-dimensional semi-stable representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ and something p-adic related to $\mathrm{GL}_2(\mathbf{Q}_p)$ as, may-be, p-adic (infinite dimensional) representations of $\mathrm{GL}_2(\mathbf{Q}_p)$.

1.3

Here is the contents of this course.

In Lecture 2, I describe the various rings of p-adic Hodge theory (which were introduced for most of them by Fontaine): B_{dR}, B_{cris}, B_{st}, etc.

In Lecture 3, I define potentially semi-stable p-adic representations, their associated filtered modules (which carry the Hodge filtration) and their associated Weil-Deligne representations. I also give many examples of filtered modules, in particular all the filtered modules coming from newforms on $\Gamma_1(N)$ where either $p \nmid N$ or $p \parallel N$.

In Lecture 4, I explain integral p-adic Hodge theory, whose aim is to describe Galois stable lattices (and not just Galois $\mathbf{Q}_p$-representations). The point is the definition of an integral structure also on the filtered module side called a strongly divisible lattice. This structure will correspond to Galois lattices. All these lattices (Galois and strongly divisible) have the virtue of being (compatibly) amenable to reduction modulo p.

In Lecture 5, I give many examples of strongly divisible lattices: examples coming from the filtered modules of Lecture 3, geometric examples, examples coming from p-divisible groups...

In Lecture 6, I explain the construction of Mazur's deformation rings following [37] and I use them to define the relevant deformation rings of p-adic Hodge theory. I state a few preliminary conjectures on the structure of the latters. Our main conjecture will deal with these rings.

In Lecture 7, following Henniart, I define what could be called a "Langlands correspondence for $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p^{\mathrm{unr}})$". More precisely, I associate to each smooth 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p^{\mathrm{unr}})$ that extends to the Weil group of $\mathbf{Q}_p$ a well defined smooth irreducible representation of $\mathrm{GL}_2(\mathbf{Z}_p)$. This holds for general p-adic local fields (not just $\mathbf{Q}_p$).

In Lecture 8, I give the heuristic explanations that led to the formulation of the main conjecture (predicting the Hilbert Samuel multiplicities of deformation rings from the decomposition modulo p of the representations of $\mathrm{GL}_2(\mathbf{Z}_p)$ defined in Lecture 7). Then I precisely formulate this main conjecture.

In Lecture 9, I give an overview of the proof of non trivial cases of the previous conjecture. This heavily uses some examples given in Lecture 5 and has applications to modular forms.

Lecture 10 is about very recent results. I first state a modulo p Langlands correspondence between certain infinite dimensional representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}_p}$ and 2-dimensional semi-simple representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$. Then I define simple and natural p-adic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ and give stricking evidence (using the previous modulo p correspondence and the computer science help of W. Stein and D. Savitt) that they may be quite deeply "linked" to irreducible 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ over $\overline{\mathbf{Q}_p}$.

It was of course impossible to include all the proofs of all the results stated or used in this text. But I have tried to include as many proofs, or sketches of proofs, or examples, or references, as my courage enabled me to. I apologize for the proofs that perhaps should be, and are not, in this course and I welcome any (constructive) criticism.

1.4

I will use the following notations: ℓ and p are prime numbers (most of the time different), $\overline{\mathbf{Q}_p}$ an algebraic closure of the field $\mathbf{Q}_p$ of p-adic rationals, $\overline{\mathbf{Z}_p}$ the ring of integers in $\overline{\mathbf{Q}_p}$, $\mathcal{O}_{\mathbf{C}_p}$ the p-adic completion of $\overline{\mathbf{Z}_p}$, $\mathbf{C}_p$ its fraction field and $\overline{\mathbf{F}_p}$ the residue field of $\overline{\mathbf{Z}_p}$ and $\mathcal{O}_{\mathbf{C}_p}$. For $[F:\mathbf{Q}_p] < +\infty$, $F \subset \overline{\mathbf{Q}_p}$, I denote by $G_F := \mathrm{Gal}(\overline{\mathbf{Q}_p}/F)$, W_F its Weil subgroup and I_F its inertia subgroup. Recall W_F is the subgroup of elements that map to a finite power of the Frobenius in $\mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$. val is the p-adic valuation normalized by $\mathrm{val}(p) = 1$. I often loosely write "modulo p" where I should write "modulo the maximal ideal of $\overline{\mathbf{Z}_p}$".

2 Lecture 2: B_{st}, $\widehat{A_{st}}$ and the like

In this lecture, I describe the 8 rings B_{dR}^+, B_{dR}, A_{cris}, B_{cris}^+, B_{cris}, B_{st}^+, B_{st} and $\widehat{A_{st}}$ all due to Fontaine (see [21]) except $\widehat{A_{st}}$ which was introduced by Kato in [30] (where it was denoted $\varprojlim P_n!$).

2.1 B_{dR}^+ and B_{dR}

Let R be the projective limit of the diagram:

$$\mathcal{O}_{\mathbf{C}_p}/p\mathcal{O}_{\mathbf{C}_p} \leftarrow \mathcal{O}_{\mathbf{C}_p}/p\mathcal{O}_{\mathbf{C}_p} \leftarrow \cdots \leftarrow \mathcal{O}_{\mathbf{C}_p}/p\mathcal{O}_{\mathbf{C}_p} \leftarrow \cdots$$

where the transition maps are $x \mapsto x^p$. This is an integral commutative perfect ring of characteristic p. It is also endowed with a natural action of $G_{\mathbf{Q}_p}$ via its action on $\mathcal{O}_{\mathbf{C}_p}$. If $x = (x^{(n)})_{n \in \mathbf{Z}_{\geq 0}} \in R$ and if $\hat{x}^{(n)} \in \mathcal{O}_{\mathbf{C}_p}$ is any lifting of $x^{(n)}$, the sequence $(\hat{x}^{(n)})^{p^n}$ converges in $\mathcal{O}_{\mathbf{C}_p}$ and we call $\hat{x}$

its limit. As R is perfect, it is tempting to consider its Witt vectors $W(R)$. Recall any element of $W(R)$ can be uniquely written $\sum_{n=0}^{+\infty} p^n[x_n]$ where $x_n \in R$ and $[x_n]$ is its multiplicative representative.

Lemma 2.1.1. *There is a $G_{\mathbf{Q}_p}$-equivariant surjection of rings:*

$$\begin{array}{cccc} \theta & : & W(R) & \to & \mathcal{O}_{\mathbf{C}_p} \\ & & \sum_{n=0}^{+\infty} p^n[x_n] & \mapsto & \sum_{n=0}^{+\infty} p^n \hat{x}_n \end{array}$$

of kernel a principal ideal generated by $p - [\underline{p}]$ where $\underline{p} := (p^{(n)}) \in R$ is such that $p^{(n)}$ is the image in $\mathcal{O}_{\mathbf{C}_p}/p$ of a compatible system of p^n-roots of p.

Proof. (sketch) The map $R \mapsto \mathcal{O}_{\mathbf{C}_p}$, $x \mapsto \hat{x}$ is clearly surjective and $G_{\mathbf{Q}_p}$-equivariant (for $y \in \mathcal{O}_{\mathbf{C}_p}$, choose a compatible system of p^n-roots of y and take its image in $\mathcal{O}_{\mathbf{C}_p}/p$). Hence, so is θ. For $n \in \mathbf{N}$, it is well known that there is a ring homomorphism $W_n(\mathcal{O}_{\mathbf{C}_p}/p\mathcal{O}_{\mathbf{C}_p}) \to \mathcal{O}_{\mathbf{C}_p}/p^n\mathcal{O}_{\mathbf{C}_p}$, $[w_0] + p[w_1] + \cdots + p^{n-1}[w_{n-1}] \mapsto \hat{w}_0^{p^n} + p\hat{w}_1^{p^n} + \cdots + p^{n-1}\hat{w}_{n-1}^{p}$ where $\hat{w}_i \in \mathcal{O}_{\mathbf{C}_p}/p^n\mathcal{O}_{\mathbf{C}_p}$ is any lifting of w_i. This induces a ring homomorphism:

$$\varprojlim_n W_n(\mathcal{O}_{\mathbf{C}_p}/p\mathcal{O}_{\mathbf{C}_p}) \longrightarrow \mathcal{O}_{\mathbf{C}_p}$$

with $[w_0] + p[w_1] + \cdots + p^{n-1}[w_{n-1}] \mapsto [w_0^p] + p[w_1^p] + \cdots + p^{n-2}[w_{n-2}^p]$ as transition maps on the left hand side. But then, one easily checks that $W(R) \xrightarrow{\sim} \varprojlim_n W_n(\mathcal{O}_{\mathbf{C}_p}/p\mathcal{O}_{\mathbf{C}_p})$ and that the induced ring homomorphism $W(R) \to \mathcal{O}_{\mathbf{C}_p}$ is exactly θ. It is clear that $\theta(p - [\underline{p}]) = 0$. Assume $\theta\big([x_0] + p(\sum_{n=1}^{+\infty}[x_n]p^{n-1})\big) = 0$, then one can deduce $x_0 = \underline{p} \cdot x_0'$ in R (using that $\mathrm{val}(\hat{x}_0) \geq 1$). Hence $\sum_{n=0}^{+\infty}[x_n]p^n = (p - [\underline{p}])[x_0'] + p(\sum_{n=0}^{+\infty}[y_n]p^n)$ with $\theta(\sum_{n=0}^{+\infty}[y_n]p^n) = 0$. Since $W(R)$ is separated and complete for the p-adic topology, a straightforward induction yields $\sum_{n=0}^{+\infty}[x_n]p^n \in (p - [\underline{p}])W(R)$. $\square$

Definition 2.1.2. *We define B_{dR}^+ to be the completion of $W(R)[\frac{1}{p}]$ with respect to the $(p - [\underline{p}])$-adic topology and B_{dR} to be its fraction field, i.e.:*

$$B_{dR}^+ := \varprojlim_n \frac{W(R)[\frac{1}{p}]}{(p - [\underline{p}])^n} \qquad B_{dR} := \mathrm{Frac}(B_{dR}^+).$$

Note that B_{dR}^+ is a complete discrete valuation ring of maximal ideal $(p - [\underline{p}])$ and residue field $B_{dR}^+/(p - [\underline{p}]) \simeq \mathbf{C}_p$, i.e. B_{dR}^+ (resp. B_{dR}) is *non canonically* isomorphic to $\mathbf{C}_p[[T]]$ (resp. $\mathbf{C}_p((T))$). Since $\theta(p - [\underline{p}]) = 0$, θ extends to a surjection $\theta : B_{dR}^+ \to \mathbf{C}_p$ and the action of $G_{\mathbf{Q}_p}$ on $W(R)$

also extends to B_{dR}^+ and B_{dR} (as this action preserves $\mathrm{Ker}(\theta)$). We define a filtration on B_{dR} by $\mathrm{Fil}^m B_{dR} := (p - [\underline{p}])^m B_{dR}^+$ if $m \in \mathbf{Z}$. In particular $\mathrm{Fil}^0 B_{dR} = B_{dR}^+$. Let $\varepsilon^{(n)}$ be the image in $\mathcal{O}_{\mathbf{C}_p}/p$ of a compatible system of p^n-roots of 1 and let $\underline{\varepsilon} := (\varepsilon^{(n)})_n \in R$. Then, $[\underline{\varepsilon}] - 1 \in \mathrm{Ker}(\theta)$ and one can check $\log([\underline{\varepsilon}])$ is a uniformizer of B_{dR}^+ (i.e. generates $\mathrm{Ker}(\theta)$). For any $g \in G_{\mathbf{Q}_p}$, $g(\log([\underline{\varepsilon}])) = \varepsilon(g)\log([\underline{\varepsilon}])$ where $\varepsilon : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$ is the p-adic cyclotomic character. In the sequel, we set $t := \log([\underline{\varepsilon}])$ (well defined up to a scalar in $\mathbf{Z}_p^\times$).

Lemma 2.1.3. *For any finite extension $F \subset \overline{\mathbf{Q}_p}$, there is a natural identification $(B_{dR}^+)^{G_F} = B_{dR}^{G_F} \simeq F$ such that the diagram:*

$$\begin{array}{ccc} F & \hookrightarrow & B_{dR}^+ \\ \| & & \downarrow \theta \\ F & \hookrightarrow & \mathbf{C}_p \end{array}$$

is commutative and $G_{\mathbf{Q}_p}$-equivariant.

Proof. We will prove that, for $m \in \mathbf{Z}_{\leq 0}$, $(\mathrm{Fil}^m B_{dR})^{G_F} \simeq F$. It will be clear from the proof that the above diagram is equivariant and commutative (using that $\theta(\mathrm{Fil}^1 B_{dR}) = 0$). Since t is a uniformizer of B_{dR}^+, we have $\mathrm{Fil}^m B_{dR}/\mathrm{Fil}^{m+1} B_{dR} \simeq \mathbf{C}_p(m) := \mathbf{C}_p \otimes_{\mathbf{Q}_p} \varepsilon^m$ for any $m \in \mathbf{Z}_p$, this isomorphism being $G_{\mathbf{Q}_p}$-equivariant. But it is a result of Tate (see [38]) that $H^0(G_F, \mathbf{C}_p(m)) = 0$ if $m \neq 0$ and $= F$ if $m = 0$. Hence $(\mathrm{Fil}^m B_{dR}/\mathrm{Fil}^{m+1} B_{dR})^{G_F} = 0$ iff $m \neq 0$ and $= F$ otherwise. Using the exact sequences:

$$0 \longrightarrow \frac{\mathrm{Fil}^m B_{dR}}{\mathrm{Fil}^{m+r} B_{dR}} \longrightarrow \frac{\mathrm{Fil}^m B_{dR}}{\mathrm{Fil}^{m+r+1} B_{dR}} \longrightarrow \frac{\mathrm{Fil}^{m+r} B_{dR}}{\mathrm{Fil}^{m+r+1} B_{dR}} \longrightarrow 0,$$

applying the functor "G_F-invariants" and taking the projective limit on r yield the result. $\square$

2.2 A_{cris}, B_{cris}^+ and B_{cris}

We call σ the Frobenius automorphism on $W(R)$.

Definition 2.2.1. *We define A_{cris} to be the p-adic completion of the divided power envelope of $W(R)$ with respect to* $\mathrm{Ker}(\theta)$*, i.e.:*

$$A_{cris} := \left\{ \sum_{n=0}^{+\infty} w_n \frac{(p - [\underline{p}])^n}{n!},\ w_n \in W(R), w_n \to 0 \right\} \subset B_{dR}^+.$$

We define $B_{cris}^+ := A_{cris}[1/p]$ and $B_{cris} := B_{cris}^+[1/t]$.

Since $\sigma(p-[\underline{p}]) \equiv (p-[\underline{p}])^p$ (p) in $W(R)$, σ extends to a Frobenius endomorphism $\varphi : A_{cris} \to A_{cris}$ (resp. on B^+_{cris} and B_{cris}). Since $G_{\mathbf{Q}_p}$ preserves the ideal $(p-[\underline{p}])W(R)$, its action extends to A_{cris} (resp. to B^+_{cris} and B_{cris}). Note that $t = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} \in A_{cris}$ since $\frac{([\varepsilon]-1)^n}{n} \in (n-1)!\frac{(p-[\underline{p}])^n}{n!}W(R)$ and $(n-1)! \to 0$. Finally, for $m \in \mathbf{Z}$, define $\mathrm{Fil}^m A_{cris} := A_{cris} \cap \mathrm{Fil}^m B_{dR}$ (resp. $\mathrm{Fil}^m B^+_{cris}$ and $\mathrm{Fil}^m B_{cris}$) and note that $G_{\mathbf{Q}_p}$ preserves $\mathrm{Fil}^m A_{cris}$. Moreover $\varphi(\mathrm{Fil}^m A_{cris}) \subset \frac{p^m}{m!}A_{cris}$. In particular, one has $\varphi(\mathrm{Fil}^m A_{cris}) \subset p^m A_{cris}$ for $0 \leq m \leq p-1$.

Before switching to B^+_{st}, we mention the cohomological interpretation of A_{cris} (a fact of high importance in the proofs of the so-called "comparison theorems").

Proposition 2.2.2. *For $m \in \mathbf{Z}_{\geq 0}$, we have:*

$$\mathrm{Fil}^m A_{cris} \simeq \varprojlim_n H^0_{cris}\left(\left(\frac{\overline{\mathbf{Z}_p}}{p\overline{\mathbf{Z}_p}}/W_n(\mathbf{F}_p)\right)_{cris}, J^{[m]}\right)$$

where $J^{[m]}$ is the "usual" sheaf of ideals on the crystalline site $\left(\frac{\overline{\mathbf{Z}_p}}{p\overline{\mathbf{Z}_p}}/W_n(\mathbf{F}_p)\right)_{cris}$. In particular $A_{cris} \simeq \varprojlim_n H^0_{cris}\left(\frac{\overline{\mathbf{Z}_p}}{p\overline{\mathbf{Z}_p}}/W_n(\mathbf{F}_p)\right)$. This identification commutes with $G_{\mathbf{Q}_p}$ and φ (=crystalline Frobenius on the right hand side).

2.3 B^+_{st} and B_{st}

On the contrary to the other rings, B^+_{st} and B_{st} (and also $\widehat{A_{st}}$, see §2.4), together with all their structures, depend upon the choice of an element $\pi \in \overline{\mathbf{Z}_p}$ such that $\mathrm{val}(\pi) > 0$. We choose such an element and let $F := \mathbf{Q}_p(\pi)$ and $F_0 \subset F$ its maximal unramified subfield.

Let $\underline{\pi} := (\pi^{(n)})_n \in R$ where $\pi^{(n)}$ is the image in $\mathcal{O}_{\mathbf{C}_p}/p$ of a compatible system of p^n-roots of π and define:

$$\log\frac{[\underline{\pi}]}{\pi} := \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\left(\frac{[\underline{\pi}]}{\pi}-1\right)^n}{n} \in B^+_{dR}$$

(recall $\frac{[\underline{\pi}]}{\pi} - 1 \in \mathrm{Ker}(\theta)$).

Definition 2.3.1. *We define $B^+_{st}(\pi)$ to be the subring of B^+_{dR} generated by B^+_{cris} and $\log\frac{[\underline{\pi}]}{\pi}$ and $B_{st}(\pi) := B^+_{st}(\pi)[1/t]$.*

Theorem 2.3.2 ([21]). *The element* $\log\frac{[\underline{\pi}]}{\pi} \in B_{dR}$ *is transcendantal over* B_{cris}*. In particular* $B_{st}^+(\pi) \simeq B_{cris}^+[\log\frac{[\underline{\pi}]}{\pi}]$.

Lemma 2.3.3. *As subrings of* B_{dR}^+ *and* B_{dR}*,* $B_{st}^+(\pi)$ *and* $B_{st}(\pi)$ *do not depend on the choice of the* $(\pi^{(n)})_n$ *(but depend on the choice of* π*).*

Proof. Replacing $\underline{\pi}$ by $\underline{\pi}\cdot\underline{\varepsilon}$ where $\underline{\varepsilon}$ is as in §2.1 changes $\log\frac{[\underline{\pi}]}{\pi}$ into $\log\frac{[\underline{\pi}]}{\pi} + \log[\underline{\varepsilon}]$. But $\log[\underline{\varepsilon}] \in B_{cris}^+$. □

We write for short B_{st}^+ (resp. B_{st}) instead of $B_{st}^+(\pi)$ (resp. $B_{st}(\pi)$) and endow now these rings with more structures.
We define $\mathrm{Fil}^m B_{st}^+ := B_{st}^+ \cap \mathrm{Fil}^m B_{dR}$ (resp. $\mathrm{Fil}^m B_{st} := B_{st} \cap \mathrm{Fil}^m B_{dR}$). We have $\log\frac{[\underline{\pi}]}{\pi} \in \mathrm{Fil}^1 B_{st}^+$, but the filtration on B_{st}^+ is NOT the convolution filtration, i.e. $\sum_{i=0}^{m}(\mathrm{Fil}^{m-i}B_{cris}^+)\big(\log\frac{[\underline{\pi}]}{\pi}\big)^i \subsetneq \mathrm{Fil}^m B_{st}^+$ if $m \geq 2$.

Example 2.3.4. Assume $\pi^2 = p$, then $\big(1+\log\frac{[\underline{\pi}]}{\pi}\big)^2 - \frac{[\underline{\pi}]^2}{p} \in \mathrm{Fil}^2 B_{st}^+$, but $\notin \sum_{i=0}^{2}(\mathrm{Fil}^{2-i}B_{cris}^+)\big(\log\frac{[\underline{\pi}]}{\pi}\big)^i$.

For any $g \in G_F$, $g\big(\log\frac{[\underline{\pi}]}{\pi}\big) = \log\frac{[\underline{\pi}]}{\pi} + \log[\underline{\varepsilon(g)}]$ where $\underline{\varepsilon(g)} := (\varepsilon^{(n)}(g))_n \in R$ is such that $\varepsilon^{(n)}(g)$ is the image in $\mathcal{O}_{\mathbf{C}_p}/p$ of $\frac{g(\pi^{(n)})}{\pi^{(n)}}$. Hence, B_{st}^+ and B_{st} as subrings of B_{dR} are preserved by G_F. We state without proof the following lemma (see [21]):

Lemma 2.3.5. *Let* $F' \subset \overline{\mathbf{Q}_p}$ *be a finite extension of* F *and* F_0' *its maximal unramified subfield.*
(i) $(B_{st}^+)^{G_{F'}} = B_{st}^{G_{F'}} = F_0'$,
(ii) the map $F' \otimes_{F_0'} B_{st} \to B_{dR}$ *is injective.*

We define $B_{st,F}^+ := F \otimes_{F_0} B_{st}^+$ (resp. $B_{st,F} = \cdots$) and $\mathrm{Fil}^m B_{st,F}^+ := B_{st,F}^+ \cap \mathrm{Fil}^m B_{dR}$ (resp. $\mathrm{Fil}^m B_{st,F} = \cdots$). Beware that $F \otimes_{F_0} \mathrm{Fil}^m B_{st}^+ \subsetneq \mathrm{Fil}^m B_{st,F}^+$ if $m \geq 1$.

Example 2.3.6. $[\underline{\pi}] - \pi \in \mathrm{Fil}^1 B_{st,F}^+$, but $\notin F \otimes_{F_0} \mathrm{Fil}^1 B_{st}^+$.

We finally endow B_{st}^+ and B_{st} with a Frobenius φ which is the already defined φ on B_{cris} and satisfies $\varphi(\log\frac{[\underline{\pi}]}{\pi}) = p\log\frac{[\underline{\pi}]}{\pi}$, and with a B_{cris}-derivation N such that $N(\log\frac{[\underline{\pi}]}{\pi}) = 1$. Note that $N\varphi = p\varphi N$ but that $N(\mathrm{Fil}^m B_{st}^+) \nsubseteq \mathrm{Fil}^{m-1} B_{st}^+$ if $m \geq 2$ (look at Example 2.3.4).

Proposition 2.3.7. *(i) The package* $(B_{st}^+, \mathrm{Fil}^{\cdot} B_{st}^+, \varphi, N, G_F)$ *doesn't depend up to isomorphism on the choice of the* $(\pi^{(n)})_n$ *(resp. with* B_{st}*).*
(ii) Furthermore, the package $(B_{st}^+, \varphi, N, G_F)$ *doesn't depend up to isomorphism on the choice of* π *in* F *(resp. with* B_{st}*).*

Proof. (i) See the proof of Lemma 2.3.3. The B_{cris}^+-linear map such that $\log\frac{[\pi\underline{\varepsilon}]}{\pi} \mapsto \log\frac{[\underline{\pi}]}{\pi} + \log[\underline{\varepsilon}]$ is compatible with all the structures.
(ii) Replace π by πw with $w \in \mathcal{O}_F^\times$ and let $\underline{w} := (w^{(n)})_n \in R$ where $w^{(n)}$ is the image in $\mathcal{O}_{\mathbf{C}_p}/p$ of a compatible system of p^n-roots of w. Let $[\overline{w}] \in \mathcal{O}_{F_0} \subset B_{st}^+$ be the Teichmüller representative of the image of w in $\overline{\mathbf{F}_p}$. Then the B_{cris}^+-linear map such that $\log\frac{[\underline{\pi w}]}{\pi w} \mapsto \log\frac{[\underline{\pi}]}{\pi} + \log\frac{[\underline{w}]}{[\overline{w}]}$ is compatible with all the structures except the filtration since $\log\frac{[\underline{w}]}{[\overline{w}]} \notin \mathrm{Fil}^1 B_{st}^+$ in general (note that $\log\frac{[\underline{w}]}{[\overline{w}]}$ is well defined in B_{cris}^+ by the usual expansion). $\square$

Note however that $B_{st}^+(\pi w)$ and $B_{st}^+(\pi)$ do not coincide in general as subrings of B_{dR}^+.

2.4 $\widehat{A_{st}}$

The ring $\widehat{A_{st}}$ came out of trying to give a cohomological definition of B_{st}^+ analogous to the one of Proposition 2.2.2. Quite surprisingly, Kato found in [30] that it was necessary for that to "enlarge" B_{st}^+ (see Lemma 2.4.2). It is also the ring $\widehat{A_{st}}$ that one has to use in order to produce lattices in semi-stable p-adic representations (see Lecture 4). We give here its brute definition, its link with B_{st}^+ and mention its cohomological definition. This ring $\widehat{A_{st}}$ will be of high importance in the sequel. We keep the notations of §2.3.

As a ring, $\widehat{A_{st}}$ is isomorphic to the p-adic completion of the divided powers polynomial ring in one variable X over A_{cris}, i.e.:

$$\widehat{A_{st}} := \left\{ \sum_{n=0}^{+\infty} a_n \frac{X^n}{n!},\ a_n \in A_{cris}, a_n \to 0 \right\}$$

(recall A_{cris} is p-adically complete). For $n \in \mathbf{N}$, let $\widehat{A_{st}}^{\geq n} \subset \widehat{A_{st}}$ be the subring of elements such that $a_0, \ldots, a_{n-1} = 0$. We define:
• $\mathrm{Fil}^m \widehat{A_{st}} := \sum_{i=0}^m \mathrm{Fil}^{m-i} A_{cris} \cdot \frac{X^i}{i!} + \widehat{A_{st}}^{\geq m+1}$
• for $g \in G_F$, $g(\sum a_n \frac{X^n}{n!}) = \sum g(a_n) \frac{g(X)^n}{n!}$ where $g(X) := [\underline{\varepsilon(g)}]X + [\underline{\varepsilon(g)}] - 1$ ($[\underline{\varepsilon(g)}]$ as in §2.3 using the choice of the $(\pi^{(n)})_n$)
• $\varphi(\sum a_n \frac{X^n}{n!}) = \sum \varphi(a_n) \frac{\varphi(X)^n}{n!}$ where $\varphi(X) := (1+X)^p - 1$
• $N(\sum a_n \frac{X^n}{n!}) = \sum a_n \frac{N(X)^n}{n!}$ where $N(X) := 1 + X$.

The following straightforward lemma sums up the relations between these structures:

Lemma 2.4.1. *(i) G_F preserves the filtration and commutes with φ, N*
(ii) $N(\mathrm{Fil}^m\widehat{A_{st}}) \subset \mathrm{Fil}^{m-1}\widehat{A_{st}}$
(iii) $N\varphi = p\varphi N$
(iv) $\varphi(\mathrm{Fil}^m\widehat{A_{st}}) \subset p^m\widehat{A_{st}}$ if $m \leq p-1$.

The link with B^+_{st} is provided by the following lemma, essentially due to Kato:

Lemma 2.4.2. *(i) We have $B^+_{cris}[\log(1+X)] \simeq \{x \in \widehat{A_{st}}[1/p] \mid N^n(x) = 0$ for some $n \in \mathbf{N}\}$ where $\log(1+X)$ is the usual expansion of log in X.*
(ii) The map $\widehat{A_{st}}[1/p] \to B^+_{dR}$, $X \mapsto \frac{[\underline{\pi}]}{\pi} - 1$ induces an isomorphism $B^+_{cris}[\log(1+X)] \xrightarrow{\sim} B^+_{st}$ which is compatible with φ, N and G_F (but only induces inclusions $\mathrm{Fil}^m(B^+_{cris}[\log(1+X)]) \subsetneq \mathrm{Fil}^m B^+_{st}$).

Proof. (i) One easily checks that $N((\log(1+X))^n) = n(\log(1+X))^{n-1}$ and that $\mathrm{Ker}(N) = B^+_{cris}$. By induction, assume $\mathrm{Ker}(N^n) = B^+_{cris} + B^+_{cris}\log(1+X) + \cdots + B^+_{cris}(\log(1+X))^{n-1}$ and let $x \in \widehat{A_{st}}[1/p]$ such that $N^{n+1}(x) = 0$. Then $N^n(x) = b \in B^+_{cris}$ and $N^n(x - \frac{b}{n!}(\log(1+X))^n) = 0$ which implies $x \in B^+_{cris} + \cdots + B^+_{cris}(\log(1+X))^n$ by induction.
(ii) is obvious if one carefully compares the definitions of φ, N, G_F and Fil on $\widehat{A_{st}}$ and B^+_{st}. See Example 2.3.4 for the last statement. □

As a summary, B^+_{st} is the part of $\widehat{A_{st}}[1/p]$ where N is nilpotent, except that its filtration is finer than the induced filtration. Before we state (for the sake of completeness) the cohomological interpretation of $\widehat{A_{st}}$, we mention without proof the following lemma (due to the author, see [7]):

Lemma 2.4.3. *Let $e := [F : F_0]$, $[n/e]$ the euclidian quotient of n by e ($n \in \mathbf{N}$) and $S := \left\{\sum_{n=0}^{+\infty} w_n \frac{u^n}{[n/e]!},\ w_n \in \mathcal{O}_{F_0}, w_n \to 0\right\}$. The continuous $\mathcal{O}_{F_0}$-linear map $S \to \widehat{A_{st}}$ such that $u \mapsto [\underline{\pi}](1+X)^{-1}$ induces an isomorphism $S \xrightarrow{\sim} \widehat{A_{st}}^{G_F}$.*

Of course, it is easily checked that $[\underline{\pi}](1+X)^{-1}$ is Galois invariant. Hence, we see that $\widehat{A_{st}}$ is naturally an S-algebra. This will somewhat motivate the following result (due to Kato), that we only state in a vague form (see [30] or [5]):

Proposition 2.4.4. *Let $S_n := S/p^nS$. We have an identification $\widehat{A_{st}} \simeq \varprojlim_n H^0_{log-cris}\left(\frac{\overline{\mathbf{Z}_p}}{p\overline{\mathbf{Z}_p}}/S_n\right)$ which is compatible with all the structures.*

Here, $H^0_{log-cris}$ is *log-crystalline* cohomology, we need to endow $\frac{\overline{\mathbf{Z}_p}}{p\overline{\mathbf{Z}_p}}$ and S_n with *log-structures* and we see $\frac{\overline{\mathbf{Z}_p}}{p\overline{\mathbf{Z}_p}}$ as an S_n-algebra via $u \mapsto \pi$. See

Lecture 5 for a little bit more (but not much) on log-schemes and log-structures.

3 Lecture 3: Potentially semi-stable representations

I define potentially semi-stable representations and give their main properties.

3.1 Definition

Let F be a finite extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}_p}$, $\mathcal{O}_F$ its ring of integers, $\mathbf{F} \subset \overline{\mathbf{F}}_p$ its residue field, $F_0 \subset F$ its maximal unramified subfield, σ the (arithmetic) Frobenius on F_0, F^{unr} the unramified closure of F inside $\overline{\mathbf{Q}_p}$ and $f := [\mathbf{F} : \mathbf{F}_p]$. To define B_{st}, we need to fix a uniformizer $\pi \in F$. If ℓ is any prime number, an ℓ-adic representation of G_F is, by definition, a continuous linear representation of G_F on a finite dimensional $\mathbf{Q}_\ell$-vector space V. For $\ell \neq p$, recall that a *semi-stable* ℓ-adic representation of G_F is an ℓ-adic representation such that the inertia acts unipotently.

Lemma 3.1.1. *Assume $\ell \neq p$. To give a semi-stable ℓ-adic representation of G_F is equivalent to give a finite dimensional $\mathbf{Q}_\ell$-vector space endowed with a continuous linear action of* $\mathrm{Gal}(F^{\text{unr}}/F)$ *(which plays the role of the Frobenius) and with a nilpotent endomorphism N (the monodromy) such that $N\varphi = p^f\varphi N$ where φ is the geometric Frobenius of* $\mathrm{Gal}(F^{\text{unr}}/F)$*.*

Proof. Let V be a semi-stable ℓ-adic representation of G_F, $(\pi_n)_{n\in\mathbf{N}}$ a compatible system of ℓ^n-roots of π and $t_\ell : I_F \to \mathbf{Z}_\ell(1) \simeq \mathbf{Z}_\ell$ the character defined by $t_\ell(g) := (\frac{g(\pi_n)}{\pi_n})_{n\in\mathbf{N}}$. Since $\mathrm{Ker}(t_\ell)$ has pro-order prime to ℓ and acts unipotently on V, it must act trivially. Hence, I_F acts through $\mathbf{Z}_\ell(1)$ and we set $N := \frac{1}{t_\ell(g)}\log(g)$ for $g \in I_F$ (this is independant of g). Since:

$$\mathrm{Gal}(F^{\text{unr}}(\pi_n, n \in \mathbf{N})/F) \simeq \mathrm{Gal}(F^{\text{unr}}(\pi_n, n \in \mathbf{N})/F^{\text{unr}}) \cdot \mathrm{Gal}(F^{\text{unr}}/F)$$

(semi-direct product), we get the action of $\mathrm{Gal}(F^{\text{unr}}/F)$ with the desired commutativity. □

Lemma 3.1.2. *(i) Let V be a p-adic representation of G_F, then:*

$$\dim_{F_0}(B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} \leq \dim_{\mathbf{Q}_p}(V).$$

(ii) If $\dim_{F_0}(B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} = \dim_{\mathbf{Q}_p}(V)$, then:

$$B_{st} \otimes_{F_0} (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} \xrightarrow{\sim} B_{st} \otimes_{\mathbf{Q}_p} V.$$

Proof. (i) Let $D_{st}(V) := (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F}$ and $\alpha_{st}(V) : B_{st} \otimes_{F_0} D_{st}(V) \to B_{st} \otimes_{\mathbf{Q}_p} V$, $b \otimes x \mapsto bx$. Since B_{st} is a domain, it is enough to prove that $\alpha_{st}(V)$ is injective after extending scalars to $\mathrm{Frac}(B_{st})$. Assume not and let n be the smallest integer such that there exists $b_1, \ldots, b_n \in \mathrm{Frac}(B_{st})^{\times}$ and $x_1, \ldots, x_n \in D_{st}(V)$ satisfying $\sum_{i=1}^n b_i \otimes x_i \mapsto 0$ and $\sum_{i=1}^n b_i \otimes x_i \neq 0$. We can assume $b_1 = 1$ and $n \geq 2$. Then $x_1 + \sum_{i=2}^n g(b_i) \otimes x_i \mapsto 0$, which implies $g(b_i) = b_i$, $\forall i$ by the minimality of n. But one can prove that $(\mathrm{Frac}(B_{st}))^{G_F} = F_0$ (this is a consequence of $\mathrm{Frac}(B_{st}) \subset B_{dR}$, $B_{dR}^{G_F} = F$ and $F \otimes_{F_0} B_{st} \hookrightarrow B_{dR}$), thus $b_i \in F_0$ which implies $\sum b_i \otimes x_i = 1 \otimes \sum b_i x_i = 0$ and is impossible by assumption.
(ii) By (i), $\alpha_{st}(V)$ is injective. Let $e_1, \ldots, e_d$ be a basis of $D_{st}(V)$ and $v_1, \ldots, v_d$ a basis of V. We have $e_1 \wedge \ldots \wedge e_d = \det \otimes v_1 \wedge \ldots \wedge v_d$ in $B_{st} \otimes \bigwedge_{\mathbf{Q}_p}^d V$ with $\det \in B_{st}$. Since $g(\det) \in \mathbf{Q}_p \det$ for any $g \in G_F$ (as G_F acts trivially on $D_{st}(V)$), we have $\det \in B_{st}^{\times}$ (see Example 3.1.4). Thus $\alpha_{st}(V)$ is also surjective. □

Definition 3.1.3. *(i) A p-adic representation V of G_F is called semi-stable if $\dim_{F_0}(B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} = \dim_{\mathbf{Q}_p} V$.*
(ii) A p-adic representation of G_F is called potentially semi-stable if it becomes semi-stable when restricted to an open subgroup of G_F.

If V is (potentially) semi-stable, then so is any $\mathbf{Q}_p$-subquotient of V (see [22]). Note that Definition 3.1.3 only uses the structure of $\mathbf{Q}_p[G_F]$-module of B_{st} and thus doesn't depend on the choice of π (see Lecture 2).

Example 3.1.4. We will give several examples later. But let us give at once the one dimensional case. Let V be a semi-stable representation of G_F of dimension 1, then I claim that $V|_{I_F} \simeq \varepsilon^i|_{I_F}$ for some $i \in \mathbf{Z}$ where ε is the p-adic cyclotomic character. Since the action of G_F on V is just a character η, we have to find under which condition $\{\lambda \in B_{st} \mid g(\lambda) = \eta^{-1}(g)\lambda, \forall g \in G_F\}$ is non zero (it is of dimension ≤ 1 by Lemma 3.1.2). But we have the following result on B_{st} (see [21]): any $\mathbf{Q}_p$-subvector space of dimension 1 in B_{st} that is preserved by Galois is contained in $W(\overline{\mathbf{F}}_p)[\frac{1}{p}]t^{\mathbf{Z}}$ where $t \in B_{cris} \subset B_{st}$ is the element defined in Lecture 2. Since $g(t) = \varepsilon(g)t$ and since the action of G_F on $W(\overline{\mathbf{F}}_p)$ is unramified, we are done.

3.2 Filtered (φ, N)-modules

Definition 3.1.3 is not very explicit. Fortunately, a recent result of Colmez and Fontaine (see [13]) gives an alternative description of semi-stable p-adic representations which is very explicit and gives a striking analogy with the ℓ-adic case. Define a filtered (φ, N)-module to be a finite dimensional F_0-vector space D endowed with:

• a σ-linear injective map $\varphi : D \to D$ (the "Frobenius")
• a linear map $N : D \to D$ such that $N\varphi = p\varphi N$ (the "monodromy")
• a decreasing filtration $(\mathrm{Fil}^i D_F)_{i\in\mathbf{Z}}$ on $D_F := F \otimes_{F_0} D$ by F-vector subspaces such that $\mathrm{Fil}^i D_F = D_F$ for $i \ll 0$ and $\mathrm{Fil}^i D_F = 0$ for $i \gg 0$.

The conditions on φ and N imply that N is nilpotent. If D is a filtered (φ, N)-module of dimension d, so is $\otimes^d_{F_0} D$ by setting $\varphi := \otimes^d \varphi$, $N := N \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes N \otimes 1 \ldots \otimes 1 + \ldots$, $\mathrm{Fil}^i(\otimes^d_F D_F) := \sum_{i_1+\ldots+i_d=i} \mathrm{Fil}^{i_1} D_F \otimes \ldots \otimes \mathrm{Fil}^{i_d} D_F$, and so is $\bigwedge^d_{F_0} D$ by taking the image structures. Since $\dim_{F_0}(\bigwedge^d_{F_0} D) = 1$, there is a unique $i_0 \in \mathbf{Z}$ such that $\mathrm{Fil}^i(\bigwedge^d_F D_F) = \bigwedge^d_F D_F$ for $i \leq i_0$ and $\mathrm{Fil}^i(\bigwedge^d_F D_F) = 0$ for $i > i_0$, and there is a unique $\alpha_0 \in \mathbf{Z}$ such that, if $e_1 \in \bigwedge^d_{F_0} D \setminus \{0\}$, $\varphi(e_1) = \lambda_0 e_1$ with $\mathrm{val}(\lambda_0) = \alpha_0$. We define (following Fontaine):

$$t_H(D) := i_0 \qquad t_N(D) := \alpha_0.$$

By definition a filtered (φ, N)-submodule of D is a filtered (φ, N)-module D' equipped with an injection $D' \hookrightarrow D$ that commutes with φ and N and for which $\mathrm{Fil}^i D'_F = D'_F \cap \mathrm{Fil}^i D_F$.

Definition 3.2.1. *A filtered (φ, N)-module D is called weakly admissible if $t_H(D) = t_N(D)$ and if $t_H(D') \leq t_N(D')$ for any filtered (φ, N)-submodule D' of D.*

Lemma 3.2.2. *Let V be a semi-stable p-adic representation of G_F. Let $D_{st}(V) := (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F}$ and define (on $D_{st}(V)$) $\varphi := \varphi_{B_{st}} \otimes \mathrm{Id}$, $N := N_{B_{st}} \otimes \mathrm{Id}$ and $\mathrm{Fil}^i D_{st}(V)_F := (\mathrm{Fil}^i B_{st,F} \otimes V) \cap D_{st}(V)_F$. Then $D_{st}(V)$ is a weakly admissible filtered (φ, N)-module.*

Before giving the proof of this lemma, we give the 1-dimensional example:

Example 3.2.3. Assume $\dim_{\mathbf{Q}_p} V = 1$. By Example 3.1.4, the action of G_F on V is a character $\eta\varepsilon^i$ where $i \in \mathbf{Z}$ and η is unramified. Hence $D_{st}(V) = F_0(\lambda t^{-i} \otimes v_1)$ where $v_1 \in V \setminus \{0\}$ and $\lambda \in W(\overline{\mathbf{F}}_p) \setminus \{0\}$ satisfies $g(\lambda) = \eta^{-1}(g)\lambda$ for $g \in G_F$ (one easily finds such a λ). This implies $D_{st}(V) = F_0 e_1$ with $\varphi(e_1) = \frac{\sigma(\lambda)}{\lambda} p^{-i} e_1$, $N(e_1) = 0$, $e_1 \in \mathrm{Fil}^{-i} D_{st}(V)_F$ and $\frac{\sigma(\lambda)}{\lambda} \in W(\mathbf{F})^\times$ (since $\sigma^f(\frac{\sigma(\lambda)}{\lambda}) = \frac{\sigma(\lambda)}{\lambda}$). In particular, $D_{st}(V)$ is clearly weakly admissible.

Proof. (sketch of proof of Lemma 3.2.2)
Let $d := \dim_{\mathbf{Q}_p} V$. If $d = 1$, this is OK by Example 3.2.3. The map $B_{st} \otimes_{F_0} \bigwedge^d_{F_0} D_{st}(V) \to B_{st} \otimes_{\mathbf{Q}_p} \bigwedge^d_{\mathbf{Q}_p} V$ is an isomorphism (since $\alpha_{st}(V)$ is an isomorphism by Lemma 3.1.2). Thus $\bigwedge^d_{F_0} D_{st}(V) \simeq D_{st}(\bigwedge^d_{\mathbf{Q}_p} V)$ which

implies $t_H(D_{st}(V)) = t_N(D_{st}(V))$ by the case $d = 1$. Let $D' \subset D_{st}(V)$ be a filtered (φ, N)-submodule of dimension d'. Then $\bigwedge_{F_0}^{d'} D' \subset \bigwedge_{F_0}^{d'} D_{st}(V)$ is a filtered (φ, N)-submodule of dimension 1. Assume $t_H(D') \geq t_N(D')$ and let Δ be the same (φ, N)-module as $\bigwedge_{F_0}^{d'} D'$ but with $t_H(\Delta) = t_N(D')$. Then, by the case $d = 1$, $\Delta = D_{st}(V')$ for some V' of dimension 1 and the morphism of filtered (φ, N)-modules $\Delta \to \bigwedge_{F_0}^{d'} D_{st}(V)$ induces a commutative diagram:

$$\begin{array}{ccc} B_{st} \otimes_{F_0} \Delta & \xrightarrow{\sim} & B_{st} \otimes_{\mathbf{Q}_p} V' \\ \downarrow & & \downarrow \\ B_{st} \otimes_{F_0} \bigwedge_{F_0}^{d'} D_{st}(V) & \xrightarrow{\sim} & B_{st} \otimes_{\mathbf{Q}_p} \bigwedge_{\mathbf{Q}_p}^{d'} V. \end{array}$$

From this diagram, we can deduce first that $V' \subset \bigwedge_{\mathbf{Q}_p}^{d'} V$ (taking $\mathrm{Fil}^0{}_{N=0}^{\varphi=1}$ on the right hand side) and second that $t_H(\Delta) = t_N(D')$ must be the induced filtration by $\bigwedge_F^{d'} D_{st}(V)_F$ (looking again on the right hand side). But this filtration is greater or equal than $t_H(D')$ ($\geq t_H(\Delta)$), thus $t_H(\Delta) = t_H(D') = t_N(D')$. Finally, we always have $t_H(D') \leq t_N(D')$. $\square$

The aforementioned result of Colmez and Fontaine is:

Theorem 3.2.4. *([10]) The functor $D_{st} : V \mapsto (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F}$ establishes an equivalence of categories between the category of semi-stable p-adic representations of G_F and the category of weakly admissible filtered (φ, N)-modules.*

The functor D_{st} does depend on the choice of π but the (φ, N)-module $D_{st}(V)$ (forgetting the filtration) doesn't by Proposition 2.3.7 (ii). When $N = 0$ on $D_{st}(V)$ (this doesn't depend on π), V is said to be crystalline and in that case $D_{st}(V)$ with its full structure doesn't depend on π. In the sequel, we will often use the contravariant functor $D^*_{st}(V) := D_{st}(V^*)$, where V^* is the dual representation of V (semi-stable/crystalline if and only if V is). If V has positive Hodge-Tate weights (i.e. if $\mathrm{Fil}^0 D^*_{st}(V)_F = D^*_{st}(V)_F$), then we have $D^*_{st}(V) \simeq \mathrm{Hom}_{G_F}(V, B^+_{st})$ and $V \simeq \mathrm{Hom}_{\varphi,N,\mathrm{Fil}^\cdot}(D^*_{st}(V), B^+_{st})$. In this last isomorphism, we mean F_0-linear maps that commute with φ, N and send $\mathrm{Fil}^i D^*_{st}(V)_F$ to $\mathrm{Fil}^i B^+_{st,F}$.

In the rest of this course, we often consider p-adic representations of G_F which are finite dimensional E-vector spaces with $E \subset \overline{\mathbf{Q}}_p$ and $[E : \mathbf{Q}_p] < \infty$, i.e. which are endowed with an injection $E \hookrightarrow \mathrm{End}_{\mathbf{Q}_p[G_F]}(V)$. Their corresponding filtered modules are free $F_0 \otimes_{\mathbf{Q}_p} E$-modules with all structures being E-linear. If $F_0 = \mathbf{Q}_p$, we again get E-vector spaces. However, if $F_0 \neq \mathbf{Q}_p$, the filtered $F_0 \otimes_{\mathbf{Q}_p} E$-module (with φ and N) can be quite complicated. In this course, we only deal with the case $F_0 = \mathbf{Q}_p$.

3.3 Potentially semi-stable representations and Weil-Deligne representations

Recall that a representation of the Weil-Deligne group is a continuous representation of W_F on a finite dimensional vector space W (over an algebraically closed field of characteristic 0 with the discrete topology) together with a nilpotent endomorphism $N : V \to V$ such that $Ng = p^{-\alpha(g)}gN$ where $g \in W_F$ and $g \mapsto \mathrm{Frob}^{\alpha(g)} \in \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$ $(\alpha(g) \in \mathbf{Z}_{\geq 0})$. Here Frob denotes the absolute arithmetic Frobenius on $\overline{\mathbf{F}}_p$. Let V be a potentially semi-stable ℓ-adic representation of G_F with $\ell =$ or $\neq p$. If $\ell \neq p$, using Lemma 3.1.1, it is easy to associate to V a representation of the Weil-Deligne group. We show here (following Fontaine) that a similar construction also exists for $\ell = p$.

Let V be a potentially semi-stable p-adic representation of G_F with coefficient in $E \subset \overline{\mathbf{Q}_p}$. Let $F' \supset F$ be a Galois extension such that $V|_{G_{F'}}$ is semi-stable and $D := D_{st}(V|_{G_{F'}}) = (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_{F'}}$ which is endowed with the residual action of $\mathrm{Gal}(F'/F)$. Thus, D is a free $F'_0 \otimes_{\mathbf{Q}_p} E$-module of rank $\dim_E V$ endowed with a nilpotent $F'_0 \otimes_{\mathbf{Q}_p} E$-endomorphism $N : D \to D$. Let W_F act $F'_0 \otimes_{\mathbf{Q}_p} E$-linearly on D via $g \mapsto \overline{g} \circ \varphi^{-\alpha(g)}$ ($\overline{g} =$ image of g in $\mathrm{Gal}(F'/F)$). Since W_F and N act linearly, if we write $D \otimes_E \overline{\mathbf{Q}_p} = \prod_{i=1}^{[F'_0:\mathbf{Q}_p]} D_i$, each D_i is naturally a representation of (W_F, N) (the commutativity condition is easily verified).

Lemma 3.3.1. *The isomorphism class of D_i as a representation of (W_F, N) doesn't depend on the choice of F' or on i.*

Proof. By a result of Deligne, there exists a $\overline{\mathbf{Q}_p}[W_F]$-automorphism $f_i : D_i \to D_i$ such that $N \circ f_i = \frac{1}{p} f_i \circ N$. Then one checks that $\varphi \circ f_i : D_i \xrightarrow{\sim} D_{i+1}$ is a $\overline{\mathbf{Q}_p}$-automorphism that commutes with both W_F and N. To prove it doesn't depend on F', it is enough to show one can replace F' by $F'' \supset F'$. But then $D_{st}(V|_{G_{F''}}) \simeq F''_0 \otimes_{F'_0} D_{st}(V|_{G_{F'}})$, hence $D_{st}(V|_{G_{F''}}) \otimes_{F''_0 \otimes E} \overline{\mathbf{Q}_p} \simeq D_{st}(V|_{G_{F'}}) \otimes_{F'_0 \otimes E} \overline{\mathbf{Q}_p}$ and the D_i's are the same. □

The isomorphism class of Lemma 3.3.1 is by definition the Weil-Deligne representation associated to V.

Example 3.3.2. Here is the filtered module corresponding *under* D^*_{st} to a potentially crystalline representation of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights $(0, k-1)$ that becomes crystalline over $F := \mathbf{Q}_p(\tilde{\pi}) = \mathbf{Q}_p(\sqrt[p]{1})$ where $\tilde{\pi}^{p-1} =$

$-p$:

$$\begin{array}{ccc} D & = & Ee_1 \oplus Ee_2 \\ \varphi(e_1) & = & \nu e_1 \\ \varphi(e_2) & = & \frac{p^{k-1}}{\nu}\mu e_2 \\ Fil^{k-1}(D_F) & = & F \otimes_{\mathbf{Q}_p} E(e_1 + \tilde{\pi}^i e_2) \\ N & = & 0 \\ g(e_1) & = & e_1 \\ g(e_2) & = & \tilde{\omega}(g)^{-i} e_2 \\ \mathrm{val}(\nu) & \in & [0, k-1] \\ \mu & \in & \mathcal{O}_E^\times \\ i & \in & \{1, \ldots, p-2\} \\ g & \in & \mathrm{Gal}(F/\mathbf{Q}_p). \end{array}$$

Here, $\tilde{\omega}$ is the Teichmüller lift of the cyclotomic character modulo p. The Weil-Deligne representation associated to this potentially crystalline representation is:

$$\begin{array}{ccc} W & = & \overline{\mathbf{Q}_p} e_1 \oplus \overline{\mathbf{Q}_p} e_2 \\ N & = & 0 \\ g(e_1) & = & \nu^{\alpha(g)} e_1 \\ g(e_2) & = & \tilde{\omega}(g)^i (\frac{p^{k-1}}{\nu}\mu)^{\alpha(g)} e_2 \end{array}$$

(we have to dualize since we have used D_{st}^*).

3.4 Examples and relation to modular forms

Here are some other important explicit examples of weakly admissible filtered modules for $F = \mathbf{Q}_p$. The corresponding representations are two dimensional over some finite extension E of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}_p}$. Since I do not want to make precise this extension, I loosely write $\overline{\mathbf{Q}_p}$.

Example 3.4.1. (i) V crystalline and decomposable:

$$\left\{ \begin{array}{ccc} \varphi(e_1) & = & p^{k-1}\mu_1 e_1 \\ \varphi(e_2) & = & \mu_2 e_2 \\ Fil^{k-1} D & = & \overline{\mathbf{Q}_p} e_1 \\ \mu_1, \mu_2 & \in & \overline{\mathbf{Z}_p}^\times \end{array} \right.$$

(ii) V crystalline and indecomposable:

$$\left\{ \begin{array}{ccc} \varphi(e_1) & = & p^{k-1}(\mu_1 e_1 + e_2) \\ \varphi(e_2) & = & \mu_2 e_2 \\ Fil^{k-1} D & = & \overline{\mathbf{Q}_p} e_1 \\ \mu_1, \mu_2 & \in & \overline{\mathbf{Z}_p}^\times \end{array} \right.$$

(iii) V crystalline and irreductible:

$$\begin{cases} \varphi(e_1) & = & p^{k-1}\mu e_2 \\ \varphi(e_2) & = & -e_1 + \nu e_2 \\ Fil^{k-1}D & = & \overline{\mathbf{Q}_p}e_1 \\ \mu & \in & \overline{\mathbf{Z}_p}^{\times} \\ \nu & \in & \mathfrak{m}_{\overline{\mathbf{Z}_p}} \end{cases}$$

(iv) V semi-stable non crystalline:

$$\begin{cases} \varphi(e_1) & = & p^{k/2}\mu e_1 \\ \varphi(e_2) & = & p^{k/2-1}\mu e_2 \\ Fil^{k-1}D & = & \overline{\mathbf{Q}_p}(e_1 - \mathcal{L}e_2) \\ N(e_1) & = & e_2 \\ N(e_2) & = & 0 \\ \mu & \in & \overline{\mathbf{Z}_p}^{\times} \\ \mathcal{L} & \in & \overline{\mathbf{Q}_p}. \end{cases}$$

We will see geometric examples of semi-stable representations in Lecture 5. Let us just say here that the cohomology of any "reasonnable" scheme over F should be (is?) a potentially semi-stable representation.

To conclude this lecture, let us describe the filtered modules coming from modular forms on $\Gamma_1(N)$ with either $(p,N)=1$ or ($p \mid N$ and $(p, \frac{N}{p})=1$). Fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$ and let f be a normalized cuspidal newform of weight $k \geq 2$ on $\Gamma_1(N)$, N being any integer ≥ 1. Deligne, using Grothendieck's étale cohomology, constructed continuous irreducible representations:

$$\rho_{f,p} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}_p})$$

satisfying the following properties (some of them were proven much later):
- $\det\rho_{f,p} = \varepsilon^{k-1}\chi$
- $\rho_{f,p}|_{G_{\mathbf{Q}_\ell}}$ is unramified if $(\ell, Np) = 1$
- if $(\ell, Np) = 1$, then the characteristic polynomial of an arithmetic Frobenius at ℓ is $X^2 - a_\ell X + \ell^{k-1}\chi(\ell)$ where $T_\ell(f) = a_\ell f$
- $\rho_{f,p}|_{G_{\mathbf{Q}_p}}$ is potentially semi-stable.

The following theorem will be studied in the other course:

Theorem 3.4.2 (Langlands, Deligne, Carayol, Saito). *Up to F-semi-simplification, $WD(\rho_{f,p}|_{G_{\mathbf{Q}_\ell}})$ doesn't depend on p.*

Recall that the semi-simplification of a Weil-Deligne representation is the semi-simplification of the underlying Weil representation together with the operator N (N still acts on the semi-simplification because $N = 0$ on any irreducible subspace for the action of the Weil group). It is a consequence of the p-adic Eichler-Shimura isomorphism (a very special case of Faltings' comparison theorems with coefficients, see [19]) that the Hodge-Tate weights of $\rho_{f,p}|_{G_{\mathbf{Q}_p}}$ are $(0, k-1)$, but in some cases, one can easily draw this from 3.4.2 (see the last remark of this lecture). Together with 3.4.2 and, e.g., Lemma 4.2.2 of [15], we deduce the complete description of $\rho_{f,p}|_{G_{\mathbf{Q}_p}}$ when $(p, N) = 1$ or $p \parallel N$:

(i) If $(p, N) = 1$, then $\rho_{f,p}|_{G_{\mathbf{Q}_p}}$ is crystalline and its filtered module (under D^*_{st}) is as in Example 3.4.1 (i) or (ii) with $\mu_1\mu_2 = \chi(p)$ and $p^{k-1}\mu_1 + \mu_2 = a_p$ if $\mathrm{val}(a_p) = 0$, or as in Example 3.4.1 (iii) with $\mu = \chi(p)$ and $\nu = a_p$ if $\mathrm{val}(a_p) > 0$ ($T_p(f) = a_p f$).

(ii) If $p \parallel N$, then there are two cases:

- if p is prime to the conductor of χ, then $\rho_{f,p}|_{G_{\mathbf{Q}_p}}$ is semi-stable (non crystalline) and its filtered module (for the choice $\pi = p$) is as in Example 3.4.1 (iv) with $\mathcal{L} = \mathcal{L}_p(f)$ and $\mu = \frac{a_p}{p^{\frac{k}{2}-1}}$ (this implies $\mathrm{val}(a_p) = \frac{k}{2} - 1$).
- if p divides the conductor of χ, then $\rho_{f,p}|_{G_{\mathbf{Q}_p}}$ becomes crystalline over $\mathbf{Q}_p(\sqrt[p]{1})$ and its filtered module is as in Example 3.3.2 with $\nu = a_p$ and μ, i such that $\mu = \chi'(p)$ and $\chi = \tilde{\omega}^i \chi'$ with $(p, \mathrm{cond}(\chi')) = 1$.

As far as I know, it is an open question to determine explicitly all the filtered modules coming from modular forms when an *arbitrary* power of p divides the level.

Remark 3.4.3. Because of the determinant, the Hodge-Tate weights are necessarily (a, b) with $a, b \geq 0$ and $a + b = k - 1$. When $(p, N) = 1$ and $\mathrm{val}(a_p) = 0$, one checks the weak admissibility condition implies $(a, b) = (0, k-1)$.

4 Lecture 4: Integral p-adic Hodge theory

We keep the same notations as for Lecture 3. We have seen how to build semi-stable representations of G_F from filtered modules. Since G_F is compact, these representations always admit Galois stable $\mathbf{Z}_p$-lattices. So one can ask if there exists corresponding integral structures on the filtered modules. This lecture gives a conjectural answer to this question as soon as the filtration (on the filtered module) doesn't "spread" too much. The general problem is still open, even conjecturally.

4.1 Where S enters

We call σ the Frobenius automorphism on $W(\mathbf{F})$. Let $E(u) \in W(\mathbf{F})[u]$ be the minimal polynomial of π (an Eisenstein polynomial of degree $e := [F : F_0]$) and S the p-adic completion of $W(\mathbf{F})[u, \frac{u^{ie}}{i!}]_{i \in \mathbf{N}}$ where u is an indeterminate. This ring is noting else than the ring S already introduced at the end of Lecture 2. Thinking about Lemma 2.4.3 of Lecture 2, we endow S with the following structures:
• a continuous σ-linear Frobenius still denoted $\sigma : S \to S$ such that $\sigma(u) = u^p$
• a continuous linear derivation $N : S \to S$ such that $N(u) = -u$
• a decreasing filtration $(\mathrm{Fil}^i S)_{i \in \mathbf{N}}$ where $\mathrm{Fil}^i S$ is the p-adic completion of $\sum_{j \geq i} S \frac{E(u)^j}{j!}$ (one checks $\frac{E(u)^j}{j!} \in S$).
Note that $N\sigma = p\sigma N$, $N(\mathrm{Fil}^{i+1} S) \subset \mathrm{Fil}^i S$ for $i \in \mathbf{N}$ and $\sigma(\mathrm{Fil}^i S) \subset p^i S$ for $i \in \{0, ..., p-1\}$. In fact, these structures are exactly the structures induced on S by $\widehat{A_{st}}$ (see Lecture 2).

Let D be a weakly admissible filtered (φ, N)-module and assume $\mathrm{Fil}^0 D_F = D_F$ (this is harmless since, up to twist, one can always assume Galois representations have positive Hodge-Tate weights). Let:

$$\mathcal{D} := S \otimes_{W(\mathbf{F})} D$$

and define :
• $\varphi := \sigma \otimes \varphi : \mathcal{D} \to \mathcal{D}$
• $N := N \otimes Id + Id \otimes N : \mathcal{D} \to \mathcal{D}$
• $\mathrm{Fil}^0 \mathcal{D} := \mathcal{D}$ and, by induction:

$$\mathrm{Fil}^{i+1} \mathcal{D} := \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^i \mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1} D_F\}$$

where $f_\pi : \mathcal{D} \twoheadrightarrow D_F$ is defined by $s(u) \otimes x \mapsto s(\pi)x$.

The filtered module $\mathcal{D}$ has the advantage over the filtered module D that all of its data are defined at the same level (no need to extend scalars to F).

Example 4.1.1. In the following examples, we have $F = F_0 = \mathbf{Q}_p$ and we choose $\pi = p$ (hence $E(u) = u - p$).
(i) Assume D is as in Example 3.4.1 (i) or (ii) (with trivial coefficients for simplicity), then one finds $\mathrm{Fil}^i \mathcal{D} = S \cdot e_1 + \mathrm{Fil}^i S \otimes D$ if $i \leq k-1$ and $\mathrm{Fil}^i \mathcal{D} = \mathrm{Fil}^{i-k+1} S \cdot e_1 + \mathrm{Fil}^i S \otimes D$ if $i > k-1$.

(ii) Assume D is as in Example 3.4.1 (iv) (with trivial coefficients), then the $\mathrm{Fil}^i\mathcal{D}$ are more involved. For instance, if $k-1 \geq 3$, one finds:

$$\begin{aligned}
\mathrm{Fil}^1\mathcal{D} &= S\cdot(e_1 - \mathcal{L}e_2) + \mathrm{Fil}^1 S\otimes D\\
\mathrm{Fil}^2\mathcal{D} &= S\cdot(e_1 - \mathcal{L}e_2 + \frac{u-p}{p}e_2) + \mathrm{Fil}^1 S\cdot(e_1-\mathcal{L}e_2) + \mathrm{Fil}^2 S\otimes D\\
\mathrm{Fil}^3\mathcal{D} &= S\cdot\left(e_1 - \mathcal{L}e_2 + \frac{u-p}{p}e_2 - \frac{1}{2}\frac{(u-p)^2}{p^2}e_2\right)\\
&\quad +\mathrm{Fil}^1 S\cdot\left(e_1-\mathcal{L}e_2+\frac{u-p}{p}e_2\right) + \mathrm{Fil}^2 S\cdot(e_1-\mathcal{L}e_2) + \mathrm{Fil}^3 S\otimes D
\end{aligned}$$

etc.

4.2 Strongly divisible lattices

Now, we define integral structures inside the $\mathcal{D}$'s:

Definition 4.2.1. *Let D be a weakly admissible filtered (φ, N)-module such that $\mathrm{Fil}^{m+1}D_F = 0$ with $m < p$. A strongly divisible lattice (or module) in $\mathcal{D}$ is an S-submodule $\mathcal{M}$ of $\mathcal{D}$ such that:*
(1) $\mathcal{M}$ is free of finite rank over S and $\mathcal{M}[\frac{1}{p}] \xrightarrow{\sim} \mathcal{D}$
(2) $\mathcal{M}$ is stable under φ and N
(3) $\varphi(\mathrm{Fil}^m\mathcal{M}) \subset p^m\mathcal{M}$ where $\mathrm{Fil}^m\mathcal{M} := \mathcal{M}\cap\mathrm{Fil}^m\mathcal{D}$.

One can show this definition doesn't depend on m (provided of course $\mathrm{Fil}^{m+1}D_F = 0$ and $m < p$). In fact, one can prove the following (see [2] and [8]):

Theorem 4.2.2. *The condition $\varphi(\mathrm{Fil}^m\mathcal{M}) \subset p^m\mathcal{M}$ in Definition 4.2.1 is equivalent to the condition $\varphi(\mathrm{Fil}^m\mathcal{M})$ spans $p^m\mathcal{M}$.*

The point is that once you have $\varphi(\mathrm{Fil}^m\mathcal{M}) \subset p^m\mathcal{M}$, the weak admissibility of D forces $\varphi(\mathrm{Fil}^m\mathcal{M})$ to actually span $p^m\mathcal{M}$.

We will spend the next lecture giving non trivial examples of such modules. So, here is a trivial example:

Example 4.2.3. Let D be the trivial filtered module (i.e. $D = F_0$ with $\mathrm{Fil}^1 D_F = 0$, $N = 0$ and obvious φ). Then S is a strongly divisible lattice in $\mathcal{D} = S[\frac{1}{p}]$.

Let $\mathcal{M}$ be a strongly divisible module in some $S\otimes_{W(\mathbf{F})} D$ with D weakly admissible as in Definition 4.2.1. Then one can associate to $\mathcal{M}$ the $\mathbf{Z}_p[G_F]$-module:

$$T^*_{st}(\mathcal{M}) := \mathrm{Hom}_{S,\varphi,N,\mathrm{Fil}^m}(\mathcal{M}, \widehat{A_{st}})$$

where one considers S-linear maps from $\mathcal{M}$ to $\widehat{A_{st}}$ that commute with φ and N and preserve Fil^m. The action of G_F is $(g \cdot f)(x) := g(f(x))$. Note that this is well defined since G_F commutes with all the other structures.

Proposition 4.2.4. *Let V be a semi-stable p-adic representation of G_F and $D := D^*_{st}(V)$. Assume $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^{m+1} D_F = 0$ with $m < p$. Let $\mathcal{M}$ be a strongly divisible lattice in $\mathcal{D} := S \otimes D$, then $T^*_{st}(\mathcal{M})$ is a Galois stable $\mathbf{Z}_p$-lattice in V.*

Proof. First, $T^*_{st}(\mathcal{M})$ is clearly a Galois stable $\mathbf{Z}_p$-lattice in $V^*_{st}(\mathcal{D}) := \mathrm{Hom}_{S,\varphi,N,\mathrm{Fil}^m}(\mathcal{D}, \widehat{A_{st}}[1/p])$. By lemma 2.4.2 (Lecture 2) and using that $D = \{x \in \mathcal{D} \mid N^n(x) = 0 \text{ for some } n \in \mathbf{N}\}$ (easy), we get that any $f \in V^*_{st}(\mathcal{D})$ sends D to B^+_{st}. Also $f(\mathrm{Fil}^m \mathcal{D}) \subset \mathrm{Fil}^m \widehat{A_{st}}[1/p]$ implies $f(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^i \widehat{A_{st}}[1/p]$ for any $i \in \mathbf{N}$ (easy again). For $f \in V^*_{st}(\mathcal{D})$, let $\overline{f} : D_F \to B^+_{dR}$ be the unique F-linear map such that the diagram:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & \widehat{A_{st}}[1/p] \\ {\scriptstyle f_\pi}\downarrow & & \downarrow \\ D_F & \xrightarrow{\overline{f}} & B^+_{dR} \end{array}$$

commutes, where the map $\widehat{A_{st}}[1/p] \to B^+_{dR}$ is that of Lemma 2.4.2 (ii). Then, $\overline{f}(\mathrm{Fil}^i D_F) \subset \mathrm{Fil}^i B^+_{dR}$ (one can check $f_\pi : \mathrm{Fil}^i \mathcal{D} \to \mathrm{Fil}^i D_F$ is surjective). We finally get a $\mathbf{Q}_p[G_F]$-linear map:

$$V^*_{st}(\mathcal{D}) \longrightarrow V^*_{st}(D) := \mathrm{Hom}_{\varphi,N,\mathrm{Fil}^\cdot}(D, B^+_{st}) = V$$

sending f to $f|_D$ which is clearly injective. It is also surjective because one can check that for any $f \in V^*_{st}(D)$, the map $\mathrm{Id} \otimes f : \mathcal{D} = S \otimes D \to S \otimes B^+_{st} \hookrightarrow \widehat{A_{st}}[1/p]$ automatically respects the filtration (and it clearly respects the rest). Hence, $T^*_{st}(\mathcal{M}) \subset V^*_{st}(\mathcal{D}) \simeq V$ is a Galois stable $\mathbf{Z}_p$-lattice. □

Definition 4.2.5. *Let D be a weakly admissible filtered module such that $\mathrm{Fil}^0 D_F = D_F$, $\mathrm{Fil}^m D_F \neq 0$ and $\mathrm{Fil}^{m+1} D_F = 0$ ($m \in \mathbf{N}$). We call D unipotent if D has no non trivial quotient $\overline{D}$ (in the category of weakly admissible filtered (φ, N)-modules; $\overline{D}_F$ has the quotient filtration) such that $\mathrm{Fil}^m \overline{D}_F = \overline{D}_F$.*

Conjecture 4.2.6. *Let V be a semi-stable p-adic representation of G_F and $D := D^*_{st}(V)$. Assume $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^{m+1} D_F = 0$.*
(i) If $m < p - 1$, the functor $\mathcal{M} \mapsto T^*_{st}(\mathcal{M})$ induces an anti-equivalence of categories between strongly divisible lattices of $\mathcal{D} = S \otimes D$ and G_F-stable lattices of V.
(ii) If $m = p - 1$ and D is unipotent, the same holds.

The following theorem sums up what is essentially known about this conjecture:

Theorem 4.2.7. *Conjecture 4.2.6 is true in the following two cases:*
(i) $m < p-1$ *and* $F = F_0$
(ii) $m < p-1$ *and* $m \leq 1$.

Case (i) is proven in [8] using the torsion theory of the next §. Case (ii) is proven in [6] and [3] using the theory of p-divisible groups (see Lecture 5).

4.3 Reducing modulo p

As with Galois lattices, it is tempting to reduce strongly divisible lattices modulo p. To do this, we first give an alternative definition of strongly divisible lattices (Theorem 4.3.2) from which we derive the definition of a category of "torsion strongly divisible modules".

For $m < p$, let $\mathcal{C}^m$ be the category of quadruples $(\mathcal{M}, \mathrm{Fil}^m\mathcal{M}, \varphi_m, N)$ where:
• $\mathcal{M}$ is an S-module
• $\mathrm{Fil}^m\mathcal{M} \subset \mathcal{M}$ is an S-submodule containing $(\mathrm{Fil}^m S)\mathcal{M}$
• $\varphi_m : \mathrm{Fil}^m\mathcal{M} \to \mathcal{M}$ is an additive map such that $\varphi_m(sx) = \sigma(s)\varphi_m(x)$ ($s \in S$, $x \in \mathrm{Fil}^m\mathcal{M}$) and $\varphi_m(sx) = \frac{\frac{\sigma}{p^m}(s)}{\frac{\sigma}{p^m}(E(u)^m)}\varphi_m(E(u)^m x)$ ($s \in \mathrm{Fil}^m S$, $x \in \mathcal{M}$)
• $N : \mathcal{M} \to \mathcal{M}$ is an additive map such that $N(sx) = N(s)x + sN(x)$, $(\mathrm{Fil}^1 S)N(\mathrm{Fil}^m\mathcal{M}) \subset \mathrm{Fil}^m\mathcal{M}$ and $\varphi_m \circ (E(u)N|_{\mathrm{Fil}^m}) = \frac{\sigma}{p}(E(u))N \circ \varphi_m$.
Of course, morphisms in $\mathcal{C}^m$ are S-linear maps preserving all these structures. For simplicity, we will just denote by $\mathcal{M}$ an object of $\mathcal{C}^m$. For any $\mathcal{M}$ in $\mathcal{C}^m$, define $\varphi : \mathcal{M} \to \mathcal{M}$ by $\varphi(x) := \frac{p^m}{\sigma(E(u)^m)}\varphi_m(E(u)^m x)$. If $\mathcal{M}$ has no p-torsion, the knowledge of $\varphi_m : \mathrm{Fil}^m\mathcal{M} \to \mathcal{M}$ is equivalent to that of φ (via $\varphi_m = \frac{\varphi}{p^m}|_{\mathrm{Fil}^m}$). For instance, $\widehat{A_{st}}$, $\widehat{A_{st}}/p^n\widehat{A_{st}}$ and $\widehat{A_{st}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p$ are objects of $\mathcal{C}^m$ for any $m < p$ (recall $\varphi(\mathrm{Fil}^m\widehat{A_{st}}) \subset p^m\widehat{A_{st}}$ if $m < p$).

Definition 4.3.1. *A sequence* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *in* $\mathcal{C}_m$ *is exact if the two sequences of* S*-modules* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *and* $0 \to \mathrm{Fil}^m\mathcal{M}' \to \mathrm{Fil}^m\mathcal{M} \to \mathrm{Fil}^m\mathcal{M}'' \to 0$ *are exact.*

Define the category of strongly divisible modules of weight $\leq m$ as the full subcategory of $\mathcal{C}^m$ of objects that are isomorphic to a strongly divisible lattice in some $S \otimes D$ with D weakly admissible as in Definition 4.2.1. One can describe directly this category:

Theorem 4.3.2. *The category of strongly divisible modules of weight $\leq m$ ($m < p$) is the full subcategory of $\mathcal{C}^m$ of objects $\mathcal{M}$ such that:*
(i) $\mathcal{M}$ is free of finite rank over S
(ii) $\mathrm{Fil}^m\mathcal{M} \cap p\mathcal{M} = p\mathrm{Fil}^m\mathcal{M}$
(iii) $\mathcal{M}$ is spanned by $\varphi_m(\mathrm{Fil}^m\mathcal{M})$.

The point is to prove that (1) $\mathcal{M}[1/p] \simeq S \otimes D$ with D a filtered (φ, N)-module (see [7]) and (2) D is weakly admissible (see [8]).

We introduce now the torsion analogue of strongly divisible modules, which is motivated by Theorem 4.3.2.

Let $\underline{\mathcal{M}}^m$ be the full subcategory of $\mathcal{C}^m$ of objects $\mathcal{M}$ such that:
• $\mathcal{M} \simeq \oplus_{n\in I}(S/p^nS)^{r_n}$ (I finite)
• $\mathcal{M}$ is spanned by $\varphi_m(\mathrm{Fil}^m\mathcal{M})$.
One can then prove in that case that $\mathrm{Fil}^m\mathcal{M} \cap p^n\mathcal{M} = p^n\mathrm{Fil}^m\mathcal{M}$ for any $n \in \mathbf{N}$. Define a functor to Galois representations by:

$$T_{st}^*(\mathcal{M}) := \mathrm{Hom}_{\mathcal{C}^m}(\mathcal{M}, \widehat{A_{st}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p).$$

Note that if $\mathcal{M}$ is a strongly divisible module of weight $\leq m$, then $\mathcal{M}/p^n\mathcal{M}$ is naturally an object of $\underline{\mathcal{M}}^m$ for any $n \in \mathbf{N}$.

Theorem 4.3.3. *Assume $F = F_0$ and $m < p - 1$.*
(i) The category $\underline{\mathcal{M}}^m$ is abelian.
(ii) The functor T_{st}^* is exact and fully faithful.
(iii) If $\mathcal{M}$ is a strongly divisible module of weight $\leq m$, then:

$$T_{st}^*(\mathcal{M})/p^nT_{st}^*(\mathcal{M}) \simeq T_{st}^*(\mathcal{M}/p^n\mathcal{M}).$$

Proof. (very rough sketch) For (i), we just refer the reader to [2]. For the exactness in (ii), it is enough to prove that $\mathrm{Ext}^1_{\mathcal{C}^m}(\mathcal{M}, \widehat{A_{st}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p) = 0$ for simple objects $\mathcal{M}$ of $\underline{\mathcal{M}}^m$. But simple objects of $\underline{\mathcal{M}}^m$ admit a simple description which was already given by Fontaine and Laffaille in [23] twenty years ago. Using this description (together with properties of $\widehat{A_{st}}$), one explicitly builds a section $\mathcal{M} \hookrightarrow \mathcal{E}$ in $\mathcal{C}^m$ to any extension $0 \to \widehat{A_{st}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p \to \mathcal{E} \to \mathcal{M} \to 0$ in $\mathcal{C}^m$. For the full faithfulness, one is reduced (by exactness and a standard devissage) to prove that:
(1) $\mathrm{Hom}_{\underline{\mathcal{M}}^m}(\mathcal{M}', \mathcal{M}'') = \mathrm{Hom}_{\mathbf{Z}_p[G_F]}(T_{st}^*(\mathcal{M}''), T_{st}^*(\mathcal{M}'))$
(2) $\mathrm{Ext}^1_{\underline{\mathcal{M}}^m}(\mathcal{M}', \mathcal{M}'') \hookrightarrow \mathrm{Ext}^1_{\mathbf{Z}_p[G_F]}(T_{st}^*(\mathcal{M}''), T_{st}^*(\mathcal{M}'))$
for $\mathcal{M}'$ and $\mathcal{M}''$ simple objects of $\underline{\mathcal{M}}^m$. (1) is done in [23] by explicit computation (as already mentionned, simple objects of $\underline{\mathcal{M}}^m$ were already known). (2) is more delicate. One must prove that any extension $0 \to \mathcal{M}' \to$

$\mathcal{M} \to \mathcal{M}'' \to 0$ in $\underline{\mathcal{M}}^m$ that splits on the Galois side already splits in $\underline{\mathcal{M}}^m$. This is done in three steps (and follows an idea of Faltings [20]). First, one checks that $pT^*_{st}(\mathcal{M}) = 0$ implies $p\mathcal{M} = 0$ (one uses for this the exactness of T^*_{st} and a direct computation of $\dim_{\mathbf{F}_p} T^*_{st}(\mathcal{M})$ when $p\mathcal{M} = 0$ which gives $\dim_{\mathbf{F}_p} T^*_{st}(\mathcal{M}) = \mathrm{rk}_{S/pS}\mathcal{M}$). Second, one proves by explicit computations on $T^*_{st}(\mathcal{M})$ that any $f \in T^*_{st}(\mathcal{M})$ is such that $f(N(\varphi_m(\mathrm{Fil}^m\mathcal{M}))) = 0$. This implies $N(\varphi_m(\mathrm{Fil}^m\mathcal{M})) = 0$. Using the relation $\varphi_m \circ (E(u)N|_{\mathrm{Fil}^m}) = \frac{\sigma}{p}(E(u))N \circ \varphi_m$, we get $N(\mathrm{Fil}^m\mathcal{M}) \subset \mathrm{Fil}^m\mathcal{M}$. This implies that $\mathcal{M}$ has a simple description, and is in fact one of the objects introduced in [23] (as are $\mathcal{M}'$ and $\mathcal{M}''$). But we are done since full faithfulness of T^*_{st} is known when restricted to such objects (this was first proved by tedious explicit computations in [23], then a more conceptual proof was given recently in [41]). (iii) is an easy corollary from (ii). $\square$

A first corollary of the above theorem is that if $\mathcal{M} \simeq \oplus_{n\in I}(S/p^nS)^{r_n}$, then $T^*_{st}(\mathcal{M}) \simeq \oplus_{n\in I}(\mathbf{Z}_p/p^n\mathbf{Z}_p)^{r_n}$ with the same r_n. A second is the full faithfulness in Theorem 4.2.7 (i) (we won't really need the essential surjectivity in the sequel, since we will only treat examples where explicit computations will directly furnish as many (isomorphism classes of) strongly divisible lattices as there are (isomorphism classes of) Galois stable lattices). I'll give several examples of applications of Theorem 4.3.3 in Lectures 8 and 9. In these applications, one has $F_0 = \mathbf{Q}_p$ and one works with additional non trivial coefficients $\mathcal{O}_E$ acting on objects of $\underline{\mathcal{M}}^m$ ($\mathcal{O}_E$ being the ring of integers of a finite extension E of $\mathbf{Q}_p$). In particular, strongly divisible modules are free $S \otimes_{\mathbf{Z}_p} \mathcal{O}_E$-modules of finite rank.

Let me finally mention that Theorem 4.3.3 should remain true more generally for $[F : F_0]m < p - 1$ (one lacks the full faithfulness).

5 Lecture 5: Various examples of strongly divisible modules

In this lecture, I will give various examples of strongly divisible modules. In §5.1, I will describe concrete and explicit examples, in §5.2, I will give examples coming from algebraic varieties and in §5.3, I will give examples coming from p-divisible groups. Only §5.1 will be used in the sequel.

5.1 Explicit examples

In this §, $S := \{\sum_{n=0}^{\infty} a_n \frac{u^n}{n!}, a_n \in \overline{\mathbf{Z}_p}, a_n \to 0, [\mathbf{Q}_p(a_n)_{n\in\mathbf{N}} : \mathbf{Q}_p] < \infty\}$ and $\pi = p$. I give examples of some strongly divisible lattices in $\mathcal{D} := S \otimes D$ for

D as in Example 3.4.1 (Lecture 3).

Proposition 5.1.1. *(i) Let D be as in Example 3.4.1 (i), (ii) or (iii) with $k \leq p$, then $\mathcal{M} := Se_1 \oplus Se_2$ is a strongly divisible lattice in $S \otimes D$.*
(ii) Let D be as in Example 3.4.1 (iv) with $k = 2$:
• *if* $\mathrm{val}(\mathcal{L}) < 1$, *then* $\mathcal{M} := S\frac{p}{\mathcal{L}}e_1 \oplus Se_2$ *is a strongly divisible lattice in* $S \otimes D$
• *if* $\mathrm{val}(\mathcal{L}) \geq 1$, *then* $\mathcal{M} := Se_1 \oplus Se_2$ *is a strongly divisible lattice in* $S \otimes D$.

Proof. (i) is obvious. For (ii), in the first case, one easily checks that $\mathrm{Fil}^1\mathcal{D} \cap \mathcal{M} = \mathrm{Fil}^1 S \cdot \mathcal{M} + S(\frac{-p}{\mathcal{L}}e_1 + pe_2)$. Since $\varphi(e_2) = \mu e_2$ and $\frac{\varphi}{p}(\frac{-p}{\mathcal{L}}e_1 + pe_2) = \mu(\frac{-p}{\mathcal{L}}e_1 + e_2)$, it is clear $\mathcal{M}$ is strongly divisible. In the second case, one has $\mathrm{Fil}^1\mathcal{D} \cap \mathcal{M} = \mathrm{Fil}^1 S \cdot \mathcal{M} + S(e_1 - \mathcal{L}e_2)$. Since $\varphi(e_2) = \mu e_2$ and $\frac{\varphi}{p}(e_1 - \mathcal{L}e_2) = \mu(e_1 - \frac{\mathcal{L}}{p}e_2)$, $\mathcal{M}$ is strongly divisible. $\square$

It is much more delicate to find strongly divisible lattices in $S \otimes D$ for D as in Example 3.4.1 with $k \geq 4$ (and $k \leq p$). We give below strongly divisible lattices in the case $k = 4$ and, at least for one of them, the complete proof that it is really strongly divisible. Recall from Lecture 4 that the filtration on $\mathcal{D} := S \otimes D$ (or just the Fil^3) is this time more involved:

$$\begin{aligned}\mathrm{Fil}^3\mathcal{D} \quad = \quad & \mathrm{Fil}^3 S \cdot \mathcal{D} + \Big\{ C_0(e_1 - \mathcal{L}e_2) + \frac{u-p}{p}\big(C_1(e_1 - \mathcal{L}e_2) + C_0 e_2\big) + \\ & \frac{(u-p)^2}{p^2}\Big(C_2(e_1 - \mathcal{L}e_2) + \big(C_1 - \frac{C_0}{2}\big)e_2\Big),\ C_i \in \overline{\mathbf{Q}_p} \Big\}.\end{aligned}$$

In the sequel, we set $\gamma := \frac{(u-p)^p}{p} \in \mathrm{Fil}^p S$ and we note that $s + \gamma \in S^\times$ if $s \in S^\times$.

Theorem 5.1.2. *Let D be as in Example 3.4.1 (iv) with $k = 4$ (and $p \geq 5$).*
(i) If $\mathrm{val}(\mathcal{L} + 3/2) = 0$ *and* $\mathrm{val}(\mathcal{L} + 2) < 1$, *then:*

$$\begin{aligned}\mathcal{M}_1 \quad &:= \quad S\Big(e_1 + \frac{\gamma}{1+\gamma}\frac{2-\mathcal{L}}{p}e_2\Big) \oplus S\frac{\mathcal{L}+2}{p}e_2 \\ \mathcal{M}_2 \quad &:= \quad S\Big(e_1 - \frac{\mathcal{L} + \frac{3}{2} - 2\gamma + \frac{1}{2}\gamma^2}{p}e_2\Big) \oplus Se_2\end{aligned}$$

are non isomorphic strongly divisible lattices in $S \otimes D$.
(ii) If $\mathrm{val}(\mathcal{L} + 3/2) = 0$ *and* $\mathrm{val}(\mathcal{L} + 2) \geq 1$, *then:*

$$\begin{aligned}\mathcal{M}_1 \quad &:= \quad S\Big(e_1 + \frac{\gamma}{1+\gamma}\frac{2-\mathcal{L}}{p}e_2\Big) \oplus Se_2 \\ \mathcal{M}_2 \quad &:= \quad S\Big(e_1 - \frac{\mathcal{L} + \frac{3}{2} - 2\gamma + \frac{1}{2}\gamma^2}{p}e_2\Big) \oplus Se_2\end{aligned}$$

are non isomorphic strongly divisible lattices in $S \otimes D$.
(iii) If $\mathrm{val}(\mathcal{L}+3/2)>0$*, then:*

$$\mathcal{M} := S\Big(e_1 + \frac{\gamma}{1+\gamma}\frac{2-\mathcal{L}}{p}e_2\Big) \oplus S\frac{e_2}{p} = Se_1 \oplus S\frac{e_2}{p}$$

is a strongly divisible lattice in $S \otimes D$.
(iv) If $\mathrm{val}(\mathcal{L}+3/2)<0$ *i.e.* $\mathrm{val}(\mathcal{L})<0$*, then:*

$$\mathcal{M} := S\Big(e_1 - \frac{\mathcal{L}+\frac{3}{2}-2\gamma+\frac{1}{2}\gamma^2}{p}e_2\Big) \oplus S\mathcal{L}e_2$$

is a strongly divisible lattice in $S \otimes D$.

Proof. (sketch) For reasons of time, length and boredom of the audience, I only give a complete proof for $\mathcal{M}_1$ (the other cases proceed in the same way, and the brave reader can find them in much greater generality in [10]). We have to prove that $\varphi(\mathrm{Fil}^3\mathcal{M}_1) \subset p^3\mathcal{M}_1$ (see Lecture 4). Let $E_1 := e_1 + \frac{\gamma}{1+\gamma}\frac{2-\mathcal{L}}{p}e_2$, $E_2 := \frac{\mathcal{L}+2}{p}e_2$ and note that $\mathcal{M}_1$ is stable under φ and N, that $e_1 - E_1 \in \mathrm{Fil}^3S\cdot\mathcal{D}$ and that $\gamma \equiv \frac{u^p}{p}(p)$. By the previous description of $\mathrm{Fil}^3\mathcal{D}$, any element of $\mathrm{Fil}^3\mathcal{D}$ can be written $x+y$ where $y \in \mathrm{Fil}^3S\cdot\mathcal{D}$ and $x = x_0 + (u-p)x_1 + (u-p)^2x_2$ with:

$$\begin{aligned} x_0 &= C_0E_1 - \frac{p\mathcal{L}C_0}{\mathcal{L}+2}E_2 \\ x_1 &= \frac{C_1}{p}E_1 - \frac{\mathcal{L}C_1 - C_0}{\mathcal{L}+2}E_2 \\ x_2 &= \frac{C_2}{p^2}E_1 - \frac{\mathcal{L}C_2 - C_1 + \frac{C_0}{2}}{p(\mathcal{L}+2)}E_2. \end{aligned}$$

Let $\alpha := \mathrm{val}(\mathcal{L}+2)$ $(0 \leq \alpha < 1)$. Since (E_1, E_2) is a basis of $\mathcal{D}$, any element of $\mathrm{Fil}^3\mathcal{D}\cap\mathcal{M}_1$ can be written $x+y$ where $y \in \mathrm{Fil}^3S\cdot\mathcal{M}_1$ and x is as above with $C_0 \in \overline{\mathbf{Z}_p}$, $C_1 \in p\overline{\mathbf{Z}_p}$, $C_2 \in p^2\overline{\mathbf{Z}_p}$, $\mathrm{val}(\mathcal{L}C_1 - C_0) \geq \alpha$ and $\mathrm{val}(\mathcal{L}C_2 - C_1 + \frac{C_0}{2}) \geq \alpha+1$. This easily implies $C_0 \in p\overline{\mathbf{Z}_p}$ and $\mathrm{val}(C_0 - 2C_1) \geq \alpha + 1$. Moreover, since $\sigma(\gamma) \in p^{p-1}S$, one checks that $\varphi(E_1) + \mu\frac{\gamma(2-\mathcal{L})}{1+\gamma}pe_2 \in p^2\mathcal{M}_1$, or equivalently $\varphi(E_1) + \mu\frac{u^p}{p}\frac{2-\mathcal{L}}{1+\frac{u^p}{p}}\frac{p^2}{\mathcal{L}+2}E_2 \in p^2\mathcal{M}_1$ (which implies $\varphi(E_1) \in p\mathcal{M}_1$). Using $\mathrm{val}(\mathcal{L}C_2 - C_1 + \frac{C_0}{2}) \geq \alpha+1$ and $C_2 \in p^2\overline{\mathbf{Z}_p}$, one finds:

$$\varphi(x) = \Big[C_0 + \Big(\frac{u^p}{p}-1\Big)C_1\Big]\varphi(E_1) + \mu\Big[\Big(\frac{u^p}{p}-1\Big)(C_0 - \mathcal{L}C_1) - \mathcal{L}C_0\Big]\frac{p^2}{\mathcal{L}+2}E_2 + p^3z$$

where $z \in \mathcal{M}_1$. Since $C_i \in p\overline{\mathbf{Z}_p}$, up to an element of $p^3\mathcal{M}_1$ one can replace $\varphi(E_1)$ by $-\mu\frac{u^p}{p}\frac{2-\mathcal{L}}{1+\frac{u^p}{p}}\frac{p^2}{\mathcal{L}+2}E_2$ and $\mathcal{L}$ by -2 in the above expression. A straightforward computation then yields:

$$\varphi(x) = \mu\Big[C_0 - 2C_1 + \frac{u^p}{u^p+p}\Big(-3C_0+6C_1+\frac{u^p}{p}(C_0-2C_1)\Big)\Big]\frac{p^2}{\mathcal{L}+2}E_2 + p^3 z$$

for some $z \in \mathcal{M}_1$. But since $\frac{C_0-2C_1}{\mathcal{L}+2} \in p\overline{\mathbf{Z}_p}$, we finally have $\varphi(x) \in p^3\mathcal{M}_1$ and we are done. □

5.2 Geometric examples

One can also realize strongly divisible modules as some cohomology groups. This section and the next are purely expository.

We keep the notations of Lectures 3,4 (so S is now as in Lecture 4) and we let X be a proper smooth scheme over $\mathrm{Spec}(F)$ admitting a proper semi-stable model $\mathcal{X}$ over $\mathcal{O}_F$ (i.e. $\mathcal{X}$ has an étale covering which is smooth over $\mathcal{O}_F[X_1, ..., X_r]/(X_1 \cdots X_r - \pi)$ for some r). Let $\mathcal{Y} := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathbf{F})$ and $\mathcal{X}_1 := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_F/p\mathcal{O}_F)$. In this situation, one can endow $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}_1$ with an extra data called a *log-structure*. We won't need the precise definition here (see [29]). Let us just say that, although the schemes $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}_1$ are not smooth, the *log-schemes* $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}_1$ (i.e. endowed with their log-structure) behave as "smooth objects". This allows to apply the techniques that worked in the smooth case, correctly modified. For $m \in \mathbf{N}$ denote by:

$$\begin{aligned} H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Z}_p) &:= \varprojlim H^m((X \times_F \overline{\mathbf{Q}}_p)_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z}) \\ H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Q}_p) &:= H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \end{aligned}$$

Grothendieck's usual p-adic étale cohomology groups of X. By [39], $H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Q}_p)$ is a semi-stable p-adic representation of G_F with Hodge-Tate weights in $\{-m, ..., 0\}$. Moreover, if $V^m := H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Q}_p)^*$ ($\mathbf{Q}_p$-dual) and $D^m := D^*_{st}(V^m)$ is the associated filtered (φ, N)-module (as in Lecture 3), then:

$$D^m \simeq H^m_{\log-\mathrm{cris}}(\mathcal{Y}/W(\mathbf{F})) \otimes F_0 \tag{1}$$

where:

$$H^m_{\log-\mathrm{cris}}(\mathcal{Y}/W(\mathbf{F})) := \varprojlim H^m_{\log-\mathrm{cris}}\big(\mathcal{Y}/\mathrm{Spec}(W_n(\mathbf{F}))\big)$$

is the log-crystalline cohomology of the log-scheme $\mathcal{Y}$ with respect to the base scheme $\mathrm{Spec}(W_n(\mathbf{F}))$ endowed with the log-structure $\big(\mathbf{N} \to W_n(\mathbf{F}), 1 \mapsto$

$0\big)$. More precisely, it is proven in [28] that this cohomology is naturally endowed with operators φ, N and that one has:

$$F \otimes_{W(\mathbf{F})} H^m_{\log-\mathrm{cris}}(\mathcal{Y}/W(\mathbf{F})) \simeq H^m_{\mathrm{dR}}(X)$$

where $H^m_{\mathrm{dR}}(X)$ is the usual de Rham cohomology of X endowed with its Hodge filtration, and it is proven in [39] that (1) is then an isomorphism of *filtered* (φ, N)*-modules.*

Now we come to S. Define:

$$\mathcal{D}^m := S \otimes_{W(\mathbf{F})} D^m$$

and endow it with the same structures as in Lecture 4, §4.1. It is shown in [28] that there is an isomorphism of $S[1/p]$-modules:

$$\mathcal{D}^m \simeq H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S) \otimes F_0$$

where:

$$H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S) := \varprojlim H^m_{\log-\mathrm{cris}}\big(\mathcal{X}_1/\mathrm{Spec}(S/p^nS)\big)$$

is the log-crystalline cohomology of the log-scheme $\mathcal{X}_1$ with respect to the base scheme $\mathrm{Spec}(S/p^nS)$ endowed with the log-structure $\big(\mathbf{N} \to S/p^nS, 1 \mapsto u\big)$. Here the log-scheme $\mathcal{X}_1$ is viewed over $\mathrm{Spec}(S/p^nS)$ via the embedding $\mathrm{Spec}(\mathcal{O}_F/p\mathcal{O}_F) \hookrightarrow \mathrm{Spec}(S/p^nS)$, $u \mapsto \pi$. Assume $m < p$ and consider:

$$T^m := \mathbf{Z}_p\text{−dual of } \big(H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Z}_p)/\text{torsion}\big)$$

which is a Galois lattice in V^m. Consider:

$$\mathcal{M}^m := H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S)/\text{torsion}.$$

One can prove that $\mathcal{M}^m \subset \mathcal{D}^m$ and that it is stable under φ and N ([28]). But what is more interesting is that $\mathcal{M}_m$ is really a strongly divisible lattice (at least in some cases) when $m < p-1$:

Theorem 5.2.1. *Assume either that F is unramified (and $m+1<p$) or that $m=1$ (and $2<p$), then $\mathcal{M}^m$ is a strongly divisible lattice in $\mathcal{D}^m$ and its associated Galois lattice is isomorphic to T^m.*

The first case is proven in [5] and the second in [18]. One can ask whether this result doesn't hold assuming only $m < p-1$, or even $m < p$.

5.3 Examples coming from p-divisible groups

Let F, $\mathcal{O}_F$, $\mathbf{F}$, π be as in the previous section and let G be a p-divisible group over $\mathcal{O}_F$. Recall that, by definition, $G = (G[n], i_n)_{n \in \mathbf{N}}$ where $G[n]$ is a finite flat commutative $\mathcal{O}_F$-group scheme killed by p^n and $i_n : G[n] \to G[n+1]$ is a group scheme homomorphism such that the sequence $0 \to G[n] \stackrel{i_n}{\to} G[n+1] \stackrel{p^n}{\to} G[n+1]$ is exact.
Let $G_1 := G \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_F/p)$. Berthelot, Breen and Messing, generalizing ideas of Grothendieck, have associated to G_1 a *crystal* $\mathbf{D}(G_1)$ (see [1]) whose evaluation on the thickening $\mathrm{Spec}(\mathcal{O}_F/p) \hookrightarrow \mathrm{Spec}(S)$, $u \mapsto \pi$ we denote by $\mathcal{M}(G)$. This is a free S-module of finite rank equipped with a σ-linear endomorphism $\varphi : \mathcal{M}(G) \to \mathcal{M}(G)$ (the crystalline Frobenius) and the data $(\mathcal{M}(G), \varphi)$ only depends on G_1. Now, let G' be a deformation of G over $\mathrm{Spec}(S)$ (S as in Lecture 4), i.e. a p-divisible group over S such that the diagram:

$$\begin{array}{ccc} G & \hookrightarrow & G' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_F) & \hookrightarrow & \mathrm{Spec}(S) \end{array}$$

is cartesian (such a G' always exists in our situation). Associated to G', we have the Hodge filtration $\mathcal{M}^1(G') \subset \mathcal{M}(G)$ which is a direct summand as an S-module. Define:

$$\mathrm{Fil}^1\mathcal{M}(G) := \mathcal{M}^1(G') + \mathrm{Fil}^1 S \cdot \mathcal{M}(G) \subset \mathcal{M}(G),$$

one can prove that $\mathrm{Fil}^1\mathcal{M}(G)$ only depends on G and not on the deformation G' and that, at least for $p > 2$, $\varphi(\mathrm{Fil}^1\mathcal{M}(G)) \subset p\mathcal{M}(G)$ and $\frac{\varphi}{p}(\mathrm{Fil}^1\mathcal{M}(G))$ generates $\mathcal{M}(G)$ over S (see e.g. [6] or [18]). This gives a contravariant functor from p-divisible groups over $\mathcal{O}_F$ to the category of triples $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \varphi)$ satisfying the above properties. It is proven in [6] that this functor is an equivalence of categories. Moreover, one has the easy lemma:

Lemma 5.3.1 ([6]). *Every $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \varphi)$ as above can be endowed with a unique additive map $N : \mathcal{M} \to \mathcal{M}$ such that:*
(i) $N(sx) = N(s)x + sN(x)$, $\forall s \in S, x \in \mathcal{M}$
(ii) $N\varphi = p\varphi N$
(iii) $N(\mathcal{M}) \subset u\mathcal{M} + \sum_{i \geq 1} \frac{u^{ie}}{i!}\mathcal{M}$.

All this finally gives a way (in theory) to obtain examples of strongly divisible modules:

Corollary 5.3.2. *Assume $p \neq 2$. There is an anti-equivalence of categories between p-divisible groups over $\mathcal{O}_F$ and strongly divisible modules $\mathcal{M}$ of weight ≤ 1 such that $N(\mathcal{M}) \subset u\mathcal{M} + \sum_{i \geq 1} \frac{u^{ie}}{i!}\mathcal{M}$.*

6 Lecture 6: Mazur's deformation theory and local deformation rings

For simplicity, we now write G_p instead of $G_{\mathbf{Q}_p}$, W_p for the Weil subgroup and I_p for the inertia subgroup. We give the main statement of deformation theory for representations of a profinite group G, namely the existence of a "universal" deformation, together with a sketch of the proof. Then we define the deformation rings that are relevant in p-adic Hodge theory (for $G = G_p$) as suitable quotients of the coefficient ring of the universal deformation. For §6.1 and §6.2, we have heavily used [37].

6.1 The main statement

Let G be a profinite group, $\mathbf{F} \subset \overline{\mathbf{F}}_p$ a finite field and $\overline{T}$ a finite dimensional $\mathbf{F}$-vector space endowed with the discrete topology and with a continuous action of G. We assume $H^1(G, \mathrm{End}_{\mathbf{F}}(\overline{T}))$ to be finite dimensional over $\mathbf{F}$ (where G acts on $\mathrm{End}_{\mathbf{F}}(\overline{T})$ by $g \cdot f := g \circ f \circ g^{-1}$). This holds for instance when G is the Galois group of a local field with finite residue field. Let $\mathcal{O}$ be a complete discrete valuation ring with residue field $\mathbf{F}$ and $\mathcal{C}$ the category of local topological $\mathcal{O}$-algebras A such that the natural map $\mathcal{O} \to A/\mathfrak{m}_A$ is surjective and the map $A \to \varprojlim_{\mathfrak{a}} A/\mathfrak{a}$ from A to its discrete artinian quotients is a topological isomorphism. If A is noetherian, this is equivalent to having a topological isomorphism $A \simeq \varprojlim A/\mathfrak{m}_A^m$.

Definition 6.1.1. *Let $A \in \mathcal{C}$.*
(i) A representation of G over A, or a A-representation, is a finitely generated free A-module T with a continuous A-linear action of G (for the product topology on $T \simeq A^n$).
(ii) A deformation of $\overline{T}$ over A, or a A-deformation, is an isomorphism class of A-representations T of G for which $T \otimes A/\mathfrak{m}_A \simeq \overline{T}$.

We denote by $\mathrm{Def}(\overline{T}, A)$ the set of A-deformations of $\overline{T}$.

Theorem 6.1.2. *(Mazur) Assume $\mathrm{End}_{\mathbf{F}[G]}(\overline{T}) = \mathbf{F}$.*
(i) There are a ring $R \in \mathcal{C}$ and a deformation $D \in \mathrm{Def}(\overline{T}, R)$ such that for any $A \in \mathcal{C}$, we have a bijection $\mathrm{Hom}_{\mathcal{C}}(R, A) \xrightarrow{\sim} \mathrm{Def}(\overline{T}, A)$, $f \mapsto D \otimes_{R,f} A$.
(ii) The pair (R, D) is unique up to unique isomorphism.
(iii) The ring R is noetherian, $\mathfrak{m}_R$-adically complete and for any $A \in \mathcal{C}$, we have a bijection $\mathrm{Hom}_{\mathcal{C}}(R, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R, A)$.

6.2 Idea of the proof

Note that (ii) of Theorem 6.1.2 follows from (i) by uniqueness of universal objects.

Fix once and for all a basis $\overline{v}_1, \dots, \overline{v}_n$ of $\overline{T}$ so that one can write the action of G on $\overline{T}$ as a continuous group homomorphism $\overline{\rho} : G \to \mathrm{GL}_n(\mathbf{F})$. For any $A \in \mathcal{C}$, let $\mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(A))$ be the set of continuous group homomorphisms $G \to \mathrm{GL}_n(A)$ such that $G \to \mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbf{F})$ is $\overline{\rho}$.

Proposition 6.2.1. *There are a ring $\tilde{R} \in \mathcal{C}$ and a map $\tilde{\rho} \in \mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(\tilde{R}))$ such that for any $A \in \mathcal{C}$, we have a bijection:*

$$\mathrm{Hom}_{\mathcal{C}}(\tilde{R}, A) \xrightarrow{\sim} \mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(A)), \tilde{f} \mapsto \big(G \to \mathrm{GL}_n(\tilde{R}) \xrightarrow{\tilde{f}} \mathrm{GL}_n(A)\big).$$

Moreover, the pair $(\tilde{R}, \tilde{\rho})$ is determined up to unique isomorphism.

Proof. (sketch) The uniqueness property follows again from the universal property. Assume first that G is finite and denote by e its identity element. Let $\mathcal{O}[G,n]$ be the commutative $\mathcal{O}$-algebra whose generators are X_{ij}^g for $g \in G$ and $1 \leq i, j \leq n$ and whose relations are $X_{ij}^e := 1$ if $i = j$, $X_{ij}^e := 0$ if $i \neq j$ and $X_{ij}^{gh} := \sum_{\ell=1}^n X_{i\ell}^g X_{\ell j}^h$ for $g, h \in G$ and $1 \leq i, j \leq n$ (for instance $\mathcal{O}[G,1]$ is the group algebra of G^{ab}). For every $\mathcal{O}$-algebra A, one has a bijection $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{O}[G,n], A) \simeq \mathrm{Hom}(G, \mathrm{GL}_n(A))$, $f \mapsto \big(g \mapsto (f(X_{ij}^g))_{i,j}\big)$. In particular, $\overline{\rho}$ gives rise to a morphism $\mathcal{O}[G,n] \to \mathbf{F}$ whose kernel is a maximal ideal $\mathfrak{m}_{\overline{\rho}}$ of $\mathcal{O}[G,n]$. Let $\tilde{R}$ be the completion of $\mathcal{O}[G,n]$ at $\mathfrak{m}_{\overline{\rho}}$, then $\tilde{R} \in \mathcal{C}$, is noetherian and the canonical map $\mathcal{O}[G,n] \to \tilde{R}$ gives a morphism $\tilde{\rho} \in \mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(\tilde{R}))$. Let $A \in \mathcal{C}$ and $\rho \in \mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(A))$, ρ corresponds to a unique $\mathcal{O}$-algebra homomorphism $f : \mathcal{O}[G,n] \to A$ and we have $f(\mathfrak{m}_{\overline{\rho}}) \subset \mathfrak{m}_A$. Since any $\mathfrak{m}_A$-Cauchy sequence in A converges (because any ideal $\mathfrak{a}$ such that $A/\mathfrak{a}$ is artinian contains some $\mathfrak{m}_A^n$), it extends uniquely to a (continuous) $\mathcal{O}$-algebra homomorphism $\tilde{f} : \tilde{R} \to A$ and the diagram:

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\rho}} & \mathrm{GL}_n(\tilde{R}) \\ \| & & \downarrow \tilde{f} \\ G & \xrightarrow{\rho} & \mathrm{GL}_n(A) \end{array}$$

commutes. This gives the isomorphism $\mathrm{Hom}_{\mathcal{C}}(\tilde{R}, A) \simeq \mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(A))$ in the case G is finite. For the general case, write $G = \varprojlim H$, H ranging over those finite quotients of G for which $\overline{\rho}$ factors through $\overline{\rho}_H : H \to \mathrm{GL}_n(\mathbf{F})$. The above construction produces a projective system $(R_H)_H$ in $\mathcal{C}$ and a compatible system of group homomorphisms $\tilde{\rho}_H : H \to \mathrm{GL}_n(R_H)$. We then define $\tilde{R} := \varprojlim R_H \in \mathcal{C}$ and $\tilde{\rho} := \varprojlim \tilde{\rho}_H$. For the last details, see [37]. $\square$

Note that the condition $\mathrm{End}_{\mathbf{F}[G]}(\overline{T}) = \mathbf{F}$ is *not* needed in the proof of Proposition 6.2.1 and that if $\mathcal{O}$ is replaced by $\mathcal{O}'$ with $\mathcal{O} \subset \mathcal{O}'$ (and $\mathbf{F} \subset \mathbf{F}' := \mathcal{O}'/\mathfrak{m}_{\mathcal{O}'}$), then $\tilde{R}$ is replaced by $\tilde{R} \otimes_{\mathcal{O}} \mathcal{O}'$.

Assume first that $\overline{T}$ is absolutely irreducible, i.e. $\overline{T} \otimes_{\mathbf{F}} \overline{\mathbf{F}}_p$ is irreducible. This implies in particular $\overline{\mathbf{F}}_p = \mathrm{End}_{\overline{\mathbf{F}}_p[G]}(\overline{T} \otimes_{\mathbf{F}} \overline{\mathbf{F}}_p) \simeq \mathrm{End}_{\mathbf{F}[G]}(\overline{T}) \otimes_{\mathbf{F}} \overline{\mathbf{F}}_p$ hence $\mathrm{End}_{\mathbf{F}[G]}(\overline{T}) = \mathbf{F}$. We omit the proof of the following lemma (see [37]):

Lemma 6.2.2. *(Serre, Carayol) Assume $\overline{T}$ is absolutely irreducible. Let $A' \subset A$ be an inclusion in $\mathcal{C}$ and $T \in \mathrm{Def}(\overline{T}, A)$. Suppose A' contains the traces of all endomorphisms of T coming from G, then there is $T' \in \mathrm{Def}(\overline{T}, A')$ such that $T \simeq T' \otimes_{A'} A$.*

This lemma says one can realize any residually irreducible representation (or deformation) over the ring generated by the traces.

Corollary 6.2.3. *Assume $\overline{T}$ is absolutely irreducible, then statement (i) of Theorem 6.1.2 holds.*

Proof. Denote by R the smallest closed sub-$\mathcal{O}$-algebra of $\tilde{R}$ that contains the traces of all matrices $\tilde{\rho}(g)$ for $g \in G$. Then $R \in \mathcal{C}$ and by 6.2.2, we can realize $\tilde{\rho}$ over R. Let D be the corresponding deformation of $\overline{T}$ over R, we have to show the map $\mathrm{Hom}_{\mathcal{C}}(R, A) \xrightarrow{\sim} \mathrm{Def}(\overline{T}, A)$ is surjective and injective. Surjectivity comes from Proposition 6.2.1 since by Nakayama's lemma, one can always find a basis of any element of $\mathrm{Def}(\overline{T}, A)$ lifting the previous fixed basis of $\overline{T}$. Let $f, f' \in \mathrm{Hom}_{\mathcal{C}}(R, A)$ giving rise to isomorphic representations. Then their traces are the same. This means f and f' agrees on the dense subring of R generated by traces, hence on R by continuity. This proves injectivity. $\square$

I explain now the general case $\mathrm{End}_{\mathbf{F}[G]}(\overline{T}) = \mathbf{F}$. The point is Lemma 6.2.2 which is not true anymore. One has to pass to a different subring of $\tilde{R}$ since it's not sufficient to consider the traces. We keep our fixed basis of $\overline{T}$. Thanks to the assumption $\mathrm{End}_{\mathbf{F}[G]}(\overline{T}) = \mathbf{F}$, we can choose $g_1, \ldots, g_r \in G$ such that the only matrices of $M_n(\mathbf{F})$ commuting with $\overline{\rho}(g_1), \ldots, \overline{\rho}(g_r)$ are the scalars and we fix $M_1, \ldots, M_r \in \mathrm{GL}_n(\mathcal{O})$ lifting $\overline{\rho}(g_1), \ldots, \overline{\rho}(g_r)$. For any $A \in \mathcal{C}$, let $M_n^0(A) := M_n(A)/A$. We have $M_n^0(A) = M_n^0(\mathcal{O}) \otimes_{\mathcal{O}} A$. One easily checks there is a split injection of $\mathcal{O}$-modules:

$$\begin{array}{rcl} i_{\mathcal{O}} : M_n^0(\mathcal{O}) & \hookrightarrow & M_n(\mathcal{O})^r \\ M & \mapsto & (MM_i - M_iM)_{1 \le i \le r} \end{array}$$

and we fix a splitting $\pi_{\mathcal{O}} : M_n(\mathcal{O})^r \twoheadrightarrow M_n^0(\mathcal{O})$ of $i_{\mathcal{O}}$. Tensoring by A, we get injections $i_A : M_n^0(A) \hookrightarrow M_n(A)^r$ and surjections $\pi_A : M_n(A)^r \twoheadrightarrow M_n^0(A)$ such that $\pi_A \circ i_A = \mathrm{Id}_{M_n^0(A)}$.

Definition 6.2.4. *(Faltings) We say $\rho \in \mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(A))$ is well placed if $\pi_A\big(\rho(g_1), \dots, \rho(g_r)\big) = \pi_A(M_1, \dots, M_r)$.*

The following lemma plays the role of Lemma 6.2.2:

Lemma 6.2.5. *(Faltings) For every $\rho \in \mathrm{Hom}_{\overline{\rho}}(G, \mathrm{GL}_n(A))$ there is $M \in \mathrm{GL}_n(A)$ reducing to $1 \in \mathrm{GL}_n(\mathbf{F})$ such that $M\rho M^{-1}$ is well placed. Moreover M is determined uniquely modulo scalars in $1 + \mathfrak{m}_A$.*

Proof. Since any $\mathfrak{m}_A$-Cauchy sequence converges in A, we can build M modulo $\mathfrak{m}_A^m$, $\mathfrak{m}_A^{m+1}$, etc. For $m = 1$, it's clear that $M = \mathrm{Id}$ works. Assume we have $M \in \mathrm{GL}_n(A)$ such that $\pi_A\big((M\rho(g_i)M^{-1})_i\big) \equiv \pi_A\big((M_i)_i\big)$ $(\mathfrak{m}_A^m)$. Changing ρ into $M\rho M^{-1}$, we see we have to find $\delta \in M_n(\mathfrak{m}_A^m)$ such that $\pi_A\big(((1+\delta)\rho(g_i)(1+\delta)^{-1})_i\big) \equiv \pi_A\big((M_i)_i\big)$ $(\mathfrak{m}_A^{m+1})$ i.e.:

$$\pi_A\big((\delta\rho(g_i) - \rho(g_i)\delta)_i\big) \equiv \pi_A\big((M_i)_i\big) - \pi_A\big((\rho(g_i))_i\big) \ (\mathfrak{m}_A^{m+1}).$$

Since $\rho(g_i) \equiv M_i$ $(\mathfrak{m}_A)$, we have:

$$\pi_A\big((\delta\rho(g_i) - \rho(g_i)\delta)_i\big) \equiv \pi_A\big((\delta M_i - M_i\delta)_i\big) \ (\mathfrak{m}_A^{m+1}).$$

But $\pi_A\big((\delta M_i - M_i\delta)_i\big) = \pi_A i_A(\delta) = \delta$ (still denoting δ the image of δ in $M_n^0(A)$). Hence, up to scalars in $1 + \mathfrak{m}_A$, we have only one possibility, namely $\delta = \pi_A\big((M_i)_i\big) - \pi_A\big((\rho(g_i))_i\big) \in M_n(\mathfrak{m}_A^m)$. $\square$

Corollary 6.2.6. *Statement (i) of Theorem 6.1.2 holds.*

Proof. By Lemma 6.2.5, let ρ be the well-placed conjugate of $\tilde{\rho}$ (see the proof of 6.2.1). Denote by R the smallest closed sub-$\mathcal{O}$-algebra of $\tilde{R}$ that contains all entries of matrices $\rho(g)$ for $g \in G$. Then $R \in \mathcal{C}$ and we can clearly realize ρ over R. Let D be the corresponding deformation of $\overline{T}$ over R, we have to show the map $\mathrm{Hom}_{\mathcal{C}}(R, A) \xrightarrow{\sim} \mathrm{Def}(\overline{T}, A)$ is surjective and injective. Surjectivity comes again from Proposition 6.2.1. Let $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(R, A)$ giving rise to isomorphic representations. Then:

$$\rho_1, \rho_2 : G \xrightarrow{\rho} \mathrm{GL}_n(R) \xrightarrow{f_1, f_2} \mathrm{GL}_n(A)$$

are both well placed and conjugate. By the unicity of M in Lemma 6.2.5 (modulo scalars), we must have $\rho_1 = \rho_2$ hence $f_1 = f_2$ by the definition of R. This proves injectivity. $\square$

If $\mathcal{O}$ is replaced by $\mathcal{O}'$ with $\mathcal{O} \subset \mathcal{O}'$ (and $\mathbf{F} \subset \mathbf{F}' := \mathcal{O}'/\mathfrak{m}_{\mathcal{O}'}$), then R is replaced by $R \otimes_{\mathcal{O}} \mathcal{O}'$. Finally, we have:

Proposition 6.2.7. *Assume statement (i) of Theorem 6.1.2 holds, then statement (iii) of Theorem 6.1.2 holds.*

Proof. (sketch) Once R is noetherian, some standard commutative algebra yields the other statements (see [37] for details). One can check that the $\mathbf{F}$-vector space $\mathrm{Hom}_{\mathcal{C}}(R, \mathbf{F}[X]/X^2)$ is finite dimensional if and only if R is noetherian. But by part (i) of Theorem 6.1.2, we have $\mathrm{Hom}_{\mathcal{C}}(R, \mathbf{F}[X]/X^2) \simeq \mathrm{Def}(\overline{T}, \mathbf{F}[X]/X^2)$, hence it's enough to prove the latter is finite dimensional. From $\mathrm{End}_{\mathbf{F}[X]/X^2}(\overline{T} \otimes_{\mathbf{F}} \mathbf{F}[X]/X^2) \simeq \mathrm{End}_{\mathbf{F}}(\overline{T}) \oplus X\mathrm{End}_{\mathbf{F}}(\overline{T})$ one gets $\mathrm{Aut}_{\mathbf{F}[X]/X^2}(\overline{T} \otimes_{\mathbf{F}} \mathbf{F}[X]/X^2) \simeq \mathrm{Aut}_{\mathbf{F}}(\overline{T}) \oplus X\mathrm{End}_{\mathbf{F}}(\overline{T})$. Thus, any element of $\mathrm{Def}(\overline{T}, \mathbf{F}[X]/X^2)$ can be written as a map $G \to \mathrm{Aut}_{\mathbf{F}}(\overline{T}) \oplus X\mathrm{End}_{\mathbf{F}}(\overline{T})$, $g \mapsto (1 + c(g)X)\overline{\rho}(g)$ where $c : G \to \mathrm{End}_{\mathbf{F}}(\overline{T})$ is a continuous 1-cocycle. Moreover, one easily checks using $\mathrm{End}_{\mathbf{F}[G]}(\overline{T}) = \mathbf{F}$ that two such maps define the same deformation if and only if the corresponding 1-cocycles differ by a coboundary. Since $H^1(G, \mathrm{End}_{\mathbf{F}}(\overline{T}))$ is finite dimensional by assumption, this proves our statement. $\square$

6.3 Some local deformation rings

Assume from now on $G = G_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and $n = 2$.

Definition 6.3.1. *A Galois type of degree* 2 *for* I_p *(or just a Galois type) is an isomorphism class of smooth representations* $\tau : I_p \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ *that extend to the Weil group* W_p.

Since I_p is compact, the smoothness implies $\tau(I_p)$ is a finite group. We will determine the structure of all Galois types of degree 2 for $p \neq 2$ in the next lecture. We consider the following data:

- k is a positive integer (hence non zero)
- $\tau : I_p \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is a Galois type
- $\overline{\rho} : G_p \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is a continuous representation such that $\mathrm{End}_{\overline{\mathbf{F}}_p[G]}(\overline{\rho}) = \overline{\mathbf{F}}_p$.

We fix $\mathcal{O} \subset \overline{\mathbf{Z}}_p$ as in §6.1 such that both τ and $\overline{\rho}$ are defined over $\mathcal{O}$ (i.e. $\overline{\rho}$ is defined over $\mathbf{F} := \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$). We denote by $R(\overline{\rho})$ the universal $\mathcal{O}$-algebra of Theorem 6.1.2 associated to $\overline{\rho}$ and by $\rho^{\mathrm{univ}} : G_p \to \mathrm{GL}_2(R(\overline{\rho}))$ the corresponding universal deformation. Recall $R(\overline{\rho})$ is a local noetherian complete $\mathcal{O}$-algebra with residue field $\mathbf{F}$.

Definition 6.3.2. *A prime ideal $\mathfrak{p}$ of $R(\overline{\rho})$ is of type (k,τ) if there exists a $\mathcal{O}$-algebra homomorphism $\iota : R(\overline{\rho}) \to \overline{\mathbf{Z}}_p$ of kernel $\mathfrak{p}$ such that the composite map:*

$$\rho : G_p \stackrel{\rho^{\mathrm{univ}}}{\longrightarrow} \mathrm{GL}_2(R(\overline{\rho})) \stackrel{\iota}{\longrightarrow} \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$$

satysfies the following properties:
(i) $\rho \otimes \mathbf{Q}_p$ is potentially semi-stable with Hodge-Tate weights $(0, k-1)$
(ii) $\mathrm{WD}(\rho \otimes \mathbf{Q}_p)|_{I_p} \simeq \tau$.

By a $\mathcal{O}$-algebra homomorphism $R(\overline{\rho}) \to \overline{\mathbf{Z}}_p$, I really mean a $\mathcal{O}$-algebra homomorphism $R(\overline{\rho}) \to \mathcal{O}' \subset \overline{\mathbf{Z}}_p$ where $\mathcal{O}'$ is a complete discrete valuation ring. Such a homomorphism is automatically continuous since $R(\overline{\rho})$ is noetherian. If $\mathfrak{p}$ is of type (k,τ), then properties (i) and (ii) are satisfied for any deformation ρ coming from an $\mathcal{O}$-algebra homomorphism $R(\overline{\rho}) \to \overline{\mathbf{Z}_p}$ of kernel $\mathfrak{p}$. Thinking in terms of representations coming from modular forms, condition (i) of Definition 6.3.2 amounts to fixing the weight whereas condition (ii) amounts roughly speaking to fixing the level (in fact the type which is more precise than the level). As one usually subdivides modular forms into subspaces indexed by the weight and the level, it is quite natural to subdivide potentially semi-stable deformations (which are usually numerous) into subsets indexed by the Hodge-Tate weights and the type.

The main deformation rings we are interested in are the following:

Definition 6.3.3. *Let $(k,\tau,\overline{\rho})$ be as above.*
(i) If there are no $\mathfrak{p}$ of type $(k,\tau,\overline{\rho})$, $R(k,\tau,\overline{\rho}) := 0$.
(ii) Otherwise, $R(k,\tau,\overline{\rho}) := \frac{R(\overline{\rho})}{\cap_{k,\tau}\mathfrak{p}}$ where the intersection is over all $\mathfrak{p}$ of type (k,τ).

The ring $R(k,\tau,\overline{\rho})$ is quite natural to introduce: it is the biggest quotient of $R(\overline{\rho})$ through which all potentially semi-stable deformations of $\overline{\rho}$ satisfying properties (i) and (ii) of 6.3.2 factor. It is a local complete noetherian flat $\mathcal{O}$-algebra with residue field $\mathbf{F}$.

Remark 6.3.4. The experienced reader will notice that these rings are not exactly the rings considered in [9] or [10] since no condition is required on the determinant of the deformations. But this is harmless and simpler: if $R(k,\tau,\overline{\rho})_{\mathrm{det}}$ denote the rings defined in *loc.cit.*, one can show that $R(k,\tau,\overline{\rho}) \simeq R(k,\tau,\overline{\rho})_{\mathrm{det}}[[D]]$ if $\det(\tau)$ is tame (this condition also appeared in [10]). So this doesn't change the Hilbert-Samuel multiplicity (see Lecture 8) when $\det(\tau)$ is tame and makes this assumption useless.

Lemma 6.3.5. *If $\mathcal{O}$ is replaced by $\mathcal{O}' \subset \overline{\mathbf{Z}_p}$ such that $\mathcal{O} \subset \mathcal{O}'$, then $R(k,\tau,\overline{\rho})$ is replaced by $R(k,\tau,\overline{\rho}) \otimes_{\mathcal{O}} \mathcal{O}'$.*

Proof. Since $R(\overline{\rho})$ is replaced by $R(\overline{\rho}) \otimes_{\mathcal{O}} \mathcal{O}'$, it is enough to prove that if $\mathfrak{p} \subset R(\overline{\rho})$ is of type (k, τ), then $\mathfrak{p} \otimes \mathcal{O}' = \cap \mathfrak{q}$ where the intersection is over all prime ideals of $R(\overline{\rho}) \otimes_{\mathcal{O}} \mathcal{O}'$ containing $\mathfrak{p}$ (these $\mathfrak{q}$ are automatically of type (k, τ)). It follows from the following general result: if $A \to B$ is a morphism of rings and $\mathfrak{p}$ a ideal of A such that $B/\mathfrak{p}B$ has no nilpotent elements, then $\mathfrak{p}B = \cap_{\mathfrak{p} \subset \mathfrak{q}} \mathfrak{q}$ where the $\mathfrak{q}$ are the prime ideals of B containing $\mathfrak{p}$. □

We now state two conjectures on $R(k, \tau, \overline{\rho})$. The first conjecture gives the Krull dimension it should have:

Conjecture 6.3.6. *If non zero then $R(k, \tau, \overline{\rho})$ is equidimensional of Krull dimension* 3.

Recall that a noetherian local ring A is equidimensional if $\dim(A) = \dim(A/\mathfrak{p})$ for any minimal prime ideal $\mathfrak{p}$ of A.

Example 6.3.7. Typically, one find rings such as $\mathcal{O}[[X, D]]$, $\dfrac{\mathcal{O}[[X, Y, D]]}{(XY - p)}$, $\dfrac{\mathcal{O}[[X, Y, D]]}{(X^2 - p(Y + 1))}$, $\mathcal{O}[[X, D]] \times_{\mathcal{O}/\mathfrak{m}_{\mathcal{O}}} \dfrac{\mathcal{O}[[X, Y, D]]}{(XY - p)}$, etc. In the last example, I mean the subring of the product ring of elements (a, b) that map to the same element in $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}$ under $\mathcal{O}[[X, D]] \to \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$, $X, D \mapsto 0$ and $\frac{\mathcal{O}[[X,Y,D]]}{(XY-p)} \to \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$, $X, Y, D \mapsto 0$.

The second conjecture states that $\mathrm{Sp}(R(k, \tau, \overline{\rho}))$ should *exactly* be the parameter space inside $\mathrm{Sp}(R(\overline{\rho}))$ of potentially semi-stable deformations of $\overline{\rho}$ satisfying (i) and (ii) of 6.3.2.

Conjecture 6.3.8. *Let $\mathfrak{p}$ be the kernel of an $\mathcal{O}$-algebra homomorphism $R(\overline{\rho}) \to \overline{\mathbf{Z}}_p$ that factors through $R(k, \tau, \overline{\rho})$, then $\mathfrak{p}$ is of type (k, τ).*

These conjectures are essentially only known when τ is scalar and $1 < k < p$ with k even (see Lecture 9).

After the Krull dimension, the next integer one can may-be associate to a noetherian local ring A is its Hilbert-Samuel multiplicity $\mu(A)$. Recall this is defined as follows: by a standard result of commutative algebra, $\mathrm{length}_A(A/\mathfrak{m}_A^n)$ is a polynomial in n when $n \gg 0$ of degree $\dim(A)$. By definition, $\mu(A)$ is $\dim(A)!$ times the leading coefficient of this polynomial. We define:

$$\mu_{\mathrm{gal}}(k, \tau, \overline{\rho}) := \mu\left(\frac{R(k, \tau, \overline{\rho})}{\mathfrak{m}_{\mathcal{O}} R(k, \tau, \overline{\rho})}\right)$$

which is easily seen using Lemma 6.3.5 to be independant of $\mathcal{O}$ as chosen before. We call $\mu_{\mathrm{gal}}(k, \tau, \overline{\rho})$ the "Galois multiplicity". The main conjecture of this course will predict the value of $\mu_{\mathrm{gal}}(k, \tau, \overline{\rho})$ (at least) for $1 < k < p$

(although I have no counter-example it shouldn't be true for just $1 < k$ and, say, $p > 2$): see Conjecture 8.3.1 of Lecture 8. The interesting feature is that this prediction involves a variant of the local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. That's why in the next lecture, I switch to local Langlands and show how to associate a smooth representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ to a Galois type of degree 2.

7 Lecture 7: Local Langlands and Henniart's theorem on the unicity of $\mathrm{GL}_2(\mathbf{Z}_p)$-types

In this section, we give an overview of the proof of a theorem of Henniart on $\mathrm{GL}_2(\mathbf{Z}_p)$-types in $\mathrm{GL}_2(\mathbf{Q}_p)$-representations (we actually work in a broader context than $\mathbf{Q}_p$). We take advantage to catch a glimpse on local Langlands correspondence for GL_2. All representations in this lecture are over $\mathbf{C}$ and we assume $p \neq 2$.

7.1 The main statement

Recall that a smooth representation (π, V) of a topological group G on a $\mathbf{C}$-vector space is by definition a map $\pi : G \to \mathrm{Aut}_{\mathbf{C}}(V)$ such that the stabilizer of any $x \in V$ is open in G (and necessarily non empty). For instance, if G is compact and π is irreducible, then π factors through a finite quotient of G.

Let F be a complete discrete valuation field of finite residue field of characteristic $p > 2$ and denote by $\mathcal{O}_F$ its ring of integers, $\mathfrak{p}_F$ its maximal ideal, ϖ_F a generator of $\mathfrak{p}_F$, W_F its Weil group and I_F the inertia subgroup. We normalize the reciprocity maps of local class field theory so that geometric Frobeniuses map to uniformizers. We let $G := \mathrm{GL}_2(F)$, $K := \mathrm{GL}_2(\mathcal{O}_F)$, $I \subset K$ the Iwahori subgroup (i.e. upper triangular matrices modulo $\mathfrak{p}_F$), $K(0) := K$ and $K(N) := 1 + M_2(\mathfrak{p}_F^N)$ ($N \geq 1$). The local Langlands correspondence for G is a "natural" bijection between the isomorphism classes of smooth irreducible representations of G and the isomorphism classes of smooth 2-dimensional representations of the Weil-Deligne group of F such that their restriction to W_F is semi-simple. If π is a smooth irreducible representation of G, we denote by $\mathrm{WD}(\pi)$ the corresponding representation of (W_F, N).

Theorem 7.1.1 (Henniart). *Let τ be a Galois type of degree 2 for I_F (same definition as for $I_p = I_{\mathbf{Q}_p}$). There exists (up to isomorphism) a unique smooth irreducible representation $\sigma(\tau)$ of K such that for any infinite* *dimensional smooth irreducible representation π of G:*

$$\pi|_K \text{ contains } \sigma(\tau) \Longleftrightarrow \mathrm{WD}(\pi)|_{I_F} \simeq \tau.$$

Remark 7.1.2. The only finite dimensional smooth representations of G are the 1-dimensional characters.

In other words, this theorem gives a way to select the π giving rise to those $\mathrm{WD}(\pi)$ such that $\mathrm{WD}(\pi)|_{I_F} \simeq \tau$. It can also be seen as defining a Langlands correspondence for smooth 2-dimensional representations of I_F that extend to W_F.

Remark 7.1.3. One can prove that any π contains $\sigma(\tau)$ with multiplicity 0 or 1.

Let us start by describing Galois types of degree 2:

Lemma 7.1.4. *Let τ be a Galois type of degree 2 for I_F. Then:*
(i) Either τ is reducible and sum of two characters of I_F that extend to W_F.
(ii) Either τ is reducible and sum of two characters of I_F that don't extend to W_F, in which case $\tau \simeq \theta \oplus \theta^{\mathrm{conj}}$ where $\theta : I_F \to \mathbf{C}^\times$ doesn't extend to W_F but extends to W_E where $[E:F] = 2$, E unramified.
(iii) Either τ is irreducible, in which case $\tau \simeq \mathrm{Ind}_{I_E}^{I_F}\theta$ where $[E:F] = 2$, E is ramified and $\theta : I_E \to \mathbf{C}^\times$ doesn't extend to I_F but extends to W_E.

Proof. We prove only (iii), the rest being obvious. Let $P_F \subset I_F$ be the wild inertia and recall $\tau(P_F)$ is hypersolvable since it is a p-group. If $\tau|_{P_F}$ is irreducible, then it is the induction of a character (see e.g. [35]) which is impossible since $\dim(\tau) = 2$ and $p \neq 2$. Hence $\tau|_{P_F} \simeq \chi_1 \oplus \chi_2$. If $\chi_1 \simeq \chi_2$, let e_1 be an eigenvector of $\tau(i)$ where $i \in I_F$ generates the tame inertia of the finite group $\tau(I_F)$, then e_1 is preserved both by P_F and i hence by I_F which is impossible by assumption. Thus $\chi_1 \neq \chi_2$. Denote by ρ an extension of τ to W_F and by (e_1, e_2) a basis of eigenvectors for $\rho|_{P_F} = \tau|_{P_F}$. Since $\rho(P_F)$ is normal in $\rho(W_F)$, one has either $\rho(w)e_1 \in \mathbf{C}e_1$ or $\rho(w)e_1 \in \mathbf{C}e_2$ for any $w \in W_F$. It is easy to deduce the result from this. $\square$

Let us now give a few useful definitions.

Definition 7.1.5. *We say two smooth irreducible representations π, π' of G are in the same component if $\mathrm{WD}(\pi)|_{I_F} \simeq \mathrm{WD}(\pi')|_{I_F}$.*

We will describe all the components in the sequel.

Definition 7.1.6. *Let σ be a smooth irreducible representation of K and s a component as in Definition 7.1.5.*

(i) We say σ is typical for s if the only smooth irreducible π such that $\pi|_K$ contains σ are in s.
(ii) We say σ is a type for s if σ is typical for s and if any $\pi \in s$ contains σ (when restricted to K).

The following proposition is straightforward:

Proposition 7.1.7. *Let η be a character of $F^\times$, π a smooth irreducible representation of G and σ a smooth irreducible representation of K. Then σ is typical (resp. a type) for the component of π if and only if $\sigma \otimes (\eta \circ \det)|_K$ is typical (resp. a type) for the component of $\pi \otimes (\eta \circ \det)$.*

Finally, recall that the (exponent of the) conductor of a 2-dimensional representation of the Weil-Deligne group of F is the (exponent of the) Artin conductor of the underlying representation of I_F plus ε where $\varepsilon = 0$ if $N = 0$ and $\varepsilon = 1$ if $N \neq 0$.

7.2 Proof of the theorem for types of case (i)

Principal and special series π are such that $\mathrm{WD}(\pi)|_{I_F}$ belongs to case (i) of Lemma 7.1.4. Special series are the twists of the special or Steinberg representation St_G. Let $B \subset G$ be the subgroup of upper triangular matrices, q the cardinality of $\mathcal{O}_F/\mathfrak{p}_F$ and $|\cdot|$ the character $|x| := q^{-\mathrm{val_F}(x)}$ where $\mathrm{val}_F(\varpi_F) = 1$. Principal series are defined as $\pi(\theta_1, \theta_2) := \mathrm{Ind}_B^G(\theta_1 \mid \cdot \mid^{1/2} \otimes \theta_2 \mid\cdot\mid^{-1/2})$ where θ_i are two characters of $F^\times$ such that $\theta_1\theta_2^{-1} \notin \{|\cdot|, |\cdot|^{-1}\}$ and $\pi(\theta|\cdot|^{1/2}, \theta|\cdot|^{-1/2}) = \pi(\theta|\cdot|^{-1/2}, \theta|\cdot|^{1/2}) := \theta \circ \det$. Here, Ind_B^G means "parabolic induction", i.e. locally constant functions $f : G \to \mathbf{C}$ such that $f(bg) = \mid b_1/b_2 \mid^{1/2} \theta_1(b_1)\theta_2(b_2)f(g)$ (where $b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in B, g \in G$), the group G acting by $(gf)(g') := f(g'g)$ and note that $\pi(\theta_1, \theta_2) \simeq \pi(\theta_2, \theta_1)$ for all θ_i. We have $\mathrm{WD}(\pi(\theta_1, \theta_2)) = \theta_1 \oplus \theta_2$ (as characters of W_F) and $\mathrm{WD}(\mathrm{St}_G \otimes (\theta \circ \det))$ is the unique non trivial extension between $\theta \mid \cdot \mid^{1/2}$ and $\theta \mid\cdot\mid^{-1/2}$. The components are:

$$\{\mathrm{St}_G \otimes (\eta_1\theta \circ \det), \pi(\eta_2, \eta_3) \otimes (\theta \circ \det), \eta_i \text{ unramified}\}$$

$$\{\pi(\varepsilon_0, 1) \otimes (\eta\theta \circ \det), \eta \text{ unramified}\}$$

where θ (resp. ε_0) is a character (resp. a ramified character) of $F^\times$. Note that $s(1) := \{\mathrm{St}_G \otimes (\eta_1 \circ \det), \pi(\eta_2, \eta_3), \eta_i \text{ unramified}\}$ is the component of the trivial representation of I_F.

For $N \in \mathbf{N}$, let $K_0(N)$ be the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \in \mathfrak{p}_F^N$ (so $K_0(0) := K$). For any character ε_0 of $\mathcal{O}_F^\times$ and any integer $N \geq \mathrm{cond}(\varepsilon_0)$

define $u_N(\varepsilon_0)$ to be the complement of $\mathrm{Ind}_{K_0(N-1)}^K \varepsilon_0$ in $\mathrm{Ind}_{K_0(N)}^K \varepsilon_0$ where $\varepsilon_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \varepsilon_0(a)$. The representations $u_N(\varepsilon_0)$ are irreducible representations of K. Recall that the (exponent of the) conductor of a smooth irreducible representation π is the smallest integer $c(\pi)$ such that $\pi^{K_1(c(\pi))} \neq 0$ where $K_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \mid d - 1 \in \mathfrak{p}_F^N \right\}$. It is also the conductor of $\mathrm{WD}(\pi)$. We will use the following result of Casselman:

Theorem 7.2.1 ([12]). *(i) Any $\pi \in s(1)$ of infinite dimension is such that $\pi|_K = \oplus_{N \geq 0} u_N(1)$ except St_G for which $\mathrm{St}_G|_K = \oplus_{N \geq 1} u_N(1)$.*
(ii) Let $\varepsilon_0 : \mathcal{O}_F^\times \to \mathbf{C}^\times$ non trivial and $s(\varepsilon_0) := \{\pi(\theta_1, \theta_2), \theta_1|_{\mathcal{O}_F^\times} = \varepsilon_0, \theta_2|_{\mathcal{O}_F^\times} = 1\}$ (the component of $\varepsilon_0 \oplus 1$), then any $\pi \in s(\varepsilon_0)$ is such that $\pi|_K = \oplus_{N \geq \mathrm{cond}(\varepsilon_0)} u_N(\varepsilon_0)$.
(iii) Let π be any smooth irreducible representation of G of conductor $c(\pi) \geq 1$ and let ε_0 be the restriction to $\mathcal{O}_F^\times$ of its central character, then:

$$\pi|_K = \pi^{K(c(\pi)-1)} \oplus \big(\oplus_{N \geq c(\pi)} u_N(\varepsilon_0) \big).$$

We will also admit that $u_1(1)$ is typical for $s(1)$ and, if $\varepsilon_0 \neq 1$, $u_{\mathrm{cond}(\varepsilon_0)}(\varepsilon_0)$ is typical for $s(\varepsilon_0)$: see [11].

Corollary 7.2.2. *(i) If $\varepsilon_0 = 1$, then $u_1(1)$ is the only smooth irreducible representation of K which is typical for $s(1)$ and which is contained in all the infinite dimensional $\pi \in s(1)$.*
(ii) If $\varepsilon_0 \neq 1$, then $u_{\mathrm{cond}(\varepsilon_0)}(\varepsilon_0)$ is a type for $s(\varepsilon_0)$ and is the only one.

Proof. (i) Taking π supercuspidal such that $\pi^{K(1)} \neq 0$ and with central character trivial on $\mathcal{O}_F^\times$ (this implies $c(\pi) = 2$, such representations exist), by Theorem 7.2.1 (iii), $\pi|_K$ contains $u_N(1)$ for $N \geq 2$. Thus $u_1(1)$ is the only possibility and it is contained in all infinite dimensional $\pi \in s(1)$ by Theorem 7.2.1 (i). (ii) By Theorem 7.2.1 (ii), $u_{\mathrm{cond}(\varepsilon_0)}(\varepsilon_0)$ is a type for $s(\varepsilon_0)$. If $\mathrm{cond}(\varepsilon_0) = 1$, one can again take a supercuspidal π with central character isomorphic to ε_0 (after restriction to $\mathcal{O}_F^\times$) and such that $\pi^{K(1)} \neq 0$. By Theorem 7.2.1 (iii), $\pi|_K$ contains $u_N(\varepsilon_0)$ for $N \geq 2$ which implies $u_1(\varepsilon_0)$ is the only type. If $\mathrm{cond}(\varepsilon_0) \geq 2$, let η be a tamely ramified non trivial character of $\mathcal{O}_F^\times$ (which exists since $q \neq 2$) and $\theta_1, \theta_2 : F^\times \to \mathbf{C}^\times$ such that $\theta_1|_{\mathcal{O}_F^\times} = \eta\varepsilon_0$ and $\theta_2|_{\mathcal{O}_F^\times} = \eta^{-1}$. Then $\pi := \pi(\theta_1, \theta_2) \notin s(\varepsilon_0)$ and $c(\pi) = \mathrm{cond}(\varepsilon_0) + 1$. By Theorem 7.2.1 (iii), $\pi(\theta_1, \theta_2)|_K$ contains $u_N(\varepsilon_0)$ for $N \geq \mathrm{cond}(\varepsilon_0) + 1$. This proves that $u_{\mathrm{cond}(\varepsilon_0)}(\varepsilon_0)$ is the only type. $\square$

Using Proposition 7.1.7, we get that the result holds for any component such that the corresponding Galois type is as in Lemma 7.1.4 (i).

7.3 Proof of the theorem for types of case (ii) and (iii)

Supercuspidals π are such that $\mathrm{WD}(\pi)|_{I_F}$ belongs to cases (ii) or (iii) of Lemma 7.1.4. This implies $\mathrm{WD}(\pi)$ is an irreducible representation of W_F. An application of the classification lemma 7.1.4 gives that any irreducible representations ρ, ρ' of W_F such that $\rho|_{I_F}\simeq \rho'|_{I_F}$ must differ by an unramified character. Thus all supercuspidal π in the same component differ by an unramified character.

The supercuspidal representations of G are all defined as $\text{c-Ind}_J^G\lambda$ where $J \subset G$ is a certain open subgroup which is compact modulo the center $F^\times$, λ a certain smooth irreducible representation of J and $\text{c-Ind}_J^G\lambda$ means "compact induction", i.e. functions $f : G \to V_\lambda$ (space of λ) with compact support modulo $F^\times$ such that $f(jg) = \lambda(j)f(g)$ $(j \in J, g \in G)$, the group G acting by $(gf)(g') := f(g'g)$. The pair (J, λ) is defined up to conjugation in G, hence we can assume the maximal compact subgroup J^0 of J is contained in K, i.e $J^0 = J \cap K$. The existence (but not the unicity!) of types in that case is also well known:

Proposition 7.3.1. *The representation $\sigma := Ind_{J\cap K}^K \lambda|_{J\cap K}$ is irreducible and is a type for the component of $\pi :=$ c-$Ind_J^G\lambda$.*

Proof. (rough sketch) First, in all cases, $\lambda|_{J\cap K}$ turns out to be still irreducible (one can check this case by case). If $g \in G$, denote by $\lambda^g(\cdot) := \lambda(g^{-1}\cdot g)$ defined on gJg^{-1}. Since $\text{c-Ind}_J^G\lambda$ is irreducible, the Mackey criterion (see e.g. [35]) tells us that any $g \in G$ such that $\lambda^g|_{gJg^{-1}\cap J}$ and $\lambda|_{gJg^{-1}\cap J}$ have commun subrepresentations (we say g intertwins λ) is in J. By a case by case check (see below for more about the representations λ), one can see the same result holds for $\lambda^0 := \lambda|_{J\cap K}$, i.e. the set of $g \in G$ that intertwins λ^0 is still J. Hence, the set of $g \in K$ that intertwins λ^0 is $J \cap K$ which implies σ is irreducible (again by the Mackey criterion). Let $\pi' = \text{c-Ind}_{J'}^G\lambda'$ be another supercuspidal representation of G in a different component. By Frobenius reciprocity, π' contains σ iff π' is a quotient of $\text{c-Ind}_{J\cap K}^G\lambda|_{J\cap K}$. But it turns out any irreducible quotient of $\text{c-Ind}_{J\cap K}^G\lambda|_{J\cap K}$ is a twist of π by an unramified character, and hence cannot be isomorphic to π'. Thus σ is a type for the component of π. □

We now prove it's the only type. For this, we need more about the pairs (J, λ).

First, let $\tau \simeq \mathrm{Ind}_{I_E}^{I_F}\theta$ be a type as in Lemma 7.1.4 (iii) with $\theta : W_E \to \mathbf{C}^\times$ and $s(\tau) := \{\pi \mid \mathrm{WD}(\pi)|_{I_F}\simeq \tau\}$ the corresponding component (which only depends on $\theta|_{I_E}$). We denote by θ^{conj} the conjugate of θ under the non trivial element of $\mathrm{Gal}(E/F)$. We say θ is minimal if its Artin conductor is the smallest one among the conductors of all its twists by characters of W_F (restricted to W_E). By an application of Hilbert 90, this turns out to be equivalent to $\mathrm{cond}(\theta^{\mathrm{conj}}\theta^{-1}) = \mathrm{cond}(\theta)$ which is, in that case, a positive even integer (see e.g. [25]). Assume θ is minimal and denote by c its (even) conductor. Choose a F-basis of E so that $E^\times \hookrightarrow G$ and denote by ι the matrix corresponding to the non trivial element of $\mathrm{Gal}(E/F)$ acting on E. Let $J := E^\times(1+\mathfrak{p}_E^{c/2}\iota)$ which is a compact open subgroup of G modulo $F^\times$ and define a character $\lambda : J \to \mathbf{C}^\times$ by $\lambda|_{E^\times} := \theta$ and $\lambda|_{1+\mathfrak{p}_E^{c/2}\iota} := 1$. It is a character because if $x, y \in 1+\mathfrak{p}_E^{c/2}\iota$, we have $xy \in (1+\mathfrak{p}_E^{c})(1+\mathfrak{p}_E^{c/2}\iota)$ and $\lambda(x)\lambda(y) = 1 = \lambda(xy)$ ($\theta|_{1+\mathfrak{p}_E^c} = 1$ by definition of c).

Proposition 7.3.2. *Let $\pi :=$ c-$\mathrm{Ind}_J^G\lambda \in s(\tau)$. Any constituant of $\pi|_K$ other than σ is also a constituant of $\pi'|_K$ for some $\pi' \notin s(\tau)$.*

Proof. Assume first $c \geq 4$ and let $\mu : E^\times \to \mathbf{C}^\times$ be a character of conductor 2 (hence ramified) which is trivial on $F^\times$. Then $\theta' := \theta\mu$ is still minimal of conductor c, and one can define λ', π' and σ' as previously by replacing everywhere θ by θ'. As $\mu|_{I_E} \not\cong \theta^{\mathrm{conj}}\theta^{-1}|_{I_E}$ (compare the conductors), $\theta'|_{I_E} \not\cong \theta^{\mathrm{conj}}|_{I_E}$ and $\mathrm{Ind}_{I_E}^{I_F}\theta' \not\cong \mathrm{Ind}_{I_E}^{I_F}\theta$. Hence π and π' are not in the same component. We are going to compare $\pi|_K$ and $\pi'|_K$. Let $K' := E^\times I$ and write $G = \coprod_g KgK'$ where $g \in \left\{\begin{pmatrix}\varpi_F^n & 0\\ 0 & 1\end{pmatrix},\ n \in \mathbf{Z}_{\geq 1}\right\}$ (the class KK' corresponding to $n = 1$ since $\begin{pmatrix}\varpi_F & 0\\ 0 & 1\end{pmatrix} \in E^\times K$). Let $\nu := \mathrm{Ind}_J^{K'}\lambda$ and $\nu' := \mathrm{Ind}_J^{K'}\lambda'$ (one checks that $J \subset K'$), then $\pi = \text{c-}\mathrm{Ind}_{K'}^G\nu$ and by the Mackey decomposition:

$$\pi|_K = \oplus_g \mathrm{Ind}_{K\cap gK'g^{-1}}^K(\nu^g|_{K\cap gK'g^{-1}})$$

(resp. with $\pi'|_K$) where the sum is for g as above and $\nu^g(\cdot) := \nu(g^{-1}\cdot g)$. For $g = \begin{pmatrix}\varpi_F & 0\\ 0 & 1\end{pmatrix}$, one gets σ (resp. σ'). For $g = \begin{pmatrix}\varpi_F^n & 0\\ 0 & 1\end{pmatrix}$ and $n \geq 2$, I claim that $\mathrm{Ind}_{K\cap gK'g^{-1}}^K(\nu^g) \simeq \mathrm{Ind}_{K\cap gK'g^{-1}}^K(\nu'^g)$. It is enough to prove $\nu^g|_{K\cap gK'g^{-1}} \simeq \nu'^g|_{K\cap gK'g^{-1}}$ or, since $\nu^g|_{K\cap gK'g^{-1}} \simeq (\nu|_{g^{-1}Kg\cap K'})^g$:

$$(\mathrm{Ind}_J^{K'}\lambda)|_{g^{-1}Kg\cap K'} \simeq (\mathrm{Ind}_J^{K'}\lambda')|_{g^{-1}Kg\cap K'}\ .$$

Note that $g^{-1}Kg\cap K' = K_0(n)$. Writing $K' = \coprod_h (g^{-1}Kg\cap K')hJ$ for some $h \in I$, we are led by the Mackey formula to compare $\oplus_h \lambda^h|_{g^{-1}Kg\cap K'\cap hJh^{-1}}$ and $\oplus_h \lambda'^h|_{g^{-1}Kg\cap K'\cap hJh^{-1}}$. Let $I(c/2) := 1 + P^{c/2} \subset J$ where P is the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_F)$ with $a, c, d \in \mathfrak{p}_F$ and note that $J = E^\times I(c/2)$ and that $I(c/2)$ is a normal subgroup of I. Hence:

$$g^{-1}Kg \cap K' \cap hJh^{-1} = K_0(n) \cap (h(1+\mathfrak{p}_E)h^{-1}I(c/2)\mathcal{O}_F^\times).$$

Let $1+x \in h(1+\mathfrak{p}_E)h^{-1} \setminus h(1+\mathfrak{p}_E^2)h^{-1}$ and $1+y \in I(c/2)$. If $z := (1+x)(1+y) \in g^{-1}Kg \cap K'$, then $z = \begin{pmatrix} 1+\varpi_F\alpha & \beta \\ \varpi_F^2 & 1+\varpi_F\delta \end{pmatrix}$, hence $\mathrm{val}_F(\det(z-1)) \geq 2$ which is impossible. Thus $g^{-1}Kg \cap K' \cap hJh^{-1} \subset h(1+\mathfrak{p}_E^2)h^{-1}I(c/2)\mathcal{O}_F^\times$. But on this group, one easily checks that λ^h and ${\lambda'}^h$ coincide (using the assumption on μ) which finishes the proof for $c \geq 4$. For $c = 2$, note that by a result of Casselman (see the proof of th.3 of [12]) the $K(2)$-invariant vectors in π are an irreducible representation of K. Since $K(2) \subset J \cap K$ and $\lambda|_{K(2)} = 1$, $\mathrm{Hom}_{K(2)}(1, \sigma)$ is non zero by Frobenius reciprocity. As σ is irreducible, one has exactly $\sigma = \pi^{K(2)}$. By Theorem 7.2.1, the complement of σ in $\pi|_K$ is $\oplus_{N\geq 3} u_N(\varepsilon_0)$ where $\varepsilon_0 := \lambda|_{\mathcal{O}_F^\times}$ is the restriction of the central character. We have seen in §7.2 that none of these constituants can be typical for $s(\tau)$. $\square$

This proves that σ is the unique possible type of $s(\tau)$. Using Proposition 7.1.7, it is clear the same result holds without assuming θ minimal.

Now, let $\tau \simeq \theta|_{I_E} \oplus \theta^{\mathrm{conj}}|_{I_E}$ be a type as in Lemma 7.1.4 (ii) with $\theta : W_E \to \mathbf{C}^\times$ and $s(\tau) := \{\pi \mid \mathrm{WD}(\pi)|_{I_F} \simeq \tau\}$ the corresponding component (which only depends on $\theta|_{I_E}$). Assume θ is minimal and denote by c its (positive) conductor. Choose a F-basis of E so that $E^\times \hookrightarrow G$ and denote by ι the matrix corresponding to the non trivial element of $\mathrm{Gal}(E/F)$ acting on E. We will only consider here the case when c is *even*, the odd case being slightly more involved, although the argument below is essentially the same: see the appendix of [10] for the general case. Let $J := E^\times(1+\mathfrak{p}_E^{c/2}\iota)$ which is a compact open subgroup of G modulo $F^\times$ and define a character $\lambda : J \to \mathbf{C}^\times$ by $\lambda|_{E^\times} := \theta$ and $\lambda|_{1+\mathfrak{p}_E^{c/2}\iota} = 1$.

Proposition 7.3.3. *Let $\pi := c\text{-}Ind_J^G\lambda \in s(\tau)$. Any constituant of $\pi|_K$ other than σ is also a constituant of $\pi'|_K$ for some $\pi' \notin s(\tau)$.*

Proof. let $\mu : E^\times \to \mathbf{C}^\times$ be a character of conductor 1 (hence tamely ramified) which is trivial on $F^\times$ (it is easy to see that such characters exist). Then $\theta' := \theta\mu$ is still minimal of conductor c, and one can define J', λ', π' and σ' as previously by replacing everywhere θ by θ'. As previously, π and π' are not in the same component and we are going to compare $\pi|_K$ and $\pi'|_K$. Writing $G = \coprod_g KgE^\times K = \coprod_g KgF^\times K$ where $g \in \left\{ \begin{pmatrix} \varpi_F^n & 0 \\ 0 & 1 \end{pmatrix},\ n \in \mathbf{Z}_{\geq 0} \right\}$ and $\pi = \text{c-Ind}_{F^\times K}^G(\mathrm{Ind}_J^{F^\times K}\lambda)$, we get as previously $\pi|_K = \oplus_g \mathrm{Ind}_{K\cap gKg^{-1}}^K(\sigma^g|_{K\cap gKg^{-1}})$ and the analogue formula for $\pi'|_K$. For $g = \begin{pmatrix} \varpi_F^n & 0 \\ 0 & 1 \end{pmatrix}$ with $n \geq 1$, I claim that $\mathrm{Ind}_{K\cap gKg^{-1}}^K(\sigma^g) \simeq \mathrm{Ind}_{K\cap gKg^{-1}}^K(\sigma'^g)$. As previously, it is enough to compare the restrictions of λ^h and λ'^h to $(g^{-1}Kg\cap K)\cap h(J\cap K)h^{-1} = (g^{-1}Kg\cap K)\cap(h\mathcal{O}_E^\times h^{-1}K(c/2))$ for $h \in K$ such that $K = \coprod_h (g^{-1}Kg \cap K)h(J\cap K)$. But any matrix in $g^{-1}Kg\cap K$ is upper triangular modulo $\mathfrak{p}_F$, hence $(g^{-1}Kg\cap K)\cap h(J\cap K)h^{-1} \subset h(1+\mathfrak{p}_E)h^{-1}K(c/2)\mathcal{O}_F^\times$. On this group, one easily checks using the assumption on μ that λ^h and λ'^h coincide. $\square$

This proves that σ is the unique possible type of $s(\tau)$. Using Proposition 7.1.7, the same result holds without assuming θ minimal.

One can actually prove the assumption $p \neq 2$ is not necessary in the case of supercuspidal representations (i.e. there is still unicity of types for the corresponding components even if $p = 2$). Following Henniart, one can conjecture that the same result holds for supercuspidal representations on GL_n for *any* $n \in \mathbf{N}$ (and no assumption on p). That is, to each n-dimensional smooth representation of I_F that extends to an irreducible representation of W_F, one should be able to associate a well defined smooth irreducible representation of $\mathrm{GL}_n(\mathcal{O}_F)$ by the same trick as in Theorem 7.1.1...

8 Lecture 8: The Deligne-Fontaine-Serre theorem and statement of the main conjecture

In Lecture 6, I have defined a Galois multiplicity and said it should be predicted by an "automorphic" recipe. In Lecture 7, I have given a fine idea of the proof of Henniart's theorem, namely that there existed a unique smooth irreducible representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathbf{Z}_p)$ that could "select" those smooth irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ giving rise to the Weil-Deligne representations with restriction to inertia $\simeq \tau$. Now, I will use $\sigma(\tau)$ to define an "automorphic multiplicity" and state the main conjecture. But first, I need a key result due to Deligne, Fontaine and Serre, and also some heuristic considerations.

8.1 The Deligne-Fontaine-Serre theorem

Denote by ω the cyclotomic character modulo p and by ω_2 Serre's fundamental character of order 2. Recall that $\omega(g) = \frac{g\left((-p)^{\frac{1}{p-1}}\right)}{(-p)^{\frac{1}{p-1}}}$ if $g \in G_p$, $\omega_2(g) = \frac{g\left((-p)^{\frac{1}{p^2-1}}\right)}{(-p)^{\frac{1}{p^2-1}}}$ if $g \in I_p$ and that we have assumed $p \neq 2$.

Lemma 8.1.1. $\dim_{\mathbf{F}_p} \mathrm{H}^1(G_p, \omega) = 2$.

Proof. (well known) The exact sequence of G_p-modules: $1 \to \mu_p(\overline{\mathbf{Q}_p}) \to \overline{\mathbf{Q}_p}^{\times} \xrightarrow{\uparrow p} \overline{\mathbf{Q}_p}^{\times} \to 1$ yields an exact sequence (since $\mathrm{H}^1(G_p, \overline{\mathbf{Q}_p}^{\times}) = 0$ by Hilbert 90): $1 \to \mathbf{Q}_p^{\times} \xrightarrow{\uparrow p} \mathbf{Q}_p^{\times} \to \mathrm{H}^1(G_p, \mathbf{F}_p(1)) \to 0$. Hence $\mathrm{H}^1(G_p, \omega) \simeq \mathbf{Q}_p^{\times}/(\mathbf{Q}_p^{\times})^p$ which is an $\mathbf{F}_p$-vector space of dimension 2 when $p > 2$ (generated for instance by p and $1-p$). $\square$

Note that the map $\mathbf{Q}_p^{\times}/(\mathbf{Q}_p^{\times})^p \to \mathrm{H}^1(G_p, \omega) \simeq \mathrm{Ext}^1_{\mathbf{F}_p[G_p]}(1, \omega)$ is given explicitely by the 1-cocycle $u \mapsto \left(g \mapsto \frac{g(\sqrt[p]{u})}{\sqrt[p]{u}} \in \mu_p(\overline{\mathbf{Q}_p})\right)$ which, up to coboundary, is independant of any choice. Hence, the Galois action on the corresponding extension factors through $\mathrm{Gal}\left(\mathbf{Q}_p[\sqrt[p]{1}, \sqrt[p]{u}]/\mathbf{Q}_p\right)$. There is a distinghished line in $\mathrm{H}^1(G_p, \omega)$ given by the image of $\mathbf{Z}_p^{\times}/(\mathbf{Z}_p^{\times})^p$ and called the "peu ramifiée" line ([34]). Since $\mathrm{H}^1(G_p, \omega \otimes \overline{\mathbf{F}_p}) \simeq \mathrm{H}^1(G_p, \omega) \otimes \overline{\mathbf{F}_p}$, we can (and will) consider extensions over $\overline{\mathbf{F}_p}$. We call an extension in $\mathrm{H}^1(G_p, \omega) \otimes \overline{\mathbf{F}_p}$ "peu ramifiée" if it is 0 or if it is coming from the peu ramifiée line (thus, there is just one up to scalar in $\overline{\mathbf{F}_p}$ and it factors through $\mathrm{Gal}\left(\mathbf{Q}_p[\sqrt[p]{1}, \sqrt[p]{1-p}]/\mathbf{Q}_p\right)$). Following Serre, we call all the other extensions in $\mathrm{H}^1(G_p, \omega) \otimes \overline{\mathbf{F}_p}$ "très ramifées".

Denote by $\underline{NMF}^1$ the category of fivefolds $(M, \mathrm{Fil}^1 M, \varphi, \varphi_1, N)$ where M is a finite dimensional $\overline{\mathbf{F}_p}$-vector space, $\mathrm{Fil}^1 M$ a $\overline{\mathbf{F}_p}$-subvector space and φ, φ_1, N three linear maps $\varphi : M \to M$, $\varphi_1 : \mathrm{Fil}^1 M \to M$, $N : M \to M$ such that $\varphi(M) + \varphi_1(\mathrm{Fil}^1 M) = M$, $N\varphi = 0$ and $N\varphi_1 = \varphi N$. One has an obvious functor "extension of scalars" $M \mapsto S \otimes M$ from $\underline{NMF}^1$ to $\underline{\mathcal{M}}^1$ (with $F = \mathbf{Q}_p$, $\pi = p$ and S as in §5.1) and one can check it induces an *equivalence of categories* between $\underline{NMF}^1$ and the subcategory of $\underline{\mathcal{M}}^1$ of objects killed by p (this would be false in higher "weights"). By Lecture 4 there is an exact fully faithful contravariant functor T^*_{st} from $\underline{NMF}^1$ to continuous representations of G_p over $\overline{\mathbf{F}_p}$. Recall that the object $\overline{\mathbf{F}_p}(1) := (\overline{\mathbf{F}_p} e_1, \overline{\mathbf{F}_p} e_1, \varphi(e_1) = 0, \varphi_1(e_1) = e_1, 0)$ is sent to ω (this is derived from Example 3.2.3 and from Theorem 4.3.3 (iii)). This can be directly seen by noticing that the corresponding Galois character factors through

$\mathrm{Gal}\big(\mathbf{Q}_p[x]/\mathbf{Q}_p\big)$ where $x \neq 0$ and $x^p = -px$: this is just the alternative definition of ω. Let $\overline{\mathbf{F}_p}(0) := (\overline{\mathbf{F}_p}e_0, 0, \varphi(e_0) = e_0, 0, 0)$ which is clearly sent to the trivial character.

Lemma 8.1.2. $\dim_{\overline{\mathbf{F}_p}} \mathrm{Ext}^1_{\underline{NMF}^1}(\overline{\mathbf{F}_p}(1), \overline{\mathbf{F}_p}(0)) = 2.$

Proof. Any extension $0 \to \overline{\mathbf{F}_p}(0) \to M \to \overline{\mathbf{F}_p}(1) \to 0$ can be written $M = \overline{\mathbf{F}_p}e_0 + \overline{\mathbf{F}_p}e_1$ with $\mathrm{Fil}^1 M = \overline{\mathbf{F}_p}e_1$, $\varphi(e_0) = e_0$, $\varphi_1(e_1) = e_1 + \lambda e_0$, $N(e_0) = 0$, $N(e_1) = \mu e_1$. Hence we see they are parametrized by $(\lambda, \mu) \in \overline{\mathbf{F}_p}^2$. □

Lemma 8.1.1 and the full faithfulness of T^*_{st} imply:

$$\mathrm{Ext}^1_{\underline{NMF}^1}(\overline{\mathbf{F}_p}(1), \overline{\mathbf{F}_p}(0)) \xrightarrow{\sim} \mathrm{Ext}^1_{\overline{\mathbf{F}_p}[G_p]}(1, \omega).$$

Lemma 8.1.3. *The extensions with $N = 0$ correspond to the peu ramifiées ones. The extensions with $N \neq 0$ correspond to the très ramifiées ones.*

Proof. (sketch) By the full faithfulness, it is enough to prove the first statement. We can assume the extension is non trivial. Up to isomorphism, we then have $M = \overline{\mathbf{F}_p}e_0 + \overline{\mathbf{F}_p}e_1$, $\mathrm{Fil}^1 M = \overline{\mathbf{F}_p}e_1$, $\varphi(e_0) = e_0$, $\varphi_1(e_1) = e_1 + e_0$ (and $N = 0$). A careful analysis of $T^*_{st}(M)$ shows that the Galois action on $T^*_{st}(M)$ factors through $\mathrm{Gal}(F/\mathbf{Q}_p)$ where F is the compositum of $\mathbf{Q}_p[x_0, x_1]$ for all $x_0, x_1 \in \overline{\mathbf{Q}_p}$ such that $x_0^p = x_0$ and $x_1^p = (-p)(x_1 + x_0)$. If $x_0 = 0$, we have seen that $\mathbf{Q}_p[x_1] = \mathbf{Q}_p[\sqrt[p]{1}]$. If $x_0 \neq 0$, the equations imply $x_0 \in [\mathbf{F}_p^\times] \subset \mathbf{Z}_p^\times$ and, replacing x_1 by $\frac{x_1}{x_0}$, $(x_1 + 1)^p = (1 - p)w$ where $w \in 1 + px_1^2\mathbf{Z}_p[x_1] = (1 + x_1\mathbf{Z}_p[x_1])^p$ ($\mathrm{val}(x_1^2) = \frac{2}{p} > \frac{1}{p-1}$ since $p > 2$). Hence $\mathbf{Q}_p[x_1]$ contains $\mathbf{Q}_p[\sqrt[p]{1-p}]$ if $x_0 \neq 0$. Since these two extensions have degree p, they are equal and $F = \mathbf{Q}_p[\sqrt[p]{1}, \sqrt[p]{1-p}]$: we are in the peu ramifié case. □

We keep the notations of Lecture 3 §3.4 and we denote by $\overline{\rho}_{f,p}$ the semisimplification modulo p of $\rho_{f,p}$ (as a representation of $G_{\mathbf{Q}}$).

Theorem 8.1.4. *(Deligne) Let f be a cuspidal newform of weight $k \geq 2$ for $\Gamma_1(N)$ with $(p, N) = 1$. Let a_p be the eigenvalue of the Hecke operator T_p and assume $\mathrm{val}(a_p) = 0$. Then $\overline{\rho}_{f,p}|_{I_p} \simeq \begin{pmatrix} \omega^{k-1} & * \\ 0 & 1 \end{pmatrix}$ with $*$ peu ramifié if $k = 2$.*

Proof. Let χ be the character of f. By Lecture 3, we know that the filtered module giving rise to $\rho_{f,p}|_{G_p}$ is $D = \overline{\mathbf{Q}_p}e_1 \oplus \overline{\mathbf{Q}_p}e_0$, $\mathrm{Fil}^{k-1} D = \overline{\mathbf{Q}_p}e_1$, $\varphi(e_1) = p^{k-1}(\mu_1 e_1 + \delta e_0)$, $\varphi(e_0) = \mu_0 e_0$ and $N = 0$ where $\delta \in \{0, 1\}$ and $\mu_i \in \overline{\mathbf{Z}_p}^\times$ is such that $p^{k-1}\mu_1 + \mu_0 = a_p$ and $\mu_1\mu_0 = \chi(p)$. Up to unramified characters, it is an extension $0 \to \overline{\mathbf{Q}_p}(0) \to D \to \overline{\mathbf{Q}_p}(k-1) \to 0$ (with obvious notations), thus we get an extension as in the lemma. The only thing that remains is the peu ramifiée condition for $k = 2$ (in the non-split case $\delta = 1$). But by Proposition 5.1.1, $Se_1 \oplus Se_0$ is a strongly divisible lattice in $S \otimes D$, the reduction of which is isomorphic to $S \otimes M$ with M as in the proof of Lemma 8.1.3 (up to unramified characters). Hence the result follows from Lemma 8.1.3. □

Theorem 8.1.5. *(Fontaine, Serre) Let f be a cuspidal newform of weight $k \geq 2$ for $\Gamma_1(N)$ with $(p, N) = 1$. Let a_p be the eigenvalue of the Hecke operator T_p and assume $\mathrm{val}(a_p) \neq 0$ and $k \leq p+1$. Then* $\overline{\rho}_{f,p}|_{I_p} \simeq \begin{pmatrix} \omega_2^{k-1} & 0 \\ 0 & \omega_2^{p(k-1)} \end{pmatrix}$.

Proof. (sketch) We give the proof for $k < p+1$ (see [16] for $k = p+1$). By Lecture 3, we know that the filtered module giving rise to $\rho_{f,p}|_{G_p}$ is $D = \overline{\mathbf{Q}_p} e_1 \oplus \overline{\mathbf{Q}_p} e_0$ with $\mathrm{Fil}^{k-1} D = \overline{\mathbf{Q}_p} e_1$, $\varphi(e_1) = p^{k-1}\chi(p)e_0$, $\varphi(e_0) = -e_1 + a_p e_0$ and $N = 0$, and by Lecture 5 we know that $Se_1 \oplus Se_0$ is a strongly divisible lattice in $S \otimes D$, the reduction of which is $S \otimes M$ with $M = \overline{\mathbf{F}_p} e_1 \oplus \overline{\mathbf{F}_p} e_0$, $\mathrm{Fil}^{k-1} M = \overline{\mathbf{F}_p} e_1$, $\varphi_{k-1}(e_1) = \overline{\chi(p)} e_0$, $\varphi_{k-1}(u^{k-1}e_0) = (-1)^k e_1$ and $N = 0$. Since it is an irreducible object in $\underline{\mathcal{M}}^{k-1}$ (this is readily checked), so is $\overline{\rho}_{f,p}|_{G_p}$ by the full faithfulness of T^*_{st}. Thus $\overline{\rho}_{f,p}|_{I_p} \simeq \begin{pmatrix} \omega_2^{a+pb} & 0 \\ 0 & \omega_2^{pa+b} \end{pmatrix}$ with $a, b \in \{0, ..., p-1\}$. Now, we use a theorem of Fontaine and Laffaille ([23],§5) which says that, in our situation, the digits $\{a, b\}$ are just the Hodge-Tate weights, i.e. $\{a, b\} = \{0, k-1\}$. □

8.2 Heuristic considerations

Fix a Galois type τ (of degree 2) and let $\sigma(\tau)$ be the associated representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ (see Lecture 7). Let $N \in \mathbf{N}$ such that $(N, p) = 1$, $\Gamma := \Gamma_1(N)$, $Y_\Gamma := Y_1(N)$ and assume Γ doesn't contain any element of finite order (e.g. $N > 4$). In that case, $\pi_1(Y_\Gamma) = \Gamma$. Let Γ act on $\sigma(\tau)$ (i.e. on its underlying vector space) via $\Gamma \hookrightarrow \mathrm{GL}_2(\mathbf{Z}_p)$ and consider $H^1(\Gamma, \sigma(\tau))$ = usual group cohomology. Then, if $\mathcal{F}_\tau$ is the corresponding local system on Y_Γ, one has $H^1(\Gamma, \sigma(\tau)) = H^1(Y_\Gamma, \mathcal{F}_\tau)$. Denote by $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))$ the image of $H^1_c(Y_\Gamma, \mathcal{F}_\tau)$ in $H^1(\Gamma, \sigma(\tau))$. Let $\mathbf{T} := \overline{\mathbf{Z}}_p[T_\ell, <\ell>]_{\ell \nmid Np}$ (polynomial algebra). Then $\mathbf{T}$ acts on $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))$ by the usual action of the Hecke and Diamond operators (see e.g. [36], we won't need the explicit definition of this action). It is known that the systems of eigenvalues of $\mathbf{T}$ acting on $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))$ are the same as the systems of eigenvalues of eigenforms f

in $\mathrm{S}_2(\Gamma_1(Np^{\mathrm{cond}(\tau)+\delta}))$ such that $\pi_{f,p}|_{\mathrm{GL}_2(\mathbf{Z}_p)}$ contains $\sigma(\tau)$. Here, $\delta = 0$ (resp. $\delta = 1$) if τ is not scalar (resp. is scalar) and if τ is not scalar, such an f is automatically new at p (f can be old at p if τ is scalar since we do not know whether the monodromy operator on $\mathrm{WD}(\pi_{f,p})$ is 0 or not).

Thus, to such an eigensystem $(a_\ell, b_\ell)_{\ell \nmid Np}$, one can associate (via eigenforms) a continuous Galois representation:

$$\rho_p : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Z}_p})$$

unramified outside Np, such that the trace (resp. determinant) of an arithmetic Frobenius at $\ell \nmid Np$ is a_ℓ (resp. ℓb_ℓ) and such that $\rho_p|_{G_p}$ is potentially semi-stable with Hodge-Tate weights $(0,1)$ and with $\mathrm{WD}(\rho_p|_{G_p})|_{I_p}$ isomorphic to τ (see Lecture 3). Let $\overline{\rho_p}$ be the reduction of ρ_p (over $\overline{\mathbf{F}}_p$).

Question: What are the possible $\overline{\rho_p}\,|_{I_p}$ associated to the systems of eigenvalues of $\mathbf{T}$ on $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))$?

(This question was studied for instance in [31]). Fix such a $\overline{\rho_p}$ and assume it is irreducible. Let $\mathfrak{m} := \mathrm{Ker}(\mathbf{T} \to \overline{\mathbf{F}}_p)$ (with $T_\ell \mapsto \overline{a_\ell}$ and $< \ell > \mapsto \overline{b_\ell}$). By Čebotarev, the knowledge of $\overline{\rho_p}$ is equivalent to that of $\mathfrak{m}$. Let

$$H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))_{\mathfrak{m}} := H^1_{\mathrm{par}}(\Gamma, \sigma(\tau)) \otimes_{\mathbf{T}} \mathbf{T}_{\mathfrak{m}}$$

and note that the systems of eigenvalues of $\mathbf{T}$ acting on $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))_{\mathfrak{m}}$ are those lifting $(\overline{a_\ell}, \overline{b_\ell})_{\ell \nmid Np}$. By assumption $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))_{\mathfrak{m}} \neq 0$ hence $H^1(\Gamma, \sigma(\tau))_{\mathfrak{m}} \neq 0$ (this is actually equivalent since $\overline{\rho_p}$ is irreducible).

Denote by $\overline{\sigma(\tau)}^{ss}$ the semi-simplification modulo p of $\sigma(\tau)$ (that is, take a $\overline{\mathbf{Z}_p}$-lattice stable by the compact group $\mathrm{GL}_2(\mathbf{Z}_p)$, reduce it modulo the maximal ideal of $\overline{\mathbf{Z}_p}$ and semi-simplify: by Brauer-Nesbitt the result doesn't depend on the choice of the lattice).

Lemma 8.2.1. *The irreducible representations of* $\mathrm{GL}_2(\mathbf{Z}_p)$ *over* $\overline{\mathbf{F}_p}$ *are given by:*

$$\sigma_{n,m} := (\mathrm{Symm}^n \overline{\mathbf{F}}_p^2) \otimes \det{}^m$$

with $n \in \{0, ..., p-1\}$ *and* $m \in \{0, ..., p-2\}$.

Proof. One easily checks these $p(p-1)$ representations are irreducible and non equivalent. The irreducible representations of $\mathrm{GL}_2(\mathbf{Z}_p)$ over $\overline{\mathbf{F}_p}$ are those of $\mathrm{GL}_2(\mathbf{F}_p)$ since the pro-p-group $\mathrm{Ker}(\mathrm{GL}_2(\mathbf{Z}_p) \to \mathrm{GL}_2(\mathbf{F}_p))$ acts trivially. The number of irreducible representations of $\mathrm{GL}_2(\mathbf{F}_p)$ in characteristic p is equal to the number of conjugacy classes of order prime to p, i.e. $p(p-1)$. $\square$

Thus one has:

$$\overline{\sigma(\tau)}^{ss} = \bigoplus_{n,m} \sigma_{n,m}^{a_{n,m}}$$

for some $a_{n,m} \in \mathbf{Z}_{\geq 0}$ (with $\sigma_{n,m}^{a_{n,m}} = 0$ if $a_{n,m} = 0$).

Lemma 8.2.2. *$H^1(\Gamma, \sigma(\tau))_{\mathfrak{m}} \neq 0$ iff $\bigoplus_{n,m|a_{n,m}\neq 0} H^1(\Gamma, \sigma_{n,m})_{\mathfrak{m}} \neq 0$.*

Proof. (sketch) Let $0 \to \sigma' \to \sigma \to \sigma'' \to 0$ be an exact sequence of $\overline{\mathbf{Z}_p}[\Gamma]$-modules, then one has an exact sequence: $0 \to H^1(\Gamma, \sigma')_{\mathfrak{m}} \to H^1(\Gamma, \sigma)_{\mathfrak{m}} \to H^1(\Gamma, \sigma'')_{\mathfrak{m}} \to 0$ (see e.g. Lemma 6.1.2 of [15]). Applying this to $0 \to \sigma(\tau) \xrightarrow{\pi} \sigma(\tau) \to \sigma(\tau) \otimes \overline{\mathbf{Z}_p}/(\pi) \to 0$ for all $\pi \in \overline{\mathbf{Z}_p}$ such that $\mathrm{val}(\pi) > 0$ yields $H^1(\Gamma, \sigma(\tau))_{\mathfrak{m}} \neq 0$ iff $H^1(\Gamma, \overline{\sigma(\tau)})_{\mathfrak{m}} \neq 0$. Applying it again to the various Jordan-Hölder sequences in $\overline{\sigma(\tau)}$ gives the result. $\square$

Lemma 8.2.3. *$H^1(\Gamma, \sigma_{n,m})_{\mathfrak{m}} \neq 0$ implies $\mathfrak{m}$ corresponds to $\overline{\rho_p}$ with:*

$$\overline{\rho_p}|_{I_p} \simeq \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m$$

or:

$$\overline{\rho_p}|_{I_p} \simeq \begin{pmatrix} \omega^{n+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega^m$$

with $*$ peu ramifié if $n = 0$.

Proof. Since $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, $\det(\gamma) = 1\ \forall \gamma \in \Gamma$ and $H^1(\Gamma, \sigma_{n,m}) \simeq H^1(\Gamma, \sigma_{n,0})$ as $\overline{\mathbf{F}}_p$-vector spaces, the difference being only in the action of $\mathbf{T}$. $(\overline{a}_\ell, \overline{b}_\ell)$ is a system of eigenvalues of $\mathbf{T}$ on $H^1(\Gamma, \sigma_{n,m})$ iff $(\ell^{-m}\overline{a}_\ell, \ell^{-2m}\overline{b}_\ell)$ is a system of eigenvalues of $\mathbf{T}$ on $H^1(\Gamma, \sigma_{n,0})$. Hence, $H^1(\Gamma, \sigma_{n,m})_{\mathfrak{m}} \neq 0$ iff $H^1(\Gamma, \sigma_{n,0})_{\mathfrak{m}'} \neq 0$ with $\mathfrak{m}'$ corresponding to $\overline{\rho_p} \otimes \omega^{-m}$. Up to twist, we are thus reduced to the case $m = 0$ and the same argument as in the proof of the previous lemma yields $H^1(\Gamma, \sigma_{n,0})_{\mathfrak{m}} \neq 0$ iff $H^1_{\mathrm{par}}(\Gamma, \mathrm{Symm}^n \overline{\mathbf{Z}_p}^2)_{\mathfrak{m}} \neq 0$. But the systems of eigenvalues of $\mathbf{T}$ on $H^1_{\mathrm{par}}(\Gamma, \mathrm{Symm}^n \overline{\mathbf{Q}_p}^2)$ are just those of weight $n+2$ cuspidal eigenforms on Γ. Since $n+2 \leq p+1$, the Deligne-Fontaine-Serre theorem tells us that $\overline{\rho_p}$ must be such that $\overline{\rho_p}|_{I_p} \simeq \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix}$ or $\overline{\rho_p}|_{I_p} \simeq \begin{pmatrix} \omega^{n+1} & * \\ 0 & 1 \end{pmatrix}$ with $*$ peu ramifié if $n = 0$. $\square$

Thus we see that $\overline{\rho_p}|_{I_p}$ must be in the list:

$$\left\{ \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m, \begin{pmatrix} \omega^{n+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega^m \text{ with } * \text{ peu ramifié if } n = 0 \right\}$$

for (n,m) such that $\sigma_{n,m}$ occurs in $\overline{\sigma(\tau)}^{ss}$. On the other end, it is easy to see (using the Deligne-Fontaine-Serre theorem and twisting for instance) that any representation in this list is the restriction to I_p of some irreducible $\overline{\rho}_{f,p}$ coming from an eigenform f on $\Gamma_1(N)$. Moreover, if $\mathfrak{m}$ is the maximal ideal that corresponds to this $\overline{\rho}_{f,p}$, $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))_{\mathfrak{m}} \neq 0$ by Lemma 8.2.2 and we see finally that the above list is *exactly* those $\overline{\rho_p}|_{I_p}$ coming from the systems of eigenvalues of $\mathbf{T}$ acting on $H^1_{\mathrm{par}}(\Gamma, \sigma(\tau))$. Equivalently, since nothing in the list depends on the level, it is also exactly the $\overline{\rho}_{f,p}|_{I_p}$ coming from weight 2 cuspidal eigenforms f such that $\overline{\rho}_{f,p}$ is irreducible and $\rho_{f,p}|_{G_p}$ satisfies the two conditions: its Hodge-Tate weights are $(0,1)$ and its associated Weil-Deligne representation restricted to I_p is isomorphic to τ.

Changing notations, we switch to our purely local situation. Let ρ be a p-adic potentially semi-stable representation of G_p of dimension 2 such that:
(i) the Hodge-Tate weights of ρ are $(0,1)$
(ii) $\mathrm{WD}(\rho)|_{I_p} \simeq \tau$.

Idea 1: the possible $\overline{\rho}|_{I_p}$ should just be the ones of the above list (that are dictated by the semi-simplification modulo p of $\sigma(\tau)$), i.e. you won't get more than what you get from cuspidal eigenforms.

This "idea" implies for instance that $R(2,\tau,\overline{\rho}) \neq 0$ if and only if $\overline{\rho}|_{I_p}$ appears in the above list (which is already something mysterious from the local point of view of p-adic Hodge theory). But one should get much more:

Idea 2: the multiplicities of $\sigma_{n,m}$ in the decomposition of $\overline{\sigma(\tau)}^{ss}$ should indicate how "big" $R(2,\tau,\overline{\rho})$ is for $\overline{\rho}|_{I_p} \simeq \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m$ or $\overline{\rho}|_{I_p} \simeq \begin{pmatrix} \omega^{n+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega^m$ (with $*$ peu ramifié if $n=0$).

All this was for weight 2. Now, for arbitrary weight k (with, say, $k \in \{2, ..., p-1\}$), one has to replace $\sigma(\tau)$ by $\sigma(\tau) \otimes \mathrm{Symm}^{k-2}\overline{\mathbf{Q}_p}^2$. We are now in position to state the main conjecture.

8.3 Statement of the main conjecture

For $\alpha \in \overline{\mathbf{F}_p}^\times$, we denote by $\mathrm{unr}(\alpha)$ the unramified character of G_p sending an arithmetic Frobenius to α.

For $\overline{\rho} : G_p \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ and $(n,m) \in \{0,...,p-1\} \times \{0,...,p-2\}$, we define:
(i) $\mu_{0,m}(\overline{\rho}) := 1$ if $\overline{\rho} \mid_{I_p} \simeq \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix} \otimes \omega^m$ or if $\overline{\rho} \mid_{I_p} \simeq \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \omega^m$ with $*$ peu ramifié and $\mu_{0,m}(\overline{\rho}) := 0$ otherwise
(ii) for $1 \leq n \leq p-2$, $\mu_{n,m}(\overline{\rho}) := 1$ if $\overline{\rho} \mid_{I_p} \simeq \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m$ or if $\overline{\rho} \mid_{I_p} \simeq \begin{pmatrix} \omega^{n+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega^m$ and $\mu_{n,m}(\overline{\rho}) := 0$ otherwise
(iii) $\mu_{p-1,m}(\overline{\rho}) := 1$ if $\overline{\rho} \simeq \begin{pmatrix} \omega^{m+1}\mathrm{unr}(\alpha) & * \\ 0 & \omega^m \mathrm{unr}(\beta) \end{pmatrix}$ with $\alpha, \beta \in \overline{\mathbf{F}}_p^\times$, $\alpha \neq \beta$ or if $\overline{\rho} \simeq \begin{pmatrix} \omega^{m+1}\mathrm{unr}(\alpha) & * \\ 0 & \omega^m \mathrm{unr}(\alpha) \end{pmatrix}$ with $\alpha \in \overline{\mathbf{F}}_p^\times$ and $*$ très ramifié or if $\overline{\rho}|_{I_p} \simeq \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix} \otimes \omega^m$, $\mu_{p-1,m}(\overline{\rho}) := 2$ if $\overline{\rho} \simeq \begin{pmatrix} \omega^{m+1}\mathrm{unr}(\alpha) & * \\ 0 & \omega^m \mathrm{unr}(\alpha) \end{pmatrix}$ with $\alpha \in \overline{\mathbf{F}}_p^\times$ and $*$ peu ramifié, and $\mu_{p-1,m}(\overline{\rho}) := 0$ otherwise.

Fix τ, $\sigma(\tau)$ as before and let $k \in \mathbf{Z}_{>1}$. Define:

$$\sigma(k,\tau) := \sigma(\tau) \otimes_{\overline{\mathbf{Q}_p}} \mathrm{Symm}^{k-2}(\overline{\mathbf{Q}_p}^2)$$

(thus $\sigma(2,\tau) \simeq \sigma(\tau)$) and let $\overline{\sigma(k,\tau)}^{ss}$ be its semi-simplification modulo p. To each integer $k > 1$, to each Galois type τ of degree 2 and to each finite representation $\overline{\rho} : G_p \to GL_2(\overline{\mathbf{F}_p})$, we define the "automorphic multiplicity" $\mu_{\mathrm{aut}}(k,\tau,\overline{\rho})$ as follows :

$$\mu_{\mathrm{aut}}(k,\tau,\overline{\rho}) := \sum_{n,m} \mu_{n,m}(\overline{\rho}) \dim_{\overline{\mathbf{F}_p}} \mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Z}_p)}\big(\sigma_{n,m}, \overline{\sigma(k,\tau)}^{ss}\big)$$

where (n,m) runs through $\{0,...,p-1\} \times \{0,...,p-2\}$.

Conjecture 8.3.1. *Assume $k \in \{2,...,p-1\}$ and $\mathrm{End}_{\overline{\mathbf{F}}_p[G_p]}(\overline{\rho}) = \overline{\mathbf{F}}_p$, then :*

$$\mu_{\mathrm{aut}}(k,\tau,\overline{\rho}) = \mu_{\mathrm{gal}}(k,\tau,\overline{\rho}).$$

The assumption $\mathrm{End}_{\overline{\mathbf{F}_p}[G_p]}(\overline{\rho}) = \overline{\mathbf{F}_p}$ is needed so that the rings $R(k,\tau,\overline{\rho})$ are defined (see Lecture 6). The assumption $k \in \{2,...,p-1\}$ is only here because in the few cases where I've been able to check this conjecture, I needed it for the computations to be carried on. It may not be a crucial assumption.

9 Lecture 9: Computations of deformation rings and proof of a special case of the main conjecture

In this lecture, I will look a little bit at the case $\tau = \mathrm{Id}$ for the special values $k = 2, 4$. I will then just state the result for the values $k = 2k'$, $1 < k < p$.

9.1 Description of lattices

The case $\tau = \mathrm{Id}$ is the case of (indecomposable) semi-stable representations V of Hodge-Tate weights $(0, k-1)$. We recall that under the functor $V \mapsto D^*_{st}(V) = D$ (with $\pi = p$) they correspond to the following weakly admissible filtered modules D (see Lecture 3):
(i) V crystalline and reducible (non split):

$$\begin{cases} \varphi(e_1) & = & p^{k-1}(\mu_1 e_1 + e_2) \\ \varphi(e_2) & = & \mu_2 e_2 \\ Fil^{k-1}D & = & \overline{\mathbf{Q}}_p e_1 \\ \mu_1, \mu_2 & \in & \overline{\mathbf{Z}}_p^{\times} \end{cases}$$

(ii) V crystalline and irreductible:

$$\begin{cases} \varphi(e_1) & = & p^{k-1}\mu e_2 \\ \varphi(e_2) & = & -e_1 + \nu e_2 \\ Fil^{k-1}D & = & \overline{\mathbf{Q}}_p e_1 \\ \mu & \in & \overline{\mathbf{Z}}_p^{\times} \\ \nu & \in & \mathfrak{m}_{\overline{\mathbf{Z}}_p} \end{cases}$$

(iii) V semi-stable non crystalline:

$$\begin{cases} \varphi(e_1) & = & p^{k/2}\mu e_1 \\ \varphi(e_2) & = & p^{k/2-1}\mu e_2 \\ Fil^{k-1}D & = & \overline{\mathbf{Q}}_p(e_1 - \mathcal{L}e_2) \\ N(e_1) & = & e_2 \\ N(e_2) & = & 0 \\ \mu & \in & \overline{\mathbf{Z}}_p^{\times} \\ \mathcal{L} & \in & \overline{\mathbf{Q}}_p. \end{cases}$$

There are no isomorphisms between these modules for different values of the parameters. We are going to compute all Galois stable $\overline{\mathbf{Z}_p}$-lattices T in those V such that the reduction modulo p satisfies $\mathrm{End}_{\overline{\mathbf{F}_p}[G_p]}(\overline{T}) = \overline{\mathbf{F}_p}$ for $k = 2$ and $k = 4$.

Lemma 9.1.1. *Up to isomorphism, there are* 1 *or* 2 *such lattices in each* V.

Proof. If $\overline{T}$ is irreducible, then T is the only lattice (up to isomorphism). If $\overline{T} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ (with $* \neq 0$), one easily checks using $\chi_1 \neq \chi_2$ and $* \neq 0$ that any other lattice giving (after reduction) a non zero extension $\begin{pmatrix} \chi_1 & *' \\ 0 & \chi_2 \end{pmatrix}$ must be isomorphic to T (and thus $* = *'$). Hence, granting Brauer-Nesbitt, the only other possibility for the reduction is a (necessarily unique) non zero extension $\begin{pmatrix} \chi_2 & * \\ 0 & \chi_1 \end{pmatrix}$. This gives at most 2 isomorphism classes. □

Let us firt recall the crystalline cases. Let k, D and V be as in examples (i), (ii) above and let $\mathcal{D} := S \otimes D$ be as in Lecture 4 (with S as in §5.1).

Proposition 9.1.2. *For any crystalline* V *as in (i) and (ii) (with* $1 < k < p$*), there is only* 1 *isomorphism class of lattices* $T \subset V$ *such that* $\mathrm{End}_{\overline{\mathbf{F}_p}[G_p]}(\overline{T}) = \overline{\mathbf{F}_p}$.

(i) If V *is reducible,* $\overline{T} \simeq \begin{pmatrix} \mathrm{unr}(\mu_1)\omega^{k-1} & * \\ 0 & \mathrm{unr}(\mu_2) \end{pmatrix}$ *with* $* \neq 0$ *and peu ramifié if* $k = 2$.

(ii) If V *is irreducible,* $\overline{T}|_{I_p} \simeq \begin{pmatrix} \omega_2^{k-1} & 0 \\ 0 & \omega_2^{p(k-1)} \end{pmatrix}$ *with* $\det(\overline{T}) = \mathrm{unr}(\mu)\omega^{k-1}$.

Proof. The existence of T follows from §8.1. The unicity follows from the proof of the previous Lemma. □

Now, let $k = 2$ and D, V be as in example (iii) above (with $\mathcal{D} = S \otimes D$):

Proposition 9.1.3. *For any semi-stable* V *as in (iii) with* $k = 2$ *(and* $p \geq 3$*), there is only* 1 *isomorphism class of lattices* $T \subset V$ *such that* $\mathrm{End}_{\overline{\mathbf{F}_p}[G_p]}(\overline{T}) = \overline{\mathbf{F}_p}$.

(i) If $\mathrm{val}(\mathcal{L}) < 1$, $\overline{T} \simeq \begin{pmatrix} \mathrm{unr}(\mu)\omega & * \\ 0 & \mathrm{unr}(\mu) \end{pmatrix}$ *with* $* \neq 0$ *and peu ramifié.*

(ii) If $\mathrm{val}(\mathcal{L}) \geq 1$, $\overline{T} \simeq \begin{pmatrix} \mathrm{unr}(\mu)\omega & * \\ 0 & \mathrm{unr}(\mu) \end{pmatrix}$ *with* $* \neq 0$ *and très ramifié.*

Proof. Take the strongly divisible lattices of Proposition 5.1.1,(ii), reduce them modulo p and use Theorem 4.3.3 (iii) and Lemma 8.1.3. □

We now go on with the case $k = 4$ and D, V as in (iii).

Theorem 9.1.4. *Let V be as in (iii) with $k=4$ (and $p \geq 5$).*
(i) If $\mathrm{val}(\mathcal{L}+3/2)=0$ and $\mathrm{val}(\mathcal{L}+2)<1$, there are 2 isomorphism classes of lattices $T_1, T_2 \subset V$ such that $\mathrm{End}_{\overline{\mathbf{F}}_p[G_p]}(\overline{T}_i)=\overline{\mathbf{F}}_p$. One has $\overline{T}_1 \simeq \begin{pmatrix} \mathrm{unr}(\frac{\mu}{3+2\mathcal{L}})\omega^2 & * \\ 0 & \mathrm{unr}((3+2\mathcal{L})\mu)\omega \end{pmatrix}$ with $* \neq 0$ and peu ramifié and $\overline{T}_2 \simeq \begin{pmatrix} \mathrm{unr}((3+2\mathcal{L})\mu)\omega & * \\ 0 & \mathrm{unr}(\frac{\mu}{3+2\mathcal{L}})\omega^2 \end{pmatrix}$ with $* \neq 0$.
(ii) If $\mathrm{val}(\mathcal{L}+3/2)=0$ and $\mathrm{val}(\mathcal{L}+2)\geq 1$, there are 2 isomorphism classes of lattices $T_1, T_2 \subset V$ such that $\mathrm{End}_{\overline{\mathbf{F}}_p[G_p]}(\overline{T}_i)=\overline{\mathbf{F}}_p$. One has $\overline{T}_1 \simeq \begin{pmatrix} \mathrm{unr}(-\mu)\omega^2 & * \\ 0 & \mathrm{unr}(-\mu)\omega \end{pmatrix}$ with $* \neq 0$ and très ramifié and $\overline{T}_2 \simeq \begin{pmatrix} \mathrm{unr}(-\mu)\omega & * \\ 0 & \mathrm{unr}(-\mu)\omega^2 \end{pmatrix}$ with $* \neq 0$.
(iii) If $\mathrm{val}(\mathcal{L}+3/2)>0$, there is 1 isomorphism class of lattices $T \subset V$ such that $\mathrm{End}_{\overline{\mathbf{F}}_p[G_p]}(\overline{T})=\overline{\mathbf{F}}_p$. One has $\overline{T}|_{I_p} \simeq \begin{pmatrix} \omega_2^{1+2p} & 0 \\ 0 & \omega_2^{2+p} \end{pmatrix}$ with $\det(\overline{T})=\mathrm{unr}(\mu^2)\omega^3$.
(iv) If $\mathrm{val}(\mathcal{L}+3/2)<0$ i.e. $\mathrm{val}(\mathcal{L})<0$, there is 1 isomorphism class of lattices $T \subset V$ such that $\mathrm{End}_{\overline{\mathbf{F}}_p[G_p]}(\overline{T})=\overline{\mathbf{F}}_p$. One has $\overline{T}|_{I_p} \simeq \begin{pmatrix} \omega_2^3 & 0 \\ 0 & \omega_2^{3p} \end{pmatrix}$ with $\det(\overline{T})=\mathrm{unr}(\mu^2)\omega^3$.

Proof. (sketch) For $k \geq 4$, V is irreducible. Then, by a general lemma of Ribet ([33]), if there is a lattice in V reducing to $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $*$ non zero, there is also one reducing to $\begin{pmatrix} \chi_2 & * \\ 0 & \chi_1 \end{pmatrix}$ with $*$ non zero. Thus one only has to prove one case of (i) and (ii). For reasons of time, length and boredom of the audience, I only give a complete proof for (i). Let T_1 be the $\overline{\mathbf{Z}}_p$-lattice corresponding to the strongly divisible module $\mathcal{M}_1$ of Theorem 5.1.2. Then I claim its reduction $\overline{T}_1$ is as in (i). For this, we compute the reduction of $\mathcal{M}_1$, keeping the notations of the proof of 5.1.2. Let:

$$\left\{ \begin{array}{l} U_1 := (u-p)(\mathcal{L}+2)\Big(E_1 - \mathcal{L}e_2 + \frac{u-p}{p}e_2\Big) \\ U_2 := p\Big(E_1 - \mathcal{L}e_2 + \frac{u-p}{p}\big(\frac{1}{2}(E_1-\mathcal{L}e_2)+e_2\big)\Big). \end{array} \right.$$

One easily checks that $U_1, U_2 \in \mathcal{M}_1 \cap \mathrm{Fil}^3\mathcal{D}$. Moreover, an easy computation gives:

$$\begin{cases} \frac{\varphi}{p^3}(U_1) & = & \mu(\frac{u^p}{p}-1)\Big((\mathcal{L}+2)E_1 - \big(1+\mathcal{L}-\frac{u^p}{p}+\frac{\gamma(2-\mathcal{L})}{1+\gamma}\big)E_2\Big)+pW_1 \\ \frac{\varphi}{p^3}(U_2) & = & \frac{\mu}{2}\Big((1+\frac{u^p}{p})E_1 - E_2\Big)+\frac{p}{\mathcal{L}+2}W_2 \end{cases}$$

where W_i are certain elements of $\mathcal{M}_1$ (which can be computed explicitely) and recall that $\mathrm{val}(\frac{p}{\mathcal{L}+2})>0$. By Theorem 4.3.3 (iii), we have to compute the reduction of $\mathcal{M}_1$. Modulo p and $\mathrm{Fil}^p S$, we get:

$$\begin{cases} \overline{U}_1 & = & (\overline{\mathcal{L}}+2)u\overline{E}_1+u^2\overline{E}_2 \\ \overline{U}_2 & = & \frac{1}{2}u\overline{E}_1 \\ \varphi_3(\overline{U}_1) & = & \overline{\mu}\big((\overline{\mathcal{L}}+1)\overline{E}_2-(\overline{\mathcal{L}}+2)\overline{E}_1\big) \\ \varphi_3(\overline{U}_2) & = & \frac{\overline{\mu}}{2}(\overline{E}_1-\overline{E}_2) \end{cases}$$

which gives:

$$\begin{cases} \varphi_3\Big(u^2\Big(\overline{E}_2-2\frac{2+\overline{\mathcal{L}}}{3+2\overline{\mathcal{L}}}\overline{E}_1\Big)\Big) & = & \overline{\mu}(3+2\overline{\mathcal{L}})\Big(\overline{E}_2-2\frac{2+\overline{\mathcal{L}}}{3+2\overline{\mathcal{L}}}\overline{E}_1\Big) \\ \varphi_3(u\overline{E}_1) & = & (-\overline{\mu})\Big(\frac{1}{3+2\overline{\mathcal{L}}}\overline{E}_1+\Big(\overline{E}_2-2\frac{2+\overline{\mathcal{L}}}{3+2\overline{\mathcal{L}}}\overline{E}_1\Big)\Big). \end{cases}$$

From this, we easily deduce the result on $\overline{T}_1$ using Lemma 8.1.3 (note that we find an object of $\underline{NMF}^1$ "twisted by $\overline{\mathbf{F}}_p(1)$"). One can prove that the lattice reducing to the other extension corresponds to the strongly divisible module $\mathcal{M}_2$ of 5.1.2,(i). $\square$

Remark 9.1.5. From the above proof, one can in fact easily deduce that $\mathrm{Fil}^3\mathcal{D}\cap\mathcal{M}_1 = SU_1\oplus SU_2+\mathrm{Fil}^3S\cdot\mathcal{M}_1$.

From Lecture 3, we obtain the following corollaries:

Corollary 9.1.6. *Let f be a cuspidal newform of weight 2 for $\Gamma_1(pN)$ with $(p,N)=1$ and $p\geq 3$. Assume p doesn't divide the conductor of the character of f and let $\mathcal{L}_p$ be the $\mathcal{L}$-invariant attached to f.*

(i) If $\mathrm{val}(\mathcal{L}_p)<1$, then $\overline{\rho}_{f,p}|_{I_p}\simeq \begin{pmatrix}\omega & * \\ 0 & 1\end{pmatrix}$ with $*$ peu ramifié.

(ii) If $\mathrm{val}(\mathcal{L}_p)\geq 1$, then $\overline{\rho}_{f,p}|_{I_p}\simeq \begin{pmatrix}\omega & * \\ 0 & 1\end{pmatrix}$ with $*$ très ramifié if non zero.

Corollary 9.1.7. *Let f be a cuspidal newform of weight 4 for $\Gamma_1(pN)$ with $(p,N)=1$ and $p\geq 5$. Assume p doesn't divide the conductor of the character of f and let $\mathcal{L}_p$ be the $\mathcal{L}$-invariant attached to f.*

(i) If $\mathrm{val}(\mathcal{L}_p+\frac{3}{2})=0$ and $\mathrm{val}(\mathcal{L}_p+2)<1$, then $\overline{\rho}_{f,p}|_{I_p}\simeq \begin{pmatrix}\omega^2 & * \\ 0 & \omega\end{pmatrix}$ with $*$

peu ramifié or $\overline{\rho}_{f,p}|_{I_p} \simeq \begin{pmatrix} \omega & * \\ 0 & \omega^2 \end{pmatrix}$.

(ii) If $\mathrm{val}(\mathcal{L}_p + \frac{3}{2}) = 0$ *and* $\mathrm{val}(\mathcal{L}_p + 2) \geq 1$, *then* $\overline{\rho}|_{I_p} \simeq \begin{pmatrix} \omega^2 & * \\ 0 & \omega \end{pmatrix}$ *with* $*$ *très ramifié or* $\overline{\rho}_{f,p}|_{I_p} \simeq \begin{pmatrix} \omega & * \\ 0 & \omega^2 \end{pmatrix}$.

(iii) If $\mathrm{val}(\mathcal{L}_p + \frac{3}{2}) > 0$, *then* $\overline{\rho}_{f,p}|_{I_p} \simeq \begin{pmatrix} \omega_2^{2+p} & 0 \\ 0 & \omega_2^{1+2p} \end{pmatrix}$.

(iv) If $\mathrm{val}(\mathcal{L}_p + \frac{3}{2}) < 0$, *then* $\overline{\rho}_{f,p}|_{I_p} \simeq \begin{pmatrix} \omega_2^3 & 0 \\ 0 & \omega_2^{3p} \end{pmatrix}$.

The previous computations allow to determine $\overline{\rho}_{f,p}\,|_{G_p}$ and not just $\overline{\rho}_{f,p}\,|_{I_p}$ but the latter is shorter to describe. The reader will find in [10] the generalization of the above results to the case k even, $k < p$. In [40], he will also find the description of $\overline{\rho}_{f,p}|_{G_p}$ for some newforms f of weight $< p$ on $\Gamma_1(pN)$ $(p \nmid N)$ when p *divides* the conductor of the character of f (with additional hypothesis). See Lecture 3 for the description of the filtered modules attached to f in that case. This suggests one more open question: can the method of [40] be adapted to the "semi-stable" setting and yield another proof of the above two corollaries using congruences between modular forms?

9.2 Deformation rings

Fix $\overline{\rho} : G_p \to \mathrm{GL}_2(\mathbf{F})$ such that $\mathrm{End}_{\mathbf{F}[G_p]}(\overline{\rho}) = \mathbf{F}$ $(\mathbf{F} \subset \overline{\mathbf{F}_p})$. For any noetherian local complete $W(\mathbf{F})$-algebra R with residue field $\mathbf{F}$, let:

$$S_R := \widehat{R{<}u{>}} = \left\{ \sum_{n=0}^{\infty} r_n \frac{u^n}{n!}, r_n \in R, r_n \to 0 \right\}$$

endowed with the R-linear extensions of φ, N and Fil^i. We explain how to (almost) compute $R(4, \mathrm{Id}, \overline{\rho})$ (for $R(2, \mathrm{Id}, \overline{\rho})$ the method is the same and is much simpler).

The first step in computing the deformation rings $R(4, \mathrm{Id}, \overline{\rho})$ is to "select" those lattices T computed in §9.1 such that $\overline{T} \simeq \overline{\rho}$. For instance, assume $\overline{\rho} \simeq \begin{pmatrix} \omega^2\mathrm{unr}(\alpha) & * \\ 0 & \omega\mathrm{unr}(\alpha) \end{pmatrix}$ with $\alpha \in \mathbf{F}^\times$, $* \neq 0$ and peu ramifié. Then the only T giving rise to $\overline{\rho}$ are those coming from the lattices $\mathcal{M}_1$ of Theorem 9.1.4 (i) for $\frac{\overline{\mu}}{3+2\overline{\mathcal{L}}} = \overline{\mu}(3 + 2\overline{\mathcal{L}}) = \alpha$ i.e. $(\overline{\mathcal{L}}, \overline{\mu}) \in \{(-1, \alpha), (-2, -\alpha)\}$. These lattices naturally form "families": consider the lattices $\mathcal{M}_1 = SE_1 \oplus SE_2$

of Theorem 9.1.4 (i) with E_i as in the proof of *loc.cit.* and $(\mathcal{L},\mu)$ such that $(\overline{\mathcal{L}},\overline{\mu}) \in \{(-1,\alpha),(-2,-\alpha)\}$. These strongly divisible lattices are defined by φ, N and Fil^3 which one can write as matrices (resp. vectors) in the basis (E_1,E_2). Replacing formally $\mathcal{L}$ by $-1+X$ and μ by $[\alpha]+D$ in the expression of these matrices and vectors, one gets a strongly divisible module "with coefficients in $W(\mathbf{F})[[X,D]]$". Concretely, this gives:

$$\begin{cases} \mathcal{M} & = & S_{W(\mathbf{F})[[X,D]]}E_1 \oplus S_{W(\mathbf{F})[[X,D]]}E_2 \\ \varphi(E_1) & = & ([\alpha]+D)\left(p^2E_1 + \left(\frac{p\varphi(\gamma)}{1+\varphi(\gamma)} - \frac{p^2\gamma}{1+\gamma}\right)\frac{3-X}{1+X}E_2\right) \\ \varphi(E_2) & = & ([\alpha]+D)pE_2 \\ N(E_1) & = & \frac{p}{1+X}\left(1+\frac{N(\gamma)}{p}\frac{3-X}{1+\gamma^2}\right)E_2 \\ N(E_2) & = & 0 \\ \mathrm{Fil}^3\mathcal{M} & = & \mathrm{Fil}^3 S_{W(\mathbf{F})[[X,D]]}\mathcal{M} + S_{W(\mathbf{F})[[X,D]]}U_1 \oplus S_{W(\mathbf{F})[[X,D]]}U_2 \\ U_1 & := & (u-p)\big((1+X)E_1+(u-pX)E_2\big) \\ U_2 & := & \frac{1}{2}(u+p)E_1 + \frac{1}{2}\big(u(3-X)-p(1+X)\big)\frac{p}{1+X}E_2. \end{cases}$$

In the same way, one can replace formally $\mathcal{L}$ by $-2+X$, μ by $-[\alpha]+D$ and $\frac{p}{2+\mathcal{L}}$ by Y (recall the condition $\mathrm{val}(2+\mathcal{L}) < 1$) in the definition of φ, N and Fil^3 and obtain a strongly divisible module over $\frac{W(\mathbf{F})[[X,Y,D]]}{(XY-p)}$. Thus, for $\overline{\rho} \simeq \begin{pmatrix} \omega^2\mathrm{unr}(\alpha) & * \\ 0 & \omega\mathrm{unr}(\alpha)\end{pmatrix}$ with $*$ peu ramifié, one gets two strongly divisible modules (one over $W(\mathbf{F})[[X,D]]$ and one over $\frac{W(\mathbf{F})[[X,Y,D]]}{(XY-p)}$) and any strongly divisible module over $W(\mathbf{F})$ giving rise to $\overline{\rho}$ is a unique specialization of one of these two. Moreover, by a mild generalization of the results of Lecture 4, one can extend the functor "strongly divisible modules $\mapsto$ Galois lattices" to the case of coefficients and associate continuous representations $G_p \to \mathrm{GL}_2(R)$ to strongly divisible modules over R (R as above). Thus, for $\overline{\rho} \simeq \begin{pmatrix} \omega^2\mathrm{unr}(\alpha) & * \\ 0 & \omega\mathrm{unr}(\alpha)\end{pmatrix}$ as above, we obtain two deformations $G_p \to \mathrm{GL}_2(W(\mathbf{F})[[X,D]])$ and $G_p \to \mathrm{GL}_2(\frac{W(\mathbf{F})[[X,Y,D]]}{(XY-p)})$ of $\overline{\rho}$ from the previous two strongly divisible modules, and any lattice in a semi-stable representation of G_p of Hodge-Tate weights $(0,3)$ deforming $\overline{\rho}$ comes from a unique specialization of one (and only one) of these two deformations.

For any $\overline{\rho}$ such that $\mathrm{End}_{\mathbf{F}[G_p]}(\overline{\rho}) = \mathbf{F}$, one can build in a similar way local complete noetherian $W(\mathbf{F})$-algebras $R_i(\overline{\rho})$ of residue field $\mathbf{F}$ and deformations $\rho_i : G_p \to \mathrm{GL}_2(R_i(\overline{\rho}))$ of $\overline{\rho}$ such that any Galois lattice as above deforming $\overline{\rho}$ factors in a unique way through one ρ_i. We give the complete description of the $R_i(\overline{\rho})$ below for each $\overline{\rho}$.

The second step in computing the $R(4, \mathrm{Id}, \overline{\rho})$ is the following theorem:

Theorem 9.2.1. *Fix $\overline{\rho}$ as above.*
(i) For any i, there is a surjection of $W(\mathbf{F})$-algebras: $R(\overline{\rho}) \twoheadrightarrow R_i(\overline{\rho})$.
(ii) For any i, this surjection factors trough $R(4, \mathrm{Id}, \overline{\rho})$.
(iii) These surjections induce an isomorphism: $R(4, \mathrm{Id}, \overline{\rho})[\frac{1}{p}] \xrightarrow{\sim} \Pi_i R_i(\overline{\rho})[\frac{1}{p}]$.

Proof. (sketch) (i) By the universal property of $R(\overline{\rho})$, we know there is a morphism of $W(\mathbf{F})$-algebras $R(\overline{\rho}) \to R_i(\overline{\rho})$. To prove it is surjective, it's enough to prove that the map $R(\overline{\rho}) \to R_i(\overline{\rho})/(p, \mathfrak{m}^2_{R_i(\overline{\rho})})$ is surjective, or that the deformation $G_p \to \mathrm{GL}_2\big(R_i(\overline{\rho})/(p, \mathfrak{m}^2_{R_i(\overline{\rho})})\big)$ cannot be defined over a $W(\mathbf{F})$-subalgebra of $R_i(\overline{\rho})/(p, \mathfrak{m}^2_{R_i(\overline{\rho})})$. Equivalently, one has to check that the corresponding object of $\underline{\mathcal{M}}^3$ over $R_i(\overline{\rho})/(p, \mathfrak{m}^2_{R_i(\overline{\rho})})$ is not coming by extension of scalars from a subobject defined over a strict $\mathbf{F}$-subalgebra. This is an explicit computation that is checked for each $R_i(\overline{\rho})$ (for instance, in the above examples, one has to prove that it's impossible to dispense with X, Y or D): see [10].
(ii) Let $x \in \cap_{4,\mathrm{Id}}\mathfrak{p} \subset R(\overline{\rho})$ (see Lecture 6) and x_i the image of x in $R_i(\overline{\rho})$. One has to prove $x_i = 0$. But for all $W(\mathbf{F})$-algebra homomorphisms $R_i(\overline{\rho}) \to \overline{\mathbf{Z}_p}$, one has $x \in \mathrm{Ker}(R(\overline{\rho}) \to R_i(\overline{\rho}) \to \overline{\mathbf{Z}_p})$, hence $x_i \mapsto 0$ for all $R_i(\overline{\rho}) \to \overline{\mathbf{Z}_p}$. This easily implies $x_i = 0$.
(iii) From (ii), we thus have a morphism of $W(\mathbf{F})$-algebras $R(4, \mathrm{Id}, \overline{\rho}) \to \Pi_i R_i(\overline{\rho})$. We first prove it is injective. Let $x \in R(\overline{\rho})$ such that $x_i = 0, \forall i$ (x_i as in (ii)). Since every $W(\mathbf{F})$-algebra morphism $R(\overline{\rho}) \to \overline{\mathbf{Z}_p}$ of kernel of type $(4, \mathrm{Id})$ factors through a (unique) $R_i(\overline{\rho})$, we see that $x \in \cap_{4,\mathrm{Id}}\mathfrak{p}$, hence $x = 0$ in $R(4, \mathrm{Id}, \overline{\rho})$. Granting $R(4, \mathrm{Id}, \overline{\rho}) \hookrightarrow \Pi_i R_i(\overline{\rho})$ and $R(4, \mathrm{Id}, \overline{\rho}) \twoheadrightarrow R_i(\overline{\rho})$, one can prove that the isomorphism (iii) is equivalent to $\coprod_i R_i(\overline{\rho})(\overline{\mathbf{Z}_p}) \xrightarrow{\sim} R(4, \mathrm{Id}, \overline{\rho})(\overline{\mathbf{Z}_p})$ (this is just some commutative algebra). We now prove this last statement. Injectivity is clear since a Galois lattice cannot come from two different specializations. For surjectivity, it is enough to prove that any morphism $R(\overline{\rho}) \to \overline{\mathbf{Z}_p}$ that factors through $R(4, \mathrm{Id}, \overline{\rho})$ has a kernel of type $(4, \mathrm{Id})$ (this is Conjecture 6.3.8!), since then it will factor through some $R_i(\overline{\rho})$. But one can consider another quotient of $R(\overline{\rho})$, namely the one that is a universal parameter for deformations of $\overline{\rho}$ over artinian local $W(\mathbf{F})$-algebras that come from objects of $\underline{\mathcal{M}}^3$ (this uses the fact $\underline{\mathcal{M}}^3$ is preserved by subobjects, quotients and direct sums). Let us call $R(\overline{\rho})/I$ this quotient. We have $I \subset \cap_{4,\mathrm{Id}}\mathfrak{p}$ by definition of being of type $(4, \mathrm{Id})$. Thus, any morphism $R(\overline{\rho}) \to \overline{\mathbf{Z}_p}$ that factors through $R(4, \mathrm{Id}, \overline{\rho})$ also factors through $R(\overline{\rho})/I$. This implies the corresponding deformation $G_p \to \mathrm{GL}_2(\overline{\mathbf{Z}_p})$ comes modulo p^n for any n from an object of $\underline{\mathcal{M}}^3$, hence comes from a strongly divisible module. Thus $\mathrm{Ker}(R(\overline{\rho}) \to \overline{\mathbf{Z}_p})$ is of type $(4, \mathrm{Id})$. This finishes the proof. □

Recall from Lecture 6 that $\mu(R_i(\overline{\rho})/p)$ is the Hilbert-Samuel multiplicity of $R_i(\overline{\rho})/p$.

Corollary 9.2.2. *For any $\overline{\rho}$ as above, $\mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho}) = \sum_i \mu(R_i(\overline{\rho})/p)$.*

Proof. This is a general result of commutative algebra. Let R and $R_1, \dots, R_r$ be noetherian local complete flat $W(\mathbf{F})$-algebras of residue field $\mathbf{F}$ and of the same Krull dimension. Assume that for each i there is a surjection (of $W(\mathbf{F})$-modules) $R \twoheadrightarrow R_i$ that induces an isomorphism $R[\frac{1}{p}] \xrightarrow{\sim} \Pi_i R_i[\frac{1}{p}]$. Then I claim that $\mu(R/(p)) = \sum_i \mu(R_i/(p))$. We prove this result by induction on r. The case $r = 1$ being trivial, we assume $r = 2$. Let $n \in \mathbf{N}$ be such that $p^n(R_1 \otimes_R R_2) = 0$, from the exact sequence of R-modules $0 \to R \to R_1 \times R_2 \to R_1 \otimes_R R_2 \to 0$, we get an exact sequence of R/p^n-modules of finite type:

$$0 \to R_1 \otimes_R R_2 \to R/p^n \to R_1/p^n \times R_2/p^n \to R_1 \otimes_R R_2 \to 0$$

from which we deduce $\mu(R/p^n) = \mu(R_1/p^n) + \mu(R_2/p^n)$ (well-behaviour of the Hilbert-Samuel multiplicity with respect to exact sequences). Since $\mu(R/p^n) = n\mu(R/p)$ (resp. with R_i), we are done for $r = 2$. For greater r, let $R' := R/(\cap_{i=1}^{r-1} I_i)$ where $I_i := \mathrm{Ker}(R \to R_i)$ and apply the induction hypothesis to R' and the case $r = 2$ to $R_1 = R'$ and $R_2 = R_r$. $\square$

Since $\mu(\mathbf{F}[[X, D]]) = 1$ and $\mu\left(\frac{\mathbf{F}[[X,Y,D]]}{(XY)}\right) = 2$, we get $\mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho}) = 1 + 2 = 3$ for $\overline{\rho} \simeq \begin{pmatrix} \omega^2 \mathrm{unr}(\alpha) & * \\ 0 & \omega \mathrm{unr}(\alpha) \end{pmatrix}$ with $*$ peu ramifié. We give now the result of all the computations in the case $\tau = \mathrm{Id}$, $k = 2, 4$. We write $R \sim \Pi_i R_i$ if $R \twoheadrightarrow R_i$ $(\forall i)$ and $R[\frac{1}{p}] \xrightarrow{\sim} \Pi_i R_i[\frac{1}{p}]$.

Theorem 9.2.3. *Let $\overline{\rho} : G_p \to \mathrm{GL}_2(\mathbf{F})$ such that $\mathrm{End}_{\mathbf{F}[G_p]}(\overline{\rho}) = \mathbf{F}$.*
(i) If $\overline{\rho}|_{I_p} \notin \left\{ \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix} \right\}$, then $R(2, \mathrm{Id}, \overline{\rho}) = 0$ and $\mu_{\mathrm{gal}}(2, \mathrm{Id}, \overline{\rho}) = 0$.
(ii) If $\overline{\rho}|_{I_p} \in \left\{ \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \text{ with } * \text{ très ramifié}, \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix} \right\}$ or if $\overline{\rho} \simeq \begin{pmatrix} \omega \mathrm{unr}(\alpha) & * \\ 0 & \mathrm{unr}(\beta) \end{pmatrix}$ with $\alpha \neq \beta$, then $R(2, \mathrm{Id}, \overline{\rho}) \simeq W(\mathbf{F})[[X, D]]$ and $\mu_{\mathrm{gal}}(2, \mathrm{Id}, \overline{\rho}) = 1$.
(iii) If $\overline{\rho} \simeq \begin{pmatrix} \omega \mathrm{unr}(\alpha) & * \\ 0 & \mathrm{unr}(\alpha) \end{pmatrix}$ with $*$ peu ramifié, then $R(2, \mathrm{Id}, \overline{\rho}) \sim W(\mathbf{F})[[X, D]] \times W(\mathbf{F})[[X, D]]$ and $\mu_{\mathrm{gal}}(2, \mathrm{Id}, \overline{\rho}) = 2$.

Theorem 9.2.4. *Let* $\overline{\rho}: G_p \to \mathrm{GL}_2(\mathbf{F})$ *such that* $\mathrm{End}_{\mathbf{F}[G_p]}(\overline{\rho}) = \mathbf{F}$.
(i) If $\overline{\rho}|_{I_p} \notin \left\{ \begin{pmatrix} \omega^3 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega^2 & * \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & * \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega_2^3 & 0 \\ 0 & \omega_2^{3p} \end{pmatrix}, \begin{pmatrix} \omega_2^{2+p} & 0 \\ 0 & \omega_2^{1+2p} \end{pmatrix} \right\}$, *then* $R(4, \mathrm{Id}, \overline{\rho}) = 0$ *and* $\mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho}) = 0$.
(ii) If $\overline{\rho}|_{I_p} \in \left\{ \begin{pmatrix} \omega^2 & * \\ 0 & \omega \end{pmatrix} \text{ with } * \text{ très ramifié}, \begin{pmatrix} \omega^3 & * \\ 0 & 1 \end{pmatrix} \right\}$, *then* $R(4, \mathrm{Id}, \overline{\rho}) \simeq W(\mathbf{F})[[X, D]]$ *and* $\mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho}) = 1$.
(iii) If $\overline{\rho}|_{I_p} \in \left\{ \begin{pmatrix} \omega & * \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega_2^{2+p} & 0 \\ 0 & \omega_2^{1+2p} \end{pmatrix} \right\}$ *or if* $\overline{\rho} \simeq \begin{pmatrix} \omega^2 \mathrm{unr}(\alpha) & * \\ 0 & \omega \mathrm{unr}(\beta) \end{pmatrix}$ *with* $\alpha \neq \beta$, *then* $R(4, \mathrm{Id}, \overline{\rho}) \sim W(\mathbf{F})[[X, D]] \times W(\mathbf{F})[[X, D]]$ *and* $\mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho}) = 2$.
(iv) If $\overline{\rho}|_{I_p} \simeq \begin{pmatrix} \omega_2^3 & 0 \\ 0 & \omega_2^{3p} \end{pmatrix}$, *then* $R(4, \mathrm{Id}, \overline{\rho}) \sim W(\mathbf{F})[[X, D]] \times W(\mathbf{F})[[X, D]] \times W(\mathbf{F})[[X, D]]$ *and* $\mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho}) = 3$.
(v) If $\overline{\rho} \simeq \begin{pmatrix} \omega^2 \mathrm{unr}(\alpha) & * \\ 0 & \omega \mathrm{unr}(\alpha) \end{pmatrix}$ *with* $*$ *peu ramifié, then* $R(4, \mathrm{Id}, \overline{\rho}) \sim W(\mathbf{F})[[X, D]] \times \frac{W(\mathbf{F})[[X,Y,D]]}{(XY-p)}$ *and* $\mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho}) = 3$.

Corollary 9.2.5. *Conjectures 6.3.6, 6.3.8 and 8.3.1 hold for* $\tau = \mathrm{Id}$, $k = 2, 4$.

Proof. For 6.3.6, it is clear from 9.2.3 and 9.2.4. We proved 6.3.8 (for $k = 4$ but $k = 2$ is similar) in the course of proving Theorem 9.2.1. By Lecture 7, Corollary 7.2.2 (i), we have $\sigma(\mathrm{Id}) = u_1(1) = \left(\mathrm{Ind}_{\Gamma_0(p)}^{\mathrm{GL}_2(\mathbf{Z}_p)} 1\right)/1$ where $\Gamma_0(p)$ is the subgroup of modulo p lower triangular matrices. Moreover, one easily checks that $\overline{\sigma(2, \mathrm{Id})} = \mathrm{Symm}^{p-1}\overline{\mathbf{F}}_p^2 = \sigma_{p-1,0}$ (see Lecture 8) and that $\overline{\sigma(4, \mathrm{Id})} = \left(\mathrm{Symm}^{p-1}\overline{\mathbf{F}}_p^2 \otimes \mathrm{Symm}^2\overline{\mathbf{F}}_p^2\right)^{\mathrm{ss}} = \sigma_{p-3,2}^2 \oplus \sigma_{2,0} \oplus \sigma_{0,1} \oplus \sigma_{p-1,1}$. Then the recipe of §8.3 immediately gives $\mu_{\mathrm{aut}}(2, \mathrm{Id}, \overline{\rho}) = \mu_{\mathrm{gal}}(2, \mathrm{Id}, \overline{\rho})$ and $\mu_{\mathrm{aut}}(4, \mathrm{Id}, \overline{\rho}) = \mu_{\mathrm{gal}}(4, \mathrm{Id}, \overline{\rho})$ for any $\overline{\rho}$. □

We refer the reader to [10] for the generalization of the above results in the case k even, $k < p$.

Let me finally mention, as an (accessible) open question, that the computations we made in §9.1 and §9.2 for k even remain to be done for k *odd*. Also, thinking about [40], it would be interesting to do the parallel computations with the filtered modules of Example 3.3.2 (i.e. for the type $\tau \simeq \tilde{\omega}^i \oplus 1$, $0 < i < p - 1$).

10 Lecture 10: Towards modulo p and p-adic Langlands correspondences for $\mathrm{GL}_2(\mathbf{Q}_p)$?

As mentionned in the introduction of this course, the title of this last Lecture is the true motivation. The conjecture we made (and proved in some cases) could be some weak by-product and is also some kind of consolation, waiting for more...

This lecture presents results of a somewhat different flavour that undoubtedly go into the direction of such correspondences. We do not provide notes for this last lecture which is a survey of [4].

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