On modular representations of $\mathrm{GL}_2(L)$ for unramified L

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Contents

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General aim:

Understand better certain smooth admissible representations of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$ associated to $\overline{r}$ (F_v :=completion of F at v).

Local factor at v associated to $\overline{r}$

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We first consider the smooth representation of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ over $\mathbb{F}$:

$$\pi(\overline{r}) := \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\Big(\overline{r}, \varinjlim_K H^1_{\text{ét}}(X_K \times_F \overline{F}, \mathbb{F})\Big) \neq 0.$$

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One doesn't know if $\pi(\overline{r})$ has a Flath decomposition as a restricted tensor product of smooth $D_w^{\times}$-representations over finite places w of F ($D_w := D \otimes_F F_w$).

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But one can still define from $\pi(\overline{r})$ in an "ad hoc" way a local factor $\pi_v(\overline{r})$ at v under technical assumptions on $\overline{r}$.

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Then one can define an "optimal" open compact subgroup K^v of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$, a certain smooth finite dim. representation M^v of K^v over $\mathbb{F}$ (a "type"), and set (B.-Diamond, Emerton-Gee-Savitt):

$$\pi_v(\overline{r}) := \mathrm{Hom}_{K^v}\big(M^v, \pi(\overline{r})\big)[\mathfrak{m}] \neq 0$$

where $[\mathfrak{m}] :=$ kernel of Hecke operators at certain places $\neq v$.

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$\pi_v(\overline{r}) =$ smooth admissible representation of $D_v^\times \cong \mathrm{GL}_2(F_v)$ over $\mathbb{F}$ with central character $\psi := \omega\det(\overline{r}_v)$ ($\omega :=$ cyclo mod p).

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Assume $F = \mathbb{Q}$ and $D = \mathrm{GL}_2$, then $\pi_v(\overline{r})$ is known. In particular:

- $\mathrm{GK}(\pi_v(\overline{r})) = 1$
- $\pi_v(\overline{r})$ is of finite length
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Definition 1 (Gelfand-Kirillov dimension)

Let π_v be a smooth admissible representation of $K_v(1)$ over $\mathbb{F}$. There exists a unique $\mathrm{GK}(\pi_v) \in \{0, \ldots, \dim_{\mathbb{Z}_p}(K_v)\}$ such that there are $a \leq b$ in $\mathbb{R}_{>0}$ with $a \leq \frac{\dim_{\mathbb{F}}(\pi_v^{K_v(n)})}{p^{n\mathrm{GK}(\pi_v)}} \leq b$ for all $n \geq 1$.

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- $\Gamma := K/K(1) \cong \mathrm{GL}_2(\mathbb{F}_q)$, $Z(1) := \text{center of } K(1)$
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For arbitrary F, D and $\overline{r}$ (as before), one has the following:

Theorem 2 (Emerton-Gee-Savitt, Le, Hu-Wang, Le-Morra-Schraen, building on B.-Paškūnas and Buzzard-Diamond-Jarvis)

The finite-dimensional Γ-representation $\pi_v(\overline{r})^{K(1)} = \pi_v(\overline{r})[\mathfrak{m}_K]$ is explicitly known, in particular is local and multiplicity free.

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If $D_v \neq \mathrm{GL}_2(\mathbb{Q}_p)$ none of the statements in Theorem 1 are known.

Hypothesis on $\overline{r}_v$

Fix an embedding $\mathbb{F}_{q^2} \hookrightarrow \mathbb{F}$ and let ω_f, ω_{2f} := associated Serre's fundamental characters of level f, $2f$ of inertia subgroup I_v.

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This strong genericity assumption on $\overline{\rho}$ is not optimized!

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- Gee-Newton proved (without the assumptions on $\overline{\rho}$) that $\mathrm{GK}(\pi_v(\overline{r})) \geq f$, so our main result is $\mathrm{GK}(\pi_v(\overline{r})) \leq f$.
- Even under the assumptions on $\overline{\rho}$, we do *not* know if $\pi_v(\overline{r})$ is of finite length or if $\pi_v(\overline{r})$ is local.

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Let:

- $I := \{g \in K, g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\} =$ Iwahori
- $I(1) := \{g \in K, g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p\} =$ pro-p Iwahori
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Corollary 1

The smooth finite-dimensional I-representation $\pi_v(\overline{r})[\mathfrak{m}_I^3]$ is multiplicity free.

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It then directly follows from Corollary 1 and Theorem 5:

Corollary 2

We have $\mathrm{GK}(\pi_v(\bar{r})) \leq f$.

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Using Gee-Newton for the reverse inequality, one gets Theorem 4.

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Let $\pi_v^\vee := \mathrm{Hom}_{\mathbb{F}}(\pi_v, \mathbb{F})$, then $\pi_v^\vee/\mathfrak{m}_I = (\pi_v^{I(1)})^\vee = \oplus_\alpha \chi_\alpha$ for some characters $\chi_\alpha : I/I(1) \to \mathbb{F}^\times$.

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Let $\mathrm{Proj}_I \chi_\alpha := \chi_\alpha \otimes_{\mathbb{F}} \mathbb{F}[[I(1)/Z(1)]]$ = projective envelope of χ_α in the category of compact $\mathbb{F}[[I/Z(1)]]$-modules.

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Thm. 5 then follows from $\mathrm{GK}(\pi_v) \leq \max_\alpha \mathrm{GK}(\mathrm{coker}(h_\alpha)^\vee)$ and:

Proof of second intermediate theorem

Let $\pi_v^\vee := \mathrm{Hom}_{\mathbb{F}}(\pi_v, \mathbb{F})$, then $\pi_v^\vee/\mathfrak{m}_I = (\pi_v^{I(1)})^\vee = \oplus_\alpha \chi_\alpha$ for some characters $\chi_\alpha : I/I(1) \to \mathbb{F}^\times$.

Let $\mathrm{Proj}_I \chi_\alpha := \chi_\alpha \otimes_{\mathbb{F}} \mathbb{F}[[I(1)/Z(1)]] =$ projective envelope of χ_α in the category of compact $\mathbb{F}[[I/Z(1)]]$-modules.

As χ_α does not appear in $\mathfrak{m}_I \pi_v^\vee/\mathfrak{m}_I^3 \pi_v^\vee$ (by assumption), one proves there exist I-equivariant maps $h_\alpha : (\mathrm{Proj}_I \chi_\alpha)^{\oplus 2f} \to \mathrm{Proj}_I \chi_\alpha$ s.t.:

- image$(h_\alpha) \subseteq \mathfrak{m}_I^2 \mathrm{Proj}_I \chi_\alpha$
- the map $(\mathrm{Proj}_I \chi_\alpha/\mathfrak{m}_I)^{\oplus 2f} \to \mathfrak{m}_I^2 \mathrm{Proj}_I \chi_\alpha/\mathfrak{m}_I^3$ is injective
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Proposition 1

We have $\mathrm{GK}(\mathrm{coker}(h_\alpha)^\vee) \leq f$ (calculation in $\mathrm{gr}_{\mathfrak{m}_I} \mathbb{F}[[I(1)/Z(1)]]$).

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Main tool: patching functor M_∞ of Emerton-Gee-Savitt (building on Taylor-Wiles, Kisin) $=$ exact functor from continuous repres. of K over finite type $W(\mathbb{F})$-modules $+$ central character lifting ψ to finite type R_∞-modules satisfying several properties (cf. E.-G.-S.).

$R_\infty =$ patched deformation ring $=$ power series ring over $W(\mathbb{F})$.

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Hence Theorem 4 (multiplicity free part) follows from:

Theorem 6

The R_∞-module $M_\infty(\mathrm{Proj}_K\sigma/\mathfrak{m}_K^2)$ is cyclic.

Equivalently $M_\infty(\mathrm{Proj}_K\sigma/\mathfrak{m}_K^2) \cong$ quotient of R_∞.

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then $\mathrm{Proj}_K\sigma/\mathfrak{m}_K^2$ is a non-split extension:

$$\mathrm{Proj}_K\sigma/\mathfrak{m}_K^2 \cong \left(\oplus_\tau (V_2^\tau \otimes_\mathbb{F} \mathrm{Proj}_\Gamma\sigma)\right) \text{——} \mathrm{Proj}_\Gamma\sigma \ .$$

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Moreover $V_2^\tau \otimes_{\mathbb{F}} \mathrm{Proj}_\Gamma\sigma \cong \mathrm{Proj}_\Gamma\sigma_{+2_\tau} \oplus \mathrm{Proj}_\Gamma\sigma \oplus \mathrm{Proj}_\Gamma\sigma_{-2_\tau}$.

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Let $Q_\tau :=$ unique quotient of $\mathrm{Proj}_K\sigma/\mathfrak{m}_K^2$ which is a non-split extension $\left(\mathrm{Proj}_\Gamma\sigma_{+2_\tau} \oplus \mathrm{Proj}_\Gamma\sigma_{-2_\tau}\right) — \mathrm{Proj}_\Gamma\sigma$.

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One can prove:

Proposition 3

(i) There is an invariant $W(\mathbb{F})$-lattice L_2^τ in $(\widetilde{V}_2^\tau \otimes_{W(\mathbb{F})} \widetilde{\mathrm{Proj}}_\Gamma\sigma)[\frac{1}{p}]$ such that $L_2^\tau/p \cong Q_\tau$.

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It is enough to prove that $M_\infty(L)$ is cyclic.

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The proof is by dévissage, using:

- $M_\infty(\sigma') \neq 0 \Leftrightarrow \sigma' \hookrightarrow \pi_v(\overline{r})[\mathfrak{m}_K]$ ($\Leftrightarrow \sigma'$ Serre weight of $\overline{\rho}$)
- $M_\infty(\mathrm{Proj}_\Gamma \sigma')$ cyclic (Hu-Wang, Le-Morra-Schraen)
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I explain why $M_\infty(L^\tau) = M_\infty(\widetilde{\mathrm{Proj}}_\Gamma \sigma) \times_{M_\infty(\mathrm{Proj}_\Gamma \sigma)} M_\infty(L_2^\tau)$ is cyclic. Proof for L can be reduced to this case by induction.

Proof of first intermediate theorem

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- R_v/J_τ parametrizes pot. cryst. lifts of $\overline{\rho}$ of same tame types but HT weights $(1, 0)$ outside embedding τ, $(2, -1)$ at τ.

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This finishes the proof of main result!

One application

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Theorem 7 (Dotto-Le, building on C.-E.-G.-G.-P.-S.)

There is a "big" patched module $\mathbf{M}_\infty$ finitely generated over $R_\infty[[\mathrm{GL}_2(\mathcal{O}_{F_v})]]$ + compatible action of $\mathrm{GL}_2(F_v)$ such that $\mathbf{M}_\infty/\mathfrak{m}_\infty \cong \pi_v(\overline{r})^\vee$.

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Corollary of our main result

For *any* map $R_\infty \to \mathcal{O}_E$ of $W(\mathbb{F})$-algebras (where $[E:\mathbb{Q}_p] < \infty$), $(\mathbf{M}_\infty \otimes_{R_\infty} \mathcal{O}_E)^\vee[1/p]$ = *non-zero* admissible unitary continuous representation of $\mathrm{GL}_2(F_v)$ over E with a unit ball lifting $\pi_v(\overline{r})$.

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Proof: The module $\mathbf{M}_\infty$ is CM over $R_\infty[[\mathrm{GL}_2(\mathcal{O}_{F_v})]]$ (Gee-Newton) + $\mathrm{GK}((\mathbf{M}_\infty/\mathfrak{m}_\infty)^\vee) = f$ (our main result) $\Rightarrow$ $\mathbf{M}_\infty$ is flat over R_∞ ("Miracle Flatness" in non-commutative setting, see Gee-Newton).

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- Hope to prove for suitable level K^v:

$$\mathrm{GK}\Big(\mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\overline{r}, \varinjlim_{K_v} H^1_{\text{ét}}(X_{K^v K_v} \times_F \overline{F}, \mathbb{F})\big)\Big) = f.$$

Need to extend previous proof to cases without multiplicity 1.