Modular representations of GL_n and tensor products of Galois representations

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Contents

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General aim:

Study certain smooth admissible representations of $\mathrm{GL}_n(F_v)$ over $\mathbb{F}$ associated to automorphic (for G) mod p Galois representations.

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$$\begin{aligned} S(U^v,\mathbb{F}) &:= \{f : G(F^+)\backslash G(\mathbb{A}_{F^+}^{\infty,v})/U^v \longrightarrow \mathbb{F},\ \text{loc. cst.}\} \\ S(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}] &:= \text{Hecke eigenspace associated to } \overline{r}. \end{aligned}$$

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$G(F_v^+)$ acts on $S(U^v,\mathbb{F})$ by right translation: $(g_v f)(g) := f(gg_v)$, preserves $S(U^v,\mathbb{F})[\mathfrak{m}_{\bar{r}}]$ = smooth admissible repres. of $G(F_v^+)$.

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Remark

$S(U^v,\mathbb{F})[\mathfrak{m}_{\overline{r}}] \neq 0 \Rightarrow \overline{r}(c\cdot c) \cong \overline{r}(\cdot)^\vee \otimes \omega^{1-n}$ where $\langle c\rangle = \mathrm{Gal}(F/F^+)$.

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Should hold as soon as $n = 2$, $F_v = \mathbb{Q}_p$. For H^1 of modular curves, no need to assume $\overline{r}_w$ irreducible (Colmez + Emerton).

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Extend it to an action of $\mathbb{F}[[X]][F]$ via:

$$F(v) := \sum_{n_1 \in N_1/\xi(p)N_1\xi(p)^{-1}} n_1 \xi(p) v, \quad v \in \pi^{N_1}.$$

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Finally, let $\mathbb{Z}_p^\times$ act on π^{N_1} via $z \cdot v := \xi(z)v$, $z \in \mathbb{Z}_p^\times$.

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Let M be a finite type $\mathbb{F}[[X]][F]$-module such that M is torsion as $\mathbb{F}[[X]]$-module and satisfies $\dim_{\mathbb{F}} M[X] < \infty$. Then $M^\vee[1/X]$ is an étale φ-module over $\mathbb{F}((X))$.

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We apply this to $M \subseteq \pi^{N_1}$ of finite type over $\mathbb{F}[[X]][F]$ preserved by $\mathbb{Z}_p^\times \cong \Gamma$ with $\dim_{\mathbb{F}} M[X] < \infty \rightsquigarrow$ get $M^\vee[1/X] =$ étale (φ, Γ)-module.

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Define the covariant functor V to ind-representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$:

$$\pi \longmapsto V(\pi) := \varinjlim_{M} V^\vee\big(M^\vee[1/X]\big)$$

where the limit is over $\mathbb{Z}_p^\times$-stable $M \subseteq \pi^{N_1}$ as above ($V^\vee(M^\vee[1/X])$ is the contravariant $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation associated to $M^\vee[1/X]$).

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There is an integer $d \geq 1$ such that:

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Remark

An étale (φ, Γ)-module D has an operator ψ. The conjecture can be restated as: if $f : (S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]^{N_1})^\vee \to D$ is a contin., Γ-equivariant, $\mathbb{F}[[X]]$-linear map sending $F^\vee$ to ψ, then f uniquely factors through the (φ, Γ)-module of the above tensor induction.

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- $U^v = \prod_{w \neq v} U^v_w$ with $\begin{cases} U^v_w \text{ max. hyperspecial if } w \text{ is inert in } F \\ \\ \end{cases}$

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Hypothesis on F, G, $\overline{r}$, U^v

Till the end of the talk: $n = 2$, $F_v/\mathbb{Q}_p$ unramified of degree $f \geq 1$.

Need the following extra assumptions (some of them standard):

- F/F^+ unramified, p inert in F^+ (the latter for simplicity)
- G quasi-split at all finite places of F^+
- $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ adequate and $\overline{r}_w$ unramified if w inert in F
- weak gener. assumptions on $\overline{r}_w$ for $w \neq v$ when $\overline{r}_w$ ramifies
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(May-be this strong genericity assumption on $\overline{r}_v$ can be improved.)

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Under the above assumptions Conjecture 1 holds, i.e. there is an integer $d \geq 1$ such that:

$$V\big(S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]\big) \cong \big(\mathrm{Ind}_{F_v}^{\otimes \mathbb{Q}_p} \overline{r}_v\big)^{\oplus d} \otimes \omega^*.$$

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Although $V\big(S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]\big)$ only depends on $\overline{r}_v$, we **do not know** if the $\mathrm{GL}_2(F_v)$-representation $S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]$ only depends on $\overline{r}_v$.

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Step 2: $\dim_{\mathbb{F}} V\big(S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]\big)$ is finite and bounded by $2^f d$.

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Theorem 3 (Emerton-Gee-Savitt, Hu-Wang, Le-Morra-Schraen, B.-H.-H.-M.-S., building on B.-Paškūnas + Buzzard-Diamond-Jarvis)

There is an integer $d \geq 1$ and an explicit representation D_0 of KZ over $\mathbb{F}$ only depending on $\bar{r}_v$ such that $S(U^v, \mathbb{F})[\mathfrak{m}_{\bar{r}}]^{K(1)} \cong D_0^{\oplus d}$.

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Theorem 4

Let π be a smooth admissible representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$ such that $(\pi^{I(1)} \hookrightarrow \pi^{K(1)}) \cong (D_0^{I(1)} \hookrightarrow D_0)^{\oplus d}$ (compatibly with $\mathfrak{n}$ and KZ). Then there is an injection $(\mathrm{Ind}_{F_v}^{\otimes \mathbb{Q}_p} \overline{r}_v)|_{I_v}^{\oplus d} \hookrightarrow V(\pi)|_{I_v}$.

Proof of Theorem 2: Step 1

Proof of Theorem 4: we compute an explicit $\mathbb{F}[[X]][F]$-submodule $M(\pi)$ in π^{N_1} preserved by $\mathbb{Z}_p^\times$ such that $V(M(\pi))|_{I_v} \cong \left(\mathrm{Ind}_{F_v}^{\otimes \mathbb{Q}_p} \overline{r}_v\right)|_{I_v}^{\oplus d}$.

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Theorem 5 (Dotto-Le + B.-H.-H.-M.-S.)

(i) There is an explicit action of $\mathfrak{n}$ on $D_0^{I(1)}$, only depending on $\overline{r}_v$, such that there is an $(\mathfrak{n}, KZ)$-equivariant isomorphism:

$$\left(S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]^{I(1)} \hookrightarrow S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]^{K(1)}\right) \cong \left(D_0^{I(1)} \hookrightarrow D_0\right)^{\oplus d}.$$

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(ii) For this action of $\mathfrak{n}$ we actually have:

$$V\left(M(S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}])\right) \cong \left(\mathrm{Ind}_{F_v}^{\otimes \mathbb{Q}_p}\overline{r}_v\right)^{\oplus d}.$$

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If π is a smooth representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$ with a central character, then $\pi^{I(1)} = \pi[\mathfrak{m}_I]$ and π is admissible if and only if $\dim_{\mathbb{F}} \pi[\mathfrak{m}_I] < \infty$.

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Theorem 6

Let π be a smooth admissible representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$ with a central character such that for any $\chi : I \to \mathbb{F}^\times$ appearing in $\pi[\mathfrak{m}_I]$:

$$[\pi[\mathfrak{m}_I] : \chi] = [\pi[\mathfrak{m}_I^3] : \chi].$$

Then $\dim_{\mathbb{F}} V(\pi) \leq \dim_{\mathbb{F}} \pi[\mathfrak{m}_I]$, in particular $V(\pi)$ is finite dimensional.

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The hyp. on π in Thm. 6 implies that the action of $\mathrm{gr}_{\mathfrak{m}_I} \Lambda_I$ on $\mathrm{gr}_{\mathfrak{m}_I} \pi^\vee$ factors through the abelian quotient $\mathbb{F}[(X_i, Y_i)_i]/(X_i Y_i)$ of $\mathrm{gr}_{\mathfrak{m}_I} \Lambda_I$.

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Hence $(\mathrm{gr}_{\mathfrak{m}_I}\pi^\vee)[1/\prod X_i]$ is generated by at most r elements over:

$$\big(\mathbb{F}[(X_i, Y_i)_i]/(X_i Y_i)\big)[1/\prod X_i] \cong \mathbb{F}[(X_i)_i][1/\prod X_i].$$

Proof of Theorem 2: Step 2

Endow $\pi^\vee[1/\prod X_i] \cong \pi^\vee \otimes_{\mathbb{F}[[N_0]]} \mathbb{F}[[N_0]][1/\prod X_i]$ with tensor product filtration for $\begin{cases} \mathfrak{m}_I\text{-adic filtration on } \pi^\vee \\ (X_0, \ldots, X_{f-1})\text{-adic filtration on } \mathbb{F}[[N_0]][1/\prod X_i]. \end{cases}$

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Let $J := \mathrm{Ker}(\mathbb{F}[[N_0]] \xrightarrow{\text{trace}} \mathbb{F}[[X]])$, hence $(\pi^\vee[1/\prod X_i])^\wedge/J$ is generated by at most r elements over $(\mathbb{F}[[N_0]][1/\prod X_i])^\wedge/J \cong \mathbb{F}((X))$.

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For any $M \subseteq \pi^{N_1}$ such that $\dim_{\mathbb{F}} M[X] < \infty$, the morphism:

$$(\pi^{N_1})^\vee \cong \pi^\vee/J \longrightarrow M^\vee[1/X]$$

factors as a surjection $(\pi^\vee[1/\prod X_i])^\wedge/J \twoheadrightarrow M^\vee[1/X]$.

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Endow $\pi^\vee[1/\prod X_i] \cong \pi^\vee \otimes_{\mathbb{F}[[N_0]]} \mathbb{F}[[N_0]][1/\prod X_i]$ with tensor product filtration for $\begin{cases} \mathfrak{m}_I\text{-adic filtration on } \pi^\vee \\ (X_0, \ldots, X_{f-1})\text{-adic filtration on } \mathbb{F}[[N_0]][1/\prod X_i]. \end{cases}$

Let $(\pi^\vee[1/\prod X_i])^\wedge :=$ corresponding completion. It is generated by at most r elements over $(\mathbb{F}[[N_0]][1/\prod X_i])^\wedge$ (look at the graded modules).

Let $J := \mathrm{Ker}(\mathbb{F}[[N_0]] \xrightarrow{\text{trace}} \mathbb{F}[[X]])$, hence $(\pi^\vee[1/\prod X_i])^\wedge/J$ is generated by at most r elements over $(\mathbb{F}[[N_0]][1/\prod X_i])^\wedge/J \cong \mathbb{F}((X))$.

For any $M \subseteq \pi^{N_1}$ such that $\dim_{\mathbb{F}} M[X] < \infty$, the morphism:

$$(\pi^{N_1})^\vee \cong \pi^\vee/J \longrightarrow M^\vee[1/X]$$

factors as a surjection $(\pi^\vee[1/\prod X_i])^\wedge/J \twoheadrightarrow M^\vee[1/X]$.

In particular $\dim_{\mathbb{F}} V(\pi) \leq \dim_{\mathbb{F}((X))} \left((\pi^\vee[1/\prod X_i])^\wedge/J\right) \leq r$. □

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Thus $\left(S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]^\vee[1/\prod X_i]\right)^\wedge / J$ is finite dimensional over $\mathbb{F}((X))$.

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Proof: $\exists$ an I-equiv. surjection $\oplus_{i=1}^{2^f d} \Lambda_I(\chi_i) \twoheadrightarrow (\mathrm{soc}_K S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}])|_I^\vee$.
Λ_I projective $\Rightarrow$ it lifts to $f : \oplus_{i=1}^{2^f d} \Lambda_I(\chi_i) \longrightarrow S(U^v, \mathbb{F})[\mathfrak{m}_{\overline{r}}]|_I^\vee$.

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The representation $S(U^v, \mathbb{F})[\mathfrak{m}_{\bar{r}}]$ satisfies the hypothesis of Theorem 6. (Only need 10 instead of $f' = \mathrm{Max}(2f, 10)$ in the bounds on the r_i.)

Thus $\big(S(U^v, \mathbb{F})[\mathfrak{m}_{\bar{r}}]^\vee[1/\prod X_i]\big)^\wedge/J$ is finite dimensional over $\mathbb{F}((X))$.

Theorem 8

We have $\dim_{\mathbb{F}((X))}\big((S(U^v, \mathbb{F})[\mathfrak{m}_{\bar{r}}]^\vee[1/\prod X_i])^\wedge/J\big) \leq 2^f d$.

Proof: $\exists$ an I-equiv. surjection $\oplus_{i=1}^{2^f d}\Lambda_I(\chi_i) \twoheadrightarrow (\mathrm{soc}_K S(U^v, \mathbb{F})[\mathfrak{m}_{\bar{r}}])|_I^\vee$. Λ_I projective $\Rightarrow$ it lifts to $f : \oplus_{i=1}^{2^f d}\Lambda_I(\chi_i) \longrightarrow S(U^v, \mathbb{F})[\mathfrak{m}_{\bar{r}}]|_I^\vee$. By an explicit computation $(\mathrm{Coker}(f)[1/\prod X_i])^\wedge = 0$. This implies we can replace $r = \dim_{\mathbb{F}} S(U^v, \mathbb{F})[\mathfrak{m}_{\bar{r}}]^{I(1)}$ by $2^f d$ in the proof of Thm. 6. □