TOWARDS A p-ADIC LANGLANDS PROGRAMME

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WARNING: These notes are informal and are not intended to be published. We apologize for the inaccuracies, flaws and English mistakes that they surely contain.

Contents

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1. Aim and contents of the course (C.B.)

1.1. **Aim of the course.** In 1998, M. Harris and R. Taylor ([25]), and slightly later G. Henniart ([26]), proved the celebrated local Langlands correspondence for GL_n (see the course of Bushnell-Henniart for the case $n = 2$, which was known much earlier).

Let p be a prime number and K, E two finite extensions of $\mathbb{Q}_p$ with E "big" (in particular containing K). Recently, Schneider and Teitelbaum introduced a theory of locally analytic and continuous representations of p-adic groups such as $\mathrm{GL}_n(K)$ over E-vector spaces (see their course and also [20]).

The aim of this course is to present some ongoing research on the (quite ambitious) project of looking for a "Langlands type local correspondence" between, on the one hand, some continuous representations of $\mathrm{W}_K := \mathrm{Weil}(\overline{\mathbb{Q}_p}/K)$ on n-dimensional E-vector spaces and, on the other hand, some continuous representations of $\mathrm{GL}_n(K)$ on infinite-dimensional E-vector spaces (although what we are going to study here is rather microscopic with respect to such a programme!). Here and throughout, E will always denote the coefficient field of the vector spaces on which the groups act. For some technical reasons, we will keep in this course E finite over $\mathbb{Q}_p$ but will always tacitly enlarge it if necessary (which explains the above "big").

Example 1.1.1. *The case $n = 1$ is just local class field theory. Let $\mathrm{W}_K \to E^\times$ be a continuous 1-dimensional representation, then, using the (continuous) reciprocity isomorphism $\tau_K : \mathrm{W}_K^{\mathrm{ab}} \xrightarrow{\sim} K^\times$, we deduce a continuous representation $K^\times \to E^\times$.*

We would like to extend the case $n = 1$ to $n > 1$. One problem is that the category of p-adic (= continuous) finite dimensional representations of W_K or $\mathrm{G}_K := \mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ is BIG, even for $n = 2$. Fontaine has defined important subcategories of p-adic representations of G_K called crystalline, semi-stable and de Rham with crystalline $\subsetneq$ semi-stable $\subsetneq$ de Rham. They are important because they are related to algebraic geometry. Moreover, these categories are now well understood. Therefore it seems natural to first look for their counterpart, if any, on the GL_n-side.

In this course, we will do so for $n = 2$, $K = \mathbb{Q}_p$ and crystalline representations. As in the ℓ-adic case (i.e. the usual local Langlands correspondence), we will have an "F-semi-simplicity" assumption. We will also need to assume that the *Hodge-Tate weights* of the 2-dimensional crystalline representation are distinct. Finally, mainly for simplicity, we will also add the small assumption that the representation is *generic* (which means that we do not want to be bothered by the trivial representation of $\mathrm{GL}_2(\mathbb{Q}_p)$; the non-generic cases behave a bit differently because of this representation). If f is an eigenform of weight $k \geq 2$ on $\Gamma_1(N)$ for some N prime to p, then the p-adic representation $\rho_f|_{\mathrm{G}_{\mathbb{Q}_p}}$ of $\mathrm{G}_{\mathbb{Q}_p}$ associated to f is known to be crystalline generic with Hodge-Tate weights $(0, k-1)$ and is conjectured to be always "F-semi-simple" (and really is if $k = 2$), so these assumptions should not be too restrictive. The case of equal Hodge-Tate weights correspond to weight 1 modular forms, that we thus disregard.

Our basic aim in this course is to define and start studying some *p-adic Banach spaces* endowed with a continuous action of $\mathrm{GL}_2(\mathbb{Q}_p)$, together with their locally analytic vectors. These Banach will be associated to 2-dimensional "F-semi-simple" generic crystalline representations of $\mathrm{G}_{\mathbb{Q}_p}$ with distinct Hodge-Tate weights. Independently of the quest for a "p-adic Langlands programme", we hope that these Banach spaces will also be useful in the future for a "representation theoretic" study of the Iwasawa theory of modular forms (see e.g. the end of §9.2).

1.2. **Why Banach spaces?** We will focus on Banach spaces rather than on locally analytic representations, although there *are* some very interesting locally analytic vectors inside the Banach (and we will study them). In order to explain why, let us first recall, without proof, some results in the classical ℓ-adic case, due to M.-F. Vignéras.

Recall that, if ℓ is any prime number and E is a finite extension of $\mathbb{Q}_\ell$, an E-Banach space is a topological E-vector space which is complete with respect to the topology given by a norm (we don't specify the norm in the data of a Banach space).

Definition 1.2.1. *Let G be a group acting E-linearly on an E-vector space V. We say a norm $\|\cdot\|$ on V is invariant (with respect to G) if $\|g\cdot v\| = \|v\|$ for all $g \in G$ and $v \in V$.*

Definition 1.2.2. *Let G be a topological group. We call a unitary G-Banach space on E any E-Banach space Π equipped with an E-linear action of G such that the map $G \times \Pi \to \Pi$, $(g, v) \mapsto g \cdot v$ ($v \in \Pi$, $g \in G$) is continuous and such that the topology on Π is given by an invariant norm.*

Unitary G-Banach spaces on E form an obvious category where the morphisms are the continuous E-linear G-equivariant maps.

Now, let us assume $\ell \neq p$ so that K is p-adic and E is ℓ-adic.

Definition 1.2.3. *A smooth representation of $\mathrm{GL}_n(K)$ on an E-vector space π is said to be integral if there exists an invariant norm on π (with respect to $\mathrm{GL}_n(K)$).*

It is easy to see that the absolutely irreducible smooth integral representations of $\mathrm{GL}_n(K)$ are exactly those absolutely irreducible smooth representations of $\mathrm{GL}_n(K)$ that correspond, under the classical local Langlands correspondence, to those F-semi-simple ℓ-adic representations of W_K that *extend* to G_K (recall that an ℓ-adic representation of W_K on a finite dimensional E-vector space is the same thing by Grothendieck's ℓ-adic monodromy theorem as a smooth representation of the Weil-Deligne group on the same vector space).

Let π be a smooth integral representation of $\mathrm{GL}_n(K)$ on an E-vector space and let $\|\cdot\|$ be an invariant norm on π. Then the completion of π with respect to $\|\cdot\|$ is obviously a unitary $\mathrm{GL}_n(K)$-Banach space on E. Conversely, let Π be a unitary $\mathrm{GL}_n(K)$-Banach space and denote by $\pi \subset \Pi$ its subspace of smooth vectors (i.e. vectors fixed by a compact open subgroup of $\mathrm{GL}_n(K)$). Then π is obviously a smooth integral representation of $\mathrm{GL}_n(K)$.

Theorem 1.2.4 (M.-F. Vignéras)**.** *The functor $\Pi \mapsto \pi$ that associates to a unitary $\mathrm{GL}_n(K)$-Banach space its subspace of smooth vectors induces an equivalence of categories between the category of unitary $\mathrm{GL}_n(K)$-Banach spaces on E which are topologically of finite length and the category of finite length smooth integral representations of $\mathrm{GL}_n(K)$ on E. An inverse functor $\pi \mapsto \Pi$ is provided by the completion with respect to any invariant norm on π. Moreover, length*(Π)= *length*(π)*.*

In particular, all the invariant norms on an integral representation π of finite length induce the same topology (in fact, Vignéras has proved that the lattices $\{v \in \pi \mid \|v\| \leq 1\}$ for an invariant norm

$\|\cdot\|$ are always finitely generated over $\mathcal{O}_E[\mathrm{GL}_2(\mathbb{Q}_p)]$, and thus are all commensurable). Moreover π is irreducible if and only Π is topologically irreducible. Thus, we see that the classical local Langlands correspondence can be reformulated as a correspondence between (F-semi-simple) ℓ-adic representations of G_K and unitary $\mathrm{GL}_n(K)$-Banach spaces on finite extensions of $\mathbb{Q}_\ell$ that are topologically absolutely irreducible. Hence, we *could* work with Banach spaces in the ℓ-adic case, although everybody agrees that smooth representations are much more convenient to handle than Banach spaces. But we know, thanks to Th.1.2.4, that it wouldn't make any difference.

We look for an analogous picture in the p-adic case, i.e. where E is a finite extension of $\mathbb{Q}_p$ and where the smooth vectors inside the unitary Banach spaces are replaced by the locally analytic vectors. However, there is no straightforward generalization of Th.1.2.4 because:
1) a strongly admissible locally analytic representation of $\mathrm{GL}_n(K)$ on E (see the course of Schneider and Teitelbaum) can have infinitely many non-equivalent invariant norms (ex: one can prove this is the case for the locally algebraic representation $\mathrm{Sym}^{k-2}E^2 \otimes_E$ Steinberg of $\mathrm{GL}_2(\mathbb{Q}_p)$ if $k > 2$)
2) thanks to a theorem of Schneider and Teitelbaum ([34]), we know that if Π is a unitary $\mathrm{GL}_n(K)$-Banach on E that is admissible (as a representation of some compact open subgroup) with locally analytic vectors π, then $\mathrm{length}(\Pi) \leq \mathrm{length}(\pi)$ (in fact, unitarity is useless here). However, in general, this is only a strict inequality (ex: the continuous Steinberg is topologically irreducible whereas the locally analytic Steinberg has length 2).

So it seems natural in the p-adic case to rather focus on Banach spaces since the number of Jordan-Hölder factors is smaller. Also, in the crystalline $\mathrm{GL}_2(\mathbb{Q}_p)$-case, we will see that the unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach spaces are somewhat closer to the Galois representations than their locally analytic vectors.

1.3. **Some notations and contents.** Here are some notations we will use: p is a prime number, $\overline{\mathbb{Q}_p}$ an algebraic closure of the field $\mathbb{Q}_p$ of p-adic rationals, $\overline{\mathbb{Z}_p}$ the ring of integers in $\overline{\mathbb{Q}_p}$, $\overline{\mathbb{F}_p}$ its residue field and $\mathrm{G}_{\mathbb{Q}_p}$ the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. We denote by $\varepsilon : \mathrm{G}_{\mathbb{Q}_p}^{\mathrm{ab}} \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character, $\mathrm{unr}(x)$ the unramified character of $\mathbb{Q}_p^\times$ sending p to x (whatever x is), val the p-adic valuation normalized by $\mathrm{val}(p) = 1$ and $|\cdot| = \frac{1}{p^{\mathrm{val}(\cdot)}}$ the p-adic norm. The inverse of the reciprocity map $\mathbb{Q}_p^\times \hookrightarrow \mathrm{G}_{\mathbb{Q}_p}^{\mathrm{ab}}$ is normalized so that p is sent to a geometric Frobenius. In particular, for $x \in \mathbb{Q}_p^\times$, we have $\varepsilon(x) = x|x|$ via this map. We let $\mathrm{B}(\mathbb{Q}_p) \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be the subgroup of upper triangular matrices. We will loosely use the same notation for a representation and its underlying E-vector space. We will use without comment basics of p-adic functional analysis ([30]) as well as Schneider and Teitelbaum's theory of locally analytic representations and the structure theorem of locally analytic principal series for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([31],[32]).

Here is a rough description of the contents of the course.

In lectures 2 to 5, the first author will introduce Fontaine's rings $\mathbb{B}_{\mathrm{cris}}$ and $\mathbb{B}_{\mathrm{dR}}$ and will define crystalline and de Rham representations of $\mathrm{G}_{\mathbb{Q}_p}$ using the theory of *weakly admissible filtered modules* (this restricted setting will be enough for our purposes). He will give the explicit classification of F-semi-simple crystalline representations of $\mathrm{G}_{\mathbb{Q}_p}$ in dimension 1 and 2 (in higher dimension, it becomes more involved). Then he will give a *second* construction of crystalline representations of $\mathrm{G}_{\mathbb{Q}_p}$ using the theory of (φ,Γ)-modules and he will describe the non trivial links between p-adic Hodge theory and (φ,Γ)-modules. He will then introduce and study the *Wach module* of a crystalline representation of $\mathrm{G}_{\mathbb{Q}_p}$ (a "smaller" module than the (φ,Γ)-module). These Wach modules will be useful in the sequel for the applications to $\mathrm{GL}_2(\mathbb{Q}_p)$.

In lectures 6 to 9, the second author will define and start studying the unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach spaces associated to 2-dimensional crystalline representations of $\mathrm{G}_{\mathbb{Q}_p}$ satisfying the above assumptions (genericity, F-semi-simplicity and distinct Hodge-Tate weights). He will first introduce the necessary material on distributions and functions on $\mathbb{Z}_p$ (functions of class $\mathcal{C}^r$, tempered distributions of order r, etc.) and will then turn to the definition of the unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space $\Pi(V)$ associated to the (2-dimensional) crystalline representation V. When V is reducible, $\Pi(V)$ is admissible, has topological length 2 and is split (as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$) if and only if V is split (as a representation of $\mathrm{G}_{\mathbb{Q}_p}$). When V is absolutely irreducible, it seems hard to prove anything on $\Pi(V)$ directly. For instance, it is not even clear in general on its definition that $\Pi(V) \neq 0$. To study $\Pi(V)$ for V irreducible, one surprisingly needs to use the theory of (φ, Γ)-modules for V. More precisely, in that case we prove that there is a "canonical" (modulo natural choices) E-linear isomorphism of vector spaces:

$$\big(\varprojlim_{\psi} D(V)\big)^{\mathrm{b}} \simeq \Pi(V)^*$$

where $D(V)$ is the (φ, Γ)-module attached to V, $\psi : D(V) \twoheadrightarrow D(V)$ is a certain canonical surjection (that will be described), b means "bounded sequences" and $\Pi(V)^*$ is the Banach dual of $\Pi(V)$. Such an isomorphism (i.e. the link with (φ, Γ)-modules) was first found by Colmez ([16]) in the case V is irreducible semi-stable non crystalline (and was inspired by work of the second author [10], [11]) but we won't speak of these cases here. The left hand side of the above isomorphism being known to be non zero, we deduce $\Pi(V)^* \neq 0$ and thus $\Pi(V) \neq 0$. The left hand side being "irreducible" and "admissible" (what we mean here is explained in the text), we also deduce that $\Pi(V)$ is topologically irreducible and admissible as a $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space. It is hoped that this isomorphism will be also useful in the future for a more advanced study of the representations $\Pi(V)$, e.g. of their locally analytic vectors and of their reduction modulo the maximal ideal of E.

Finally, in the last lecture, the first author will state a conjecture (contained in [9]) relating the semi-simplifications of $\Pi(V)$ and V modulo the maximal ideal of E and will prove in some cases this conjecture using his Wach modules and previous computations of the second author.

We didn't include all the proofs of all the results stated or used in this text. But we have tried to include as many proofs, or sketches of proofs, or examples, or references, as our energy enabled us to. We apologize for the proofs that perhaps should be, and are not, in this course.

2. p-adic Hodge theory (L.B.)

The aim of this lecture is to explain what crystalline representations are and to classify them (at least in dimensions 1 and 2). Let us start with a p-adic representation V, that is a finite dimensional E-vector space endowed with a continuous linear action of the group $\mathrm{G}_{\mathbb{Q}_p}$. For the time being, we will consider the underlying $\mathbb{Q}_p$-vector space of V and p-adic representations will be $\mathbb{Q}_p$-vector spaces. We want to single out certain subcategories of the category of all p-adic representations. There is a general strategy for cutting out subcategories of the category of p-adic representations. Suppose that we are given a topological $\mathbb{Q}_p$-algebra B which is equipped with a continuous linear action of $\mathrm{G}_{\mathbb{Q}_p}$. We say that V is B-admissible if $B \otimes_{\mathbb{Q}_p} V = B^{\dim_{\mathbb{Q}_p}(V)}$ as $B[\mathrm{G}_{\mathbb{Q}_p}]$-modules. The set of all B-admissible representations is then a subset of the set of all p-adic representations. Furthermore, if B is equipped with extra structures which commute with the action of $\mathrm{G}_{\mathbb{Q}_p}$ then $D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{\mathrm{G}_{\mathbb{Q}_p}}$ inherits those structures and provides us with invariants which we can use to classify B-admissible representations. In this lecture, we will define a ring of periods $\mathbb{B}_{\mathrm{cris}}$ which has two extra structures: a filtration and a Frobenius map φ. Hence, a $\mathbb{B}_{\mathrm{cris}}$-admissible representation V gives rise to a filtered φ-module $D_{\mathrm{cris}}(V)$ and we will see that one can actually recover the representation V from $D_{\mathrm{cris}}(V)$.

2.1. **Some rings of periods.** Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$ for the p-adic topology and let

$$\widetilde{\mathbb{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \cdots) \mid (x^{(i+1)})^p = x^{(i)}\},$$

and let $\widetilde{\mathbb{E}}^+$ be the set of $x \in \widetilde{\mathbb{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. If $x = (x^{(i)})$ and $y = (y^{(i)})$ are two elements of $\widetilde{\mathbb{E}}$, we define their sum $x + y$ and their product xy by:

$$(x+y)^{(i)} = \lim_{j \to +\infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \quad \text{and} \quad (xy)^{(i)} = x^{(i)} y^{(i)},$$

which makes $\widetilde{\mathbb{E}}$ into a field of characteristic p. If $x = (x^{(n)})_{n \geq 0} \in \widetilde{\mathbb{E}}$, let $v_E(x) = v_p(x^{(0)})$. This is a valuation on $\widetilde{\mathbb{E}}$ for which $\widetilde{\mathbb{E}}$ is complete; the ring of integers of $\widetilde{\mathbb{E}}$ is $\widetilde{\mathbb{E}}^+$.

Let $\epsilon = (\epsilon^{(i)}) \in \widetilde{\mathbb{E}}^+$ where $\epsilon^{(0)} = 1$ and $\epsilon^{(i)}$ is a primitive p^ith root of 1. It is easy to see that $\mathbb{F}_p((\epsilon - 1)) \subset \widetilde{\mathbb{E}} = \widetilde{\mathbb{E}}^+[(\epsilon - 1)^{-1}]$ and one can show that $\widetilde{\mathbb{E}}$ is a field which is the completion of the algebraic (non-separable!) closure of $\mathbb{F}_p((\epsilon - 1))$, so it is really a familiar object.

Let $\widetilde{\mathbb{A}}^+$ be the ring $W(\widetilde{\mathbb{E}}^+)$ of Witt vectors with coefficients in $\widetilde{\mathbb{E}}^+$ and let

$$\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[1/p] = \{ \sum_{k \gg -\infty} p^k [x_k],\ x_k \in \widetilde{\mathbb{E}}^+ \}$$

where $[x] \in \widetilde{\mathbb{A}}^+$ is the Teichmüller lift of $x \in \widetilde{\mathbb{E}}^+$. The topology of $\widetilde{\mathbb{A}}^+$ is defined by taking the collection of open sets $\{([\overline{\pi}]^k, p^n)\widetilde{\mathbb{A}}^+\}_{k,n \geq 0}$ as a family of neighborhoods of 0, so that the natural map $\prod \widetilde{\mathbb{E}}^+ \to \widetilde{\mathbb{A}}^+$ which to $(x_k)_k$ assigns $\sum p^k [x_k]$ is a homeomorphism when $\widetilde{\mathbb{E}}^+$ is given its valued field topology. This is not the p-adic topology of $\widetilde{\mathbb{A}}^+$, which makes the map $(x_k)_k \mapsto \sum p^k [x_k]$ homeomorphism when $\widetilde{\mathbb{E}}^+$ is given the discrete topology.

The ring $\widetilde{\mathbb{B}}^+$ is endowed with a map $\theta : \widetilde{\mathbb{B}}^+ \to \mathbb{C}_p$ defined by the formula

$$\theta \left(\sum_{k \gg -\infty} p^k [x_k] \right) = \sum_{k \gg -\infty} p^k x_k^{(0)}.$$

The absolute Frobenius $\varphi : \widetilde{\mathbb{E}}^+ \to \widetilde{\mathbb{E}}^+$ lifts by functoriality of Witt vectors to a map $\varphi : \widetilde{\mathbb{B}}^+ \to \widetilde{\mathbb{B}}^+$. It's easy to see that $\varphi(\sum p^k [x_k]) = \sum p^k [x_k^p]$ and that φ is bijective.

Recall that $\epsilon = (\epsilon^{(i)})_{i \geq 0} \in \widetilde{\mathbb{E}}^+$ where $\epsilon^{(i)}$ is as above, and define $\pi = [\epsilon] - 1$, $\pi_1 = [\epsilon^{1/p}] - 1$, $\omega = \pi / \pi_1$ and $q = \varphi(\omega) = \varphi(\pi)/\pi$. One can easily show that $\ker(\theta : \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p})$ is the principal ideal generated by ω.

Here is a simple proof: obviously, the kernel of the induced map $\theta : \widetilde{\mathbb{E}}^+ \to \mathcal{O}_{\mathbb{C}_p}/p$ is the ideal of $x \in \widetilde{\mathbb{E}}^+$ such that $v_E(x) \geq 1$. Let y be any element of $\widetilde{\mathbb{A}}^+$ killed by θ whose reduction modulo p satisfies $v_E(\overline{y}) = 1$. The map $y\widetilde{\mathbb{A}}^+ \to \ker(\theta)$ is then injective, and surjective modulo p; since both sides are complete for the p-adic topology, it is an isomorphism. Now, we just need to observe that the element ω is killed by θ and that $v_E(\overline{\omega}) = 1$.

2.2. **The rings $\mathbb{B}_{\mathrm{cris}}$ and $\mathbb{B}_{\mathrm{dR}}$.** Using this we can define $\mathbb{B}_{\mathrm{dR}}$; let $\mathbb{B}_{\mathrm{dR}}^+$ be the ring obtained by completing $\widetilde{\mathbb{B}}^+$ for the $\ker(\theta)$-adic topology, so that

$$\mathbb{B}_{\mathrm{dR}}^+ = \varprojlim_n \widetilde{\mathbb{B}}^+ / (\ker(\theta))^n.$$

In particular, since $\ker(\theta) = (\omega)$, every element $x \in \mathbb{B}_{\mathrm{dR}}^+$ can be written (in many ways) as a sum $x = \sum_{n=0}^{+\infty} x_n \omega^n$ with $x_n \in \widetilde{\mathbb{B}}^+$. The ring $\mathbb{B}_{\mathrm{dR}}^+$ is then naturally a $\mathbb{Q}_p$-algebra. Let us construct an interesting element of this ring; since $\theta(1-[\epsilon])=0$, the element $1-[\epsilon]$ is "small" for the topology of $\mathbb{B}_{\mathrm{dR}}^{+}$ and the following series

$$-\sum_{n=1}^{+\infty}\frac{(1-[\epsilon])^n}{n}$$

will converge in $\mathbb{B}_{\mathrm{dR}}^{+}$, to an element which we call t. Of course, one should think of t as $t=\log([\epsilon])$. For instance, if $g\in \mathrm{G}_{\mathbb{Q}_p}$, then

$$g(t)=g(\log([\epsilon]))=\log([g(\epsilon^{(0)},\epsilon^{(1)},\cdots)])=\log([\epsilon^{\varepsilon(g)}])=\varepsilon(g)t$$

so that t is a "period" for the cyclotomic character ε.

We now set $\mathbb{B}_{\mathrm{dR}}=\mathbb{B}_{\mathrm{dR}}^{+}[1/t]$, which is a field that we endow with the filtration defined by $\mathrm{Fil}^i\,\mathbb{B}_{\mathrm{dR}}=t^i\mathbb{B}_{\mathrm{dR}}^{+}$. This is the natural filtration on $\mathbb{B}_{\mathrm{dR}}$ coming from the fact that $\mathbb{B}_{\mathrm{dR}}^{+}$ is a complete discrete valuation ring. By functoriality, all the rings we have defined are equipped with a continuous linear action of $\mathrm{G}_{\mathbb{Q}_p}$. One can show that $\mathbb{B}_{\mathrm{dR}}^{\mathrm{G}_{\mathbb{Q}_p}}=\mathbb{Q}_p$, so that if V is a p-adic representation, then $D_{\mathrm{dR}}(V)=(\mathbb{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V)^{\mathrm{G}_{\mathbb{Q}_p}}$ is naturally a filtered $\mathbb{Q}_p$-vector space. Note that $\dim_{\mathbb{Q}_p}(D_{\mathrm{dR}}(V))\leq d=\dim_{\mathbb{Q}_p}(V)$ in general.

Definition 2.2.1. *We say that V is* de Rham *if* $\dim_{\mathbb{Q}_p}D_{\mathrm{dR}}(V)=d$.

If V is a de Rham representation, the *Hodge-Tate weights* of V are the integers h such that $\mathrm{Fil}^{-h}\,D_{\mathrm{dR}}(V)\neq \mathrm{Fil}^{-h+1}\,D_{\mathrm{dR}}(V)$. The multiplicity of h is $\dim \mathrm{Fil}^{-h}\,D_{\mathrm{dR}}(V)/\,\mathrm{Fil}^{-h+1}\,D_{\mathrm{dR}}(V)$ so that V has d Hodge-Tate weights.

One unfortunate feature of $\mathbb{B}_{\mathrm{dR}}^{+}$ is that it is too coarse a ring: there is no natural extension of the natural Frobenius $\varphi:\widetilde{\mathbb{B}}^{+}\to\widetilde{\mathbb{B}}^{+}$ to a continuous map $\varphi:\mathbb{B}_{\mathrm{dR}}^{+}\to\mathbb{B}_{\mathrm{dR}}^{+}$. For example, if $\tilde{p}\in\widetilde{\mathbb{E}}^{+}$ is an element such that $\tilde{p}^{(0)}=p$, then $\theta([\tilde{p}^{1/p}]-p)\neq 0$, so that $[\tilde{p}^{1/p}]-p$ is invertible in $\mathbb{B}_{\mathrm{dR}}^{+}$, and so $1/([\tilde{p}^{1/p}]-p)\in\mathbb{B}_{\mathrm{dR}}^{+}$. But if φ is a natural extension of $\varphi:\widetilde{\mathbb{B}}^{+}\to\widetilde{\mathbb{B}}^{+}$, then one should have $\varphi(1/([\tilde{p}^{1/p}]-p))=1/([\tilde{p}]-p)$, and since $\theta([\tilde{p}]-p)=0$, $1/([\tilde{p}]-p)\notin\mathbb{B}_{\mathrm{dR}}^{+}$. One would still like to have a Frobenius map, and there is a natural way to complete $\widetilde{\mathbb{B}}^{+}$ (where one avoids inverting elements like $[\tilde{p}^{1/p}]-p$) such that the completion is still endowed with a Frobenius map.

Recall that the topology of $\widetilde{\mathbb{B}}^{+}$ is defined by taking the collection of open sets $\{([\overline{\pi}]^k,p^n)\widetilde{\mathbb{A}}^{+}\}_{k,n\geq 0}$ as a family of neighborhoods of 0. The ring $\mathbb{B}_{\mathrm{cris}}^{+}$ is defined as being

$$\mathbb{B}_{\mathrm{cris}}^{+}=\{\sum_{n\geq 0}a_n\frac{\omega^n}{n!}\text{ where }a_n\in\widetilde{\mathbb{B}}^{+}\text{ is sequence converging to }0\},$$

and $\mathbb{B}_{\mathrm{cris}}=\mathbb{B}_{\mathrm{cris}}^{+}[1/t]$.

One could replace ω by any generator of $\ker(\theta)$ in $\widetilde{\mathbb{A}}^{+}$. The ring $\mathbb{B}_{\mathrm{cris}}$ injects canonically into $\mathbb{B}_{\mathrm{dR}}$ ($\mathbb{B}_{\mathrm{cris}}^{+}$ is naturally defined as a subset of $\mathbb{B}_{\mathrm{dR}}^{+}$) and, in particular, it is endowed with the induced Galois action and filtration, as well as with a continuous Frobenius φ, extending the map $\varphi:\widetilde{\mathbb{B}}^{+}\to\widetilde{\mathbb{B}}^{+}$. For example, $\varphi(t)=pt$. Let us point out once again that φ does not extend continuously to $\mathbb{B}_{\mathrm{dR}}$. One also sets $\widetilde{\mathbb{B}}_{\mathrm{rig}}^{+}=\cap_{n=0}^{+\infty}\varphi^n(\mathbb{B}_{\mathrm{cris}}^{+})$.

Definition 2.2.2. *We say that a representation V of $\mathrm{G}_{\mathbb{Q}_p}$ is* crystalline *if it is $\mathbb{B}_{\mathrm{cris}}$-admissible.*

Note that V is crystalline if and only if it is $\widetilde{\mathbb{B}}_{\mathrm{rig}}^{+}[1/t]$-admissible (the periods of crystalline representations live in finite dimensional $\mathbb{Q}_p$-vector subspaces of $\mathbb{B}_{\mathrm{cris}}$, stable by φ, and so in fact in $\cap_{n=0}^{+\infty}\varphi^n(\mathbb{B}_{\mathrm{cris}}^{+})[1/t]$); this is equivalent to requiring that the $\mathbb{Q}_p$-vector space

$$D_{\mathrm{cris}}(V)=(\mathbb{B}_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}V)^{\mathrm{G}_{\mathbb{Q}_p}}=(\widetilde{\mathbb{B}}_{\mathrm{rig}}^{+}[1/t]\otimes_{\mathbb{Q}_p}V)^{\mathrm{G}_{\mathbb{Q}_p}}$$

be of dimension $d = \dim_{\mathbb{Q}_p}(V)$. Then $D_{\mathrm{cris}}(V)$ is endowed with a Frobenius φ induced by that of $\mathbb{B}_{\mathrm{cris}}$ and $(\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{G}_{\mathbb{Q}_p}} = D_{\mathrm{dR}}(V) = D_{\mathrm{cris}}(V)$ so that a crystalline representation is also de Rham and $D_{\mathrm{cris}}(V)$ is a filtered $\mathbb{Q}_p$-vector space.

If V is an E-linear representation, then we can do the above constructions (of $D_{\mathrm{cris}}(V)$ and $D_{\mathrm{dR}}(V)$) by considering V as a $\mathbb{Q}_p$-linear representation, and then one can check that $D_{\mathrm{cris}}(V)$ and $D_{\mathrm{dR}}(V)$ are themselves E-vector spaces and that: (1) $\varphi : D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V)$ is E-linear (2) the filtration on $D_{\mathrm{dR}}(V)$ is given by E-vector spaces. The Hodge-Tate weights of V are then by convention taken to be with multiplicity $\dim_E \mathrm{Fil}^{-h} D_{\mathrm{dR}}(V)/\mathrm{Fil}^{-h+1} D_{\mathrm{dR}}(V)$ and so V has $\dim_E(V)$ Hodge-Tate weights.

Remark: one can also define a ring $\mathbb{B}_{\mathrm{st}}$ and study semistable representations, but we will not do so in this course.

2.3. **Crystalline representations in dimensions 1 and 2.** It is not true that every filtered φ-module D over E arises as the D_{cris} of some crystalline representation V. If D is a filtered φ-module of dimension 1, and e is some basis, define $t_N(D)$ as the valuation of the coefficient of φ in that basis. Define $t_H(D)$ as the largest integer h such that $\mathrm{Fil}^h D \neq 0$. If D is a filtered φ-module of dimension ≥ 1, define $t_N(D) := t_N(\det D)$ and $t_H(D) := t_H(\det D)$. Note that if $D = D_{\mathrm{cris}}(V)$, then $t_H(D)$ is minus the sum of the Hodge-Tate weights. We say that D is an admissible filtered φ-module if $t_N(D) = t_H(D)$ and if for every subspace D' of D stable under φ, we have $t_N(D') - t_H(D') \geq 0$.

One can prove that if V is a crystalline representation, then the filtered φ-module $D_{\mathrm{cris}}(V)$ is admissible and conversely, Colmez and Fontaine proved the following (cf [17]):

Theorem 2.3.1. *If D is admissible, then there exists a crystalline representation V of $\mathrm{G}_{\mathbb{Q}_p}$ such that $D_{\mathrm{cris}}(V) = D$. The functor $V \mapsto D_{\mathrm{cris}}(V)$ is then an equivalence of categories, from the category of E-linear crystalline representations to the category of filtered φ-modules over E.*

This allows us to make a list of all crystalline representations: we just have to make a list of all possible admissible filtered φ-modules. We will now do this in dimensions 1 and 2. Let $D = D_{\mathrm{cris}}(V)$ be an admissible filtered φ-module.

If $\dim_E(D) = 1$ we can write $D = E \cdot e$ where $\varphi(e) = p^n \lambda e$ with $\lambda \in \mathcal{O}_E^*$ and if h is the Hodge-Tate weight of V, then $\mathrm{Fil}^{-h}(D) = D$ and $\mathrm{Fil}^{-h+1}(D) = 0$. The admissibility condition in this case is nothing more than $n = -h$. We can then explicitly describe the representation V. Let μ_λ be the unramified character of $\mathrm{G}_{\mathbb{Q}_p}$ sending σ to λ. One can then check that $V = \varepsilon^n \mu_\lambda$. If $\lambda = 1$ and $E = \mathbb{Q}_p$, then $V = \varepsilon^n$ is called $\mathbb{Q}_p(n)$ and it has the following property: if V is a de Rham representation with Hodge-Tate weights $h_1, \cdots, h_d$, then $V(n) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ has Hodge-Tate weights $h_1 + n, \cdots, h_d + n$. The representation $V(n)$ is called a *twist* of V.

If $\dim_E(D) = 2$ we can once again make a list of the possible D's. However, we won't be able to make a list of the corresponding V's because in general, it's very hard to give "explicitly" a p-adic representation. This is why the theory of filtered φ-modules is so useful.

Given a crystalline representation V, we can always twist it by a suitable power of the cyclotomic character, so that its Hodge-Tate weights are $0, k-1$ with $k \geq 1$, so now we assume this. Suppose furthermore that the Hodge-Tate weights of V are distinct, so that $k \geq 2$, and that $\varphi : D \to D$ is semi-simple (as we said in the introduction, we're only interested in those cases).

We can then enlarge E enough so that it contains the eigenvalues of φ, which we will call α^{-1} and β^{-1}. We then have $D = Ee_\alpha \oplus Ee_\beta$ with $\varphi(e_\alpha) = \alpha^{-1} e_\alpha$ and $\varphi(e_\beta) = \beta^{-1} e_\beta$. If $\alpha = \beta$, we choose any basis of D.

We have $t_H(D) = -(k-1)$ and $t_N(D) = -\operatorname{val}(\alpha) - \operatorname{val}(\beta)$ so if D is admissible, then $\operatorname{val}(\alpha) + \operatorname{val}(\beta) = k-1$. But this is not the only condition. The filtration on D is given as follows: $\operatorname{Fil}^i D$ is D if $i \leq -(k-1)$, it's a line $\Delta = E\delta$ if $-(k-2) \leq i \leq 0$ and it's 0 if $i \geq 1$ (by definition of the Hodge-Tate weights). Note in particular that $t_H(e_\alpha)$ and $t_H(e_\beta)$ are both either $-(k-1)$ or 0.

The admissibility condition says that $-\operatorname{val}(\alpha) = t_N(e_\alpha) \geq t_H(e_\alpha) \geq -(k-1)$ and similarly for $\operatorname{val}(\beta)$. This implies that $(k-1) \geq \operatorname{val}(\alpha) \geq 0$ and $(k-1) \geq \operatorname{val}(\beta) \geq 0$. Suppose that $\operatorname{val}(\beta) \leq \operatorname{val}(\alpha)$ (switch α and β if necessary). There are three cases to consider.

(1) $0 < \operatorname{val}(\beta) \leq \operatorname{val}(\alpha)$. If δ were e_α or e_β, then Δ would be a sub-φ-module of D and the admissibility condition says that $-\operatorname{val}(\alpha)$ or $-\operatorname{val}(\beta)$ should be $\geq t_H(\delta) = 0$ which is a contradiction. Therefore δ cannot be e_α nor e_β. In particular, this implies that $\alpha \neq \beta$. We can then rescale e_α and e_β so that $\delta = e_\alpha + e_\beta$.

(2) $\operatorname{val}(\beta) = 0$ and $\operatorname{val}(\alpha) = (k-1)$. Then δ can be e_β or (up to rescaling as previously) $e_\alpha + e_\beta$. Suppose it is $e_\alpha + e_\beta$. Then Ee_α is a subobject of D which is itself admissible, so it corresponds to a subrepresentation of V so that V is reducible. It is however non-split, because D has no other admissible subobjects.

(3) $\operatorname{val}(\beta) = 0$ and $\operatorname{val}(\alpha) = (k-1)$ and $\delta = e_\beta$. In this case, D is the direct sum of the two admissible objects Ee_α and Ee_β and so V is the direct sum of an unramified representation and another unramified representation twisted by $\mathbb{Q}_p(k-1)$.

As an exercise, one can classify the remaining cases (when $k = 1$ or φ is not semisimple). Finally, note that in the special case $\alpha + \beta = 0$, it is possible to give an "explicit" description of V.

To conclude this section, we note that if V is a representation which comes from a modular form, whose level N is not divisible by p, then V is crystalline and we know exactly what $D_{\mathrm{cris}}(V)$ is in terms of the modular form. This makes the approach via filtered φ-modules especially interesting for concrete applications.

3. (φ, Γ)-modules (L.B.)

In this lecture, we will explain what (φ, Γ)-modules are. The idea is similar to the one from the previous lecture: we construct a ring of periods and we use it to classify representations. In this case, we can classify all p-adic representations, as well as the $\mathrm{G}_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattices in them. The objects we use for this classification (the (φ, Γ)-modules) are unfortunately much more complicated than filtered φ-modules and not very well understood. In the next lecture, we will explain the links between the theory of filtered φ-modules and the theory of (φ, Γ)-modules for crystalline representations.

Once more, we will deal with $\mathbb{Q}_p$-linear representations and then at the end say what happens for E-linear ones (the advantage of doing this is that we can avoid an "extra layer" of notations).

3.1. **Some more rings of periods.** Let us go back to the field $\widetilde{\mathbb{E}}$ which was defined in the previous lecture. As we pointed out then, $\widetilde{\mathbb{E}}$ contains $\mathbb{F}_p((\epsilon-1))$ and is actually isomorphic to the completion of $\mathbb{F}_p((\epsilon-1))^{\mathrm{alg}}$. Let $\mathbb{E}$ be the separable closure of $\mathbb{F}_p((\epsilon-1))$ (warning: *not* the completion of the separable closure, which by a theorem of Ax is actually equal to $\widetilde{\mathbb{E}}$) and write $\mathbb{E}_{\mathbb{Q}_p}$ for $\mathbb{F}_p((\epsilon-1))$ (the reason for this terminology will be clearer later on). Write $\mathrm{G}_{\mathbb{E}}$ for the Galois group $\operatorname{Gal}(\mathbb{E}/\mathbb{E}_{\mathbb{Q}_p})$ (a more correct notation for $\mathrm{G}_{\mathbb{E}}$ would be $\mathrm{G}_{\mathbb{E}_{\mathbb{Q}_p}}$ but this is a bit awkward).

Let U be an $\mathbb{F}_p$-representation of $\mathrm{G}_{\mathbb{E}}$, that is a finite dimensional ($=d$) $\mathbb{F}_p$-vector space U with a continuous linear action of $\mathrm{G}_{\mathbb{E}}$. Since U is a finite set, this means that $\mathrm{G}_{\mathbb{E}}$ acts through an open subgroup. We can therefore apply "Hilbert 90" which tells us that the $\mathbb{E}$-representation $\mathbb{E} \otimes_{\mathbb{F}_p} U$ is isomorphic to $\mathbb{E}^d$ and therefore that $D(U) := (\mathbb{E} \otimes_{\mathbb{F}_p} U)^{\mathrm{G}_{\mathbb{E}}}$ is a d-dimensional $\mathbb{E}_{\mathbb{Q}_p}$-vector space and that $\mathbb{E} \otimes_{\mathbb{F}_p} U = \mathbb{E} \otimes_{\mathbb{E}_{\mathbb{Q}_p}} D(U)$. In addition, $D(U)$ is a *φ-module over* $\mathbb{E}_{\mathbb{Q}_p}$, meaning a vector space over $\mathbb{E}_{\mathbb{Q}_p}$ with a semilinear Frobenius φ such that $\varphi^* D(U) = D(U)$.

It is therefore possible to recover U from $D(U)$ by the formula $U = (\mathbb{E} \otimes_{\mathbb{E}_{\mathbb{Q}_p}} D(U))^{\varphi=1}$ and as a consequence, we get the following standard theorem going back at least to Katz:

Theorem 3.1.1. *The functor $U \mapsto D(U)$ is an equivalence of categories from the category of $\mathbb{F}_p$-representations of $\mathrm{G}_{\mathbb{E}}$ to the category of φ-modules over $\mathbb{E}_{\mathbb{Q}_p}$.*

Now let $\mathbb{A}_{\mathbb{Q}_p}$ be the p-adic completion of $\mathbb{Z}_p[[X]][1/X]$ inside $\widetilde{\mathbb{A}}$ where we have written X for $\pi = [\epsilon] - 1$. We can identify $\mathbb{A}_{\mathbb{Q}_p}$ with the set of power series $\sum_{i=-\infty}^{\infty} a_i X^i$ where $a_i \in \mathbb{Z}_p$ and $a_{-i} \to 0$ as $i \to \infty$. The ring $\mathbb{A}_{\mathbb{Q}_p}$ is a Cohen ring for $\mathbb{E}_{\mathbb{Q}_p}$. In particular, $\mathbb{B}_{\mathbb{Q}_p} = \mathbb{A}_{\mathbb{Q}_p}[1/p]$ is a field and we let $\mathbb{B}$ denote the p-adic completion of the maximal unramified extension of $\mathbb{B}_{\mathbb{Q}_p}$ in $\widetilde{\mathbb{B}}$ so that $\mathbb{A} = \mathbb{B} \cap \widetilde{\mathbb{A}}$ is a Cohen ring for $\mathbb{E}$.

Let T be a $\mathbb{Z}_p$-representation of $\mathrm{G}_{\mathbb{E}}$, that is a finite type $\mathbb{Z}_p$-module with a continuous linear action of $\mathrm{G}_{\mathbb{E}}$. We can lift the equivalence of categories given by theorem 3.1.1 to get the following one: the functor $T \mapsto D(T) := (\mathbb{A} \otimes_{\mathbb{Z}_p} T)^{\mathrm{G}_{\mathbb{E}}}$ is an equivalence of categories from the category of $\mathbb{Z}_p$-representations of $\mathrm{G}_{\mathbb{E}}$ to the category of φ-modules over $\mathbb{A}_{\mathbb{Q}_p}$.

Finally, by inverting p we get:

Theorem 3.1.2. *The functor $V \mapsto D(V) := (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{\mathrm{G}_{\mathbb{E}}}$ is an equivalence of categories from the category of $\mathbb{Q}_p$-representations of $\mathrm{G}_{\mathbb{E}}$ to the category of étale φ-modules over $\mathbb{B}_{\mathbb{Q}_p}$.*

Notice the appearance of "étale" in the last statement. We say that a φ-module D over $\mathbb{B}_{\mathbb{Q}_p}$ is *étale* if we can write $D = \mathbb{B}_{\mathbb{Q}_p} \otimes_{\mathbb{A}_{\mathbb{Q}_p}} D_0$ where D_0 is a φ-module over $\mathbb{A}_{\mathbb{Q}_p}$ (which includes the requirement $\varphi^* D_0 = D_0$).

This way, we have a convenient way of studying representations of $\mathrm{G}_{\mathbb{E}}$ using φ-modules over various rings. In the next paragraph, we will answer the question: what does this have to do with representations of $\mathrm{G}_{\mathbb{Q}_p}$? The answer is given by an amazing construction of Fontaine and Wintenberger.

3.2. **The theory of the field of norms.** Recall that we have chosen a primitive p^nth root of unity $\epsilon^{(n)}$ for every n. Set $F_n = \mathbb{Q}_p(\epsilon^{(n)})$ and $F_\infty = \cup_{n \geq 0} F_n$. The Galois group $\Gamma_{\mathbb{Q}_p}$ of $F_\infty/\mathbb{Q}_p$ is isomorphic to $\mathbb{Z}_p^*$ (by the cyclotomic character). In this paragraph, we will explain a striking relation between $H_{\mathbb{Q}_p}$, the Galois group of $\overline{\mathbb{Q}_p}/F_\infty$ and $\mathrm{G}_{\mathbb{E}}$ the Galois group of $\mathbb{E}/\mathbb{E}_{\mathbb{Q}_p}$. Indeed, Fontaine and Wintenberger proved that we have a natural isomorphism $H_{\mathbb{Q}_p} \simeq \mathrm{G}_{\mathbb{E}}$. We will now recall the main points of their construction.

Let K be a finite extension of $\mathbb{Q}_p$ and let $K_n = K(\epsilon^{(n)})$ as above. Let $\mathcal{N}_K$ be the set $\varprojlim_n K_n$ where the transition maps are given by $N_{K_n/K_{n-1}}$, so that $\mathcal{N}_K$ is the set of sequences $(x^{(0)}, x^{(1)}, \cdots)$ with $x^{(n)} \in K_n$ and $N_{K_n/K_{n-1}}(x^{(n)}) = x^{(n-1)}$. If we define a ring structure on $\mathcal{N}_K$ by

$$(xy)^{(n)} = x^{(n)} y^{(n)} \quad \text{and} \quad (x+y)^{(n)} = \lim_{m \to +\infty} N_{K_{n+m}/K_n}(x^{(n+m)} + y^{(n+m)}),$$

then $\mathcal{N}_K$ is actually a field, called the *field of norms* of K_∞/K. It is naturally endowed with an action of H_K. Furthermore, for every finite Galois extension L/K, $\mathcal{N}_L/\mathcal{N}_K$ is a finite Galois extension whose Galois group is $\mathrm{Gal}(L_\infty/K_\infty)$, and every finite Galois extension of $\mathcal{N}_K$ is of this kind so that the absolute Galois group of $\mathcal{N}_K$ is naturally isomorphic to H_K.

We will now describe $\mathcal{N}_{\mathbb{Q}_p}$. There is a map $\mathcal{N}_{\mathbb{Q}_p} \to \widetilde{\mathbb{E}}$ which sends $(x^{(n)})$ to $(y^{(n)})$ where $y^{(n)} = \lim (x^{(n+m)})^{p^m}$ and it is possible to show that this map is a ring homomorphism and defines an isomorphism from $\mathcal{N}_{\mathbb{Q}_p}$ to $\mathbb{E}_{\mathbb{Q}_p}$ (the point is that by ramification theory, the $N_{K_n/K_{n-1}}$ maps are pretty close to the $x \mapsto x^p$ map). As a consequence of this and similar statements for the $\mathcal{N}_K$'s, we see that there is a natural bijection between separable extensions of $\mathcal{N}_{\mathbb{Q}_p}$ and separable extensions of $\mathbb{E}_{\mathbb{Q}_p}$. Finally, we get out of this that $H_{\mathbb{Q}_p} \simeq \mathrm{G}_{\mathbb{E}}$.

Let us now go back to the main results of the previous paragraph: we constructed an equivalence of categories from the category of $\mathbb{Z}_p$-representations of $\mathrm{G}_{\mathbb{E}}$ to the category of φ-modules over $\mathbb{A}_{\mathbb{Q}_p}$ and by inverting p an equivalence of categories from the category of $\mathbb{Q}_p$-representations of $\mathrm{G}_{\mathbb{E}}$ to the category of étale φ-modules over $\mathbb{B}_{\mathbb{Q}_p}$. Since $H_{\mathbb{Q}_p} \simeq \mathrm{G}_{\mathbb{E}}$, we get equivalences of categories from the category of $\mathbb{Z}_p$-representations (resp. $\mathbb{Q}_p$-representations) of $H_{\mathbb{Q}_p}$ to the category of φ-modules over $\mathbb{A}_{\mathbb{Q}_p}$ (resp over $\mathbb{B}_{\mathbb{Q}_p}$).

Finally, if we start with a $\mathbb{Z}_p$-representation or $\mathbb{Q}_p$-representation of $\mathrm{G}_{\mathbb{Q}_p}$, then what we get is a φ-module over $\mathbb{A}_{\mathbb{Q}_p}$ (resp over $\mathbb{B}_{\mathbb{Q}_p}$) which has a residual action of $\mathrm{G}_{\mathbb{Q}_p}/H_{\mathbb{Q}_p} \simeq \Gamma_{\mathbb{Q}_p}$. This is an étale (φ,Γ)-module: it is a module over $\mathbb{A}_{\mathbb{Q}_p}$ (resp over $\mathbb{B}_{\mathbb{Q}_p}$) with a Frobenius φ (required to be étale) and an action of $\Gamma_{\mathbb{Q}_p}$ which commutes with φ. Note that on $\mathbb{A}_{\mathbb{Q}_p}$, we have

$$\varphi(X) = \varphi([\epsilon]-1) = ([\epsilon^p]-1) = (1+X)^p - 1$$

and also

$$\gamma(X) = \gamma([\epsilon]-1) = ([\epsilon^{\varepsilon(\gamma)}]-1) = (1+X)^{\varepsilon(\gamma)} - 1.$$

Theorem 3.2.1. *The functors $T \mapsto D(T)$ and $V \mapsto D(V)$ are equivalences of categories from the category of $\mathbb{Z}_p$-representations (resp. $\mathbb{Q}_p$-representations) of $\mathrm{G}_{\mathbb{Q}_p}$ to the category of étale (φ,Γ)-modules over $\mathbb{A}_{\mathbb{Q}_p}$ (resp over $\mathbb{B}_{\mathbb{Q}_p}$).*

This theorem is proved in [23], in pretty much the way which we recalled.

Example 3.2.2. *For example, let $V = \mathbb{Q}_p(r) = \mathbb{Q}_p \cdot e_r$. The restriction of V to $H_{\mathbb{Q}_p}$ is trivial, so the underlying φ-module of V is trivial: $D(V) = \mathbb{B}_{\mathbb{Q}_p} \cdot e_r$ and $\varphi(e_r) = e_r$. The action of $\Gamma_{\mathbb{Q}_p}$ is then given by $\gamma(e_r) = \varepsilon(\gamma)^r e_r$. In the lecture on Wach modules, we will see non-trivial examples of (φ,Γ)-modules.*

To conclude this paragraph, note that if E is a finite extension of $\mathbb{Q}_p$, and V is an E-linear representation (or T is an $\mathcal{O}_E$-representation) then $D(V)$ (resp $D(T)$) is a (φ,Γ)-module over $E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathbb{Q}_p}$ (resp $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_{\mathbb{Q}_p}$) and the resulting functors are again equivalences of categories.

3.3. **(φ,Γ)-modules and the operator ψ.** Let V be a p-adic representation of $\mathrm{G}_{\mathbb{Q}_p}$ and let T be a lattice of V (meaning that T is a free $\mathbb{Z}_p$-module of rank $d = \dim_{\mathbb{Q}_p}(V)$ such that $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ and such that T is stable under $\mathrm{G}_{\mathbb{Q}_p}$).

In this paragraph, we will introduce an operator $\psi : D(T) \to D(T)$ which will play a fundamental role in the sequel. Note that $\varphi : \mathbb{B} \to \mathbb{B}$ is not surjective, actually $\mathbb{B}/\varphi(\mathbb{B})$ is an extension of degree p of local fields, whose residual extension is purely inseparable. This makes it possible to define a left inverse ψ of φ by the formula:

$$\varphi(\psi(x)) = \frac{1}{p}\operatorname{Tr}_{\mathbb{B}/\varphi(\mathbb{B})}(x)$$

and then some ramification theory shows that $\psi(\mathbb{A}) \subset \mathbb{A}$. Obviously, $\psi(\varphi(x)) = x$ and more generally $\psi(\lambda\varphi(x)) = \psi(\lambda)x$, and also ψ commutes with the action of $\mathrm{G}_{\mathbb{Q}_p}$ on $\mathbb{B}$. Therefore, we get an induced map $\psi : D(T) \to D(T)$ which is a left inverse for φ. One can give a slightly more explicit description of ψ. Given $y \in D(T)$, there exist uniquely determined $y_0, \cdots, y_{p-1}$ such that $y = \sum_{i=0}^{p-1} \varphi(y_i)(1+X)^i$ and then $\psi(y) = y_0$. Finally, if $y = y(X) \in \mathbb{A}_{\mathbb{Q}_p}$, we can give a slight variant of the definition of ψ:

$$\varphi(\psi(y)) = \frac{1}{p} \sum_{\eta^p=1} y((1+X)\eta - 1).$$

The $\mathbb{Z}_p$-module $D(T)^{\psi=1}$ will be of particular importance to us. It turns out to be of independent interest in Iwasawa theory. Indeed, it is possible to give explicit maps $h_n^1 : D(T)^{\psi=1} \to H^1(F_n, T)$ for all $n \geq 0$, and these maps are compatible with each other and corestriction (meaning that $\mathrm{cor}_{F_{n+1}/F_n} \circ h_{n+1}^1 = h_n^1$) so that we get a map $D(T)^{\psi=1} \to \varprojlim_n H^1(F_n, T)$ and this map is, by a theorem of Fontaine, an isomorphism. The $\mathbb{Z}_p[[\Gamma_{\mathbb{Q}_p}]]$-module $\varprojlim_n H^1(F_n, T)$ is denoted by $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T)$ and is called the Iwasawa cohomology of T. Explaining this further is beyond the scope of the course, but it is an important application of the theory of (φ, Γ)-modules. One use we will make of the previous discussion is the following: for any T, $D(T)^{\psi=1} \neq \{0\}$ (see corollary 9.2.3 for another application). Indeed, in Iwasawa theory one proves that $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T)$ is a $\mathbb{Z}_p[[\Gamma_{\mathbb{Q}_p}]]$-module whose torsion free part is of rank $\mathrm{rk}_{\mathbb{Z}_p}(T)$.

Note that in a different direction, Cherbonnier and Colmez have proved that $D(T)^{\psi=1}$ contains a basis of $D(T)$ on $\mathbb{A}_{\mathbb{Q}_p}$. We see that therefore, the kernel of $1-\psi$ on $D(T)$ is rather large. On the other hand, Cherbonnier and Colmez have proved that $D(T)/(1-\psi)$ is very small: indeed, if V has no quotient of dimension 1, then $D(T)/(1-\psi)$ is finite (and so $D(V)/(1-\psi) = 0$). The reason is that we have an identification $D(T)/(1-\psi) = H^2_{\mathrm{Iw}}(\mathbb{Q}_p, T)$. We will use this to prove the following proposition due to Colmez which will be useful later on:

Proposition 3.3.1. *Let V be an E-linear representation such that $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V$ has no quotient of dimension 1. If $P \in E[X]$, then $P(\psi) : D(V) \to D(V)$ is surjective.*

Proof. By enlarging E if necessary, we can assume that $P(X)$ splits completely in E and so we only need to show the proposition when $P(X) = X - \alpha$ with $\alpha \in E$. There are several cases to consider:

(1) $\alpha = 0$. In this case for any $x \in D(V)$, we can write $x = \psi(\varphi(x))$.
(2) $v_p(\alpha) < 0$. In this case, we use the fact that $D(T)$ is preserved by ψ so that the series $(1-\alpha^{-1}\psi)^{-1} = 1+\alpha^{-1}\psi+(\alpha^{-1}\psi)^2+\cdots$ converges and since $(\psi-\alpha)^{-1} = -\alpha^{-1}(1-\alpha^{-1}\psi)^{-1}$ we're done.
(3) $v_p(\alpha) > 0$. By the same argument as the previous one, $(1-\alpha\varphi)^{-1}$ converges on $D(V)$. If $x \in D(V)$, we can therefore write $\varphi(x) = (1-\alpha\varphi)y$ and taking ψ, we get $x = (\psi - \alpha)y$.
(4) $v_p(\alpha) = 0$. Let μ_α be the unramified character sending the frobenius to α. Then $D(V)/(\psi - \alpha) = D(V \otimes \mu_\alpha)/(1-\psi)$ and as we have recalled above, $D(V \otimes \mu_\alpha)/(1-\psi) = 0$ if $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V$ has no quotient of dimension 1.

□

4. (φ, Γ)-modules and p-adic Hodge theory (L.B.)

In this chapter, we study the link between p-adic Hodge theory (the theory of $\mathbb{B}_{\mathrm{cris}}$ and $\mathbb{B}_{\mathrm{dR}}$) and the theory of (φ, Γ)-modules. The latter theory provides a very flexible way of studying p-adic representations, since to describe p-adic representations, one only needs to give two matrices over $\mathbb{B}_{\mathbb{Q}_p}$ (one for φ and one for a topological generator of $\Gamma_{\mathbb{Q}_p}$ (which is procyclic if $p \neq 2$)) satisfying some simple conditions (which say that φ and $\Gamma_{\mathbb{Q}_p}$ commute and that φ is étale).

4.1. **Overconvergent representations.** In order to link p-adic Hodge theory and the theory of (φ, Γ)-modules, we will apply the usual strategy: construct more rings of periods. Let $x \in \widetilde{\mathbb{B}}$ be given and write $x = \sum_{k \gg -\infty} p^k [x_k]$ with $x_k \in \widetilde{\mathbb{E}}$. Note that if $y \in \widetilde{\mathbb{E}}$, then $[y] \in \mathbb{B}_{\mathrm{dR}}^+$ but still, there is no reason for the series $x = \sum_{k \gg -\infty} p^k [x_k]$ to converge in $\mathbb{B}_{\mathrm{dR}}^+$ because the x_k's could "grow too fast" (recall that $\widetilde{\mathbb{E}}$ is a valued field). We will therefore impose growth conditions on the x_k's. Choose $r \in \mathbb{R}_{>0}$ and define

$$\widetilde{\mathbb{B}}^{\dagger,r} = \{x \in \widetilde{\mathbb{B}},\ x = \sum_{k \gg -\infty} p^k [x_k],\ k + \frac{p-1}{pr} v_E(x_k) \to +\infty\}.$$

If $r_n = p^{n-1}(p-1)$ for some $n \geq 0$, then the definition of $\widetilde{\mathbb{B}}^{\dagger,r_n}$ boils down to requiring that $\sum_{k \gg -\infty} p^k x_k^{(n)}$ converge in $\mathbb{C}_p$, which in turn is equivalent to requiring that the series $\varphi^{-n}(x) = \sum_{k \gg -\infty} p^k [x_k^{1/p^n}]$ converge in $\mathbb{B}_{\mathrm{dR}}^+$. For example $X \in \mathbb{B}^{\dagger,r}$ for all r's and $\varphi^{-n}(X) = [\epsilon^{1/p^n}] - 1 = \epsilon^{(n)} e^{t/p^n} - 1 \in \mathbb{B}_{\mathrm{dR}}^+$.

Let $\mathbb{B}^{\dagger,r} = \widetilde{\mathbb{B}}^{\dagger,r} \cap \mathbb{B}$ and let $\mathbb{B}^{\dagger} = \cup_{r>0} \mathbb{B}^{\dagger,r}$. This is the subring of elements of $\mathbb{B}$ which are "related to $\mathbb{B}_{\mathrm{dR}}$" in a way. We say that a p-adic representation V of $\mathrm{G}_{\mathbb{Q}_p}$ is *overconvergent* if $D(V)$ has a basis which is made up of elements of $D^{\dagger}(V) = (\mathbb{B}^{\dagger} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$. If there is such a basis, then by finiteness it will actually be in $D^{\dagger,r}(V) = (\mathbb{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$ for r large enough.

We therefore need to know which p-adic representations are overconvergent, and this is given to us by a theorem of Cherbonnier and Colmez (cf [12]):

Theorem 4.1.1. *Any p-adic representation V of $\mathrm{G}_{\mathbb{Q}_p}$ is overconvergent.*

This theorem is quite hard, and we will not comment its proof. For de Rham representations, one can give a simpler proof, based on the results of Kedlaya.

If V is overconvergent and if there is a basis of $D(V)$ in $D^{\dagger,r}(V)$, then $D^{\dagger,r}(V)$ is a $\mathbb{B}_{\mathbb{Q}_p}^{\dagger,r} := (\mathbb{B}^{\dagger,r})^{H_{\mathbb{Q}_p}}$-module. Warning: $\varphi(\mathbb{B}^{\dagger,r}) \subset \mathbb{B}^{\dagger,pr}$ so one has to be careful about the fact that $D^{\dagger,r}(V)$ is not a φ-module. In any case, we need to know what $\mathbb{B}_{\mathbb{Q}_p}^{\dagger,r}$ looks like. The answer turns out to be very nice. A power series $f(X) \in \mathbb{B}_{\mathbb{Q}_p}$ belongs to $\mathbb{B}_{\mathbb{Q}_p}^{\dagger,r}$ if and only if it is convergent on the annulus $0 < v_p(z) \leq 1/r$ (the fact that $f(X) \in \mathbb{B}_{\mathbb{Q}_p}$ then implies that it is bounded on that annulus). In particular, one can think "geometrically" about the elements of $\mathbb{B}_{\mathbb{Q}_p}^{\dagger,r}$ and this turns out to be very convenient.

Given $r > 0$, let $n(r)$ be the smallest n such that $p^{n-1}(p-1) \geq r$ so that $\varphi^{-n}(x)$ will converge in $\mathbb{B}_{\mathrm{dR}}^+$ for all $x \in \widetilde{\mathbb{B}}^{\dagger,r}$ and $n \geq n(r)$. We then have a map $\varphi^{-n} : D^{\dagger,r}(V) \to (\mathbb{B}_{\mathrm{dR}}^+ \otimes V)^{H_{\mathbb{Q}_p}}$ and the image of this map is in $(\mathbb{B}_{\mathrm{dR}}^+ \otimes V)^{H_{\mathbb{Q}_p}}$ if $n \geq n(r)$. This is used by Cherbonnier and Colmez to prove a number of reciprocity laws, but it is not enough for our purposes, which is to reconstruct $D_{\mathrm{cris}}(V)$ from $D^{\dagger}(V)$ if V is crystalline.

We do point out, however, that

Proposition 4.1.2. *If V is de Rham, then the image of the map $\varphi^{-n} : D^{\dagger,r}(V) \to (\mathbb{B}_{\mathrm{dR}}^+ \otimes V)^{H_{\mathbb{Q}_p}}$ lies in $F_n((t)) \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V)$ and so if V is an E-linear de Rham representation, then the image of the map $\varphi^{-n} : D^{\dagger,r}(V) \to (\mathbb{B}_{\mathrm{dR}}^+ \otimes V)^{H_{\mathbb{Q}_p}}$ lies in $(E \otimes_{\mathbb{Q}_p} F_n((t))) \otimes_E D_{\mathrm{dR}}(V)$.*

This will be used later on.

4.2. **A large ring of periods.** In this paragraph, we take up the task of reconstructing $D_{\mathrm{cris}}(V)$ from $D^{\dagger}(V)$ if V is crystalline. For this purpose, we will need a few more rings of periods. Let

$\mathbb{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}$ be the ring of power series $f(X)=\sum_{i=-\infty}^{\infty}a_iX^i$ where $a_i\in\mathbb{Q}_p$ and $f(X)$ converges on the annulus $0<v_p(z)\leq 1/r$ (but is not assumed to be bounded anymore). For example, $t=\log(1+X)$ belongs to $\mathbb{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}$ but not to $\mathbb{B}^{\dagger,r}_{\mathbb{Q}_p}$. Let $\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}=\cup_{r>0}\mathbb{B}^{\dagger,r}_{\mathrm{rig},\mathbb{Q}_p}$. Note that this ring is often given a different name, namely $\mathcal{R}_{\mathbb{Q}_p}$. It is the "Robba ring". As Colmez says, $\mathcal{R}_{\mathbb{Q}_p}$ is its "first name" and $\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}$ is its "last name". Similarly, let $\mathbb{B}^{\dagger}_{\mathbb{Q}_p}=\cup_{r>0}\mathbb{B}^{\dagger,r}_{\mathbb{Q}_p}=(\mathbb{B}^{\dagger})^{H_{\mathbb{Q}_p}}$. The first name of that ring is $\mathcal{E}^{\dagger}_{\mathbb{Q}_p}$.

The main result which we have in sight is the following:

Theorem 4.2.1. *If V is a p-adic representation of $\mathrm{G}_{\mathbb{Q}_p}$, then*

$$\left(\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}}D^{\dagger}(V)\right)^{\Gamma_{\mathbb{Q}_p}}$$

is a φ-module and as a φ-module, it is isomorphic to $D_{\mathrm{cris}}(V)$. If V is crystalline, then in addition we have

$$\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{Q}_p}\left(\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}}D^{\dagger}(V)\right)^{\Gamma_{\mathbb{Q}_p}}=\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}}D^{\dagger}(V).$$

The proof of this result is quite technical, and as one suspects, it involves introducing more rings of periods, so we will only sketch the proof. Let us mention in passing that there are analogous results for semistable representations.

Recall that $D^{\dagger}(V)=(\mathbb{B}^{\dagger}\otimes_{\mathbb{Q}_p}V)^{H_{\mathbb{Q}_p}}$ and that $D_{\mathrm{cris}}(V)=(\widetilde{\mathbb{B}}^{+}_{\mathrm{rig}}[1/t]\otimes_{\mathbb{Q}_p}V)^{\mathrm{G}_{\mathbb{Q}_p}}$ as we have explained in paragraph 2.2. The main point is then to construct a big ring $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$ which contains both $\widetilde{\mathbb{B}}^{\dagger}$ (and hence $\mathbb{B}^{\dagger}$) and $\widetilde{\mathbb{B}}^{+}_{\mathrm{rig}}$. This way, we have inclusions

$$D_{\mathrm{cris}}(V)\subset\left(\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}[1/t]\otimes_{\mathbb{Q}_p}V\right)^{\mathrm{G}_{\mathbb{Q}_p}}\tag{1}$$

and

$$\left(\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}}D^{\dagger}(V)\right)^{\Gamma_{\mathbb{Q}_p}}\subset\left(\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}[1/t]\otimes_{\mathbb{Q}_p}V\right)^{\mathrm{G}_{\mathbb{Q}_p}}.\tag{2}$$

We then need to prove that these two inclusions are equalities. There are essentially two steps. the first is to use the fact that all the above are finite dimensional $\mathbb{Q}_p$-vector spaces stable under φ and that φ^{-1} tends to "regularize" functions (think of the fact that $\varphi(\widetilde{\mathbb{B}}^{\dagger,r})=\widetilde{\mathbb{B}}^{\dagger,rp}$). The "most regular elements" of $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$ are those which are in $\widetilde{\mathbb{B}}^{+}_{\mathrm{rig}}$ which explains why inclusion (1) is an equality.

For inclusion (2), we first remark that

$$(\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}[1/t]\otimes_{\mathbb{Q}_p}V)^{H_{\mathbb{Q}_p}}=\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}}D^{\dagger}(V),$$

where $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}=(\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}})^{H_{\mathbb{Q}_p}}$ and in a way, the ring $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}$ is the completion of $\cup_{m\geq 0}\varphi^{-m}(\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p})$ in $\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$. Inclusion (2) then becomes the statement that:

$$\left(\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}}D^{\dagger}(V)\right)^{\Gamma_{\mathbb{Q}_p}}=\left(\widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]\otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}}D^{\dagger}(V)\right)^{\Gamma_{\mathbb{Q}_p}},$$

and the proof of this is a "φ-decompletion" process, similar to the one used by Cherbonnier and Colmez to prove that all p-adic representations are overconvergent, and about which we shall say no more.

4.3. **Crystalline representations.** In the previous paragraph, we explained the ideas of the (admittedly technical) proof of the facts that if V is a crystalline representation, then

$$D_{\mathrm{cris}}(V) = \left(\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}} D^{\dagger}(V)\right)^{\Gamma_{\mathbb{Q}_p}}$$

and

$$\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V) = \mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}} D^{\dagger}(V).$$

In this paragraph, we will explain one important consequence of this theorem, namely that p-adic representations are of finite height. First, let us explain what this means. Let $\mathbb{A}^+ = \widetilde{\mathbb{A}}^+ \cap \mathbb{A}$ and $\mathbb{B}^+ = \widetilde{\mathbb{B}}^+ \cap \mathbb{B}$. since $\mathbb{B}^+ \subset \widetilde{\mathbb{B}}^+$, we have a natural map $\mathbb{B}^+ \to \mathbb{B}^+_{\mathrm{dR}}$ and the rings $\mathbb{B}^+$ are "even better behaved" than the $\mathbb{B}^{\dagger,r}$'s. In addition, they are stable under φ. We say that a p-adic representation V is of *finite height* if $D(V)$ has a basis made of elements of $D^+(V) := (\mathbb{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$. This last space is a module over $\mathbb{B}^+_{\mathbb{Q}_p}$ and one can prove that $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$ which is indeed a very nice ring. We will explain the proof of Colmez' theorem (cf [13]):

Theorem 4.3.1. *Every crystalline representation of* $\mathrm{G}_{\mathbb{Q}_p}$ *is of finite height.*

This means that we can study crystalline representations effectively using $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$-modules which we will do in the next lecture.

Recall that Cherbonnier and Colmez have proved that $D(V)^{\psi=1}$ contains a basis of $D(V)$. The strategy for our proof that if V is crystalline, then V is of finite height is to prove that if V is crystalline, and S is the set

$$S = X,\ \varphi(X),\ \varphi^2(X), \cdots,$$

then $D(V)^{\psi=1} \subset (\mathbb{B}^+[S^{-1}] \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$. This will show that $D(V)$ has a basis of elements which live in $(\mathbb{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$. This proof is due to Colmez; there is a different proof, due to the first author, which relies on a result of Kedlaya. We will use the fact (due to Cherbonnier) that $D(V)^{\psi=1} \subset (\mathbb{B}^{\dagger} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$.

For simplicity, we will assume that φ is semi-simple on $D_{\mathrm{cris}}(V)$, and we will consider V as an E-linear crystalline representation, where E contains the eigenvalues of φ on $D_{\mathrm{cris}}(V)$. Suppose then that $D_{\mathrm{cris}}(V) = \oplus E \cdot e_i$ where $\varphi(e_i) = \alpha_i^{-1} e_i$. If $x \in D^{\dagger}(V)$, then we can write $x = \sum x_i \otimes e_i$ with $x_i \in E \otimes_{\mathbb{Q}_p} \mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t]$ and if $\psi(x) = x$ then, since ψ acts by φ^{-1} on $D_{\mathrm{cris}}(V)$, we have $x_i \in (E \otimes_{\mathbb{Q}_p} \mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t])^{\psi=1/\alpha_i}$.

Let $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ be the ring of power series $f(X) = \sum_{i \geq 0} a_i X^i$ which converge on the open unit disk. The first name of that ring is $\mathcal{R}^+_{\mathbb{Q}_p}$. By an argument of p-adic analysis, which we will give in paragraph 4.4, we can prove that for any $\alpha \in E$, we have $(E \otimes_{\mathbb{Q}_p} \mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t])^{\psi=\alpha} = (E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t])^{\psi=\alpha}$. We conclude that $D(V)^{\psi=1} \subset \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V)$. Finally, $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \subset \widetilde{\mathbb{B}}^+_{\mathrm{rig}}$ so that we have $D(V)^{\psi=1} \subset (\widetilde{\mathbb{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$. Let us recall that $D(V)^{\psi=1} \subset (\mathbb{B}^{\dagger} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$. This means that the periods of the elements of $D(V)^{\psi=1}$ live in $\widetilde{\mathbb{B}}^+_{\mathrm{rig}}[1/t] \cap \mathbb{B}^{\dagger}$ and one can easily prove (using the definition of all these rings) that $\widetilde{\mathbb{B}}^+_{\mathrm{rig}}[1/t] \cap \mathbb{B}^{\dagger} \subset \mathbb{B}^+[S^{-1}]$. Indeed, one should think of $\widetilde{\mathbb{B}}^+_{\mathrm{rig}}$ as "holomorphic (algebraic) functions on the open unit disk" and of $\mathbb{B}^{\dagger}$ as "bounded (algebraic) functions on some annulus". The statement that $\widetilde{\mathbb{B}}^+_{\mathrm{rig}}[1/t] \cap \mathbb{B}^{\dagger} \subset \mathbb{B}^+[S^{-1}]$ then says that a function which is both meromorphic with poles at the $\epsilon^{(n)} - 1$ and bounded toward the external boundary is the quotient of a bounded holomorphic function on the disk (an element of $\mathbb{B}^+$) by $\varphi^n(X^k)$ for k, n large enough. Therefore, V is of finite height.

In other words, the (φ, Γ)-module of a crystalline representation has a very special form: one can choose a basis such that the matrices of φ and $\gamma \in \Gamma_{\mathbb{Q}_p}$ have coefficients in $\mathbb{B}^+_{\mathbb{Q}_p}$ (in $E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathbb{Q}_p}$ if V is E-linear).

4.4. **Eigenvectors of ψ.** In this paragraph, we investigate further the action of the operator ψ on certain rings of power series. The goal is to explain the proof of the following proposition:

Proposition 4.4.1. *If $\alpha \in E^*$, then:*

$$(E \otimes_{\mathbb{Q}_p} \mathbb{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}[1/t])^{\psi=\alpha} = (E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t])^{\psi=\alpha}.$$

Proof. Since $\psi(t^{-h}f) = p^h t^{-h}\psi(f)$, it is enough to prove that

$$(E \otimes_{\mathbb{Q}_p} \mathbb{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})^{\psi=\alpha} = (E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=\alpha}$$

whenever $v_p(\alpha) < 0$ (if we are given f such that $\psi(f) = \alpha f$, then $\psi(t^h f) = p^{-h}\alpha \cdot t^h f$ so to make $v_p(\alpha) < 0$ we just need to take $h \gg 0$) To avoid cumbersome formulas, we will do the case $E = \mathbb{Q}_p$ where the whole argument already appears.

Recall that we have the following rings of power series: $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and $\mathbb{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ and $\mathbb{B}^\dagger_{\mathbb{Q}_p}$ and $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$ and $\mathbb{B}_{\mathbb{Q}_p}$ as well as $\mathbb{A}_{\mathbb{Q}_p}$ and $\mathbb{A}^+_{\mathbb{Q}_p} = \mathbb{Z}_p[[X]]$. Note that each of these rings is stable under ψ. Let $\mathbb{A}^-_{\mathbb{Q}_p}$ be the set of $f(X) = \sum a_i X^i \in \mathbb{A}_{\mathbb{Q}_p}$ such that $a_i = 0$ if $i \geq 0$ (this is not a ring!) so that $\mathbb{A}_{\mathbb{Q}_p} = \mathbb{A}^-_{\mathbb{Q}_p} \oplus \mathbb{A}^+_{\mathbb{Q}_p}$ and let $\mathbb{B}^-_{\mathbb{Q}_p} = \mathbb{A}^-_{\mathbb{Q}_p}[1/p]$.

Let us first prove that $\psi(\mathbb{A}^-_{\mathbb{Q}_p}) \subset \mathbb{A}^-_{\mathbb{Q}_p}$. To see this, recall that $q = \varphi(X)/X = 1 + [\epsilon] + \cdots + [\epsilon]^{p-1}$ so that if $\ell \geq 1$, then $\psi(q^\ell)$ is a polynomial in $[\epsilon]$ of degree $\leq \ell - 1$ (since $\psi([\epsilon^i]) = 0$ if $p \nmid i$ and $[\epsilon^{i/p}]$ otherwise) and so $\psi(q^\ell)$ is a polynomial in $X = [\epsilon] - 1$ of degree $\leq \ell - 1$. We'll leave it as an exercise to show that its constant term is $p^{\ell-1}$ (this will be useful later). Now observe that

$$\psi(X^{-\ell}) = \psi(q^\ell/\varphi(X)^\ell) = X^{-\ell}\psi(q^\ell),$$

so that $\psi(X^{-\ell}) \in \mathbb{A}^-_{\mathbb{Q}_p}$ and consequently $\psi(\mathbb{A}^-_{\mathbb{Q}_p}) \subset \mathbb{A}^-_{\mathbb{Q}_p}$.

If $f \in \mathbb{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ then we can write $f = f^- + f^+$ with $f^+ \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and $f^- \in \mathbb{B}^-_{\mathbb{Q}_p}$. In order to prove our main statement, and since ψ commutes with $f \mapsto f^\pm$, we therefore only need to study the action of ψ on $\mathbb{B}^-_{\mathbb{Q}_p}$. But since $\psi(\mathbb{A}^-_{\mathbb{Q}_p}) \subset \mathbb{A}^-_{\mathbb{Q}_p}$, ψ cannot have any eigenvalues α with $v_p(\alpha) < 0$ on $\mathbb{B}^-_{\mathbb{Q}_p}$, so we are done. $\square$

Exercise: using this, determine all the eigenvalues of ψ on $\mathbb{B}^-_{\mathbb{Q}_p} \cap \mathbb{B}^\dagger_{\mathbb{Q}_p}$.

4.5. **A review of the notation.** The point of this paragraph is to review the notation for the various rings that have been introduced so far. There are some general guidelines for understanding "who is who". The rings $\mathbb{B}_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{cris}}$ do not follow these patterns though.

Most rings are $\mathbb{E}^*_*$ or $\mathbb{A}^*_*$ or $\mathbb{B}^*_*$. An "$\mathbb{E}$" indicates a ring of characteristic p, an "$\mathbb{A}$" a ring of characteristic 0 in which p is not invertible, and a "$\mathbb{B}$" a ring of characteristic 0 in which p is invertible.

The superscripts are generally + or † (or nothing). A "+" indicates objects defined on the whole unit disk (in a sense) while a "†" indicates objects defined on an annulus (of specified radius if we have a "$\dagger, r$") and nothing generally indicates objects defined "on the boundary". A "+" also means that $X = \pi$ is not invertible. So a "$\dagger$" indicates that X is invertible but not too much (as on an annulus).

The subscript "rig" denotes holomorphic-like growth conditions toward the boundary, while no subscript means that we ask for the much stronger condition "bounded". The subscript "$\mathbb{Q}_p$" means we take invariants under $H_{\mathbb{Q}_p}$.

A tilde "$\sim$" means that φ is invertible (eg in the algebraic closure $\widetilde{\mathbb{E}}$ of $\mathbb{E}_{\mathbb{Q}_p}$) while no tilde means that φ is not invertible (eg the separable closure $\mathbb{E}$ of $\mathbb{E}_{\mathbb{Q}_p}$).

Finally, as we said, $\mathbb{B}_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{cris}}$ do not follow these patterns, in part because they are not defined "near the boundary of the open disk" like the other ones. It is better to use the ring $\widetilde{\mathbb{B}}^+_{\mathrm{rig}}[1/t]$ than the ring $\mathbb{B}_{\mathrm{cris}}$ which is linked to crystalline cohomology but has no other advantage. The notation $\widetilde{\mathbb{B}}^+_{\mathrm{rig}}$ is consistent with our above explanations. Note that this ring is also called $\mathbb{B}^+_{\mathrm{cont}}$ by some authors (with inconsistent notation).

5. Crystalline representations and Wach modules (L.B.)

This lecture will be a little different from the previous three: indeed we have (finally!) defined all of the period rings which we needed and constructed most of the objects which describe, in various ways, crystalline representations. It remains to construct the Wach module associated to a crystalline representation V, and then to prove some technical statements about the action of ψ on this module, which will be crucial for the applications to representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

5.1. **Wach modules.** Recall from the previous lecture that if V is a crystalline representation, then it is of finite height, meaning that we can write $D(V) = \mathbb{B}_{\mathbb{Q}_p} \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} D^+(V)$ where $D^+(V) = (\mathbb{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$ is a free $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$-module of rank d. If V is an E-linear representation, then we have that $D^+(V)$ is a free $E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathbb{Q}_p} = E \otimes_{\mathcal{O}_E} \mathcal{O}_E[[X]]$-module of rank $\dim_E(V)$. Once again, we will deal with $\mathbb{Q}_p$-linear representations in this chapter and at the end explain how everything remains true for E-linear objects in an obvious way.

The starting point for this paragraph is that all crystalline representations are of finite height, but there are finite height representations which are not crystalline (for example: a non-integer power of the cyclotomic character). Wach has characterized which representations V of finite height are crystalline: they are the ones for which there exists a $\mathbb{B}^+_{\mathbb{Q}_p}$-submodule N of $D^+(V)$ which is stable under $\Gamma_{\mathbb{Q}_p}$ and such that: (1) $D(V) = \mathbb{B}_{\mathbb{Q}_p} \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} N$ (2) there exists $r \in \mathbb{Z}$ such that $\Gamma_{\mathbb{Q}_p}$ acts trivially on $(N/X \cdot N)(r)$.

Suppose that the Hodge-Tate weights of V are ≤ 0. One can then take $r = 0$ above and one can further ask that $N[1/X] = D^+(V)[1/X]$. In this case, N is uniquely determined by the above requirements and we call it the Wach module $N(V)$ associated to V. It is then stable by φ. So to summarize, the Wach module $N(V)$ associated to V satisfies the following properties:

Proposition 5.1.1. *The Wach module $N(V) \subset D(V)$ has the following properties if the weights of V are ≤ 0:*

(1) it is a $\mathbb{B}^+_{\mathbb{Q}_p}$-module free of rank d
(2) it is stable under the action of $\Gamma_{\mathbb{Q}_p}$ and $\Gamma_{\mathbb{Q}_p}$ acts trivially on $N(V)/X \cdot N(V)$
(3) $N(V)$ is stable under φ.

Suppose that the Hodge-Tate weights of V are in the interval $[-h; 0]$ and let $\varphi^*(N(V))$ be the $\mathbb{B}^+_{\mathbb{Q}_p}$-module generated by $\varphi(N(V))$. One can show that $N(V)/\varphi^*(N(V))$ is killed by $(\varphi(X)/X)^h$. Recall that we write q for the important element $\varphi(X)/X$.

Note that the Wach module of $V(-h)$ is simply $X^h N(V)e_{-h}$ (here e_1 is a basis of $\mathbb{Q}_p(1)$ and $e_k = e_1^{\otimes k}$). If the Hodge-Tate weights of V are no longer ≤ 0 it is still possible to define the Wach module of V by analogy with the above formula: we set $N(V) = X^{-r}N(V(-r))e_r$ where r is large enough so that $V(-r)$ has negative Hodge-Tate weights. This obviously does not depend on the choice of r. The module $N(V) \subset D(V)$ is then stable under $\Gamma_{\mathbb{Q}_p}$ and $\Gamma_{\mathbb{Q}_p}$ acts trivially on $N(V)/XN(V)$ but $N(V)$ is no longer stable under φ. What is true is the following:

Proposition 5.1.2. *If in prop 5.1.1, the Hodge-Tate weights of V are in the interval $[a;b]$ with b not necessarily ≤ 0, then $\varphi(X^b N(V)) \subset X^b N(V)$ and $X^b N(V)/\varphi(X^b N(V))$ is killed by q^{b-a}.*

We finish this paragraph with a few examples. If $V = \mathbb{Q}_p(r)$ then $N(V) = X^{-r}\mathbb{B}^+_{\mathbb{Q}_p} e_r$ which means that $N(V)$ is a $\mathbb{B}^+_{\mathbb{Q}_p}$-module of rank 1 with a basis n_r such that $\varphi(n_r) = q^{-r} n_r$ and $\gamma(n_r) = (\gamma(X)/X)^{-r}\varepsilon(\gamma)^r n_r$ if $\gamma \in \Gamma_{\mathbb{Q}_p}$.

Suppose now that V is the representation attached to a supersingular elliptic curve (with $a_p = 0$), twisted by $\mathbb{Q}_p(-1)$ to make its weights ≤ 0. This representation is crystalline and its Hodge-Tate weights are -1 and 0 (and the eigenvalues of φ on $D_{\mathrm{cris}}(V)$ are $\pm p^{1/2}$). The Wach module $N(V)$ is of rank 2 over $\mathbb{B}^+_{\mathbb{Q}_p}$ generated by e_1 and e_2 with $\varphi(e_1) = qe_2$ and $\varphi(e_2) = -e_1$. The action of $\gamma \in \Gamma_{\mathbb{Q}_p}$ is given by:

$$\gamma(e_1) = \frac{\log^+(1+X)}{\gamma(\log^+(1+X))} e_1 \quad \text{and} \quad \gamma(e_2) = \frac{\log^-(1+X)}{\gamma(\log^-(1+X))} e_2$$

where

$$\log^+(1+X) = \prod_{n \geq 0} \frac{\varphi^{2n+1}(q)}{p} \quad \text{and} \quad \log^-(1+X) = \prod_{n \geq 0} \frac{\varphi^{2n}(q)}{p},$$

so that $t = \log(1+X) = X \log^+(1+X) \log^-(1+X)$.

Finally, we point out that if V is an E-linear representation then $N(V)$ is a free $E \otimes \mathcal{O}_E[[X]]$-module satisfying all the properties given previously.

5.2. **The weak topology.** Until now, we have largely ignored issues of topology, but for the sequel it will be important to distinguish between two topologies on $D(V)$. There is the *strong topology* which is the p-adic topology defined by choosing a lattice T of V and decreeing the $p^\ell D(T)$ to be neighborhoods of 0. This does not depend on the choice of T. Then, there is the *weak topology* which Colmez has defined for all p-adic representations V, but which we will only define for crystalline ones, using the Wach module. Choose a lattice T in V and define the Wach module associated to T to be $N(T) = N(V) \cap D(T)$. One can prove that $N(T)$ is a free $\mathbb{Z}_p[[X]]$-module of rank d such that $N(V) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} N(T)$.

For each $k \geq 0$, we define a semi-valuation ν_k on $D(T)$ as follows: if $x \in D(T)$ then $\nu_k(x)$ is the largest integer $j \in \mathbb{Z} \cup \{+\infty\}$ such that $x \in X^j N(T) + p^k D(T)$. The weak topology on $D(T)$ is then the topology defined by the set $\{\nu_k\}_{k \geq 0}$ of all those semi-valuations. Remark that the weak topology induces on $N(T)$ the (p, X)-adic topology. The weak topology on $D(V)$ is the inductive limit topology on $D(V) = \cup_{\ell \geq 0} D(p^{-\ell}T)$. This does not depend on the choice of T. Concretely, if we have a sequence $(v_n)_n$ of elements of $D(V)$, and that sequence is bounded for the weak topology, then there is a $\mathrm{G}_{\mathbb{Q}_p}$-stable lattice T of V such that $v_n \in D(T)$ for every $n \geq 0$ and furthermore, for every $k \geq 0$, there exists $f(k) \in \mathbb{Z}$ such that $v_n \in X^{-f(k)} N(T) + p^k D(T)$.

In lecture 8, we will use the weak topology on $D(T)$.

5.3. **The operator ψ and Wach modules.** In this lecture, we assume that V is a crystalline representation with ≥ 0 Hodge-Tate weights. The goal of this paragraph is to show that

Theorem 5.3.1. *If V is a crystalline representation with ≥ 0 Hodge-Tate weights and if T is a $\mathrm{G}_{\mathbb{Q}_p}$-stable lattice in V, then*

$$\psi(N(T)) \subset N(T)$$

and more generally, if $\ell \geq 1$, then

$$\psi(X^{-\ell}N(T)) \subset p^{\ell-1}X^{-\ell}N(T) + X^{-\ell+1}N(T).$$

Proof. Say that the weights of V are in $[0,h]$. We know that $\varphi(X^hN(T)) \subset X^hN(T)$ and that $X^hN(T)/\varphi^*(X^hN(T))$ is killed by q^h. If $y \in N(T)$, then $q^hX^hy \in \varphi^*(X^hN(T)) = \varphi(X^h)\varphi^*(N(T))$ and since by definition $q^hX^h = \varphi(X^h)$ this shows that $y \in \varphi^*(N(T))$ and therefore $N(T) \subset \varphi^*(N(T))$. If $y \in \varphi^*(N(T))$ then by definition we can write $y = \sum y_i\varphi(n_i)$ with $y_i \in \mathcal{O}_E[[X]]$ and $n_i \in N(T)$ and $\psi(y) = \sum \psi(y_i)n_i \in N(T)$. This proves that $\psi(N(T)) \subset N(T)$.

Now choose $y \in N(T)$ and $\ell \geq 1$. Once again, since $N(T) \subset \varphi^*(N(T))$, we can write $y = \sum y_i\varphi(n_i)$ with $y_i \in \mathcal{O}_E[[X]]$ and $n_i \in N(T)$ so that $\psi(X^{-\ell}y) = \sum \psi(X^{-\ell}y_i)n_i$. Therefore, in order to prove that

$$\psi(X^{-\ell}N(T)) \subset p^{\ell-1}X^{-\ell}N(T) + X^{-\ell+1}N(T),$$

it is enough to prove that

$$\psi(X^{-\ell}\mathcal{O}_E[[X]]) \subset p^{\ell-1}X^{-\ell}\mathcal{O}_E[[X]] + X^{-\ell+1}\mathcal{O}_E[[X]],$$

which we did in §4.4. □

5.4. **The module $D^0(T)$.** The goal of this paragraph is to explain the proof of the following proposition:

Proposition 5.4.1. *If V is crystalline irreducible and $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V$ has no quotient of* $\dim 1$ *(e.g. V is absolutely irreducible), then there exists a* unique *non-zero $\mathbb{Q}_p$-vector subspace $D^0(V)$ of $D(V)$ possessing an $\mathbb{Z}_p$-lattice $D^0(T)$ which is a compact (for the induced weak topology) $\mathbb{Z}_p[[X]]$-submodule of $D(V)$ preserved by ψ and $\Gamma_{\mathbb{Q}_p}$ with ψ surjective.*

If V has a quotient of $\dim 1$, then such an $\mathbb{Z}_p[[X]]$-module exists but is not unique. For example, if $V = \mathbb{Q}_p$, the we can take either $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$ or $X^{-1}\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$. If V is not crystalline, it still exists (this is a result of Herr) but in our case, the situation is greatly simplified if we assume V to be crystalline. Indeed, in this case $D^0(V)$ is free of rank d over $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$.

Proof. First, we'll prove the existence of such a module. Suppose that the Hodge-Tate weights of V are in $[-h;0]$. We can always twist V to assume this, because the underlying φ-module of $D(V)$ is unchanged by twists. Choose a $\mathrm{G}_{\mathbb{Q}_p}$-stable lattice T in V. Our assumption implies that $N(T) \subset \varphi^*(N(T))$ and that $\varphi^*(X^hN(T)) \subset X^hN(T)$ as we've seen previously. As a consequence, $\psi(N(T)) \subset N(T)$ and $X^hN(T) \subset \psi(X^hN(T))$. Consider the sequence

$$X^hN(T),\ \psi(X^hN(T)),\ \psi^2(X^hN(T)), \cdots$$

It is increasing and all its terms are contained in $N(T)$. Since $\mathbb{Z}_p[[X]]$ is noetherian, this implies that the sequence is eventually constant, and so there exists m_0 such that if $m \geq m_0$, then $\psi^{m+1}(X^hN(T)) = \psi^m(X^hN(T))$. One can then set $D^0_1(T) = \psi^{m_0}(X^hN(T))$ and $D^0_1(V) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} D^0_1(T)$. This proves the existence of one module $D^0_1(T)$ (and the associated $D^0_1(V)$) satisfying the conditions of the proposition. Before we prove the uniqueness statement, we want to set $D^0(V)$ to be the largest such module and first we need to prove some properties of those modules satisfying the conditions of the proposition.

Suppose therefore that some module $M(V)$ satisfies the condition of the proposition, that is we can write $M(V) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M(T)$ with $\psi : M(T) \to M(T)$ surjective and $M(T)$ compact in $D(T)$ for the induced weak topology. This last property means that for any $k \geq 0$, there exists $f(k)$ such that $M(T) \subset X^{-f(k)}N(T) + p^k D(T)$. Recall that in proposition 5.3.1, we've proved that if $f(k) \geq 1$, then:

$$\psi(X^{-f(k)}N(T)) \subset p^{f(k)-1}X^{-f(k)}N(T) + X^{-f(k)+1}N(T).$$

Since $\psi(M(T)) = M(T)$, we have

$$M(T) = \psi^m(M(T)) \subset p^{m(f(k)-1)}X^{-f(k)}N(T) + X^{-f(k)+1}N(T) + p^k D(T)$$

and if m is large enough, we finally get that $M(T) \subset X^{-f(k)+1}N(T) + p^k D(T)$. What this shows is that we can take $f(k) = 1$ for all $k \geq 0$ and so $M(T) \subset X^{-1}N(T) + p^k D(T)$ for all $k \geq 0$ which implies that $M(T) \subset X^{-1}N(T)$. Therefore, any such $M(T)$ is contained in $\cap_{m\geq 0}\psi^m(X^{-1}N(T))$.

On the other hand, if we set $D^0(T) = \cap_{m\geq 0}\psi^m(X^{-1}N(T))$, then obviously $D^0(T)$ satisfies the conditions of the proposition, and it contains $D^0_1(T)$ constructed previously so it is certainly non-zero, and $D^0(V)$ is therefore a free $\mathbb{B}^+_{\mathbb{Q}_p}$-module of rank d contained in $X^{-1}N(V)$.

Any other $M(T)$ satisfying the conditions of the proposition is included in $D^0(T)$. If $M(V)$ was a free $\mathbb{B}^+_{\mathbb{Q}_p}$-module of rank $< d$, then we would have a submodule of $N(V)$ stable under ψ and Γ. One can prove that it would then have to be stable under φ (this is easy to see in dimension $d = 2$, less so in higher dimensions) and this would correspond to a subrepresentation of V, which we assumed to be irreducible. Therefore, any such $M(V)$ is a $\mathbb{B}^+_{\mathbb{Q}_p}$-module of rank d contained in $X^{-1}N(V)$.

Recall that by proposition 3.3.1, for any polynomial P, the map $P(\psi) : D(V) \to D(V)$ is surjective. By continuity, we can find a set $\Omega \subset D(T)$ bounded for the weak topology and $\ell \geq 0$ such that $P(\psi)(\Omega) \supset p^\ell D^0(T)$. This means that for any $k \geq 0$, there exists $f(k)$ such that

$$p^\ell D^0(T) = \psi^m(p^\ell D^0(T)) \subset \psi^m \circ P(\psi)(X^{-f(k)}N(T) + p^k D(T))$$

for all $m \geq 0$, and just as above this proves that $p^\ell D^0(T) \subset P(\psi)X^{-1}N(T)$ and then that

$$p^\ell D^0(T) \subset P(\psi) \cap_{m\geq 0} \psi^m(X^{-1}N(T)) = P(\psi)(D^0(T)),$$

so that $\psi : D^0(V) \to D^0(V)$ is surjective.

We now prove the uniqueness of an $M(V)$ satisfying the conditions of the proposition. By the previous discussion, $D^0(V)/M(V)$ is a finite dimensional $\mathbb{Q}_p$-vector space stable under ψ and such that the operator $P(\psi)$ is surjective for all $P \in \mathbb{Q}_p[X]$ which is impossible unless $D^0(V) = M(V)$. $\square$

5.5. **Wach modules and $D_{\mathrm{cris}}(V)$.** We end this lecture with two ways of recovering $D_{\mathrm{cris}}(V)$ from the Wach module $N(V)$ of a crystalline representation V.

Let V be a crystalline representation. Recall that $N(V) \subset D(V)$ is not necessarily stable under φ, but that in any case $\varphi(N(V)) \subset N(V)[1/q]$. We can define a filtration on $N(V)$ in the following way: $\mathrm{Fil}^i N(V) = \{x \in N(V),\ \varphi(x) \in q^i N(V)\}$. The results of 5.1 show that if the weights of V are in $[a; b]$, then $\mathrm{Fil}^{-b} N(V) = N(V)$.

Note also that since X and $q = \varphi(X)/X$ are coprime, we have a natural identification $N(V)/X \cdot N(V) = N(V)[1/q]/X \cdot N(V)[1/q]$ and we endow $N(V)/X \cdot N(V)$ with the induced filtration from $N(V)$, and the Frobenius induced from $N(V)/X \cdot N(V) = N(V)[1/q]/X \cdot N(V)[1/q]$. This makes $N(V)/X \cdot N(V)$ into a filtered φ-module. The main result is then (see [5, §III.4]):

Theorem 5.5.1. *The two filtered φ-modules $N(V)/X \cdot N(V)$ and $D_{\mathrm{cris}}(V)$ are isomorphic.*

We shall now prove a technical result which will be used later on. Suppose now that the weights of V are ≥ 0. We've seen that both $N(V)$ and $X^{-1}N(V)$ are stable under ψ, so we get a map $\psi : X^{-1}N(V)/N(V) \to X^{-1}N(V)/N(V)$.

Proposition 5.5.2. *If we identify $X^{-1}N(V)/N(V)$ with $D_{\mathrm{cris}}(V)$ by the map $X^{-1}y \mapsto \overline{y}$, then the map $\psi : X^{-1}N(V)/N(V) \to X^{-1}N(V)/N(V)$ coincides with the map $\varphi^{-1} : D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V)$.*

Proof. Indeed, given $y \in N(V)$, choose $z \in N(V)[1/q]$ such that $y - \varphi(z) \in X \cdot N(V)[1/q]$. We then have $\psi(y/X) = \psi(\varphi(z)/X + w)$ with $w \in N(V)$ and $\psi(\varphi(z)/X) = z/X$ so that $\psi(y/X) \equiv z$ mod $N(V)[1/q]$. $\square$

Now, we will give a different relation between $D_{\mathrm{cris}}(V)$ and $N(V)$. Recall that we have an isomorphism

$$\mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V) \simeq \mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{B}^{\dagger}_{\mathbb{Q}_p}} D^{\dagger}(V)$$

and in particular an inclusion

$$N(V) \subset D^{\dagger}(V) \subset \mathbb{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V).$$

One can then prove that:

Proposition 5.5.3. *If V is crystalline, then*

$$N(V) \subset \mathbb{B}^{+}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V),$$

and

$$\mathbb{B}^{+}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{B}^{+}_{\mathbb{Q}_p}} N(V) = \mathbb{B}^{+}_{\mathrm{rig},\mathbb{Q}_p}[1/t] \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V).$$

Furthermore, if the weights of V are ≥ 0, then

$$N(V) \subset \mathbb{B}^{+}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V).$$

Exercise: work out explicitly all the above identifications for the Wach modules given at the end of 5.1.

6. Preliminaries of p-adic analysis (C.B.)

We give here some classical material of p-adic analysis which will crucially be used in the coming lectures. The results can be found in [28], [14], [29] or any treatise on p-adic analysis. Thus, we do not provide most of the proofs.

6.1. Functions of class $\mathcal{C}^r$.

Fix E a finite extension of $\mathbb{Q}_p$. We have extensively seen in Schneider and Teitelbaum's course what a locally analytic function $f : \mathbb{Q}_p \to E$ is. Let r be an element in $\mathbb{R}^+$. Here, we study the larger class of functions $f : \mathbb{Z}_p \to E$ which are *of class $\mathcal{C}^r$*. For $f : \mathbb{Z}_p \to E$ any function and $n \in \mathbb{Z}_{\geq 0}$, we first set:

$$a_n(f) := \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n-i).$$

Definition 6.1.1. *A function $f : \mathbb{Z}_p \to E$ is of class $\mathcal{C}^r$ if $n^r|a_n(f)| \to 0$ in $\mathbb{R}^+$ when $n \to +\infty$.*

Of course, if f is of class $\mathcal{C}^r$, then it is also of class $\mathcal{C}^s$ for any $s \leq r$. Functions of class $\mathcal{C}^0$ are functions f such that $a_n(f) \to 0$ in E, which can easily be seen to be equivalent to f continuous (see e.g. [15],§V.2.1). Hence, a function of class $\mathcal{C}^r$ for $r \in \mathbb{R}^+$ is necessarily at least continuous. Moreover, one can write ([15],§V.2.1):

$$f(z) = \sum_{n=0}^{+\infty} a_n(f) \binom{z}{n} \tag{3}$$

where $z \in \mathbb{Z}_p$ and $\binom{z}{n} := \frac{z(z-1)\cdots(z-n+1)}{n!}$. The E-vector space of functions of class $\mathcal{C}^r$ is a Banach for the norm $\|f\|_r := \sup_n\big((n+1)^r|a_n(f)|\big)$ which contains locally analytic functions on $\mathbb{Z}_p$, hence also locally polynomial functions. We denote this Banach by $\mathcal{C}^r(\mathbb{Z}_p, E)$.

Theorem 6.1.2. *Let d be an integer such that $r - 1 < d$, then the E-vector space of locally polynomial functions $f : \mathbb{Z}_p \to E$ of degree at most d is dense in $\mathcal{C}^r(\mathbb{Z}_p, E)$.*

This theorem is a consequence of the Amice-Vélu condition as will be seen in §6.2 (Cor.6.2.7).

We now state results on the spaces $\mathcal{C}^r(\mathbb{Z}_p, E)$ when $r = n$ is in $\mathbb{Z}_{\geq 0}$ (although this is actually probably a useless assumption for some of the results below).

For $n \in \mathbb{Z}_{\geq 0}$ and $f : \mathbb{Z}_p \to E$ any function, define $f^{[n]}(z, h_1, \cdots, h_n) : \mathbb{Z}_p \times (\mathbb{Z}_p - \{0\})^n \to E$ by induction as follows:

$$\begin{aligned} f^{[0]}(z) &:= f(z) \\ f^{[n]}(z, h_1, \cdots, h_n) &:= \frac{f^{[n-1]}(z + h_n, h_1, \cdots, h_{n-1}) - f^{[n-1]}(z, h_1, \cdots, h_{n-1})}{h_n}. \end{aligned}$$

Theorem 6.1.3. *A function $f : \mathbb{Z}_p \to E$ is of class $\mathcal{C}^n$ if and only if the functions $f^{[i]}$ for $0 \leq i \leq n$ extend continuously on $\mathbb{Z}_p^{i+1}$. Moreover, the norm $\sup_{0\leq i\leq n}\sup_{(z,h_1,\cdots,h_i)\in\mathbb{Z}_p^{i+1}}|f^{[i]}(z, h_1, \cdots, h_i)|$ is equivalent to the norm $\|\cdot\|_n$.*

See [15],§V.3.2 or [28],§54. Note that, in particular, any function f of class $\mathcal{C}^n$ admits a continuous derivative $f' := f^{[1]}(z, 0)$.

Theorem 6.1.4. *Assume $n \geq 1$. The derivative map $f \mapsto f'$ induces a continuous topological surjection of Banach spaces $\mathcal{C}^n(\mathbb{Z}_p, E) \twoheadrightarrow \mathcal{C}^{n-1}(\mathbb{Z}_p, E)$ which admits a continuous section.*

Note that, with our definition, it is not transparent that $f' \in \mathcal{C}^{n-1}(\mathbb{Z}_p, E)$ (and indeed, this requires a proof). We refer to [28],§§78 to 81. The topological surjection obviously follows from the existence of a continuous section. Let us at least explicitly give such a section $P_n : \mathcal{C}^{n-1}(\mathbb{Z}_p, E) \hookrightarrow \mathcal{C}^n(\mathbb{Z}_p, E)$:

$$P_n f(z) := \sum_{j=0}^{+\infty} \sum_{i=0}^{n-1} \frac{f^{(i)}(z_j)}{(i+1)!}(z_{j+1} - z_j)^{i+1}$$

where $z_j := \sum_{l=0}^{j-1} a_l p^l$ if $z = \sum_{l=0}^{+\infty} a_l p^l$ with $a_l \in \{0, \cdots, p-1\}$. The fact $P^n f \in \mathcal{C}^n(\mathbb{Z}_p, E)$ if $f \in \mathcal{C}^{n-1}(\mathbb{Z}_p, E)$ is proved in [28],§81. The fact P^n is continuous is proved in [28],§79 for $n = 1$ and in [29],§11 for arbitrary n.

By an obvious induction, we obtain:

Corollary 6.1.5. *The n-th derivative map $f \mapsto f^{(n)}$ induces a continuous topological surjection of Banach spaces $\mathcal{C}^n(\mathbb{Z}_p, E) \twoheadrightarrow \mathcal{C}^0(\mathbb{Z}_p, E)$ which admits a continuous section.*

Let d be an integer such that $d \geq n$. We have already seen that the closure in $\mathcal{C}^n(\mathbb{Z}_p, E)$ of the vector subspace of locally polynomial functions of degree at most d is $\mathcal{C}^n(\mathbb{Z}_p, E)$ itself (Th.6.1.2). We we will need an analogous result when $d < n$:

Theorem 6.1.6. *Assume $n \geq 1$ and $d < n$. The closure in $\mathcal{C}^n(\mathbb{Z}_p, E)$ of the vector subspace of locally polynomial functions of degree at most d is the closed subspace of $\mathcal{C}^n(\mathbb{Z}_p, E)$ of functions f such that $f^{(d+1)} = 0$.*

This is proved in [28],§68 for $n = 1$ (and $d = 0$) and in [29],§8 for n and d arbitrary.

6.2. **Tempered distributions of order r.** We let $\mathcal{R}_E$ be the Robba ring of power series with coefficients in E converging on an open annulus and $\mathcal{R}_E^+ \subset \mathcal{R}_E$ be the subring of power series converging on the open (unit) disk. We thus have $\mathcal{R}_E^+ = \{\sum_{n=0}^{+\infty} a_n X^n \mid a_n \in E,\ \lim_n |a_n| r^n = 0\ \forall\ r \in [0,1[\}$ that we equip with the natural Fréchet topology given by the collection of norms $\sup_n(|a_n| r^n)$ for $0 < r < 1$. We start by recalling Amice's famous result:

Theorem 6.2.1. *The map:*

$$\mu \mapsto \sum_{n=0}^{+\infty} \mu\Big(\binom{z}{n}\Big) X^n \tag{4}$$

induces a topological isomorphism between the dual of the E-vector space of locally analytic functions $f : \mathbb{Z}_p \to E$ and $\mathcal{R}_E^+$.

We now fix $r \in \mathbb{R}^+$.

Definition 6.2.2. *We define the space of tempered distributions of order r on $\mathbb{Z}_p$ as the Banach dual of the Banach space $\mathcal{C}^r(\mathbb{Z}_p, E)$.*

People sometimes also say "tempered distributions of order $\leq r$". A tempered distribution of order r being also locally analytic (as $\mathcal{C}^r(\mathbb{Z}_p, E)$ contains locally analytic functions), the space of tempered distributions of order r corresponds by Th.6.2.1 to a subspace of $\mathcal{R}_E^+$.

Definition 6.2.3. *An element $w = \sum_{n=0}^{+\infty} a_n X^n \in \mathcal{R}_E^+$ is of order r if $n^{-r}|a_n|$ is bounded (in $\mathbb{R}^+$) when n varies.*

The following corollary immediately follows from Def.6.1.1 and (3):

Corollary 6.2.4. *The map (4) induces a topological isomorphism between tempered distributions of order r and the subspace of $\mathcal{R}_E^+$ of elements w of order r endowed with the norm $\|w\|_r := \sup_n((n+1)^{-r}|a_n|)$.*

Let μ be a locally analytic distribution, that is, an element of the dual of the E-vector space of locally analytic functions $f : \mathbb{Z}_p \to E$. If $f : \mathbb{Z}_p \to E$ is a locally analytic function which is 0 outside of $a + p^n\mathbb{Z}_p$ ($a \in \mathbb{Z}_p$, n an integer), we use the convenient notation $\int_{a+p^n\mathbb{Z}_p} f(z)\mu(z) := \mu(f)$. For any $r \in \mathbb{R}^+$, we denote $\|\mu\|_r := \sup_n\big((n+1)^{-r}|\int_{\mathbb{Z}_p}\binom{z}{n}\mu(z)|\big) \in \mathbb{R}^+ \cup \infty$. We define another norm $\|\mu\|'_r$ as follows:

$$\|\mu\|'_r := \sup_{a\in\mathbb{Z}_p}\sup_{j,n\in\mathbb{Z}_{\geq 0}} p^{n(j-r)} \left|\int_{a+p^n\mathbb{Z}_p}(z-a)^j\mu(z)\right| \in \mathbb{R}^+ \cup \infty.$$

Lemma 6.2.5. *The two norms $\|\cdot\|_r$ and $\|\cdot\|'_r$ are equivalent (when defined).*

Proof. We are going to use an intermediate norm. For $w = \sum_{m=0}^{+\infty} a_m X^m \in \mathcal{R}_E^+$, define:

$$\|w\|''_r := \sup_n\big(p^{-nr}\sup_m(|a_m[\tfrac{m}{p^n}]!|)\big) \in \mathbb{R}^+ \cup \infty$$

where $[\frac{m}{p^n}]$ is the largest integer smaller than $\frac{m}{p^n}$. Then one can prove that there exists $c_1, c_2 \in \mathbb{R}^+$ such that $c_1\|w\|_r \leq \|w\|''_r \leq c_2\|w\|_r$ (this is purely an exercise in $\mathcal{R}_E^+$: see [15],Lem.V.3.19 and note that $|[\frac{m}{p^n}]!|$ is $p^{-\frac{m}{(p-1)p^n}}$ up to a bounded scalar). If $\|\mu\|_r'' := \|w\|_r''$ when w is the Amice transform of μ, it is thus enough to prove that $\|\cdot\|_r''$ is equivalent to $\|\cdot\|_r'$. Fix $n \in \mathbb{Z}_{\geq 0}$ and let:

$$\begin{aligned} \|\mu\|'_{r,n} &:= p^{-nr}\mathrm{sup}_{a\in\mathbb{Z}_p}\mathrm{sup}_{j\in\mathbb{Z}_{\geq 0}}\left|\int_{a+p^n\mathbb{Z}_p}\Big(\frac{z-a}{p^n}\Big)^j\mu(z)\right| \\ \|\mu\|''_{r,n} &:= p^{-nr}\mathrm{sup}_{m\in\mathbb{Z}_{\geq 0}}\left(\left|\mu\Big(\binom{z}{m}\Big[\frac{m}{p^n}\Big]!\Big)\right|\right) \end{aligned}$$

(so that $\|\mu\|'_r = \mathrm{sup}_n\|\mu\|'_{r,n}$ and $\|\mu\|''_r = \mathrm{sup}_n\|\mu\|''_{r,n}$). Note that one can replace $a \in \mathbb{Z}_p$ by $a \in \{1 \cdots, p^n\}$ in the definition of $\|\mu\|'_{r,n}$. One can prove that each function $\binom{z}{m}[\frac{m}{p^n}]!$ can be written as a (finite) linear combination of the functions $\mathbf{1}_{a+p^n\mathbb{Z}_p}\big(\frac{z-a}{p^n}\big)^j$ (for $j \in \mathbb{Z}_{\geq 0}$ and $a \in \{1 \cdots, p^n\}$) with coefficients in $\mathbb{Z}_p$. Conversely, one can also write the function $\mathbf{1}_{a+p^n\mathbb{Z}_p}\big(\frac{z-a}{p^n}\big)^j$ as an (infinite) linear combination of the functions $\binom{z}{m}[\frac{m}{p^n}]!$ with coefficients in $\mathbb{Z}_p$ converging toward 0 (see [15],§V.3.1). This implies that $\|\mu\|'_{r,n}$ is in $\mathbb{R}^+$ if and only if $\|\mu\|''_{r,n}$ is in $\mathbb{R}^+$ and, if so, $\|\mu\|'_{r,n} = \|\mu\|''_{r,n}$. Likewise with $\|\mu\|''_r$ and $\|\mu\|'_r$. This finishes the proof. □

Lemma 6.2.6. *Let μ be a linear form on the space of locally polynomial functions $f : \mathbb{Z}_p \to E$ of degree at most $d \in \mathbb{Z}_{\geq 0}$. Assume there exists a constant $C_\mu \in E$ such that $\forall a \in \mathbb{Z}_p$, $\forall j \in \{0, \cdots, d\}$ and $\forall n \in \mathbb{Z}_{\geq 0}$:*

$$\int_{a+p^n\mathbb{Z}_p}(z-a)^j\mu(z) \in C_\mu p^{n(j-r)}\mathcal{O}_E. \tag{5}$$

If $d > r - 1$, there is a unique way to extend μ as a locally analytic distribution on $\mathbb{Z}_p$ such that $\|\mu\|'_r \in \mathbb{R}^+$ (i.e. is bounded).

Proof. We give the idea of the proof in the case $d = 0$ and $r < 1$, leaving the technical details and the general case (which is analogous) to the reader. We will first define $\int_{a+p^n\mathbb{Z}_p}(z-a)^j\mu(z)$ by induction on j. For $j = 1$ (it is known for $j = 0$), we have:

$$\begin{aligned} &\int_{a+p^n\mathbb{Z}_p}(z-a)\mu(z) = \sum_{b\equiv a(p^n)}\int_{b+p^{n+1}\mathbb{Z}_p}(z-a)\mu(z) = \sum_{b\equiv a(p^n)}\int_{b+p^{n+1}\mathbb{Z}_p}(z-b)\mu(z) + \\ &\sum_{b\equiv a(p^n)}(b-a)\int_{b+p^{n+1}\mathbb{Z}_p}\mu(z) = \sum_{b\equiv a(p^n)}\sum_{c\equiv b(p^{n+1})}\int_{c+p^{n+2}\mathbb{Z}_p}(z-c)\mu(z) + \\ &\sum_{b\equiv a(p^n)}\sum_{c\equiv b(p^{n+1})}(b-c)\int_{c+p^{n+2}\mathbb{Z}_p}\mu(z) + \sum_{b\equiv a(p^n)}(b-a)\int_{b+p^{n+1}\mathbb{Z}_p}\mu(z) = \cdots \end{aligned}$$

We have

$$(b-c)\int_{c+p^{n+2}\mathbb{Z}_p}\mu(z) \in p^{n+1}C_\mu p^{-r(n+2)}\mathcal{O}_E = p^{(n+1)(1-r)}p^{-r}C_\mu\mathcal{O}_E$$

and $(a-b)\int_{b+p^{n+1}\mathbb{Z}_p}\mu(z) \in p^{n(1-r)}p^{-r}C_\mu\mathcal{O}_E$. Decomposing again $c + p^{n+2}\mathbb{Z}_p$, we see that we get a converging sum of terms indexed by N of the type $p^N\int_{\alpha+p^{N+1}\mathbb{Z}_p}\mu(z) \in p^{N(1-r)}p^{-r}C_\mu\mathcal{O}_E$ with N growing together with a remainder which is a sum of terms of the type $\int_{\alpha+p^{N+1}\mathbb{Z}_p}(z-\alpha)\mu(z)$. Since $\int_{\alpha+p^{N+1}\mathbb{Z}_p}(z-\alpha)\mu(z)$ must be p-adically very small if μ extends thanks to the condition $\|\mu\|'_r \in \mathbb{R}^+$ (which in particular implies $\int_{\alpha+p^{N+1}\mathbb{Z}_p}(z-\alpha)\mu(z) \in \|\mu\|'_r p^{(N+1)(1-r)}\mathcal{O}_E$), we immediately see that we can canonically extend μ on locally polynomial functions of degree at most 1 by setting

$\int_{a+p^n\mathbb{Z}_p}(z-a)\mu(z)$:=the above converging sum. Replacing $(z-a)$ by $(z-a)^2$, etc., an easy induction shows that μ extends in a similar way to all locally polynomial functions and by continuity to all locally analytic functions and moreover that $\|\mu\|'_r \in \mathbb{R}^+$. Unicity also follows obviously from the above decomposition and the condition $\|\mu\|'_r \in \mathbb{R}^+$. $\square$

The condition (5) in Lem.6.2.6 is sometimes called the "Amice-Vélu condition".

Corollary 6.2.7. *1) When $d > r-1$, the subspace of locally polynomial functions $f : \mathbb{Z}_p \to E$ of degree at most d is dense in $\mathcal{C}^r(\mathbb{Z}_p, E)$.*
2) Let μ be as in Lem.6.2.6 with $d > r-1$, then μ uniquely extends as a tempered distribution on $\mathbb{Z}_p$ of order r.

Proof. 1) If it is not dense, there exists a non zero μ which is tempered of order r and vanishes on locally polynomial functions of degree at most d. Because of Lem.6.2.6, μ vanishes on all locally polynomial functions, hence in particular on $\binom{z}{n}$ for any n, hence μ has 0 as Amice transform which is impossible by Th.6.2.1 since $\mu \neq 0$. 2) The unicity follows from 1) and the existence follows from Lem.6.2.6, Lem.6.2.5 and Cor.6.2.4. $\square$

Assuming Th.6.1.6 and using Th.6.1.3, one can also prove that, when r is a positive integer and $d < r$, any linear form μ as in Lem.6.2.6 extends uniquely to a continuous linear form on the closed subspace of $\mathcal{C}^r(\mathbb{Z}_p, E)$ of functions f such that $f^{(d+1)} = 0$. We leave this as an exercise to the reader.

7. Crystalline representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (C.B.)

We define and start studying unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach spaces $\Pi(V)$ associated to the 2-dimensional crystalline representations V of $\mathrm{G}_{\mathbb{Q}_p}$ as in the introduction. When V is reducible, we prove that $\Pi(V)$ is admissible of topological length 2.

7.1. **Preliminary Banach spaces.** Let V be a crystalline representation of $\mathrm{G}_{\mathbb{Q}_p}$ on a two-dimensional E-vector space with distinct Hodge-Tate weights. Replacing V by $V \otimes_E \varepsilon^n$ for some $n \in \mathbb{Z}$, we can assume that the Hodge-Tate weights of V are $(0, k-1)$ with $k \in \mathbb{Z}$, $k \geq 2$. Assuming moreover that V is F-semi-simple, i.e. that the Frobenius φ on $\mathrm{D}_{\mathrm{cris}}(V)$ is semi-simple, we have seen in Lecture 2 that there are two elements $\alpha, \beta \in \mathcal{O}_E$ such that $\alpha \neq \beta$, $\mathrm{val}(\beta) \leq \mathrm{val}(\alpha)$, $\mathrm{val}(\alpha) + \mathrm{val}(\beta) = k-1$ and $\mathrm{D}_{\mathrm{cris}}(V) = D(\alpha,\beta) = Ee_\alpha \oplus Ee_\beta$ with $\varphi(e_\alpha) = \alpha^{-1}e_\alpha$, $\varphi(e_\beta) = \beta^{-1}e_\beta$ and:

1) V absolutely irreducible: $\mathrm{val}(\alpha) > 0$, $\mathrm{val}(\beta) > 0$, $\mathrm{Fil}^i D(\alpha,\beta) = D(\alpha,\beta)$ if $i \leq -(k-1)$, $\mathrm{Fil}^i D(\alpha,\beta) = E(e_\alpha + e_\beta)$ if $-(k-2) \leq i \leq 0$ and $\mathrm{Fil}^i D(\alpha,\beta) = 0$ if $i > 0$,

2) V reducible and non-split: $\mathrm{val}(\alpha) = k-1$, $\mathrm{val}(\beta) = 0$, $\mathrm{Fil}^i D(\alpha,\beta) = D(\alpha,\beta)$ if $i \leq -(k-1)$, $\mathrm{Fil}^i D(\alpha,\beta) = E(e_\alpha + e_\beta)$ if $-(k-2) \leq i \leq 0$ and $\mathrm{Fil}^i D(\alpha,\beta) = 0$ if $i > 0$,

3) V reducible and split: $\mathrm{val}(\alpha) = k-1$, $\mathrm{val}(\beta) = 0$, $\mathrm{Fil}^i D(\alpha,\beta) = D(\alpha,\beta)$ if $i \leq -(k-1)$, $\mathrm{Fil}^i D(\alpha,\beta) = Ee_\beta$ if $-(k-2) \leq i \leq 0$ and $\mathrm{Fil}^i D(\alpha,\beta) = 0$ if $i > 0$.

The genericity hypothesis alluded to in the introduction just means that we moreover want $\alpha \neq p\beta$. Note that $D(\alpha,\beta) \simeq D(\beta,\alpha)$ when $\mathrm{val}(\alpha) = \mathrm{val}(\beta)$.

We define now preliminary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach spaces $B(\alpha)$, $B(\beta)$, $L(\alpha)$, $L(\beta)$ and, when $\mathrm{val}(\alpha) = k-1$, $N(\alpha)$.

Let $B(\alpha)$ be the following Banach space. Its underlying E-vector space is the vector space of function $f : \mathbb{Q}_p \to E$ such that $f|_{\mathbb{Z}_p}$ is of class $\mathcal{C}^{\mathrm{val}(\alpha)}$ and $(\alpha p\beta^{-1})^{\mathrm{val}(z)}z^{k-2}f(1/z)|_{\mathbb{Z}_p}$ can be extended as a function of class $\mathcal{C}^{\mathrm{val}(\alpha)}$ on $\mathbb{Z}_p$. As a vector space we thus have:

$$B(\alpha) \simeq \mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E) \oplus \mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E),\ f \mapsto f_1 \oplus f_2 \tag{6}$$

where, for $z \in \mathbb{Z}_p$, $f_1(z) := f(pz)$ and $f_2(z) := (\alpha p\beta^{-1})^{\mathrm{val}(z)}z^{k-2}f(1/z)$. Hence $B(\alpha)$ is a Banach space for the norm:

$$\|f\| := \mathrm{Max}\big(\|f_1\|_{\mathcal{C}^{\mathrm{val}(\alpha)}}, \|f_2\|_{\mathcal{C}^{\mathrm{val}(\alpha)}}\big).$$

We endow $B(\alpha)$ with an E-linear action of $\mathrm{GL}_2(\mathbb{Q}_p)$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(f)(z) = \alpha^{-\mathrm{val}(ad-bc)}(\alpha p\beta^{-1})^{\mathrm{val}(-cz+a)}(-cz+a)^{k-2}f\Big(\frac{dz-b}{-cz+a}\Big) \tag{7}$$

and to check that this action induces a continuous map $\mathrm{GL}_2(\mathbb{Q}_p)\times B(\alpha) \to B(\alpha)$ is a simple exercise that we leave to the reader (recall that by the Banach-Steinhaus Theorem (see [30]) it is enough to check that the map $\mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{Hom}_E(B(\alpha), B(\alpha))$ is continuous, where the right hand side is endowed with the weak topology of pointwise convergence). Likewise, we define $B(\beta)$ with a continuous action of $\mathrm{GL}_2(\mathbb{Q}_p)$. Note that we have a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant injection:

$$LA(\alpha) := \Big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{unr}(\alpha^{-1}) \otimes x^{k-2}\mathrm{unr}(p\beta^{-1})\Big)^{\mathrm{an}} \hookrightarrow B(\alpha) \tag{8}$$

given by:

$$\big(h : \mathrm{GL}_2(\mathbb{Q}_p) \to E\big) \mapsto \big(z \in \mathbb{Q}_p \mapsto h(\begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix})\big) \tag{9}$$

where the left hand side is a locally analytic principal series in the sense of Schneider and Teitelbaum ([31]) with left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ by right translation on functions. We thus see that the Banach $B(\alpha)$ is nothing else than:

$$\text{“}\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{unr}(\alpha^{-1}) \otimes x^{k-2}\mathrm{unr}(p\beta^{-1})\big)^{\mathcal{C}^{\mathrm{val}(\alpha)}}.\text{”}$$

We have analogous properties with $B(\beta)$ and $LA(\beta)$ by interchanging α and β.

When $\mathrm{val}(\alpha) = k-1$, let $N(\alpha) \subsetneq B(\alpha)$ be the closed E-vector subspace of functions f such that $f_1^{(k-1)} = f_2^{(k-1)} = 0$ ($k-1$-derivative) so that we have an exact sequence of Banach spaces:

$$0 \to N(\alpha) \to B(\alpha) \to \mathcal{C}^0(\mathbb{Z}_p, E)^2 \to 0 \tag{10}$$

where the third map sends (f_1, f_2) to $(f_1^{(k-1)}, f_2^{(k-1)})$. Note that this is well defined since f_1 and f_2 are $\mathcal{C}^{k-1}$ on $\mathbb{Z}_p$ (the topological surjection on the right follows from Cor.6.1.5). Now, let us look at the action of $\mathrm{GL}_2(\mathbb{Q}_p)$. When χ_1, χ_2 are two characters on $\mathbb{Q}_p^\times$ with values in $\mathcal{O}_E^\times$ (i.e. two integral characters), we denote by:

$$\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1 \otimes \chi_2\big)^{\mathcal{C}^0}$$

the Banach of continuous functions $h : \mathrm{GL}_2(\mathbb{Q}_p) \to E$ such that:

$$h(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \chi_1(a)\chi_2(d)h(g)$$

equipped with the topology of the norm $\|h\| := \mathrm{Sup}_{g\in \mathrm{GL}_2(\mathbb{Z}_p)}|h(g)|$ and with an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ by right translation. It is easily seen to be a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach. We can also describe this Banach as before using (9) in terms of continuous functions $f : \mathbb{Q}_p \to E$ satisfying a certain continuity assumption at infinity.

Lemma 7.1.1. *Assume* $\mathrm{val}(\alpha) = k-1$ *(hence* $\mathrm{val}(\beta) = 0$*), then the map* $f \mapsto f^{(k-1)}$ *induces an exact sequence of* $\mathrm{GL}_2(\mathbb{Q}_p)$*-Banach spaces on* E*:*

$$0 \to N(\alpha) \to B(\alpha) \to \left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} x^{k-1}\mathrm{unr}(\alpha^{-1}) \otimes x^{-1}\mathrm{unr}(p\beta^{-1})\right)^{\mathcal{C}^0} \to 0.$$

Proof. This is a standard and formal computation which is the same as in the locally analytic context and that we leave to the reader. Note that the characters in the right hand side parabolic induction are $\mathcal{O}_E^\times$-valued. □

Let $L(\alpha)$ (resp. $L(\beta)$) be the following closed subspace of $B(\alpha)$ (resp. $B(\beta)$): it is the closure of the E-vector subspace generated by the functions z^j and $(\alpha p\beta^{-1})^{\mathrm{val}(z-a)}(z-a)^{k-2-j}$ for $a \in \mathbb{Q}_p$ and $j \in \mathbb{Z}$, $0 \le j < \mathrm{val}(\alpha)$ (resp. $0 \le j < \mathrm{val}(\beta)$ with the convention $L(\beta) = 0$ if $\mathrm{val}(\beta) = 0$). The reader can check using (7) that $L(\alpha)$ (resp. $L(\beta)$) is preserved by $\mathrm{GL}_2(\mathbb{Q}_p)$. In order for $L(\alpha)$ (resp. $L(\beta)$) to be a subspace of $B(\alpha)$ (resp. $B(\beta)$), it is enough to check (using (6) and up to a translation on z):

Lemma 7.1.2. *For* $0 \le j < \mathrm{val}(\alpha)$*, the function* $z \mapsto (\alpha p\beta^{-1})^{\mathrm{val}(z)} z^{k-2-j}$ *is of class* $\mathcal{C}^{\mathrm{val}(\alpha)}$ *on* $\mathbb{Z}_p$*. Likewise interchanging* α *and* β*.*

Proof. Fix j as in the statement and let $f(z) := (\alpha p\beta^{-1})^{\mathrm{val}(z)} z^{k-2-j}$ which is easily seen to extend to a continuous function on $\mathbb{Z}_p$ by setting $f(z) = 0$ (look at the valuation of $f(z)$ for $z \neq 0$). Let f_0 be the zero function on $\mathbb{Z}_p$ and, for $n \in \mathbb{Z}$, $n > 0$, consider on $\mathbb{Z}_p$ the functions $f_n(z) := (\alpha p\beta^{-1})^{\mathrm{val}(z)} z^{k-2-j}$ if $\mathrm{val}(z) < n$ and $f_n(z) := 0$ otherwise. It is clear that f_n is of class $\mathcal{C}^{\mathrm{val}(\alpha)}$ on $\mathbb{Z}_p$ since it is locally polynomial. If we can prove that $f_{n+1} - f_n \to 0$ in $\mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E)$ when $n \to +\infty$, we deduce that $\sum_{n=0}^{\infty}(f_{n+1} - f_n) \in \mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E)$ since $\mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E)$ is complete. But this function is clearly f (as can be checked for any $z \in \mathbb{Z}_p$). If B is any Banach space, we have a closed embedding $B \hookrightarrow (B^*)^*$ (see [30],Lem.9.9), hence, to check that the sequence $(f_{n+1} - f_n)_n$ converges toward 0 in B, we can do it inside $(B^*)^*$. It is thus enough to prove that for any tempered distribution μ on $\mathbb{Z}_p$ of order $\mathrm{val}(\alpha)$, we have:

$$\sup_{\mu} \frac{\left|\int_{\mathbb{Z}_p}(f_{n+1}(z) - f_n(z))\mu(z)\right|}{\|\mu\|_{\mathrm{val}(\alpha)}} \longrightarrow 0 \text{ when } n \to +\infty.$$

But:

$$\begin{aligned}\int_{\mathbb{Z}_p}(f_{n+1}(z) - f_n(z))\mu(z) &= (\alpha p\beta^{-1})^n\Big(\int_{p^n\mathbb{Z}_p} z^{k-2-j}\mu(z) \quad - \int_{p^{n+1}\mathbb{Z}_p} z^{k-2-j}\mu(z)\Big)\\ &\in \|\mu\|'_{\mathrm{val}(\alpha)}{}^{-1} p^{n(2\mathrm{val}(\alpha)-k+2)} p^{n(k-2-j-\mathrm{val}(\alpha))}\mathcal{O}_E\end{aligned}$$

using that $\mathrm{val}(\alpha)+\mathrm{val}(\beta) = k-1$ and that μ is of order $\mathrm{val}(\alpha)$ (see §6.2). This implies $\int_{\mathbb{Z}_p}(f_{n+1}(z) - f_n(z))\mu(z) \in C_\mu \|\mu\|_{\mathrm{val}(\alpha)}{}^{-1} p^{n(\mathrm{val}(\alpha)-j)}\mathcal{O}_E$ hence the result since $j < \mathrm{val}(\alpha)$. □

If $\mathrm{val}(\alpha) = k-1$, note that the functions f in $L(\alpha)$ give rise to pairs (f_1, f_2) with zero $(k-1)$-derivative, hence $L(\alpha) \subseteq N(\alpha) \subsetneq B(\alpha)$ in that case. If $\mathrm{val}(\alpha) < k-1$, the authors do not know how to prove directly that $L(\alpha)$ is distinct from $B(\alpha)$.

7.2. **Definition of $\Pi(V)$ and first results.** 1) Assume V absolutely irreducible, or equivalently $0 < \mathrm{val}(\alpha) < k-1$. Then we define:

$$\Pi(V) := B(\alpha)/L(\alpha)$$

with the induced action of $\mathrm{GL}_2(\mathbb{Q}_p)$. We will prove in the next lectures that $\Pi(V)$ is always non zero, but this will use (φ,Γ)-modules.

2) Assume V is reducible and non-split. This implies $\mathrm{val}(\alpha)=k-1$ and $\mathrm{val}(\beta)=0$. We define again:

$$\Pi(V):=B(\alpha)/L(\alpha)\neq 0$$

with the induced action of $\mathrm{GL}_2(\mathbb{Q}_p)$. By Lem.7.1.1, we have an exact sequence:

$$0\to\frac{N(\alpha)}{L(\alpha)}\to\Pi(V)\to\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}x^{k-1}\mathrm{unr}(\alpha^{-1})\otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathcal{C}^0}\to 0. \tag{11}$$

We will identify below the Banach $N(\alpha)/L(\alpha)$.

3) Assume V is reducible and split. This implies $\mathrm{val}(\alpha)=k-1$ and $\mathrm{val}(\beta)=0$. Then we define:

$$\begin{aligned}\Pi(V)\ &:=\ B(\beta)\oplus\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}x^{k-1}\mathrm{unr}(\alpha^{-1})\otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathcal{C}^0}\\ &=\ \big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{unr}(\beta^{-1})\otimes x^{k-2}\mathrm{unr}(p\alpha^{-1})\big)^{\mathcal{C}^0}\oplus\\ &\qquad\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}x^{k-1}\mathrm{unr}(\alpha^{-1})\otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathcal{C}^0}.\end{aligned}$$

In that case, $\Pi(V)$ can also be written:

$$\begin{gathered}\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{unr}(\beta^{-1})\otimes\varepsilon^{k-2}\mathrm{unr}(p^{k-1}\alpha^{-1})\big)^{\mathcal{C}^0}\oplus\\ \big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\varepsilon(\varepsilon^{k-2}\mathrm{unr}(p^{k-1}\alpha^{-1}))\otimes\varepsilon^{-1}\mathrm{unr}(\beta^{-1})\big)^{\mathcal{C}^0}\end{gathered}$$

where one can see some symmetry $(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1\otimes\chi_2)^{\mathcal{C}^0}\oplus(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\varepsilon\chi_2\otimes\varepsilon^{-1}\chi_1)^{\mathcal{C}^0}$.

We will now try to "justify" these definitions, or at least explain where they are coming from. We also start studying $\Pi(V)$.

Let $\pi(\alpha):=\mathrm{Sym}^{k-2}E^2\otimes_E\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{unr}(\alpha^{-1})\otimes\mathrm{unr}(p\beta^{-1})$ (the parabolic induction is here the usual smooth one): $\pi(\alpha)$ is an irreducible locally algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ in the sense of D. Prasad (see appendix to [32]). Likewise, we define $\pi(\beta):=\mathrm{Sym}^{k-2}E^2\otimes_E\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{unr}(\beta^{-1})\otimes\mathrm{unr}(p\alpha^{-1})$. There are closed $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embeddings $\pi(\alpha)\hookrightarrow LA(\alpha)$ (resp. $\pi(\beta)\hookrightarrow LA(\beta)$) identifying the source with the subspace of functions $f:\mathbb{Q}_p\to E$ such that $f|_{\mathbb{Z}_p}$ and $(\alpha p\beta^{-1})^{\mathrm{val}(z)}z^{k-2}f(1/z)|_{\mathbb{Z}_p}$ (resp. $(\beta p\alpha^{-1})^{\mathrm{val}(z)}z^{k-2}f(1/z)|_{\mathbb{Z}_p}$) are locally polynomial functions on $\mathbb{Z}_p$ of degree at most $k-2$ (using (9)). Moreover, the usual intertwining operator between smooth generic principal series induces a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism:

$$I:\pi(\alpha)\simeq\pi(\beta). \tag{12}$$

See (24) and (25) later in the text for an explicit description of I.

Remark 7.2.1. The smooth principal series $\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{unr}(\alpha^{-1})\otimes\mathrm{unr}(p\beta^{-1})$ is natural to introduce here since it is exactly the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ that corresponds under the local Langlands correspondence (slightly twisted) to the unramified representation of $\mathrm{W}_{\mathbb{Q}_p}$ sending an arithmetic Frobenius to $\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}=$ matrix of φ^{-1} on $D(\alpha,\beta)$.

Theorem 7.2.2. *1) Assume* $\mathrm{val}(\alpha) < k-1$ *(hence* $\mathrm{val}(\beta) > 0$*) then* $B(\alpha)/L(\alpha)$ *(resp.* $B(\beta)/L(\beta)$*) is isomorphic to the completion of* $\pi(\alpha)$ *(resp.* $\pi(\beta)$*) with respect to any* $\mathcal{O}_E$*-lattice of* $\pi(\alpha)$ *(resp.* $\pi(\beta)$*) which is finitely generated over* $\mathrm{GL}_2(\mathbb{Q}_p)$.
2) Assume $\mathrm{val}(\alpha) = k-1$ *(hence* $\mathrm{val}(\beta) = 0$*) , then* $N(\alpha)/L(\alpha)$ *(resp.* $B(\beta)$*) is isomorphic to the completion of* $\pi(\alpha)$ *(resp.* $\pi(\beta)$*) with respect to any* $\mathcal{O}_E$*-lattice of* $\pi(\alpha)$ *(resp.* $\pi(\beta)$*) which is finitely generated over* $\mathrm{GL}_2(\mathbb{Q}_p)$.

Note that we don't know *a priori* in 1) that such a lattice exists in $\pi(\alpha)$ or $\pi(\beta)$. In case not, the statement means $B(\alpha)/L(\alpha) = 0 = B(\beta)/L(\beta)$. Before giving the proof of Theorem 7.2.2, we mention the following four corollaries:

Corollary 7.2.3. *Assume* $\mathrm{val}(\alpha) < k-1$*, then* $B(\alpha)/L(\alpha)$ *and* $B(\beta)/L(\beta)$ *are unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-Banach spaces and we have a commutative* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant diagram:*

$$\begin{array}{ccc} B(\alpha)/L(\alpha) & \simeq & B(\beta)/L(\beta) \\ \uparrow & & \uparrow \\ \pi(\alpha) & \overset{I}{\simeq} & \pi(\beta) \end{array}$$

where I *is the intertwining operator of (12).*

Proof. The statement follows from Th.7.2.2 together with the fact that the isomorphism I obviously sends an $\mathcal{O}_E$-lattice of $\pi(\alpha)$ of finite type over $\mathrm{GL}_2(\mathbb{Q}_p)$ to an $\mathcal{O}_E$-lattice of $\pi(\beta)$ of finite type over $\mathrm{GL}_2(\mathbb{Q}_p)$. □

We will see later on that $B(\alpha)/L(\alpha)$ and $B(\beta)/L(\beta)$ are topologically irreducible and admissible.

Corollary 7.2.4. *Assume* $\mathrm{val}(\alpha) = k-1$*, then we have a commutative* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant diagram:*

$$\begin{array}{ccc} N(\alpha)/L(\alpha) & \simeq & B(\beta) \\ \uparrow & & \uparrow \\ \pi(\alpha) & \overset{I}{\simeq} & \pi(\beta) \end{array}$$

where I *is the intertwining operator of (12). Moreover,* $N(\alpha)/L(\alpha)$ *is a topologically irreducible admissible unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-Banach and* $B(\alpha)/L(\alpha)$ *is an admissible unitary* $\mathrm{GL}_2(\mathbb{Q}_p)$*-Banach of topological length* 2 *which is a non-trivial extension of the Banach* $\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} x^{k-1}\mathrm{unr}(\alpha^{-1}) \otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathcal{C}^0}$ *by the Banach* $N(\alpha)/L(\alpha) \simeq B(\beta)$.

Proof. The first part follows from Th.7.2.2 as previously. The unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach $B(\beta)$ is admissible and topologically irreducible. The admissibility is true for any Banach $\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\big)^{\mathcal{C}^0}$ with integral χ_i and follows from the fact that restriction of functions to $\mathrm{GL}_2(\mathbb{Z}_p)$ induces an isomorphism with $\big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Z}_p)}^{\mathrm{GL}_2(\mathbb{Z}_p)} \chi_1 \otimes \chi_2\big)^{\mathcal{C}^0}$, the dual of which is $E \otimes (\mathcal{O}_E[[\mathrm{GL}_2(\mathbb{Z}_p)]] \otimes_{\mathcal{O}_E[[\mathrm{B}(\mathbb{Z}_p)]]} \mathcal{O}_E)$ (the map $\mathrm{B}(\mathbb{Z}_p) \to \mathcal{O}_E$ being $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$) which is obviously of finite type over $E \otimes_{\mathcal{O}_E} \mathcal{O}_E[[\mathrm{GL}_2(\mathbb{Z}_p)]]$ (this proof is extracted from [33]). The irreducibility can be proved as follows: if B is a closed non zero $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant E-vector subspace in $B(\beta)$, then B is also admissible and $B^{\mathrm{an}} \neq 0$ by the density of locally analytic vectors (see Schneider and Teitelbaum's course). But the locally analytic vectors in $B(\beta)$ are just $LA(\beta)$ hence $\pi(\beta) \subset B$ since $\pi(\beta)$ is the unique irreducible subobject of $LA(\beta)$. But $\pi(\beta)$ is dense by Prop.7.2.2, hence $B = B(\beta)$. This proves the statement on $N(\alpha)/L(\alpha)$. The proof that $B(\alpha)/L(\alpha)$ is admissible of topological length 2 is analogous. The exact sequence is non-split because the analogous exact sequence with locally analytic vectors is non-split thanks to Cor.7.2.6 below (and the structure of locally analytic principal series for $\mathrm{GL}_2(\mathbb{Q}_p)$). Finally, let us prove that $B(\alpha)/L(\alpha)$ is unitary. We will use the fact that the surjection $B(\alpha)/L(\alpha) \twoheadrightarrow \big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} x^{k-1}\mathrm{unr}(\alpha^{-1}) \otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathcal{C}^0}$ admits a continuous section (not $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant of course). This follows from (6), (10) and the fact that, for any positive integer n, the n^{th}-derivation map $\mathcal{C}^n(\mathbb{Z}_p, E) \twoheadrightarrow \mathcal{C}^0(\mathbb{Z}_p, E)$ admits a continuous section (Cor.6.1.5). Thus, as a Banach space, we can write $B(\alpha)/L(\alpha)$ as:

$$B(\beta) \oplus \big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} x^{k-1}\mathrm{unr}(\alpha^{-1}) \otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathcal{C}^0}. \tag{13}$$

Since the matrix $\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}$ acts continuously on $B(\alpha)/L(\alpha)$, replacing the unit ball $B(\beta)^0$ of $B(\beta)$ by $p^{-n}B(\beta)^0$ for a convenient integer n, we can assume, using (13) and the fact that the two Jordan-Hölder factors of $B(\alpha)/L(\alpha)$ are unitary, that the unit ball of $B(\alpha)/L(\alpha)$ is preserved by $\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}$ and $\mathrm{GL}_2(\mathbb{Z}_p)$, hence by $\mathrm{GL}_2(\mathbb{Q}_p)$ using the Cartan decomposition and the integrality of the central character. □

Corollary 7.2.5. *Assume* $\mathrm{val}(\alpha) < k-1$ *and* $B(\alpha)/L(\alpha) \neq 0$ *(this fact will be proved later on), then we have a continuous* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant injection:*

$$LA(\alpha) \oplus_{\pi(\alpha)} LA(\beta) \hookrightarrow (B(\alpha)/L(\alpha))^{\mathrm{an}}$$

where $\pi(\alpha)$ *embeds into* $LA(\beta)$ *via the intertwining (12).*

Proof. We have a continuous equivariant injection $LA(\alpha) \hookrightarrow (B(\alpha)/L(\alpha))^{\mathrm{an}}$ by (8) (it is injective since, otherwise, $\pi(\alpha)$ would necessarily map to zero and we know this is not true because of Th.7.2.2). Likewise with β. The corollary then follows from Cor.7.2.3. □

Because of Cor.7.2.6 below, I am tempted to conjecture that the injection in Cor.7.2.5 is actually a topological isomorphism. The statement of Cor.7.2.5 (without the description of the completion of $\pi(\alpha)$ as $B(\alpha)/L(\alpha)$) was also noted in [19],Prop.2.5.

Corollary 7.2.6. *Assume* $\mathrm{val}(\alpha) = k-1$*, then we have a topological* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant isomorphism:*

$$LA(\alpha) \oplus_{\pi(\alpha)} LA(\beta) \xrightarrow{\sim} (B(\alpha)/L(\alpha))^{\mathrm{an}}$$

where $\pi(\alpha)$ *embeds into* $LA(\beta)$ *via the intertwining (12).*

Proof. Since all the Banach in (11) are admissible by Cor.7.2.4, a theorem of Schneider and Teitelbaum tells us that we have an exact sequence of locally analytic representations from the sequence (11):

$$0 \to LA(\beta) \to \Big(\frac{B(\alpha)}{L(\alpha)}\Big)^{\mathrm{an}} \to \big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} x^{k-1}\mathrm{unr}(\alpha^{-1}) \otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathrm{an}} \to 0 \tag{14}$$

where we have used $(N(\alpha)/L(\alpha))^{\mathrm{an}} \simeq B(\beta)^{\mathrm{an}} \simeq LA(\beta)$. But we also have a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant injection $LA(\alpha) \hookrightarrow (B(\alpha)/L(\alpha))^{\mathrm{an}}$ as in Cor.7.2.5, hence a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map $LA(\alpha) \oplus_{\pi(\alpha)} LA(\beta) \to (B(\alpha)/L(\alpha))^{\mathrm{an}}$. To prove it is a topological bijection, note that from the exact sequence of locally analytic representations:

$$0 \to \pi(\alpha) \to LA(\alpha) \to \big(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} x^{k-1}\mathrm{unr}(\alpha^{-1}) \otimes x^{-1}\mathrm{unr}(p\beta^{-1})\big)^{\mathrm{an}} \to 0,$$

we deduce an exact sequence similar to (14) with $(B(\alpha)/L(\alpha))^{\mathrm{an}}$ replaced by $LA(\alpha) \oplus_{\pi(\alpha)} LA(\beta)$), together with an obvious commutative diagram between the two exact sequences. □

Proof of Th.7.2.2: Note first that the completion of $\pi(\alpha)$ doesn't depend on which lattice of finite type is chosen (if any) since all these lattices are commensurable and thus give rise to equivalent invariant norms. Moreover, this completion is also the same as the completion with respect to any $\mathcal{O}_E$-lattice finitely generated over $\mathrm{B}(\mathbb{Q}_p)$. Indeed, any $\mathcal{O}_E$-lattice of finite type over $\mathrm{GL}_2(\mathbb{Q}_p)$ is of finite type over $\mathrm{B}(\mathbb{Q}_p)$ since $\mathrm{GL}_2(\mathbb{Q}_p) = \mathrm{B}(\mathbb{Q}_p)\mathrm{GL}_2(\mathbb{Z}_p)$ and the $\mathrm{GL}_2(\mathbb{Z}_p)$-span of any vector in $\pi(\alpha)$ is finite dimensional. Conversely, one can check using the compacity of $\mathrm{GL}_2(\mathbb{Z}_p)$ that any $\mathcal{O}_E$-lattice of finite type over $\mathrm{B}(\mathbb{Q}_p)$ is contained in an $\mathcal{O}_E$-lattice of finite type over $\mathrm{GL}_2(\mathbb{Q}_p)$ (and is thus commensurable to it). If $\pi(\alpha)$ admits an $\mathcal{O}_E$-lattice preserved by $\mathrm{B}(\mathbb{Q}_p)$, then one can check that the following $\mathcal{O}_E$-submodule:

$$\sum_{j=0}^{k-2}\mathcal{O}_E[\mathrm{B}(\mathbb{Q}_p)]z^j\mathbf{1}_{\mathbb{Z}_p}+\sum_{j=0}^{k-2}\mathcal{O}_E[\mathrm{B}(\mathbb{Q}_p)](\alpha p\beta^{-1})^{\mathrm{val}(z)}z^j\mathbf{1}_{\mathbb{Q}_p-p\mathbb{Z}_p}$$

(viewing $\pi(\alpha)$ as embedded into $LA(\alpha)$) is necessarily an $\mathcal{O}_E$-lattice of $\pi(\alpha)$ which is finitely generated over $\mathrm{B}(\mathbb{Q}_p)$. If $\pi(\alpha)$ doesn't admit any $\mathcal{O}_E$-lattice preserved by $\mathrm{B}(\mathbb{Q}_p)$, then one can check that the above $\mathcal{O}_E$-submodule is the E-vector space $\pi(\alpha)$. The dual of the sought after completion of $\pi(\alpha)$ is thus isomorphic to:

$$\begin{aligned}\{\mu\in\pi(\alpha)^*\ \mid\ \forall g\in\mathrm{B}(\mathbb{Q}_p),\forall j\in\{0,\cdots,k-2\},|\mu(g(z^j\mathbf{1}_{\mathbb{Z}_p}))|\leq 1\\ \text{and } |\mu(g((\alpha p\beta^{-1})^{\mathrm{val}(z)}z^j\mathbf{1}_{\mathbb{Q}_p-p\mathbb{Z}_p}))|\leq 1\}\otimes_{\mathcal{O}_E}E.\end{aligned}\tag{15}$$

Granting the fact that the central character is integral, we can even take $g\in\begin{pmatrix}1&\mathbb{Q}_p\\0&\mathbb{Q}_p^\times\end{pmatrix}$ in (15). For $f\in\pi(\alpha)$, seeing f as a function on $\mathbb{Q}_p$ via (9), we write $\int_{\mathbb{Q}_p}f(z)\mu(z)$ instead of $\mu(f)$ in the sequel. A short computation then gives that the conditions in (15) are equivalent to the existence for each μ of a constant $C_\mu\in E$ such that $\forall a\in\mathbb{Q}_p$, $\forall j\in\{0,\cdots,k-2\}$ and $\forall n\in\mathbb{Z}$, $n>\mathrm{val}(a)$ if $a\neq 0$:

$$\int_{a+p^n\mathbb{Z}_p}(z-a)^j\mu(z)\ \in\ C_\mu p^{n(j-\mathrm{val}(\alpha))}\mathcal{O}_E\tag{16}$$

$$\int_{\mathbb{Q}_p-(a+p^n\mathbb{Z}_p)}\Big(\frac{\alpha p}{\beta}\Big)^{\mathrm{val}(z-a)}(z-a)^{k-2-j}\mu(z)\ \in\ C_\mu p^{n(\mathrm{val}(\alpha)-j)}\mathcal{O}_E\tag{17}$$

where $\int_U f(z)\mu(z):=\int_{\mathbb{Q}_p}f(z)\mathbf{1}_U(z)\mu(z)$ for any open $U\subset\mathbb{Q}_p$. Let us now first assume $\mathrm{val}(\alpha)<k-1$. From (16), we deduce (modifying C_μ if necessary):

$$\int_{(a+p^{n-1}\mathbb{Z}_p)-(a+p^n\mathbb{Z}_p)}\Big(\frac{\alpha p}{\beta}\Big)^{\mathrm{val}(z-a)}(z-a)^{k-2-j}\mu(z)\in C_\mu p^{n(\mathrm{val}(\alpha)-j)}\mathcal{O}_E.\tag{18}$$

Writing $\mathbb{Q}_p-(a+p^n\mathbb{Z}_p)=\mathbb{Q}_p-(a+p^{n-1}\mathbb{Z}_p)\amalg(a+p^{n-1}\mathbb{Z}_p)-(a+p^n\mathbb{Z}_p)$, using (18) for the right hand side and decomposing again $\mathbb{Q}_p-(a+p^{n-1}\mathbb{Z}_p)=\mathbb{Q}_p-(a+p^{n-2}\mathbb{Z}_p)\amalg(a+p^{n-2}\mathbb{Z}_p)-(a+p^{n-1}\mathbb{Z}_p)$, we see by induction that $\int_{\mathbb{Q}_p-(a+p^n\mathbb{Z}_p)}\big(\frac{\alpha p}{\beta}\big)^{\mathrm{val}(z-a)}(z-a)^{k-2-j}\mu(z)$ for $j>\mathrm{val}(\alpha)$ is uniquely determined and that (17) for $j>\mathrm{val}(\alpha)$ is a consequence of (16). Furthermore, we see by the same argument that (17) for $j=\mathrm{val}(\alpha)$ (if $\mathrm{val}(\alpha)$ is a positive integer) and all a follows from (17) for $j=\mathrm{val}(\alpha)$, $a=0$ and from (16) (develop $(z-a)^{k-2-j}$ when $n\ll 0$). Moreover, we know by Lem.6.2.6 that (16) for $j>\mathrm{val}(\alpha)$ is a consequence of (16) for $j\leq\mathrm{val}(\alpha)$. Hence, (16) and (17) are equivalent to: (16) for $j\leq\mathrm{val}(\alpha)$ and $a\in\mathbb{Q}_p$ + (17) for $j<\mathrm{val}(\alpha)$ and $a\in\mathbb{Q}_p$ + (17) for $j=\mathrm{val}(\alpha)$ and $a=0$. Any $\mu\in\pi(\alpha)^*$ is equivalent to the data of a pair (μ_1,μ_2) where, if $f\in\pi(\alpha)$, $\int_{\mathbb{Q}_p}f(z)\mu(z)=\int_{\mathbb{Z}_p}f(pz)\mu_1(z)+\int_{\mathbb{Z}_p}(\alpha p\beta^{-1})^{\mathrm{val}(z)}f(1/z)\mu_2(z)$. Assuming $j\leq\mathrm{val}(\alpha)$, we then see by an easy computation (left to the reader) that [(16) for $j = \mathrm{val}(\alpha)$ + (16) for $j < \mathrm{val}(\alpha)$ and $a \in \mathbb{Q}_p^\times$ + (16) for $j < \mathrm{val}(\alpha)$, $a = 0$ and $n \geq 0$ + (17) for $j = \mathrm{val}(\alpha)$ and $a = 0$ + (17) for $j < \mathrm{val}(\alpha)$, $a = 0$ and $n \leq 0$] is equivalent to μ_1 and μ_2 satisfying condition (5) of Lem.6.2.6 with $r = \mathrm{val}(\alpha)$ and $d = k-2$. Hence μ_1 and μ_2 are in fact tempered distributions on $\mathbb{Z}_p$ of order $\mathrm{val}(\alpha)$ by Cor.6.2.7. Thus we can integrate *any* function in $B(\alpha)$ against μ, in particular the functions of $L(\alpha)$. By making n tends to $-\infty$, we see that (16) for $j < \mathrm{val}(\alpha)$ and $a = 0$ implies μ cancels the functions $z^j \in L(\alpha)$. By making n tends to $+\infty$, we see that (17) for $j < \mathrm{val}(\alpha)$ implies μ cancels the functions $\left(\frac{\alpha p}{\beta}\right)^{\mathrm{val}(z-a)}(z-a)^{k-2-j} \in L(\alpha)$. Now, the reader can check that μ_1 and μ_2 being tempered of order $\mathrm{val}(\alpha)$ together with these cancellations is in fact *equivalent* to (16) for $j \leq \mathrm{val}(\alpha)$ and $a \in \mathbb{Q}_p$ + (17) for $j < \mathrm{val}(\alpha)$ and $a \in \mathbb{Q}_p$ + (17) for $j = \mathrm{val}(\alpha)$ and $a = 0$. This gives a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism between the dual of the completion of $\pi(\alpha)$ in (15) and $(B(\alpha)/L(\alpha))^*$. A closer examination shows that this is a topological isomorphism of Banach spaces and thus $(B(\alpha)/L(\alpha))^*$ is a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space. Using the closed embedding $B(\alpha)/L(\alpha) \hookrightarrow ((B(\alpha)/L(\alpha))^*)^*$, we see that the $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach $B(\alpha)/L(\alpha)$ is unitary since it is closed in a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach. This means that the canonical map $\pi(\alpha) \to B(\alpha)/L(\alpha)$ induces a continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism from the completion of $\pi(\alpha)$ (with respect to any finite type lattice) to $B(\alpha)/L(\alpha)$. This morphism is, as we have seen, an isomorphism on the duals for the strong topologies, hence also for the weak topologies (using the theory of [33]). By biduality, we finally get 1) for $B(\alpha)/L(\alpha)$ and also for $B(\beta)/L(\beta)$ by interchanging α and β. The proof of 2) is similar (and actually simpler since we only have to deal with $j < \mathrm{val}(\alpha)$), the only difference being that here, the conditions (16) and (17) only allow you to integrate functions in $N(\alpha)$ against μ (see the end of §6.2).

8. Representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and (φ,Γ)-modules I (C.B.)

We now focus on the study of $\Pi(V)$ when V is irreducible. Quite surprisingly, this will crucially use the (φ,Γ)-module $D(V)$ associated to V. We define here a topological space $\left(\varprojlim_\psi D(V)\right)^{\mathrm{b}}$ and give a first explicit description of it using the Wach module $N(T)$ associated to a Galois lattice in V. We then give preliminary results on distributions on $\mathbb{Q}_p$ naturally arising from $\left(\varprojlim_\psi D(V)\right)^{\mathrm{b}}$ via Amice transform.

8.1. **Back to (φ,Γ)-modules.**

Let V be as in case 1) of §7.1, $\Pi(V)$ as in §7.2 and $D(V)$ be the (φ,Γ)-module associated to V in §3.1. Recall that we have defined in §3.3 a map $\psi : D(V) \twoheadrightarrow D(V)$ which is a left-inverse of φ (and hence a surjection). Define the following E-vector space:

$$\left(\varprojlim_\psi D(V)\right)^{\mathrm{b}} := \{(v_n)_{n\in\mathbb{Z}_{\geq 0}} \mid v_n \in D(V),\ \psi(v_{n+1}) = v_n, (v_n)_n \text{ a bounded sequence in } D(V)\},$$

(recall that *bounded* means bounded for the weak topology which was defined in §5.2). We equip the above space with the following structures:

1) a left $\mathcal{O}_E[[X]]$-module structure, by $s((v_n)_n) := (\varphi^n(s)v_n)_n$, if $s \in \mathcal{O}_E[[X]]$
2) a bijection ψ given by $\psi((v_n)_n) := (\psi(v_n))_n$
3) an action of Γ given $\gamma((v_n)_n) := (\gamma v_n)_n$.

The aim of this lecture and the next one is to prove a "canonical" topological isomorphism:

Theorem 8.1.1. *Assume that V is absolutely irreducible. There is an isomorphism of topological E-vector spaces (for the weak topologies on both sides):*

$$\big(\varprojlim_{\psi} D(V)\big)^{\mathrm{b}} \simeq \Pi(V)^*$$

such that the action of $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ *on* $\Pi(V)^*$ *corresponds to* $(v_n)_n \mapsto (\psi(v_n))_n$*, the action of* $\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{Z}_p^\times \end{pmatrix}$ *to that of* $\Gamma \simeq \mathbf{Z}_p^\times$ *and the action of* $\begin{pmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ *to the multiplication by* $(1+X)^{\mathbf{Z}_p}$*.*

Such an isomorphism was inspired by an analogous isomorphism due to Colmez in the case V is semi-stable non crystalline ([16]).

Before proving this theorem, we will need to establish a number of preliminary results. Let T be a $\mathrm{G}_{\mathbf{Q}_p}$-stable lattice in V. First of all, because V is assumed to be crystalline, we can write $D(T) = \mathbb{A}_{\mathbf{Q}_p} \otimes_{\mathbf{Z}_p[[X]]} N(T)$ where $N(T)$ is the Wach module which we have constructed in §5. This is a free $\mathcal{O}_E[[X]]$-module of finite rank and thus it is equipped with the (p,X)-adic topology. The first result which we will need in order to prove theorem 8.1.1 is the following one:

Proposition 8.1.2. *Let T be a $\mathrm{G}_{\mathbf{Q}_p}$-stable lattice in a crystalline, irreducible representation V with Hodge-Tate weights 0 and $r > 0$ and let $\varprojlim_{\psi} N(T)$ denote the set of ψ-compatible sequences of elements of $N(T)$. Then the natural map:*

$$\big(\varprojlim_{\psi} N(T)\big) \otimes_{\mathcal{O}_E} E \to \big(\varprojlim_{\psi} D(V)\big)^{\mathrm{b}}$$

is a topological isomorphism.

Proof. The main content of this proposition is the explicit description of the topology of $D(V)$ in terms of $N(T)$ which we have given in §5 and its interaction with ψ. Recall that for each $k \geq 0$, we define a semi-valuation ν_k on $D(T)$ as follows: if $x \in D(T)$ then $\nu_k(x)$ is the largest integer $j \in \mathbf{Z} \cup \{+\infty\}$ such that $x \in X^j N(T) + p^k D(T)$. The weak topology on $D(T)$ is then the topology defined by the set $\{\nu_k\}_{k \geq 0}$ of all those semi-valuations. The weak topology on $D(V)$ is the inductive limit topology on $D(V) = \cup_{\ell \geq 0} D(p^{-\ell} T)$. Concretely, if we have a sequence $(v_n)_n$ of elements of $D(V)$, and that sequence is bounded for the weak topology, then there is a $\mathrm{G}_{\mathbf{Q}_p}$-stable lattice T of V such that $v_n \in D(T)$ for every $n \geq 0$ and furthermore, for every $k \geq 0$, there exists $f(k) \in \mathbf{Z}$ such that $v_n \in X^{-f(k)} N(T) + p^k D(T)$. Recall that we have proved (cf. Th.5.3.1) that if V is crystalline with positive Hodge-Tate weights, and $\ell \geq 1$, then:

$$\psi(X^{-\ell} N(T)) \subset p^{\ell-1} X^{-\ell} N(T) + X^{-(\ell-1)} N(T)$$

by iterating this m times we get:

$$\psi^m(X^{-\ell} N(T)) \subset p^{m(\ell-1)} X^{-\ell} N(T) + X^{-(\ell-1)} N(T).$$

Choose $k \geq 0$ and $\ell \geq 2$ such that $v_n \in X^{-\ell} N(T) + p^k D(T)$. Since $v_n = \psi^m(v_{n+m})$ for every $m \geq 1$, we see that $v_n \in p^{m(\ell-1)} X^{-\ell} N(T) + X^{-(\ell-1)} N(T) + p^k D(T)$ for all m and by taking m large enough we get that $v_n \in X^{-(\ell-1)} N(T) + p^k D(T)$ so that if $\ell \geq 2$ then we may replace it by $\ell - 1$. Hence, for every $k \geq 0$, we have $v_n \in X^{-1} N(T) + p^k D(T)$. This being true for every k, we get $v_n \in X^{-1} N(T)$. We will now prove that $v_n \in N(T)$ if V is irreducible (it is in fact enough to assume that the φ-module $D_{\mathrm{cris}}(V)$ has no part of slope 0). Indeed, both $X^{-1} N(V)$ and $N(V)$ are stable by ψ and we have seen in proposition 5.5.2 that the map $\psi : \pi^{-1} N(V)/N(V) \to \pi^{-1} N(V)/N(V)$ coincides, via the identification $\pi^{-1} N(V)/N(V) = D_{\mathrm{cris}}(V)$ given by $\pi^{-1} x \mapsto \overline{x}$, with the map

$\varphi^{-1}: D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V)$. If $D_{\mathrm{cris}}(V)$ has no part of slope 0, then $\varphi^{-m}(x) \to 0$ as $m \to \infty$ and so for every ℓ there exists m such that $\psi^m(X^{-1}N(T)) \subset p^\ell X^{-1}N(T) + N(T)$. Consequently, we have $v_n \in N(T)$ for all $n \geq 0$. To finish the proof of the proposition, note that $\varprojlim_\psi N(T)$ is compact and that the map:

$$\varprojlim_{\psi} N(T) \to \big(\varprojlim_{\psi} D(T)\big)^{\mathrm{b}}$$

is a continuous isomorphism, so that it is a topological isomorphism. The proposition follows by inverting p. $\square$

Next, recall that in Prop.5.5.3 we have seen that if V is a crystalline irreducible representation whose Hodge-Tate weights are ≥ 0, then we have an injective map $N(T) \hookrightarrow \mathcal{R}^+_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(V)$. Now let $V = V_{cris}(D(\alpha,\beta))$, V irreducible. Given a sequence $(v_n)_n \in \varprojlim_\psi N(T)$ we can write $v_n = w_{\alpha,n} \otimes e_\alpha + w_{\beta,n} \otimes e_\beta$. Recall that we write $\mathcal{R}^+_E$ for $E \otimes_{\mathbb{Q}_p} \mathcal{R}^+_{\mathbb{Q}_p}$.

Proposition 8.1.3. *Given $(v_n)_n \in (\varprojlim_\psi N(T)) \otimes_{\mathcal{O}_E} E$, the sequences $(w_{\alpha,n})_n$ and $(w_{\beta,n})_n$ of elements of $\mathcal{R}^+_E$ defined above satisfy the following properties:*
1) $\forall\ n \geq 0$, $w_{\alpha,n}$ (resp. $w_{\beta,n}$) is of order $\mathrm{val}(\alpha)$ *(resp.* $\mathrm{val}(\beta)$*) and $\|w_{\alpha,n}\|_{\mathrm{val}(\alpha)}$ (resp. $\|w_{\beta,n}\|_{\mathrm{val}(\beta)}$) is bounded independently of n*
2) $\forall\ n \geq 0$ and $\forall\ m \geq 1$, we have

$$\varphi^{-m}(w_{\alpha,n} \otimes e_\alpha + w_{\beta,n} \otimes e_\beta) \in \mathrm{Fil}^0((\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)[[t]] \otimes_E D(\alpha,\beta))$$

3) $\forall\ n \geq 1$, $\psi(w_{\alpha,n}) = \alpha^{-1} w_{\alpha,n-1}$ and $\psi(w_{\beta,n}) = \beta^{-1} w_{\beta,n-1}$.

Proof. We can always change T in order to have $(v_n)_n \in \varprojlim_\psi N(T)$ which we now assume. Let $c_\alpha : N(T) \to \mathcal{R}^+_E$ be the map which to $x = x_\alpha \otimes e_\alpha + x_\beta \otimes e_\beta$ assigns x_α.

To prove that $c_\alpha(N(T))$ is of order $\leq \mathrm{val}(\alpha)$, we use the following characterization of elements of order r in $\mathcal{R}^+_E$. Recall that $\mathcal{R}^+_E$ is included in a larger ring $\widetilde{\mathcal{R}}^+_E := \cap_{n \geq 0} \varphi^n(\mathbb{B}^+_{\mathrm{cris}})$ on which φ is a bijection and which has a topology defined by the family of semi-norms $\|\cdot\|_D$ where D runs over all closed disks of radii < 1. Fix one such closed disk $D = D(0,\rho)$ of radius $\rho < 1$. We say that $f \in \widetilde{\mathcal{R}}^+_E$ is of order r if the sequence $\{p^{-mr}\|\varphi^{-m} f\|_D\}$ is bounded independently of m. We can then define $\|f\|_{D,r} := \sup_m(p^{-mr}\|\varphi^{-m}f\|_D)$. The norms $\|\cdot\|_{D,r}$ (which depend on the choice of D) are equivalent to the norms $\|\cdot\|_r$ defined in §6.1. Indeed, if D_r has radius r, then $\|\varphi^{-m}f\|_{D_r} = \|f\|_{D_{r^{1/p^m}}}$ and one recovers the definition of order as "order of growth".

Now if $x \in N(T)$ and $x = x_\alpha \otimes e_\alpha + x_\beta \otimes e_\beta$ then:

$$\varphi^{-m}(x) = \varphi^{-m}(x_\alpha)\alpha^{-m} \otimes e_\alpha + \varphi^{-m}(x_\beta)\beta^{-m} \otimes e_\beta$$

and the set of $\varphi^{-m}(x)$ is bounded for any $\|\cdot\|_D$ norm (because $x \in \mathbb{A}^+[1/X] \otimes_{\mathbb{Z}_p} T$) so that the sets $\{\varphi^{-m}(x_\alpha)\alpha^{-m}\}$ and $\{\varphi^{-m}(x_\beta)\beta^{-m}\}$ are both bounded. It thus follows that x_α is of order $\mathrm{val}(\alpha)$ and x_β of order $\mathrm{val}(\beta)$. We have proved that the image of c_α is made up of elements of order $\mathrm{val}(\alpha)$ and it is bounded in the $\|\cdot\|_{\mathrm{val}(\alpha)}$ norm because $N(T)$ is an $\mathcal{O}_E[[X]]$-module of finite type. The same holds with β in place of α of course. For the second point, we use the fact that $v_n \in N(T)$ so that for all $m \geq 1$ we have $\varphi^{-m}(v_n) \in B^+_{dR} \otimes_{\mathbb{Q}_p} V$. On the other hand by the results of §4.1 (see Prop.4.1.2) we have $\varphi^{-m}(v_n) \in (\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)((t)) \otimes_E D(\alpha,\beta)$ and therefore:

$$\begin{aligned}\varphi^{-m}(v_n) \in (B^+_{dR} \otimes_{\mathbb{Q}_p} V) \cap \big((\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)((t)) \otimes_E D(\alpha,\beta)\big) \\ = \mathrm{Fil}^0\big((\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)((t)) \otimes_E D(\alpha,\beta)\big).\end{aligned}$$

Since the weights of the filtration on $D(\alpha,\beta)$ are negative, we can replace $(\mathbb{Q}_p(\mu_{p^m})\otimes_{\mathbb{Q}_p}E)((t))$ by $(\mathbb{Q}_p(\mu_{p^m})\otimes_{\mathbb{Q}_p}E)[[t]]$ in the Fil^0. Finally, the third point is obvious since $v_{n-1}=\psi(v_n)$ and ψ acts as φ^{-1} on $D_{cris}(V)$. □

Proposition 8.1.4. *Conversely, if we are given two sequences $(w_{\alpha,n})_n$ and $(w_{\beta,n})_n$ of elements of $\mathcal{R}_E^+$ satisfying properties 1)-3) of Prop.8.1.3 and if we set $v_n := w_{\alpha,n}\otimes e_\alpha + w_{\beta,n}\otimes e_\beta$, then $(v_n)_n \in (\varprojlim_\psi N(T))\otimes_{\mathcal{O}_E} E$.*

Proof. It is enough to show that there exists a lattice T such that $v_n\in N(T)$ because condition 3) of Prop.8.1.3 ensures that the sequence $(v_n)_n$ is ψ-compatible. Choose any lattice U. By the results of §4 and §5 (more specifically proposition 5.5.3), we know that $v_n\in\mathcal{R}_E^+[1/t]\otimes_{\mathcal{O}_E[[X]]}N(U)$.

Condition 2) of proposition 8.1.3 then implies that $v_n\in\mathcal{R}_E^+[1/X]\otimes_{\mathcal{O}_E[[X]]}N(U)$, because the filtration condition for $m\ge 1$ is equivalent to an order-of-vanishing condition at $\epsilon^{(m)}-1$, and implies that $v_n\in\mathcal{R}_E^+[\varphi^{m-1}(q)/t]\otimes_{\mathcal{O}_E[[X]]}N(U)$. By using this for each $m\ge 1$, we get that $v_n\in\mathcal{R}_E^+[1/X]\otimes_{\mathcal{O}_E[[X]]}N(U)$.

The ψ-compatibility allows us to get rid of the denominators in X. Finally, condition 1) tells us that there exists ℓ independent of n such that $v_n\in p^{-\ell}N(U)$ so we can take $T=p^{-\ell}U$. □

It is important to notice that the map $\psi : N(T)\to N(T)$ is *not* surjective. Recall that we have in particular proved in Prop.5.4.1 the following result (we actually proved more, and we proved it for $\mathbb{Q}_p$-linear representations, but as usual the E-linear result follows immediately):

Proposition 8.1.5. *If V is crystalline absolutely irreducible, then there exists a* unique *non-zero E-vector subspace $D^0(V)$ of $D(V)$ possessing an $\mathcal{O}_E$-lattice $D^0(T)$ which is a compact $\mathcal{O}_E[[X]]$-submodule of $D(V)$ preserved by ψ and Γ with ψ surjective.*

In fact, as we saw, the lattice $D^0(T)$ is of finite type over $\mathcal{O}_E[[X]]$. The following corollary will be used in §9.3 to prove the irreducibility of $\Pi(V)$ (for V irreducible):

Corollary 8.1.6. *Let $\mathcal{M}\subseteq\varprojlim_\psi N(T)$ be a closed non zero $\mathcal{O}_E[[X]]$-submodule which is stable by ψ, ψ^{-1} and Γ, then:*

$$\mathcal{M}\otimes_{\mathcal{O}_E}E=(\varprojlim_{\psi}N(T))\otimes_{\mathcal{O}_E}E=\big(\varprojlim_{\psi}D(V)\big)^{\mathrm{b}}.$$

Proof. Note first that since $\varprojlim_\psi N(T)$ is compact, so is $\mathcal{M}$ (being closed in a compact set). For $m\in\mathbb{Z}_{\ge 0}$, denote by $\mathrm{pr}_m : \varprojlim_\psi N(T)\to N(T)$, $(v_n)_n\mapsto v_m$ and define $M_m := \mathrm{pr}_m(\mathcal{M})$. As $\mathcal{M}\neq 0$, there exists m such that $M_m\neq 0$. Using that $\mathcal{M}$ is stable by ψ and ψ^{-1}, it is straightforward to check that M_m doesn't depend on m and we denote it by M. Via $M=M_0$, it is an $\mathcal{O}_E[[X]]$-submodule of $N(T)$ (necessarily of finite type since $N(T)$ is) stable by Γ and ψ and such that ψ is surjective. Moreover, the canonical map $\mathcal{M}\to\varprojlim_\psi M$ is easily checked to be an isomorphism since $\mathcal{M}$ is both dense in $\varprojlim_\psi M$ and compact (the density follows from $M=\mathrm{pr}_m(\mathcal{M})\ \forall\, m$). Doing the same with $\varprojlim_\psi N(T)$, we find another (finite type) $\mathcal{O}_E[[X]]$-submodule M^0 of $N(T)$ containing M stable by Γ and ψ with ψ surjective and such that $\varprojlim_\psi N(T)=\varprojlim_\psi M^0$. By Prop.8.1.5, we have thus $M\otimes_{\mathcal{O}_E}E=M^0\otimes_{\mathcal{O}_E}E$ and there exists $a\in E^\times$ such that $M\subset M^0\subset aM$. Hence, we have $(\varprojlim_\psi M)\otimes_{\mathcal{O}_E}E\simeq(\varprojlim_\psi M^0)\otimes_{\mathcal{O}_E}E$ which finishes the proof. □

8.2. **Back to distributions.** Recall from §6.2 that $\mu \mapsto \sum_{n=0}^{+\infty} \int_{\mathbb{Z}_p} \binom{z}{n}\mu(z)X^n$ gives a bijection between $\mathcal{R}_E^+$ and "E-valued" locally analytic distributions on $\mathbb{Z}_p$ (some people write "$\mu \mapsto \int_{\mathbb{Z}_p}(1+X)^z\mu(z)$") which induces a bijection between elements of $\mathcal{R}_E^+$ of order r and tempered distributions of order r. We will need the following easy lemma (we leave its proof to the reader):

Lemma 8.2.1. *If $w \in \mathcal{R}_E^+$ corresponds to μ, then $\psi(w) \in \mathcal{R}_E^+$ corresponds to the unique locally analytic distribution $\psi(\mu)$ on $\mathbb{Z}_p$ such that:*

$$\int_{\mathbb{Z}_p} f(z)\psi(\mu)(z) := \int_{p\mathbb{Z}_p} f(z/p)\mu(z)$$

for any locally analytic function f on $\mathbb{Z}_p$.

We have an analogous lemma for tempered distributions of order r and functions of class $\mathcal{C}^r$ (as one easily checks that ψ preserves elements in $\mathcal{R}_E^+$ of order r). In particular, if $(\mu_n)_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of tempered distributions on $\mathbb{Z}_p$ of order r such that $\psi(\mu_{n+1}) = \mu_n$, we can define a linear form μ on the E-vector space $\mathrm{LPol}_{\mathrm{c},k-2}$ of locally polynomial functions $f : \mathbb{Q}_p \to E$ with compact support of degree less than $k-2$ as follows:

$$\int_{\mathbb{Q}_p} f(z)\mu(z) = \int_{p^{-N}\mathbb{Z}_p} f(z)\mu(z) := \int_{\mathbb{Z}_p} f(z/p^N)\mu_N(z)$$

where $N \in \mathbb{Z}_{\geq 0}$ is such that $\mathrm{supp}(f) \subset p^{-N}\mathbb{Z}_p$. One readily checks using Lem.8.2.1 that it doesn't depend on such an N.

Let $(v_n)_n = (w_{\alpha,n} \otimes e_\alpha + w_{\beta,n} \otimes e_\beta)_n \in (\varprojlim_\psi N(T)) \otimes_{\mathcal{O}_E} E$. Recall this means (Prop.8.1.3 and Prop.8.1.4):
1) $\forall\ n \geq 0$, $w_{\alpha,n}$ (resp. $w_{\beta,n}$) is of order $\mathrm{val}(\alpha)$ (resp. $\mathrm{val}(\beta)$) and $\|w_{\alpha,n}\|_{\mathrm{val}(\alpha)}$ (resp. $\|w_{\beta,n}\|_{\mathrm{val}(\beta)}$) is bounded independently of n
2) $\forall\ n \geq 0$, $\forall\ m \geq 1$, $\varphi^{-m}(w_{\alpha,n} \otimes e_\alpha + w_{\beta,n} \otimes e_\beta) \in \mathrm{Fil}^0((\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)[[t]] \otimes_E D(\alpha,\beta))$
3) $\forall\ n \geq 1$, $\psi(w_{\alpha,n}) = \alpha^{-1}w_{\alpha,n-1}$, $\psi(w_{\beta,n}) = \beta^{-1}w_{\beta,n-1}$.

Lemma 8.2.2. *Let $w_\alpha,\ w_\beta \in \mathcal{R}_E^+$, $m \in \mathbb{N}$ and $\mu_\alpha,\ \mu_\beta$ the locally analytic distributions on $\mathbb{Z}_p$ corresponding to $w_\alpha,\ w_\beta$. The condition:*

$$\varphi^{-m}(w_\alpha \otimes e_\alpha + w_\beta \otimes e_\beta) \in \mathrm{Fil}^0((\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)[[t]] \otimes_E D(\alpha,\beta))$$

is equivalent to the equalities:

$$\alpha^m \int_{\mathbb{Z}_p} z^j \zeta_{p^m}^z \mu_\alpha(z) = \beta^m \int_{\mathbb{Z}_p} z^j \zeta_{p^m}^z \mu_\beta(z)$$

for all $j \in \{0, \cdots, k-2\}$ and all primitive p^m-roots ζ_{p^m} of 1.

Proof. Here $\zeta_{p^m}^z$ is considered as a locally constant function $\mathbb{Z}_p \to E$ (enlarging E is necessary). Fixing ζ_{p^m} a primitive root of 1, we have:

$$\varphi^{-m}(X) = (\zeta_{p^m} \otimes 1)e^{t/p^m} - 1 = (\zeta_{p^m} \otimes 1)(e^{t/p^m} - 1) + \zeta_{p^m} \otimes 1 - 1$$

in $(\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)[[t]]$ (see previous lectures). Writing $w_\alpha = \sum_{i=0}^{+\infty} \alpha_i X^i$ and $w_\beta = \sum_{i=0}^{+\infty} \beta_i X^i$, the condition on Fil^0 is equivalent to:

$$\alpha^m \sum_{i=0}^{+\infty} \alpha_i \varphi^{-m}(X)^i e_\alpha + \beta^m \sum_{i=0}^{+\infty} \beta_i \varphi^{-m}(X)^i e_\beta \in$$
$$(\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)[[t]](e_\alpha + e_\beta) \oplus (e^{t/p^m} - 1)^{k-1}((\mathbb{Q}_p(\mu_{p^m}) \otimes_{\mathbb{Q}_p} E)[[t]]e_\alpha)$$

noting that $e^{t/p^m}-1$ generates $\mathrm{gr}^1(\mathbb{Q}_p[[t]])=\mathbb{Q}_p\bar{t}$. Assuming E contains $\mathbb{Q}_p(\mu_{p^m})$, we have:

$$\mathbb{Q}_p(\mu_{p^m})\otimes_{\mathbb{Q}_p}E=\prod_{\mathbb{Q}_p(\mu_{p^m})\hookrightarrow E}E$$

and writing the binomial expansion of $\varphi^{-m}(X)^i=(\zeta_{p^m}(e^{t/p^m}-1)+\zeta_{p^m}-1)^i$, a simple calculation gives that the above condition is equivalent to the equalities:

$$\alpha^m\sum_{i=j}^{+\infty}\alpha_i\binom{i}{j}(\zeta_{p^m}-1)^{i-j}=\beta^m\sum_{i=j}^{+\infty}\beta_i\binom{i}{j}(\zeta_{p^m}-1)^{i-j}\tag{19}$$

for $j\in\{0,\cdots,k-2\}$ and all primitive p^m-roots ζ_{p^m} of 1. Using $\alpha_i=\int_{\mathbb{Z}_p}\binom{z}{i}\mu_\alpha(z)$ and the elementary Mahler expansion:

$$\binom{z}{j}\zeta_{p^m}^{z-j}=\sum_{i=j}^{+\infty}\binom{i}{j}(\zeta_{p^m}-1)^{i-j}\binom{z}{i}$$

we obtain:

$$\begin{aligned}\sum_{i=j}^{+\infty}\alpha_i\binom{i}{j}(\zeta_{p^m}-1)^{i-j}&=\int_{\mathbb{Z}_p}\Big(\sum_{i=j}^{+\infty}\binom{z}{i}\binom{i}{j}(\zeta_{p^m}-1)^{i-j}\Big)\mu_\alpha(z)\\&=\zeta_{p^m}^{-j}\int_{\mathbb{Z}_p}\binom{z}{j}\zeta_{p^m}^z\mu_\alpha(z)\end{aligned}$$

and likewise with β_i and μ_β. Using (19), we easily get the result. $\square$

For $n\in\mathbb{Z}_{\geq 0}$, let $\mu_{\alpha,n}$ be the tempered distribution on $\mathbb{Z}_p$ of order $\mathrm{val}(\alpha)$ associated to $\alpha^n w_{\alpha,n}$. We have $\psi(\mu_{\alpha,n})=\mu_{\alpha,n-1}$ for $n\in\mathbb{N}$ and can thus define $\mu_\alpha\in\mathrm{LPol}_{c,k-2}^*$ (algebraic dual of $\mathrm{LPol}_{c,k-2}$) as before:

$$\int_{p^{-N}\mathbb{Z}_p}f(z)\mu_\alpha(z):=\int_{\mathbb{Z}_p}f(z/p^N)\mu_{\alpha,N}(z).\tag{20}$$

Likewise with β instead of α. By Lem.8.2.2, we have that condition 2) above on $(w_{\alpha,n},w_{\beta,n})_n$ is equivalent to:

$$\alpha^{m-n}\int_{\mathbb{Z}_p}z^j\zeta_{p^m}^z\mu_{\alpha,n}(z)=\beta^{m-n}\int_{\mathbb{Z}_p}z^j\zeta_{p^m}^z\mu_{\beta,n}(z)\tag{21}$$

for $j\in\{0,\cdots,k-2\}$, $n\geq 0$ and $m\geq 1$.

Fixing embeddings $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$, $\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}_p}$ and using the identification $\mathbb{Q}_p/\mathbb{Z}_p=\mathbb{Z}[1/p]/\mathbb{Z}$, we can "see" $e^{2i\pi z}$ in $\overline{\mathbb{Q}_p}$ for any $z\in\mathbb{Q}_p$.

Corollary 8.2.3. *Condition 2) above is equivalent to the equalities in $\overline{\mathbb{Q}_p}$:*

$$\int_{p^{-N}\mathbb{Z}_p}z^je^{2i\pi zy}\mu_\alpha(z)=\Big(\frac{\alpha}{\beta}\Big)^{\mathrm{val}(y)}\int_{p^{-N}\mathbb{Z}_p}z^je^{2i\pi zy}\mu_\beta(z)$$

for $j\in\{0,\cdots,k-2\}$, $y\in\mathbb{Q}_p^\times$ and $N>\mathrm{val}(y)$.

Proof. This follows from (20) and (21) noting that the algebraic number $e^{2i\pi y/p^N}$ is a primitive $p^{N-\mathrm{val}(y)}$-root of 1. $\square$

9. Representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and (φ,Γ)-modules II (C.B.)

We still assume V absolutely irreducible as in §7.1. The aim of this section is to construct a topological isomorphism $(\varprojlim_\psi D(V))^{\mathrm{b}} \simeq \Pi(V)^*$, more precisely a topological isomorphism:

$$(\varprojlim_{\psi} N(T)) \otimes_{\mathcal{O}_E} E \simeq (B(\beta)/L(\beta))^*.$$

We then deduce from this isomorphism that $\Pi(V)$ is non zero, topologically irreducible and admissible. These three statements were conjectured in [9] and [10] (and already known in some cases by a "reduction modulo p" method, see [9] and Lecture 10). We also deduce a link with the Iwasawa cohomology $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ of V.

9.1. **The map $\Pi(V)^* \to (\lim_\psi D(V))^{\mathrm{b}}$.** We first construct a map from $(B(\beta)/L(\beta))^*$ to the space $(\varprojlim_\psi N(T)) \otimes_{\mathcal{O}_E} E$. Note that any f in $\pi(\alpha)$ or $\pi(\beta)$ which has compact support as a function on $\mathbb{Q}_p$ via (9) is an element in $\mathrm{LPol}_{\mathrm{c},k-2}$. Recall that $I : \pi(\alpha) \simeq \pi(\beta)$ is the intertwining operator of (12).

Lemma 9.1.1. *Let $\mu_\alpha, \mu_\beta \in \mathrm{LPol}^*_{\mathrm{c},k-2}$. The following statements are equivalent:*
1) For $j \in \{0, \cdots, k-2\}$, $y \in \mathbb{Q}_p^\times$ and $N > \mathrm{val}(y)$, we have:

$$\int_{p^{-N}\mathbb{Z}_p} z^j e^{2i\pi zy} \mu_\alpha(z) = \left(\frac{\alpha}{\beta}\right)^{\mathrm{val}(y)} \int_{p^{-N}\mathbb{Z}_p} z^j e^{2i\pi zy} \mu_\beta(z). \tag{22}$$

2) For any $f \in \pi(\alpha)$ with compact support (as a function on $\mathbb{Q}_p$ via (9)) such that $I(f) \in \pi(\beta)$ has compact support (as a function on $\mathbb{Q}_p$ via (9)), we have:

$$\int_{\mathbb{Q}_p} f(z)\mu_\alpha(z) = \frac{1-\frac{\beta}{\alpha}}{1-\frac{\alpha}{p\beta}} \int_{\mathbb{Q}_p} I(f)(z)\mu_\beta(z). \tag{23}$$

Proof. Note that $\alpha \neq \beta$ and $\alpha \neq p\beta$ (see §7.1) hence the constant in (23) is well defined and non zero. Recall the intertwining $I^{\mathrm{sm}} : \mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{unr}(\alpha^{-1}) \otimes \mathrm{unr}(p\beta^{-1}) \simeq \mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{unr}(\beta^{-1}) \otimes \mathrm{unr}(p\alpha^{-1})$ is given by:

$$I^{\mathrm{sm}}(h) = \left(g \mapsto \int_{\mathbb{Q}_p} h\Big(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} g\Big) dx\right) \tag{24}$$

where dx is Haar measure on $\mathbb{Q}_p$ (with values in $\mathbb{Q} \subset E$). In terms of locally constant functions on $\mathbb{Q}_p$ via (9), we thus get:

$$I^{\mathrm{sm}} : f \mapsto \left(z \mapsto \int_{\mathbb{Q}_p} (\alpha p \beta^{-1})^{\mathrm{val}(x)} f(z + x^{-1}) dx = \int_{\mathbb{Q}_p} (\beta p \alpha^{-1})^{\mathrm{val}(x)} f(z+x) dx\right). \tag{25}$$

This is in fact an *algebraic* intertwining. This means that we shouldn't bother too much about convergence problems in $\int_{\mathbb{Q}_p} (\beta p\alpha^{-1})^{\mathrm{val}(x)} f(z+x)dx$ since, *in fine*, we can always replace the infinite sums around 0 or $-\infty$ by well defined algebraic expressions in $\beta p \alpha^{-1}$. Let $j \in \{0, \cdots, k-2\}$ and $f_j : \mathbb{Q}_p \to E$ be a locally constant function with compact support such that $I^{\mathrm{sm}}(f_j)$ also has compact support. Let $\hat{f}_j$ be the usual Fourier transform of f_j which is again a locally constant function on $\mathbb{Q}_p$ with compact support. Recall $\hat{f}_j(x) = \int_{\mathbb{Q}_p} f_j(z) e^{-2i\pi zx} dz$ and $f_j(z) = \int_{\mathbb{Q}_p} \hat{f}_j(x) e^{2i\pi zx} dx$ (see the end of §8.2 for $e^{-2i\pi zx}$) where dx, dz is Haar measure on $\mathbb{Q}_p$. The fact that $I^{\mathrm{sm}}(f_j)$ has compact support is easily checked with (25) to be equivalent to $\hat{f}_j(0) = 0$ (for $|z| \gg 0$, we have $I^{\mathrm{sm}}(f_j)(z) = (\beta p\alpha^{-1})^{\mathrm{val}(z)} \int_{\mathbb{Q}_p} f_j(x)dx = (\beta p\alpha^{-1})^{\mathrm{val}(z)}\hat{f}_j(0))$. Let $N \in \mathbb{N}$ be such that f_j and $I^{\mathrm{sm}}(f_j)$ have support in $p^{-N}\mathbb{Z}_p$ and such that $\hat{f}_j|_{p^N\mathbb{Z}_p} = 0$. For $z \in p^{-N}\mathbb{Z}_p$, we have:

$$\begin{aligned} I(z^j f_j)(z) &= z^j \int_{p^{-N}\mathbb{Z}_p} (\beta p\alpha^{-1})^{\mathrm{val}(x)} f_j(z+x)dx \\ &= z^j \int_{p^{-N}\mathbb{Z}_p} (\beta p\alpha^{-1})^{\mathrm{val}(x)} \Big(\int_{\mathbb{Q}_p - p^N\mathbb{Z}_p} \hat{f}_j(y) e^{2i\pi y(z+x)} dy \Big) dx \\ &= z^j \int_{\mathbb{Q}_p - p^N\mathbb{Z}_p} \hat{f}_j(y) e^{2i\pi zy} \Big(\int_{p^{-N}\mathbb{Z}_p} (\beta p\alpha^{-1})^{\mathrm{val}(x)} e^{2i\pi xy} dx \Big) dy. \end{aligned}$$

Decomposing $p^{-N}\mathbb{Z}_p = p^{-N}\mathbb{Z}_p^\times \amalg p^{-N+1}\mathbb{Z}_p^\times \amalg \cdots$, a straightforward computation yields for $N > \mathrm{val}(y)$:

$$\int_{p^{-N}\mathbb{Z}_p} (\beta p\alpha^{-1})^{\mathrm{val}(x)} e^{2i\pi xy} dx = \sum_{i=-N}^{+\infty} (\beta p\alpha^{-1})^i \int_{p^i\mathbb{Z}_p^\times} e^{2i\pi xy} dx = \frac{1 - \frac{\alpha}{p\beta}}{1 - \frac{\beta}{\alpha}} \Big(\frac{\alpha}{\beta}\Big)^{\mathrm{val}(y)}$$

hence:

$$I(z^j f_j)(z) = \frac{1 - \frac{\alpha}{p\beta}}{1 - \frac{\beta}{\alpha}} \int_{\mathbb{Q}_p - p^N\mathbb{Z}_p} \hat{f}_j(y) z^j e^{2i\pi zy} \Big(\frac{\alpha}{\beta}\Big)^{\mathrm{val}(y)} dy \tag{26}$$

for $z \in p^{-N}\mathbb{Z}_p$ and $I(z^j f_j)(z) = 0$ otherwise. Likewise, we have:

$$z^j f_j(z) = \int_{\mathbb{Q}_p - p^N\mathbb{Z}_p} \hat{f}_j(y) z^j e^{2i\pi zy} dy \tag{27}$$

for $z \in p^{-N}\mathbb{Z}_p$ and $z^j f_j(z) = 0$ otherwise. Note that (26) and (27) are in fact finite sums over the same finite set of values of y (as $\hat{f}_j$ is locally constant with compact support). Replacing $I(z^j f_j)(z)$ and $z^j f_j(z)$ in (28) below by the finite sums (26) and (27), we then easily deduce that the following equalities for various f_j as above with support in $p^{-N}\mathbb{Z}_p$:

$$\int_{p^{-N}\mathbb{Z}_p} I(z^j f_j(z)) \mu_\beta(z) = \frac{1 - \frac{\alpha}{p\beta}}{1 - \frac{\beta}{\alpha}} \int_{p^{-N}\mathbb{Z}_p} z^j f_j(z) \mu_\alpha(z) \tag{28}$$

are equivalent to the equalities in 1) (final details here are left to the reader). This finishes the proof. □

Let $\mu_\beta \in (B(\beta)/L(\beta))^* \subset \pi(\beta)^*$. Define a sequence of locally analytic distributions on $\mathbb{Z}_p$, $(\mu_{\beta,N})_{N\in\mathbb{Z}_{\geq 0}}$, as follows:

$$\int_{\mathbb{Z}_p} f(z)\mu_{\beta,N}(z) := \int_{p^{-N}\mathbb{Z}_p} f(p^N z)\mu_\beta(z).$$

One obviously has $\psi(\mu_{\beta,N}) = \mu_{\beta,N-1}$ by Lem.8.2.1.

Lemma 9.1.2. *The distributions $\mu_{\beta,N}$ are tempered of order $\mathrm{val}(\beta)$ and $\|\beta^{-N}\mu_{\beta,N}\|'_{\mathrm{val}(\beta)}$ is bounded independently of N (see §6.2 for the definition of this norm).*

Proof. The fact that $\mu_{\beta,N}$ is tempered of order $\mathrm{val}(\beta)$ is easy and left to the reader. Going back to the proof of Th.7.2.2, we know that μ_β satisfies in particular (16) and (17). Hence, for $a \in \mathbb{Z}_p$, $0 \leq j \leq k-2$ and $n \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned}\beta^{-N}\int_{a+p^n\mathbb{Z}_p}(z-a)^j\mu_{\beta,N}(z) &= \beta^{-N}p^{Nj}\int_{p^{-N}a+p^{n-N}\mathbb{Z}_p}(z-p^{-N}a)^j\mu_\beta(z)\\ &\in C_{\mu_\beta}p^{-N\mathrm{val}(\beta)}p^{Nj}p^{(n-N)(j-\mathrm{val}(\beta))}\mathcal{O}_E\\ &\in C_{\mu_\beta}p^{n(j-\mathrm{val}(\beta))}\mathcal{O}_E.\end{aligned}$$

For any locally analytic distribution μ on $\mathbb{Z}_p$ and any $r \in \mathbb{R}^+$ such that $r < k-1$, define:

$$\|\mu\|'_{r,k} := \sup_{a\in\mathbb{Z}_p}\sup_{j\in\{0,\cdots,k-2\}}\sup_{n\in\mathbb{Z}_{\geq 0}}p^{n(j-r)}\left|\int_{a+p^n\mathbb{Z}_p}(z-a)^j\mu(z)\right|.$$

An examination of the proof of Lem.6.2.6 shows that the norms $\|\cdot\|'_r$ and $\|\cdot\|'_{r,k}$ are in fact equivalent (see also [15],§V.3.6). This implies:

$$\|\beta^{-N}\mu_{\beta,N}\|'_{\mathrm{val}(\beta)} \leq c\|\beta^{-N}\mu_{\beta,N}\|'_{\mathrm{val}(\beta),k} \leq c|C_{\mu_\beta}|$$

for a convenient constant $c \in \mathbb{R}^+$. $\square$

By Cor.7.2.3, the intertwining $I : \pi(\alpha) \simeq \pi(\beta)$ extends "by continuity" to an intertwining $\hat{I} : B(\alpha)/L(\alpha) \simeq B(\beta)/L(\beta)$ and we define:

$$\mu_\alpha := \frac{1-\frac{\beta}{\alpha}}{1-\frac{\alpha}{p\beta}}\mu_\beta \circ \hat{I} \in (B(\alpha)/L(\alpha))^*.$$

By restriction to $\mathrm{LPol}_{\mathrm{c},k-2}$, μ_α and μ_β define elements of $\mathrm{LPol}^*_{\mathrm{c},k-2}$ satisfying (23) by definition, hence (22) by Lem.9.1.1. As in Lem.9.1.2, μ_α gives rise to a bounded sequence of tempered distributions $\mu_{\alpha,N}$ on $\mathbb{Z}_p$. From Cor.8.2.3 and Lem.9.1.2, we get that Amice transform produces from $(\mu_{\alpha,N})_N$ and $(\mu_{\beta,N})_N$ a sequence $(v_N)_N = (w_{\alpha,N}\otimes e_\alpha + w_{\beta,N}\otimes e_\beta)_N$ satisfying conditions 1), 2), 3) before Lem.8.2.2, hence $(v_N)_N \in (\varprojlim_\psi N(T)) \otimes_{\mathcal{O}_E} E$. This defines a linear map $(B(\beta)/L(\beta))^* \to (\varprojlim_\psi N(T)) \otimes_{\mathcal{O}_E} E$. The continuity follows from the fact that the function "Amice transform" from the space of tempered distributions of order r on $\mathbb{Z}_p$ with norm smaller than 1 endowed with the weak topology of pointwise convergence to the subspace of $\mathcal{R}_E^+$ of elements w of order r such that $\|w\|_r \leq 1$ endowed with its compact topology induced by that of $\mathcal{R}_E^+$ is a topological isomorphism. Final details here are left to the reader.

9.2. **The map** $(\lim_\psi D(V))^{\mathrm{b}} \to \Pi(V)^*$**.** We now construct a map from $(\varprojlim_\psi N(T)) \otimes_{\mathcal{O}_E} E$ to $(B(\beta)/L(\beta))^*$. Let $(v_n)_n = (w_{\alpha,n}\otimes e_\alpha + w_{\beta,n}\otimes e_\beta)_n \in (\varprojlim_\psi N(T)) \otimes_{\mathcal{O}_E} E$ and define μ_α and μ_β as in §8.2. They are elements of $\mathrm{Pol}^*_{\mathrm{c},k-2}$ satisfying (22) by Cor.8.2.3, hence (23) by Lem.9.1.1.

Lemma 9.2.1. *There is a unique way to extend μ_β and μ_α as elements of respectively $B(\beta)^*$ and $B(\alpha)^*$ such that for any $f \in \pi(\alpha)$ (seen as a function on $\mathbb{Q}_p$ via (9)):*

$$\int_{\mathbb{Q}_p} f(z)\mu_\alpha(z) = \frac{1-\frac{\beta}{\alpha}}{1-\frac{\alpha}{p\beta}}\int_{\mathbb{Q}_p} I(f)(z)\mu_\beta(z). \tag{29}$$

Proof. An easy computation using (25) gives:

$$I(z^j\mathbf{1}_{\mathbb{Z}_p}) = \frac{1-\frac{1}{p}}{1-\frac{\beta}{\alpha}}z^j\mathbf{1}_{\mathbb{Z}_p} + z^j\Big(\frac{p\beta}{\alpha}\Big)^{\mathrm{val}(z)}(1-\mathbf{1}_{\mathbb{Z}_p}) \tag{30}$$

$$I\big(z^j\Big(\frac{p\alpha}{\beta}\Big)^{\mathrm{val}(z)}(1-\mathbf{1}_{p\mathbb{Z}_p})\big) = z^j\mathbf{1}_{p\mathbb{Z}_p} + \frac{1-\frac{1}{p}}{1-\frac{\beta}{\alpha}}z^j\Big(\frac{p\beta}{\alpha}\Big)^{\mathrm{val}(z)}(1-\mathbf{1}_{p\mathbb{Z}_p}). \tag{31}$$

Since $\int_{\mathbb{Q}_p} z^j\mathbf{1}_{\mathbb{Z}_p}\mu_\alpha(z)$ and $\int_{\mathbb{Q}_p} z^j\mathbf{1}_{\mathbb{Z}_p}\mu_\beta(z)$ are well defined, we see using (29) and (30) that

$$\int_{\mathbb{Q}_p} z^j\Big(\frac{p\beta}{\alpha}\Big)^{\mathrm{val}(z)}(1-\mathbf{1}_{\mathbb{Z}_p})\mu_\beta(z)$$

is uniquely determined. Then, using (31) (and (29)), we see that $\int_{\mathbb{Q}_p} z^j\big(\frac{p\alpha}{\beta}\big)^{\mathrm{val}(z)}(1-\mathbf{1}_{p\mathbb{Z}_p})\mu_\alpha(z)$ is also uniquely determined. One readily checks that this defines unique extensions of μ_β and μ_α as linear forms on respectively $\pi(\beta)$ and $\pi(\alpha)$. To check that μ_β is in fact in $B(\beta)^*$, it is enough by Cor.6.2.7 to check that the relevant Amice-Vélu condition (as in Lem.6.2.6) is satisfied by μ_β on each copy of $\mathbb{Z}_p$ via (6). We already know it for the first one since $w_{\beta,0}$ is of order $\mathrm{val}(\beta)$. For the second one, a straightforward computation via the definition of f_2 in (6) (exchanging α and β) shows that we have to find a constant $C_{\mu_\beta}\in E$ such that:

$$\int_{a^{-1}+p^{n-2\mathrm{val}(a)}\mathbb{Z}_p}\Big(\frac{p\beta}{\alpha}\Big)^{\mathrm{val}(z)}z^{k-2-j}(1-az)^j\mu_\beta(z)\in C_{\mu_\beta}p^{n(j-\mathrm{val}(\beta))}\mathcal{O}_E \tag{32}$$

for $a\in\mathbb{Z}_p-\{0\}$, $j\in\{0,\cdots,k-2\}$ and $n>\mathrm{val}(a)$, and:

$$\int_{\mathbb{Q}_p-p^{-n}\mathbb{Z}_p}\Big(\frac{p\beta}{\alpha}\Big)^{\mathrm{val}(z)}z^{k-2-j}\mu_\beta(z)\in C_{\mu_\beta}p^{n(j-\mathrm{val}(\beta))}\mathcal{O}_E \tag{33}$$

for $j\in\{0,\cdots,k-2\}$ and $n\in\mathbb{Z}_{\geq 0}$. From condition 1) before Lem.8.2.2, we know there exists $C_{\mu_\beta}\in E$ such that $\beta^{-N}\int_{a+p^n\mathbb{Z}_p}(z-a)^j\mu_{\beta,N}(z)\in C_{\mu_\beta}p^{n(j-\mathrm{val}(\beta))}\mathcal{O}_E$ for $a\in\mathbb{Z}_p$ and $N,n\in\mathbb{Z}_{\geq 0}$. For $a\in\mathbb{Q}_p^\times$, let $u_a:=p^{\mathrm{val}(a)}a^{-1}\in\mathbb{Z}_p^\times$. By (20), the left hand side of (32) is equal to (up to a unit in $\mathcal{O}_E$):

$$\Big(\frac{p\beta}{\alpha}\Big)^{-\mathrm{val}(a)}p^{(j-k+2)\mathrm{val}(a)}\int_{u_a+p^{n-\mathrm{val}(a)}\mathbb{Z}_p}z^{k-2-j}(z-u_a)^j\mu_{\beta,\mathrm{val}(a)}(z)$$

and we see, writing $z^{k-2-j}=(z-u_a+u_a)^{k-2-j}=\cdots$ and using the above bound with $N=\mathrm{val}(a)$, that it belongs to:

$$C_{\mu_\beta}p^{(\mathrm{val}(\alpha)-\mathrm{val}(\beta)-1)\mathrm{val}(a)}p^{(j-k+2)\mathrm{val}(a)}p^{\mathrm{val}(\beta)\mathrm{val}(a)}p^{(n-\mathrm{val}(a))(j-\mathrm{val}(\beta))}\mathcal{O}_E$$

which is exactly $C_{\mu_\beta}p^{n(j-\mathrm{val}(\beta))}\mathcal{O}_E$ using $\mathrm{val}(\alpha)+\mathrm{val}(\beta)=k-1$. This proves (32). To prove (33), one uses the equality (similar to (30)):

$$\begin{aligned}\Big(\frac{p\beta}{\alpha}\Big)^{\mathrm{val}(z)}z^{k-2-j}(1-\mathbf{1}_{p^{-n}\mathbb{Z}_p}) &= p^{-n}I(z^{k-2-j}\mathbf{1}_{p^{-n}\mathbb{Z}_p})-\\ &\quad\frac{1-\frac{1}{p}}{1-\frac{\beta}{\alpha}}\Big(\frac{\alpha}{p\beta}\Big)^n z^{k-2-j}\mathbf{1}_{p^{-n}\mathbb{Z}_p}\end{aligned}$$

together with (29), the equalities:

$$\begin{aligned} p^{-n}\int_{p^{-n}\mathbb{Z}_p} z^{k-2-j}\mu_\alpha(z) &= p^{(j-k+1)n}\int_{\mathbb{Z}_p} z^{k-2-j}\mu_{\alpha,n}(z) \\ \left(\frac{\alpha}{p\beta}\right)^n\int_{p^{-n}\mathbb{Z}_p} z^{k-2-j}\mu_\beta(z) &= \left(\frac{\alpha}{p\beta}\right)^n p^{(j-k+2)n}\int_{\mathbb{Z}_p} z^{k-2-j}\mu_{\beta,n}(z)\end{aligned}$$

and the fact $\alpha^{-n}\int_{\mathbb{Z}_p} z^{k-2-j}\mu_{\alpha,n}(z)$ and $\beta^{-n}\int_{\mathbb{Z}_p} z^{k-2-j}\mu_{\beta,n}(z)$ are bounded independently of n. The proof that μ_α extends as a distribution on $B(\alpha)$ is similar. $\square$

By Lem.9.2.1, we have a continuous injective map $(\varprojlim_\psi N(T))\otimes_{\mathcal{O}_E} E\hookrightarrow B(\beta)^*$, $(w_{\alpha,n}\otimes e_\alpha + w_{\beta,n}\otimes e_\beta)_n\mapsto \mu_\beta$ (the continuity follows as in §9.1 from the continuity of Amice transform and the injectivity is straightforward using Lem.9.2.1). With the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ deduced from (7) on the dual $B(\beta)^*$, it is an easy and pleasant exercise on Amice transform that we leave to the reader (using e.g. the formula $\int_{\mathbb{Z}_p}(1+X)^z\mu(z)$) to check that the induced action of $\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix}$ on $(\varprojlim_\psi N(T))\otimes_{\mathcal{O}_E} E$ is indeed as in Th.8.1.1. In particular, it preserves $\varprojlim_\psi N(T)$.

Lemma 9.2.2. *The distribution $\mu_\beta\in B(\beta)^*$ defined in Lem.9.2.1 sits in $(B(\beta)/L(\beta))^*$.*

Proof. The non zero compact $\mathcal{O}_E$-submodule of $B(\beta)^*$ which is the image of $\varprojlim_\psi N(T)$ is preserved by the action of $\mathrm{B}(\mathbb{Q}_p)$ since $\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix}$ generates $\begin{pmatrix}1 & \mathbb{Q}_p\\ 0 & \mathbb{Q}_p^\times\end{pmatrix}$ and the central character is integral. Let:

$$\pi(\beta)^0 := \left\{ f\in\pi(\beta),\ \int_{\mathbb{Q}_p} f(z)\mu(z)\in\mathcal{O}_E\ \forall\ \mu\in\varprojlim_\psi N(T)\right\}.$$

We have that $\pi(\beta)^0\subset\pi(\beta)$ is thus preserved by $\mathrm{B}(\mathbb{Q}_p)$. By [30],Lem.13.1(iii), we also have that $\pi(\beta)^0\otimes_{\mathcal{O}_E} E=\pi(\beta)$ (using that $B(\beta)^*\hookrightarrow\pi(\beta)^*$ and thus that $\varprojlim_\psi N(T)$ is bounded inside $\pi(\beta)^*$). By the same proof as for Th.7.2.2 (using that any $\mu\in\varprojlim_\psi N(T)$ is such that $\forall\ g\in\mathrm{B}(\mathbb{Q}_p)$, $\forall\ f\in\pi(\beta)^0$, $|\mu(g(f))|\leq 1$), we deduce that any $\mu\in B(\beta)^*$ which is in the image of $\varprojlim_\psi N(T)$ cancels $L(\beta)$. Whence the result. $\square$

By Lem.9.2.2, the above continuous injection $(\varprojlim_\psi N(T))\otimes_{\mathcal{O}_E} E\hookrightarrow B(\beta)^*$ takes values in the subspace $(B(\beta)/L(\beta))^*$. It is easily checked that this map is an inverse to the previous one (§9.1), thus finishing the proof of Th.8.1.1. In particular, we have an isomorphism:

$$D(V)^{\psi=1}=\big((\varprojlim_\psi D(V))^{\mathrm{b}}\big)^{\psi=1}\simeq \Pi(V)^{*\begin{pmatrix}1 & 0\\ 0 & p^{\mathbb{Z}}\end{pmatrix}}.$$

Using Fontaine's canonical isomorphism (see e.g. [15],§4.3):

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V):=\big(\varprojlim_n H^1(\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p(\mu_{p^n})),T)\big)\otimes_{\mathcal{O}_E} E\simeq D(V)^{\psi=1} \tag{34}$$

for any Galois lattice T in V, we deduce the quite surprising corollary:

Corollary 9.2.3. *We have an isomorphism of $\mathcal{O}_E[[\mathbb{Z}_p^\times]]$-modules:*

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)\simeq \Pi(V)^{*\begin{pmatrix}1 & 0\\ 0 & p^{\mathbb{Z}}\end{pmatrix}}$$

where the $\mathcal{O}_E[[\mathbb{Z}_p^\times]]$-module structure on the left hand side is coming from the action of Γ and on the right hand side from the action of $\begin{pmatrix} 1 & 0 \\ 0 & \mathbb{Z}_p^\times \end{pmatrix}$.

When V is reducible (as in 2) and 3) of §7.1), it is not true that we have an isomorphism $(\varprojlim_\psi D(V))^{\mathrm{b}} \simeq \Pi(V)^*$ (although it is "almost true"). But if $\alpha \neq p^{k-1}$ and $\beta \neq 1$ (which always holds if V is coming from a modular form by the Weil conjectures), one can prove that there is still an isomorphism of $\mathcal{O}_E[[\mathbb{Z}_p^\times]]$-modules $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \simeq \Pi(V)^{*\begin{pmatrix} 1 & 0 \\ 0 & p^{\mathbb{Z}} \end{pmatrix}}$.

9.3. **Irreducibility and admissibility.** We deduce from Th.8.1.1 that, for V irreducible as before, $\Pi(V)$ is non zero, topologically irreducible and admissible.

Corollary 9.3.1. *The Banach $B(\alpha)/L(\alpha)$ and $B(\beta)/L(\beta)$ are non zero.*

Proof. This follows from $(\varprojlim_\psi D(V))^{\mathrm{b}} \neq 0$ which follows from $D(V)^{\psi=1} \neq 0$ which follows e.g. from (34). □

Corollary 9.3.2. *The Banach $B(\alpha)/L(\alpha)$ and $B(\beta)/L(\beta)$ are topologically irreducible.*

Proof. Assume there exists a (necessarily unitary) $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach V together with a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant topological surjection $B(\alpha)/L(\alpha) \twoheadrightarrow V$. We can find norms on the two Banach such that we have a surjection of unit balls $(B(\alpha)/L(\alpha))^0 \twoheadrightarrow V^0$. By duality, we get a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant continuous injection of compact $\mathcal{O}_E$-modules:

$$\mathrm{Hom}_{\mathcal{O}_E}(V^0, \mathcal{O}_E) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_E}((B(\alpha)/L(\alpha))^0, \mathcal{O}_E).$$

We have seen that $\varprojlim_\psi N(T)$ is a compact $\mathcal{O}_E$-lattice of $(B(\alpha)/L(\alpha))^*$ which is preserved by $\mathrm{B}(\mathbb{Q}_p)$. Hence, multiplying by a scalar in $E^\times$ if necessary, we get a continuous $\mathrm{B}(\mathbb{Q}_p)$-equivariant injection of compact $\mathcal{O}_E$-modules $\mathrm{Hom}_{\mathcal{O}_E}(V^0, \mathcal{O}_E) \hookrightarrow \varprojlim_\psi N(T)$. If $V \neq 0$, Cor.8.1.6 together with the translation of the action of $\mathrm{B}(\mathbb{Q}_p)$ in terms of ψ and Γ imply $\mathrm{Hom}_{\mathcal{O}_E}(V^0, \mathcal{O}_E) \otimes E \xrightarrow{\sim} (B(\alpha)/L(\alpha))^*$, which implies $B(\alpha)/L(\alpha) \xrightarrow{\sim} V$. Hence $B(\alpha)/L(\alpha)$ (and $B(\beta)/L(\beta)$ by Cor.7.2.3) is topologically irreducible. □

We see from the previous proof that $B(\alpha)/L(\alpha)$ and $B(\beta)/L(\beta)$ are even topologically irreducible as $\mathrm{B}(\mathbb{Q}_p)$-representations.

Corollary 9.3.3. *The Banach $B(\alpha)/L(\alpha)$ and $B(\beta)/L(\beta)$ are admissible.*

Proof. We don't know if the compact $\mathcal{O}_E$-module $\varprojlim_\psi N(T)$ is preserved by $\mathrm{GL}_2(\mathbb{Z}_p)$ inside

$$(\varprojlim_\psi D(V))^{\mathrm{b}} \simeq (B(\alpha)/L(\alpha))^*,$$

but we can replace it by $\mathcal{M} := \cap_{g \in \mathrm{GL}_2(\mathbb{Z}_p)} g(\varprojlim_\psi N(T)) \subset \varprojlim_\psi N(T)$ which is a non zero compact $\mathcal{O}_E[[X]]$-submodule as in Cor.8.1.6 (since it is easily checked to be preserved by $B(\mathbb{Q}_p)$ using the Iwasawa decomposition of $\mathrm{GL}_2(\mathbb{Q}_p)$). Note that there are now two natural structures of $\mathcal{O}_E[[X]]$-module on $\mathcal{M}$: the first is the one already defined and the second is:

$$(\lambda, v) \in \mathcal{O}_E[[X]] \times \mathcal{M} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v.$$

Equivalently, the first is such that multiplication by $(1+X)^{\mathbb{Z}_p}$ corresponds to the action of $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ and the second is such that multiplication by $(1+X)^{\mathbb{Z}_p}$ corresponds to the action of $\begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix}$. We have seen that $\mathcal{M} = \varprojlim_{\psi} M$ (see proof of Cor.8.1.6) with M of finite type over $\mathcal{O}_E[[X]]$, $M \subset N(T)$ and ψ surjective on M. Let $\mathrm{pr}_0 : \mathcal{M} \twoheadrightarrow M$ be the projection on the first component and define $\mathcal{N} := \mathrm{Ker}(\mathrm{pr}_0) \subsetneq \mathcal{M}$. The map $\mathcal{N} \mapsto M$, $v \mapsto \mathrm{pr}_0(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v)$ is injective: if v maps to 0, its associated distribution $\mu \in B(\alpha)^* \simeq \mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E)^* \oplus \mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E)^*$ (see (6)) is zero on both copies of $\mathcal{C}^{\mathrm{val}(\alpha)}(\mathbb{Z}_p, E)$ hence is zero in $(B(\alpha)/L(\alpha))^*$. Thinking in terms of distributions again, we also see that $\mathcal{N}$ is an $\mathcal{O}_E[[X]]$-module for the first structure but only a $\varphi(\mathcal{O}_E[[X]])$-module for the second structure (recall from (7) and Th.8.1.1 that multiplication by $\varphi(1+X) = (1+X)^p$ on distributions correspond to the change of variables $z \mapsto z+p$ on functions). Moreover, for this second action, the above injection $\mathcal{N} \hookrightarrow M$ is $\varphi(\mathcal{O}_E[[X]])$-linear. Since M is of finite type over $\mathcal{O}_E[[X]]$, hence over $\varphi(\mathcal{O}_E[[X]])$, we thus get that the $\varphi(\mathcal{O}_E[[X]])$-module $\mathcal{N}$ for the second action of $\varphi(\mathcal{O}_E[[X]])$ is of finite type. Now fix elements $(e_1, \cdots, e_m) \in \mathcal{M}$ (resp. $(f_1, \cdots, f_n) \in \mathcal{N}$) such that $\mathrm{pr}_0(e_i)$ (resp. f_i) generate M over $\mathcal{O}_E[[X]]$ (resp. $\mathcal{N}$ over $\varphi(\mathcal{O}_E[[X]])$). Let $v \in \mathcal{M}$. There exist $\lambda_1, \cdots, \lambda_m$ in $\mathcal{O}_E[[X]]$ such that $v - \sum \lambda_i e_i \in \mathcal{N}$. Then there exist $\mu_1, \cdots, \mu_n$ in $\varphi(\mathcal{O}_E[[X]])$ such that $v - \sum \lambda_i e_i = \sum \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mu_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f_i$. Since the λ_i correspond to the action of elements in the group algebra of $\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ and the $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mu_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ correspond to the action of elements in the group algebra of $\begin{pmatrix} 1 & 0 \\ p\mathbb{Z}_p & 1 \end{pmatrix}$, we see in particular that $\mathcal{M}$ is of finite type over the group algebra of $\mathrm{GL}_2(\mathbb{Z}_p)$, whence the admissibility. □

10. Reduction modulo p (L.B.)

In the ℓ-adic case ($\ell \neq p$), Vignéras has proved that one can reduce the classical local Langlands correspondence over $\overline{\mathbb{Q}_\ell}$ modulo the maximal ideal of $\overline{\mathbb{Z}_\ell}$ and thus obtain a new correspondence between semi-simple representations over $\overline{\mathbb{F}_\ell}$ ([36]).

It is tempting to do the same here with V and $\Pi(V)$. That is, for any V as in §7.1, we have associated a non zero unitary admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach $\Pi(V)$. Let $\Pi(V)^0 \subset \Pi(V)$ (resp. $T \subset V$) be any unit ball (resp. any $\mathcal{O}_E$-lattice) which is preserved by $\mathrm{GL}_2(\mathbb{Q}_p)$ (resp. $\mathrm{G}_{\mathbb{Q}_p}$). In this last lecture, we want to state and prove some cases of a conjecture relating the semi-simplification of $\Pi(V)^0 \otimes_{\mathcal{O}_E} \mathbb{F}_E$ and the semi-simplification of $T \otimes_{\mathcal{O}_E} \mathbb{F}_E$.

10.1. **Statement of the conjecture.** In order to state the conjecture, we need to make lists of some 2-dimensional $\overline{\mathbb{F}_p}$-representations of $\mathrm{G}_{\mathbb{Q}_p}$ and of some $\overline{\mathbb{F}_p}$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let us start with those of $\mathrm{G}_{\mathbb{Q}_p}$. Let ω be the mod p cyclotomic character seen as a character of $\mathbb{Q}_p^\times$ via class field theory, and let μ_{-1} be the unramified quadratic character of $\mathbb{Q}_p^\times$. We write $\mathrm{I}_{\mathbb{Q}_p}$ for the inertia subgroup of $\mathrm{G}_{\mathbb{Q}_p}$. Let $\omega_2 : \mathrm{I}_{\mathbb{Q}_p} \to \overline{\mathbb{F}_p}^\times$ be Serre's fundamental character of level 2 and for $s \in \{0, \cdots, p\}$, let $\mathrm{ind}(\omega_2^s)$ be the unique (irreducible) representation of $\mathrm{G}_{\mathbb{Q}_p}$ whose determinant is ω^s and whose restriction to the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2})$ is $\omega_2^s \oplus \omega_2^{ps}$. We know that as r runs over $0, \dots, p-1$ and as η runs over all characters $\eta : \mathbb{Q}_p^\times \to \overline{\mathbb{F}_p}^\times$, the representations $\rho(r, \eta) := (\mathrm{ind}(\omega_2^r)) \otimes \eta$ run over all irreducible 2-dimensional $\overline{\mathbb{F}_p}$-representations of $\mathrm{G}_{\mathbb{Q}_p}$ and that the only isomorphisms are:

$$\begin{aligned}\rho(r,\eta) &\simeq \rho(r,\eta\mu_{-1})\\ \rho(r,\eta) &\simeq \rho(p-1-r,\eta\omega^r)\\ \rho(r,\eta) &\simeq \rho(p-1-r,\eta\omega^r\mu_{-1})\end{aligned}$$

Now, we turn to representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. For $r \in \{0,\cdots,p-1\}$, let $\mathrm{Sym}^r\overline{\mathbb{F}_p}^2$ denote the natural representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ acting through $\mathrm{GL}_2(\mathbb{F}_p)$ which we extend to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ by sending p to 1. Let

$$\mathrm{ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^r\overline{\mathbb{F}_p}^2$$

be the $\overline{\mathbb{F}_p}$-vector space of functions $f : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{Sym}^r\overline{\mathbb{F}_p}^2$ which are compactly supported modulo $\mathbb{Q}_p^\times$ and such that $f(kg) = \mathrm{Sym}^r(k)(f(g))$ with the left action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

For $r \in \{0,\cdots,p-1\}$, and $\lambda \in \overline{\mathbb{F}_p}$, let

$$\pi(r,\lambda,\eta) := \left[\left(\mathrm{ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^r\overline{\mathbb{F}_p}^2\right)/(T-\lambda)\right] \otimes (\eta\circ\det),$$

where T is a “Hecke operator” which corresponds to the double coset

$$\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p).$$

Those representation are irreducible, and all smooth irreducible $\overline{\mathbb{F}_p}$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ having a central character are given by (see [1],[2],[8]):

1. $\eta\circ\det$ where $\eta : \mathbb{Q}_p^\times \to \overline{\mathbb{F}_p}^\times$ is a smooth character.
2. the $\mathrm{Sp}\otimes(\eta\circ\det)$ where Sp is the “Special” representation.
3. the $\pi(r,\lambda,\eta)$ where $\eta : \mathbb{Q}_p^\times \to \overline{\mathbb{F}_p}^\times$ is a smooth character and $r \in \{0,\cdots,p-1\}$ and $\lambda \in \overline{\mathbb{F}_p}\setminus\{-1,1\}$.

When $\lambda \neq 0$, the $\pi(r,\lambda,\eta)$ have no non trivial intertwinings (unless $\lambda = \pm 1$, in which case there are some “easy” ones, see [2]) and when $\lambda = 0$, the only isomorphisms are:

$$\begin{aligned}\pi(r,0,\eta) &\simeq \pi(r,0,\eta\mu_{-1})\\ \pi(r,0,\eta) &\simeq \pi(p-1-r,0,\eta\omega^r)\\ \pi(r,0,\eta) &\simeq \pi(p-1-r,0,\eta\omega^r\mu_{-1}).\end{aligned}$$

This certainly suggests that there is a correspondence between the representations ρ of $\mathrm{G}_{\mathbb{Q}_p}$ and the representations π of $\mathrm{GL}_2(\mathbb{Q}_p)$. More precisely, we define the following “correspondence” ([8]):

Definition 10.1.1. *If $\lambda = 0$, then*

$$\pi(r,0,\eta) \leftrightarrow \rho(r,\eta)$$

and if $\lambda \neq 0$, then

$$\pi(r,\lambda,\eta)^{\mathrm{ss}} \oplus \pi([p-3-r],\lambda^{-1},\omega^{r+1}\eta)^{\mathrm{ss}} \leftrightarrow \begin{pmatrix}\mathrm{unr}(\lambda^{-1})\omega^{r+1} & 0\\ 0 & \mathrm{unr}(\lambda)\end{pmatrix}\otimes\eta$$

where $\mathrm{unr}(\lambda)$ is the unramified character sending Frobenius to λ, ss denotes the semisimplification and $[x]$ is the integer in $\{0,\cdots,p-2\}$ which is congruent to $x \mod p-1$.

On the other hand, we have associated in the previous chapters a unitary representation $\Pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ to a crystalline representation V of $\mathrm{G}_{\mathbb{Q}_p}$. Let $V \mod p$ denotes the semisimplification of T/pT where T is some Galois invariant lattice of V and let $\overline{\Pi}(V)$ be the semisimplification of $\Pi^0(V)/p$ where $\Pi^0(V)$ is some $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant unit ball of $\Pi(V)$.

Conjecture 10.1.2. *The representation $\Pi^0(V)/p$ is of finite length (so that $\overline{\Pi}(V)$ is well-defined) and $\overline{\Pi}(V)$ corresponds to $V \mod p$ under the correspondence of definition 10.1.1.*

One application of this conjecture is that it is in principle easier to compute $\overline{\Pi}(V)$ than to compute $V \mod p$ given $D_{\mathrm{cris}}(V)$. Note that using [2] and the definition of $\Pi(V)$, this conjecture is easily checked to be true when V is reducible. We thus only consider V irreducible in the sequel.

For $a_p \in E$ with positive valuation, let V_{k,a_p} be the crystalline representation with Hodge-Tate weights $(0, k-1)$ such that $D_{\mathrm{cris}}(V_{k,a_p}^*) = D_{k,a_p} = Ee_1 \oplus Ee_2$ where:

$$\begin{cases} \varphi(e_1) = p^{k-1} e_2 \\ \varphi(e_2) = -e_1 + a_p e_2 \end{cases} \quad \text{and} \quad \mathrm{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leq 0, \\ Ee_1 & \text{if } 1 \leq i \leq k-1, \\ 0 & \text{if } i \geq k. \end{cases}$$

In the notation of the previous lectures, we have $V_{k,a_p} = V(\alpha, \beta)$ where α and β are the roots of the polynomial $X^2 - a_p X + p^{k-1} = 0$. The description with k and a_p is a bit more convenient for computations associated to Wach modules.

It is possible to compute $\overline{\Pi}(V_{k,a_p})$ "by hand" for small values of k and one deduces from 10.1.2 some predictions for what $V_{k,a_p} \mod p$ should be. Indeed, we have the following theorem (see [9, théorème 1.4]):

Theorem 10.1.3. *Suppose that* $\mathrm{val}(a_p) > 0$.

(1) If $2 \leq k \leq p+1$, then $\overline{\Pi}(V_{k,a_p}) = \pi(k-2, 0, 1)$.
(2) If $k = p+2$ and $p \neq 2$, then
  (a) if $\mathrm{val}(a_p) < 1$, then $\overline{\Pi}(V_{k,a_p}) = \pi(p-2, 0, \omega) \simeq \pi(1, 0, 1)$.
  (b) if $\mathrm{val}(a_p) \geq 1$, then $\overline{\Pi}(V_{k,a_p}) = \pi(p-2, \lambda, \omega)^{\mathrm{ss}} \oplus \pi(p-2, \lambda^{-1}, \omega)^{\mathrm{ss}}$ where $\lambda^2 - \overline{(a_p/p)}\lambda + 1 = 0$.

If $p+3 \leq k \leq 2p$ and $p \neq 2$, then
  (a) if $\mathrm{val}(a_p) < 1$, then $\overline{\Pi}(V_{k,a_p}) = \pi(2p-k, 0, \omega^{k-1-p}) \simeq \pi(k-1-p, 0, 1)$.
  (b) if $\mathrm{val}(a_p) = 1$, then $\overline{\Pi}(V_{k,a_p}) = \pi(k-3-p, \lambda, \omega)^{\mathrm{ss}} \oplus \pi(2p-k, \lambda^{-1}, \omega^{k-1-p})^{\mathrm{ss}}$ where $\lambda = \overline{(k-1)a_p/p}$.
  (c) if $\mathrm{val}(a_p) > 1$, then $\overline{\Pi}(V_{k,a_p}) = \pi(k-3-p, 0, \omega)$.

In the remainder of this chapter, we will prove some of the formulas for $V_{k,a_p} \mod p$ which are predicted by the above theorem (via Conj.10.1.2). In particular, we will explain the proof of the following theorem:

Theorem 10.1.4. *(1) If $k \leq p+1$, then $V_{k,a_p} \mod p = \mathrm{ind}(\omega_2^{k-1})$.*
(2) If $k = p+2$ and $\mathrm{val}(a_p) > 1$ then

$$V_{k,a_p} \mod p = \mathrm{unr}(\sqrt{-1})\omega \oplus \mathrm{unr}(-\sqrt{-1})\omega.$$

(3) If $p+3 \leq k \leq 2p-1$ and $\mathrm{val}(a_p) > 1$ then $V_{k,a_p} \mod p = \mathrm{ind}(\omega_2^{k-1})$.

When $a_p = 0$, one can explicitly describe $V_{k,0}$ in terms of Lubin-Tate characters and so we can also describe $V_{k,0} \mod p$ for all k. When $k \leq p$, then the theory of Fontaine-Laffaille gives us an explicit description of $V_{k,a_p} \mod p$ regardless of the valuation of a_p and the theorem is then straightforward. Our method of proof for theorem 10.1.4 is to show that once we fix k, then if we vary a_p a little bit, $V_{k,a_p} \mod p$ does not change. In order to do this, we compute the Wach module associated to V_{k,a_p} as explicitly as possible.

10.2. **Lifting $D_{\mathrm{cris}}(V)$.** In this paragraph, we'll explain the general strategy for constructing the Wach module $N(V)$ attached to a crystalline representation V if we are given the filtered φ-module $D_{\mathrm{cris}}(V)$. Recall from theorem 5.5.1 in §5.5 that $N(V)/X \cdot N(V)$ is a filtered φ-module isomorphic to $D_{\mathrm{cris}}(V)$. Furthermore, the functor $V \mapsto D_{\mathrm{cris}}(V)$ is fully faithful, so if we can construct *some* Wach module N such that $N/X \cdot N = D_{\mathrm{cris}}(V)$, then we necessarily have $N = N(V)$.

Note that both the φ-module structure and the filtration on N/XN depend solely on the map $\varphi : N \to N$ and not at all on the action of $\Gamma_{\mathbb{Q}_p}$. Furthermore, given an admissible filtered φ-module D, it is easy to construct a φ-module N over $\mathcal{O}_E[[X]]$ such that $N/XN = D$. The problem then is that it is very hard in general to find an action of $\Gamma_{\mathbb{Q}_p}$ which commutes with φ.

Therefore, in order to construct the Wach module $N(V)$ starting from $D_{\mathrm{cris}}(V)$, one has to find a "lift" of φ which is such that we can then define an action of $\Gamma_{\mathbb{Q}_p}$ which will commute with that lift. I do not know of any systematic way of doing this in general.

If $a_p = 0$, then we can carry out this program easily enough and the resulting Wach module was given in §5.1 when $k = 2$. Let us generalize it to arbitrary $k \geq 2$. The Wach module $N(V_{k,0}^*)$ is of rank 2 over $\mathbb{B}_{\mathbb{Q}_p}^+$ generated by e_1 and e_2 with $\varphi(e_1) = q^{k-1}e_2$ and $\varphi(e_2) = -e_1$. The action of $\gamma \in \Gamma_{\mathbb{Q}_p}$ is given by:

$$\gamma(e_1) = \left(\frac{\log^+(1+X)}{\gamma(\log^+(1+X))}\right)^{k-1} e_1$$
$$\gamma(e_2) = \left(\frac{\log^-(1+X)}{\gamma(\log^-(1+X))}\right)^{k-1} e_2$$

where

$$\log^+(1+X) = \prod_{n\geq 0} \frac{\varphi^{2n+1}(q)}{p} \quad \text{and} \quad \log^-(1+X) = \prod_{n\geq 0} \frac{\varphi^{2n}(q)}{p},$$

as previously.

10.3. **Continuity of the Wach module.** One can also think about the above constructions in terms of matrices. Given a Wach module N, we can fix a basis and let $P \in \mathrm{M}(d, \mathbb{B}_{\mathbb{Q}_p}^+)$ be the matrix of Frobenius in that basis. The problem is then to construct an action of $\Gamma_{\mathbb{Q}_p}$, that is for every $\gamma \in \Gamma_{\mathbb{Q}_p}$, we need to construct a matrix $G \in \mathrm{GL}(d, \mathbb{B}_{\mathbb{Q}_p}^+)$ such that $P\varphi(G) = G\gamma(P)$. In practice we need do this only for a topological generator γ of $\Gamma_{\mathbb{Q}_p}$ (which is procyclic for $p \neq 2$).

In this paragraph, we use this approach to show that if we know a Wach module N and we "change" P slightly, then this defines another Wach module. More to the point, we have the following proposition:

Proposition 10.3.1. *Let N be a Wach module and let P and G be the matrices of φ and $\gamma \in \Gamma_{\mathbb{Q}_p}$. Define $\alpha(r) = \mathrm{vam}(1 - \varepsilon(\gamma)) \cdot (1 - \varepsilon(\gamma)^2) \cdots (1 - \varepsilon(\gamma)^r)$. If $H_0 \in \mathrm{M}(d, p^{\alpha(k-1)+C}\mathbb{A}_{\mathbb{Q}_p}^+)$ with $C \geq 0$, then there exists $H \in \mathrm{M}(d, p^C\mathbb{A}_{\mathbb{Q}_p}^+)$ such that if we set $Q = (\mathrm{Id} + H)P$, then the φ-module defined by the matrix Q admits a commuting action of $\Gamma_{\mathbb{Q}_p}$.*

The proof of this rests on the following technical lemma:

Lemma 10.3.2. *If N is a Wach module and if $G \in \mathrm{GL}(d, \mathbb{A}_{\mathbb{Q}_p}^+)$ is the matrix of $\gamma \in \Gamma_{\mathbb{Q}_p}$, and if $H_0 \in \mathrm{M}(d, p^{\alpha(k-1)+C}\mathbb{A}_{\mathbb{Q}_p}^+)$ with $C \geq 0$, then there exists $H \in \mathrm{M}(d, p^C\mathbb{A}_{\mathbb{Q}_p}^+)$ such that:*

(1) $H \equiv H_0 \mod X$.

(2) $(\mathrm{Id} + H)^{-1}G\gamma(\mathrm{Id} + H) \equiv G \mod X^{k-1}$.

The idea of the proof of proposition 10.3.1 is then that if P, G are the matrices of φ and γ on N, then the lemma above and the fact that $P\varphi(G) = G\gamma(P)$ imply that $G - (\mathrm{Id} + H)P\varphi(G)\gamma(P(\mathrm{Id} + H))^{-1} \equiv 0 \mod X^{k-1}$ and one can then construct a matrix for γ by successive approximations starting from G (see [7, §3.1] for more details).

The proof of lemma 10.3.2 is also by successive approximations and we sketch it below.

Proof. If we write explicitly the condition (2) of the proposition, then we get:

$$(\mathrm{Id} + XG_1 + X^2G_2 + \cdots)\gamma(H_0 + XH_1 + \cdots) = (H_0 + XH_1 + \cdots)(\mathrm{Id} + XG_1 + X^2G_2 + \cdots)$$

and bearing in mind that $\gamma(X^r) \equiv \varepsilon(\gamma)^r X^r \mod X^{r+1}$, we can solve the above equation $\mod X^r$ in r successive steps. The first is to find H_1 such that $(1 - \varepsilon(\gamma))H_1 = G_1H_0 - H_0G_1$ so that

$$H_1 \in \mathrm{M}(d, p^{\alpha(k-1)+C-v_p(1-\varepsilon(\gamma))}\mathbb{A}_{\mathbb{Q}_p}^+).$$

The second is to find H_2 such that $(1 - \varepsilon(\gamma)^2)H_2 =$ (some $\mathbb{Z}_p$-linear combination of products of G_0, G_1, G_2, H_0, H_1) so that

$$H_2 \in \mathrm{M}(d, p^{\alpha(k-1)+C-v_p(1-\varepsilon(\gamma))-v_p(1-\varepsilon(\gamma)^2)}\mathbb{A}_{\mathbb{Q}_p}^+).$$

Eventually, we get $H_0, \cdots, H_{k-1} \in \mathrm{M}(d, p^C\mathbb{A}_{\mathbb{Q}_p}^+)$ and we can then set $H = H_0 + XH_1 + \cdots + X^{k-1}H_{k-1}$. □

We can then apply proposition 10.3.1 to the matrices

$$P = \begin{pmatrix} 0 & -1 \\ q^{k-1} & 0 \end{pmatrix} \qquad \text{and} \qquad H_0 = \begin{pmatrix} 0 & 0 \\ -a_p & 0 \end{pmatrix}$$

to get a Wach module N_{k,a_p} as soon as the valuation of a_p is large enough, and a straightforward computation shows that $N_{k,a_p}/XN_{k,a_p}$ is then isomorphic to $D_{\mathrm{cris}}(V_{k,a_p}^*)$.

An explicit computation of $\alpha(k-1)$ and some refinements of the computations above specific to this case then give us the proof of theorem 10.1.4.

Remark 10.3.3. *To conclude, let us point out that using the description of $\Pi(V)^*$ with (φ, Γ)-modules given in the previous lectures, we can prove that $\Pi^0(V)/p$ is of finite length in general and we can also prove conjecture 10.1.2 in many cases, including all the cases when T/pT is irreducible (forthcoming work).*

References

[1] Barthel L., Livné R., *Modular representations of* GL_2 *of a local field: the ordinary, unramified case*, J. of Number Theory 55, 1995, 1-27.

[2] Barthel L., Livné R., *Irreducible modular representations of* GL_2 *of a local field*, Duke Math. J. 75, 1994, 261-292.

[3] Berger L., *Représentations p-adiques et équations différentielles*, Inv. Math. 148, 2002, 219-284.

[4] Berger L., *Bloch and Kato's exponential map: three explicit formulas*, Doc. Math. Extra Volume: Kazuya Kato's Fiftieth Birthday, 2003, 99-129.

[5] Berger L., *Limites de représentations cristallines*, to appear in Comp. Math.

[6] Berger L., *Équations différentielles p-adiques et (φ, N)-modules filtrés*, preprint.

[7] Berger L., Li H., Zhu H. J., *Construction of some families of 2-dimensional crystalline representations*, Math. Ann. 329, 2004, 365-377.

[8] Breuil C., *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *I*, Comp. Math. 138, 2003, 165-188.

[9] Breuil C., *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *II*, J. Institut Math. Jussieu 2, 2003, 23-58.

[10] Breuil C., *Invariant $\mathcal{L}$ et série spéciale p-adique*, to appear in Ann. Scient. E.N.S.

[11] Breuil C., *Série spéciale p-adique et cohomologie étale complétée*, available online at: http://www.ihes.fr/~breuil/publications.html.

[12] Cherbonnier F., Colmez P., *Représentations p-adiques surconvergentes*, Inv. Math. 133, 581-611, 1998.

[13] Colmez P., *Représentations cristallines et représentations de hauteur finie*, J. Reine Angew. Math. 514, 1999, 119-143.

[14] Colmez P., *Arithmétique de la fonction zêta*, available online at: http://math.polytechnique.fr/xups/vol02.html

[15] Colmez P., *La conjecture de Birch et Swinnerton-Dyer p-adique*, Séminaire Bourbaki 919, juin 2003.

[16] Colmez P., *in preparation.*

[17] Colmez P., Fontaine J.-M., *Construction des représentations p-adiques semi-stables*, Inv. Math. 140, 2000, 1-43.

[18] Deligne P., *Formes modulaires et représentations de* GL_2, Modular functions of one variable II, Lecture Notes 349, 1973, 55-105.

[19] Emerton M., *p-adic L-functions and completions of representations of p-adic reductive groups*, preprint 2004.

[20] Emerton M., *Locally analytic representation theory of p-adic reductive groups: A summary of some recent developments*, preprint 2003.

[21] Fontaine J-M., *Le corps des périodes p-adiques*, Astérisque 223, 1994, 59-111.

[22] Fontaine J-M., *Représentations p-adiques semi-stables*, Astérisque 223, 1994, 113-184.

[23] Fontaine J-M., *Représentations p-adiques des corps locaux I*, The Grothendieck Festschrift II, Progr. Math. 87, Birkhäuser, 1990, 249-309.

[24] Fontaine J-M., Wintenberger J-P., *Le "corps des normes" de certaines extensions algébriques de corps locaux*, C. R. Acad. Sci. Paris 288, 1979, 367-370.

[25] Harris M., Taylor R., *On the geometry and cohomology of some simple Shimura varieties*, Ann. Math. Studies 151, Princeton Univ. Press, 2001.

[26] Henniart G., *Une preuve simple des conjectures de Langlands locales pour* GL_n *sur un corps p-adique*, Inv. Math. 139, 2000, 439-455.

[27] Mazur B., Tate J., Teitelbaum J., *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Inv. Math. 84, 1986, 1-48.

[28] Schikhof W., *An introduction to p-adic analysis*, Cambridge Studies in Advanced Math. 4, Cambridge University Press, 1984.

[29] Schikhof W., *Non-archimedean calculus*, preprint, university of Nijmegen, 1978.

[30] Schneider P., *Nonarchimedean Functional Analysis*, Springer-Verlag, 2001.

[31] Schneider P., Teitelbaum J., *Locally analytic distributions and p-adic representation theory, with applications to* GL_2, J. Amer. Math. Soc. 15, 2002, 443-468.

[32] Schneider P., Teitelbaum J. (with an appendix by D. Prasad), *$U(\mathfrak{g})$-finite locally analytic representations*, Representation Theory 5, 2001, 111-128.

[33] Schneider P., Teitelbaum J., *Banach space representations and Iwasawa theory*, Israel J. Math. 127, 2002, 359-380.

[34] Schneider P., Teitelbaum J., *Algebras of p-adic distributions and admissible representations*, Inv. Math. 153, 2003, 145-196.

[35] Serre J.-P., *Sur les représentations modulaires de* $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. 54, 1987, 179-230.

[36] Vignéras M.-F., *Correspondance de Langlands semi-simple pour* $\mathrm{GL}_n(F)$ *modulo* $\ell \neq p$, Inv. Math. 144, 2001, 177-223.

[37] Wach N., *Représentations cristallines de torsion*, Comp. Math. 108, 1997, 185-240.

[38] Wintenberger J-P., *Le corps des normes de certaines extensions infinies des corps locaux; applications*, Ann. Scient. E.N.S. 16, 1983, 59-89.