Towards the locally analytic socle for GL_n

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1 Lecture 1

I first thank Ruochuan Liu for inviting me to the B.I.C.M.R. and giving me the opportunity to give these lectures.

In all the talks, p is a prime number, E is a finite extension of $\mathbb{Q}_p$ which is assumed "sufficiently large" (the precise meaning of this being clear in the context) and ϖ_E a uniformizer of the ring of integers $\mathcal{O}_E$ of E. We normalize local class field theory so that uniformizers correspond to *geometric* Frobeniuses. A topological representation is said to be irreducible if it has no nonzero strict closed invariant subspace.

1.1 Introduction and motivation

The motivation underlying these lectures is to try to make progress on the locally analytic representations occuring in the p-adic Langlands program for $\mathrm{GL}_n(\mathbb{Q}_p)$ with $n > 2$. Let me start with a quite "concrete" problem. Let G be a unitary algebraic group over $\mathbb{Q}$ which is the unitary group $U(n, 0)$ over $\mathbb{R}$ and GL_n over $\mathbb{Q}_p$, and let $U^p \subset G(\mathbb{A}^{\infty,p})$ be a compact open subgroup ($\mathbb{A}^{\infty,p}$ = finite adèles of $\mathbb{Q}$ outside p). Let:

$$S(U^p) := \varinjlim_{U_p} \{f : G(\mathbb{Q})\backslash G(\mathbb{A}^{\infty,p})/U^pU_p \longrightarrow E\}$$

where the inductive limit is over compact open subgroups $U_p \subset G(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p)$. Let $\widehat{S}(U^p)$ be its p-adic completion, that is, the completion of $S(U^p)$ with respect to the norm $\max_g\{|f(g)|\}$. This is a p-adic Banach space over E and the natural action of $\mathrm{GL}_n(\mathbb{Q}_p)$ on $S(U^p)$ by right translation extends to $\widehat{S}(U^p)$ by continuity. Moreover the above norm is invariant under this action (i.e. this is a unitary Banach space representation of $\mathrm{GL}_n(\mathbb{Q}_p)$). Finally there is also an action of some Hecke operators outside p which commute with that of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Let $\pi = \otimes'_\ell \pi_\ell$ be a classical cuspidal automorphic representation of $G(\mathbb{A}^\infty)$ which, say, occurs in $\varinjlim_{U^p} S(U^p)$ and let $\widehat{S}(U^p)[\pi] \subseteq \widehat{S}(U^p)$ be the closed subspace which is the eigenspace for the system of eigenvalues of the Hecke operators acting on $(\otimes'_{\ell\neq p}\pi_\ell)^{U^p}$. The $\mathrm{GL}_n(\mathbb{Q}_p)$-action preserves $\widehat{S}(U^p)[\pi]$ (since it commutes with Hecke) and we can consider its *locally analytic vectors* $\widehat{S}(U^p)[\pi]^{\mathrm{an}}$ which are still preserved by $\mathrm{GL}_n(\mathbb{Q}_p)$. The aforementioned "concrete problem" is the following:

Problem 1.1.1. Describe explicitly $\widehat{S}(U^p)[\pi]^{\mathrm{an}}$ as a representation of $\mathrm{GL}_n(\mathbb{Q}_p)$.

It obviously contains $S(U^p)[\pi]$ (which a finite direct sum of copies of π_p) but is strictly bigger since π_p is never a Banach space as it is infinite dimensional. So far, we know almost nothing about $\widehat{S}(U^p)[\pi]^{\mathrm{an}}$ when $n > 2$, for instance we don't know whether it depends only on $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ (where ρ_π is the n-dimensional global p-adic Galois representation associated to π by work of many people), nor even whether it determines $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$, though the latter at least is strongly expected (note that π_p itself is very far from determining $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ in general).

The $\widehat{S}(U^p)$ are the most simple examples of more general p-adic Banach spaces representations $\widehat{H}^d(U^p)$ loosely called "completed étale cohomology groups" which were defined by Emerton as follows:

$$\widehat{H}^d(U^p) := E \otimes_{\mathcal{O}_E} \varprojlim_n \Big(\varinjlim_{U_p} H^d_{\acute{e}t}\big(X(U^pU_p)_{\overline{\mathbb{Q}}}, \mathcal{O}_E\big)/(\varpi_E^n)\Big)$$

where $d \in \mathbb{Z}_{\geq 0}$ and $(X(U^pU_p))_{U_p}$ is a tower of P.E.L. Shimura varieties. Then all the previous considerations and questions apply to the locally analytic $\mathrm{GL}_n(\mathbb{Q}_p)$-representations $\widehat{H}^d(U^p)[\pi]^{\mathrm{an}}$. When $d = 1$ and $X(U^pU_p)$ are the modular curves, then the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations $\widehat{H}^1(U^p)[\pi]^{\mathrm{an}}$ are completely understood in many cases thanks to the p-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$ and in particular work of Colmez and Emerton (and other people).

The aim of these lectures is rather modest: try to guess as many constituents (up to multiplicity) in the $\mathrm{GL}_n(\mathbb{Q}_p)$-socle of $\widehat{H}^d(U^p)[\pi]^{\mathrm{an}}$ as possible (though we don't mention $\widehat{H}^d(U^p)[\pi]^{\mathrm{an}}$ in the lectures, these are the spaces we have in mind). More precisely, we associate to many of the π (such that π_p is a principal series) a finite length semi-simple multiplicity free locally analytic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ containing π_p, only depending on $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$, and for which we *hope* that it is contained in $\widehat{H}^d(U^p)[\pi]^{\mathrm{an}}$ (thus in its $\mathrm{GL}_n(\mathbb{Q}_p)$-socle). Though this representation is still far from determining $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ in general, it captures however many properties of $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ which can't be seen in π_p. Neither the assumption that π_p is a principal series nor the one that the ground field is $\mathbb{Q}_p$ are in fact necessary, but hopefully make these notes easier to read.

In fact, we work with purely local data on the Galois side using Fontaine's theory. Starting from a rank n diagonalizable (for simplicity) Weil representation $\underline{D}$ over E, that we see as a Deligne-Fontaine module, that is, a "φ-filtered module without its filtration", a collection of distinct Hodge-Tate weights $\underline{h} = h_1 < \cdots < h_n$, and an exhaustive separated filtration $\underline{\mathrm{Fil}}^{\cdot}$ on $\underline{D}$ of weights $\underline{h}$, we associate a finite length semi-simple locally analytic $\mathrm{GL}_n(\mathbb{Q}_p)$-representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$. The constituents of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ are the $\mathrm{GL}_n(\mathbb{Q}_p)$-socles of some locally analytic principal series, and to define them precisely we rely heavily on important recent work of Orlik and Strauch that will be recalled. We then prove the following theorem in the last lecture:

Theorem 1.1.2. *If $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ has an $\mathcal{O}_E$-lattice which is $\mathrm{GL}_n(\mathbb{Q}_p)$-invariant, then the Hodge filtration $\underline{\mathrm{Fil}}^{\cdot}$ is weakly admissible.*

Recall that to a filtration on $\underline{D}$ which is weakly admissible, the theorem of Colmez and Fontaine associates an n-dimensional crystabelline representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E. Thus, the above statement can be read as: if $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ satisfies a strong integral condition (has an invariant lattice), then $(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ corresponds to a Galois representation. Note that, if $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ indeed occurs inside some $\widehat{H}^d(U^p)$, then it has an invariant lattice, namely the one induced by its intersection with the image of $\varprojlim \big(\varinjlim H^d_{\acute{e}t}\big(X(U^pU_p)_{\overline{\mathbb{Q}}}, \mathcal{O}_E\big)/(\varpi_E^n)\big)$. When $n = 2$ and $\underline{\mathrm{Fil}}^{\cdot}$ is weakly admissible, then by a Theorem of R. Liu (and also Xie, Zhang) and (independently) Colmez, the representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ is the $\mathrm{GL}_2(\mathbb{Q}_p)$-socle of the locally analytic representation corresponding to the associated 2-dimensional crystabelline representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ by the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (and thus in particular always admits an invariant lattice).

I now briefly describe the contents of these lectures. In the (second half of) the first lecture, I give a quick review of locally analytic representations of p-adic analytic groups and related material (mainly due to Schneider and Teitelbaum). In the first half of the second lecture, I give a review of generalized Verma modules (for an arbitrary split connected reductive algebraic group G over $\mathbb{Q}_p$) and of useful abelian categories $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ introduced by Bernstein, Gelfand (I.) and Gelfand (S.) that contain them. Then, following Orlik and Strauch, I use these results in the second half to define important locally analytic representations $\mathcal{F}^G_P(M, \pi_P)$ of $G(\mathbb{Q}_p)$ (I mainly skip the proofs here, as they go beyond the material of these lectures). In the third lecture, I prove useful properties of the representations $\mathcal{F}^G_P(M, \pi_P)$, for instance that under good conditions they are the $G(\mathbb{Q}_p)$-socle of locally analytic parabolic inductions. Then I illustrate all the previous results by explicitly decomposing certain locally analytic principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_3(\mathbb{Q}_p)$. In the first half of the fourth lecture, I recall some necessary conditions due to Emerton that must satisfy locally analytic parabolic inductions which possess invariant lattices. Then I prove that these conditions still hold when one considers instead the $G(\mathbb{Q}_p)$-representations $\mathcal{F}_P^G(M, \pi_P)$ (which are not parabolic inductions in general). In the second half, I quickly review Deligne-Fontaine modules $\underline{D}$ and recall how one can associate to them (and to distinct Hodge-Tate weights $\underline{h}$) an irreducible locally algebraic representation $\pi(\underline{D}, \underline{h})$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ using the classical local Langlands correspondence. In the fifth lecture, I associate to $\underline{D}$ and $\underline{h}$ many other locally analytic irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ of the form $\mathcal{F}_P^G(M, \pi_P)$ that are denoted $C(w^{\mathrm{alg}}, w)$ (where w^{alg}, w are elements in the Weyl group of GL_n), I study their intertwinings and I define the $\mathrm{GL}_n(\mathbb{Q}_p)$-representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}})$ for any filtration $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ on $\underline{D}$ of Hodge-Tate weights $\underline{h}$. Finally, in the sixth and last lecture, I prove Theorem 1.1.2 above, then give explicit examples of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}})$ for $\mathrm{GL}_3(\mathbb{Q}_p)$, and finally end with several open questions.

1.2 Quick review of locally analytic representations

Recall that a locally convex E-vector space is a topological E-vector space V such that each $v \in V$ has a basis of open and closed neighbourhoods of the form $v + L$ where L ranges over a family $\mathcal{F}$ of generating $\mathcal{O}_E$-submodules of V (i.e. $L \otimes_{\mathcal{O}_E} E = V$) such that:
(i) for any L in $\mathcal{F}$ and any $\lambda \in E^\times$, there exists L' in $\mathcal{F}$ such that $L' \subseteq \lambda L$
(ii) for any L, L' in $\mathcal{F}$, there exists L'' in $\mathcal{F}$ such that $L'' \subseteq L \cap L'$.

We say that a Hausdorff locally convex E-vector space V is *of compact type* if it can be written as an inductive limit $V = \varinjlim_{n \in \mathbb{Z}_{>0}} V_n$ of Banach spaces V_n over E (with the locally convex final topology, i.e. the finest locally convex topology making all maps $V_n \to V$ continuous) such that all the maps $V_n \to V_{n+1}$ are continuous, injective and compact, that is, the image of a unit ball of V_n is contained in a compact of V_{n+1} (recall that E is a finite extension of $\mathbb{Q}_p$ and hence is locally compact). Any closed E-subvector space of a Hausdorff locally convex E-vector space of compact type (with induced topology) is again of compact type.

Example 1.2.1. Let V be an E-vector space of countable dimension. We can endow V with the finest locally convex topology, i.e. the one where $\mathcal{F}$ is the family of *all* generating $\mathcal{O}_E$-submodules of V. Any E-linear map from V to another locally convex E-vector space is automatically continuous. It is also the finest locally convex topology making all injections $W \hookrightarrow V$ continuous where W is a finite dimensional subvector space of V equipped with its canonical (Banach) topology. It is thus a Hausdorff locally convex E-vector space of compact type.

Let M be a p-adic analytic variety, e.g. $M = G(\mathbb{Q}_p)$ or $M = G(\mathbb{Z}_p)$ where G is a split connected reductive algebraic group over $\mathbb{Q}_p$ or $\mathbb{Z}_p$. We assume that any covering of M can be refined into a covering by disjoint open subsets (this holds for instance if M is compact or if M is as above by a result of Féaux de Lacroix). If U is an open subset of M which is isomorphic to a closed ball of $\mathbb{Q}_p^m$ of center $\underline{a} \in \mathbb{Q}_p^m$ and V a Banach space over E, recall that a function $f : U \to V$ is called *analytic* if there are $n \in \mathbb{Z}_{>0}$ and vectors $(v_{\underline{d}})_{\underline{d}=(d_1,\cdots,d_m)\in\mathbb{Z}_{\geq 0}^m}$ in V such that for all $\underline{z} := (z_1, \cdots, z_m) \in \mathbb{Q}_p^m$ such that $\underline{a} + \underline{z} \in U$, $f(\underline{a} + \underline{z})$ is a convergent sum in V :

$$f(\underline{a} + \underline{z}) = \sum_{\underline{d}\in\mathbb{Z}_{\geq 0}^m} v_{\underline{d}} \underline{z}^{\underline{d}}$$

where $\underline{z}^{\underline{d}} := z_1^{d_1} \cdots z_m^{d_m}$. We denote by $C^{\mathrm{rig}}(U, V)$ this E-vector space, which doesn't depend on the choice of $\underline{a}$ (as center of U) and is obviously a Banach space, a norm being given by $\sup_{\underline{d}}(\|v_{\underline{d}}\| r^{\sum_{i=1}^m d_i})$ if r is the radius of U.

Exercice 1.2.2. If $U' \subsetneq U$ are as above and if V is finite dimensional over E, prove that the restriction to U' induces a compact injection $C^{\mathrm{rig}}(U, V) \hookrightarrow C^{\mathrm{rig}}(U', V)$.

Now let $V \cong \varinjlim V_n$ be a Hausdorff locally convex E-vector space of compact type, a function $f : M \to V$ is called *locally analytic* if there is an open disjoint covering $M = \amalg_i U_i$ of M by closed balls U_i as above and positive integers $(n_i)_i$ such that $f|_{U_i} \in C^{\mathrm{rig}}(U_i, V_{n_i})$ for all i. Denote by $C^{\mathrm{an}}(M, V)$ the E-vector space of locally analytic functions. One obviously has:

$$C^{\mathrm{an}}(M, V) = \varinjlim \prod_i C^{\mathrm{rig}}(U_i, V_{n_i}),$$

the inductive limit being taken over all $(U_i)_i$ and $(n_i)_i$ as before. One endows $\prod_i C^{\mathrm{rig}}(U_i, V_{n_i})$ with the direct product locally convex topology (i.e. the coarsest locally convex topology making all projections continuous) and $C^{\mathrm{an}}(M, V)$ with the corresponding locally convex final topology. It is a Hausdorff locally convex E-vector space.

Lemma 1.2.3. *If M is compact and V is as in Example 1.2.1 (that is, each V_n is of finite dimension), then $C^{\mathrm{an}}(M, V)$ is of compact type.*

Proof. Let $\mathcal{U}_n = (U_{i,n})$ for $n \in \mathbb{Z}_{>0}$ be a cofinal system (for inclusion) of finite disjoint coverings of M where the radius r_n of the $U_{i,n}$ is strictly decreasing when n increases. Then $C^{\mathrm{an}}(M, V) = \varinjlim_n \prod_i C^{\mathrm{rig}}(U_{i,n}, V_n)$ and the statement easily follows from Exercise 1.2.2. $\square$

Definition 1.2.4 (Schneider-Teitelbaum)**.** *Assume that M is a p-adic analytic group. A locally analytic representation of M (over E) is a Hausdorff locally* *convex E-vector space V of compact type equipped with an M-action by continuous E-linear endomorphisms such that, for each $v \in V$, the orbit map $m \in M \mapsto mv \in V$ is in $C^{\mathrm{an}}(M,V)$.*

One can show that the canonical map $M \times V \to V$ is continuous. In fact, Schneider and Teitelbaum have a more general definition (they do not require the underlying space V to be of compact type), but the above special case will be enough for our purpose. Locally analytic representations on Hausdorff locally convex E-vector spaces of compact type obviously form a category with morphisms being the continuous applications that commute with the group action. We now give the main example of locally analytic representations for these lectures.

Example 1.2.5. Let G be a split connected reductive algebraic group over $\mathbb{Q}_p$ and $P \subseteq G$ a parabolic subgroup. Let $G(\mathbb{Q}_p)$ and $P(\mathbb{Q}_p)$ be the corresponding p-adic analytic groups of $\mathbb{Q}_p$-points. Let V_P be a locally analytic representation of $P(\mathbb{Q}_p)$ on a locally convex E-vector space which is as in Example 1.2.1. We define:

$$\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V_P\big)^{\mathrm{an}} := \{f \in C^{\mathrm{an}}(G(\mathbb{Q}_p), V_P),\ f(pg) = p(f(g)) \\ \forall\, p \in P(\mathbb{Q}_p),\ \forall\, g \in G(\mathbb{Q}_p)\}.$$

This is a closed subspace of $C^{\mathrm{an}}(G(\mathbb{Q}_p), V_P)$ and we endow it with the induced topology. Choose a locally analytic section $s : G(\mathbb{Q}_p)/P(\mathbb{Q}_p) \hookrightarrow G(\mathbb{Q}_p)$ and let $S := s(G(\mathbb{Q}_p)/P(\mathbb{Q}_p)) \subseteq G(\mathbb{Q}_p)$ which is a compact p-adic analytic variety. Then the restriction to S induces an isomorphism of topological spaces $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V_P\big)^{\mathrm{an}} \cong C^{\mathrm{an}}(S, V_P)$, in particular $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V_P\big)^{\mathrm{an}}$ is of compact type by Lemma 1.2.3. We endow $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V_P\big)^{\mathrm{an}}$ with a left action of $G(\mathbb{Q}_p)$ given by $(g \cdot f)(h) := f(hg)$ $(g, h \in G(\mathbb{Q}_p))$. One can check that, for any $f \in \big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V_P\big)^{\mathrm{an}}$, the orbit map:

$$G(\mathbb{Q}_p) \to C^{\mathrm{an}}(S, V_P),\ g \mapsto \big(s \mapsto f(sg)\big)$$

is locally analytic. Hence $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V_P\big)^{\mathrm{an}}$ is a locally analytic representation of $G(\mathbb{Q}_p)$. In particular, if $P = B$ is a Borel subgroup of G and $V_B : B(\mathbb{Q}_p) \twoheadrightarrow T(\mathbb{Q}_p) \to E^\times$ is a locally analytic character where T is the split torus of B, the representations $\big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V_B\big)^{\mathrm{an}}$ are called locally analytic principal series.

Let L_P be the Levi subgroup of P. In the rest of these lectures, we will only consider a special case of Example 1.2.5: the case where the $P(\mathbb{Q}_p)$-representation V_P is locally algebraic of the form $V_P \cong W \otimes_E \pi_P$ where W is a finite dimensional algebraic representation of $P(\mathbb{Q}_p)$ over E and π_P is a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E that we view as a $P(\mathbb{Q}_p)$-representation via $P(\mathbb{Q}_p) \twoheadrightarrow L_P(\mathbb{Q}_p)$.

The continuous dual $D(M,E)$ of $C^{\mathrm{an}}(M,E)$ is called the E-vector space of locally analytic distributions on M. When M is a p-adic analytic group, it is naturally endowed with the structure of a noncommutative E-algebra (the product comes from $D(M,E) \times D(M,E) \to D(M \times M, E) \to D(M,E)$ where the second map is induced by the product map $M \times M \to M$ in the group M). Note that it contains the $\mathbb{Q}_p$-Lie algebra $\mathfrak{m}$ of M by ($f \in C^{\mathrm{an}}(M,E)$, $\mathfrak{x} \in \mathfrak{m}$):

$$\mathfrak{x} \mapsto \Big(f \mapsto \frac{d}{dt} f\big(\exp(-t\mathfrak{x})\big)|_{t=0} \in E\Big) \in D(M,E).$$

A locally analytic representation of M, hence also its continuous dual, can be endowed with a structure of a module over $D(M,E)$.

Finally, let us mention that Schneider and Teitelbaum managed to define a full subcategory of the category of locally analytic representations of a p-adic analytic group M on Hausdorff locally convex E-vector spaces of compact type called *admissible representations* (they actually first define the continuous duals of admissible representations by imposing conditions on the $D(M,E)$-modules). The main feature is that this category of admissible locally analytic representations is *abelian*, and kernels and images are the algebraic kernels and images with the induced subspace topology. It won't be necessary in these lectures to have the precise definition of an admissible locally analytic representation of M, let it suffice here to say that the above representations $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W \otimes_E \pi_P\big)^{\mathrm{an}}$ (with $W \otimes_E \pi_P$ as before), as well as all their subquotients, are admissible.

2 Lecture 2

2.1 Quick review of Verma modules

We review here the category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and some of its most interesting objects: generalized Verma modules.

We fix once and for all a split connected reductive algebraic group G over $\mathbb{Q}_p$, a Borel subgroup B of G with split torus $T \subseteq B$ and unipotent radical N, and a parabolic subgroup $P \subseteq G$ containing B. We have $P = L_P N_P$ where L_P is the Levi subgroup of P and N_P its unipotent radical. We denote by $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{l}_P$, $\mathfrak{n}_P$, $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{t}$ the $\mathbb{Q}_p$-Lie algebras of the p-adic analytic groups $G(\mathbb{Q}_p)$, $P(\mathbb{Q}_p)$, $L_P(\mathbb{Q}_p)$, $N_P(\mathbb{Q}_p)$, $B(\mathbb{Q}_p)$, $N(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$. Finally we denote by $U(\mathfrak{g})$, $U(\mathfrak{p})$, $U(\mathfrak{l}_P)$, $U(\mathfrak{n}_P)$, $U(\mathfrak{b})$, $U(\mathfrak{n})$ and $U(\mathfrak{t})$ their respective universal enveloping algebra.

Recall that a (finite dimensional) algebraic representation of $\mathfrak{l}_P$ over E is the same thing as an algebraic representation of $L_P(\mathbb{Q}_p)$ over E, that is, comes from the derived action of $L_P(\mathbb{Q}_p)$ on a unique algebraic representation. For instance, if $P = B$, an algebraic character of $\mathfrak{t}$ (= a weight) obviously comes from an algebraic character of $T(\mathbb{Q}_p)$. The category of algebraic representations of $\mathfrak{l}_P$ (or of $L_P(\mathbb{Q}_p)$) over E is semi-simple and each object can also be seen as an $U(\mathfrak{l}_P) \otimes_{\mathbb{Q}_p} E$-module, or even as an $U(\mathfrak{p}) \otimes_{\mathbb{Q}_p} E$-module via the surjection $U(\mathfrak{p}) \twoheadrightarrow U(\mathfrak{l}_P)$.

Definition 2.1.1. *Let W be an irreducible algebraic representation of $\mathfrak{l}_P$ over E. A generalized Verma module is an $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$-module of the form:*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W.$$

In order to study irreducible constituents of (generalized) Verma modules, Bernstein, Gelfand (I.) and Gelfand (S.) introduced a very nice and convenient artinian category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ of $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$-modules that contains all Verma modules as in Definition 2.1.1, all of their subquotients and also all algebraic (finite dimensional) representations of $\mathfrak{g}$ over E.

Definition 2.1.2 (Bernstein, Gelfand, Gelfand)**.** *We let $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be the full subcategory of the category of linear representations of $\mathfrak{g}$ on E-vector spaces made out of representations M such that:*
(i) M is a finite type $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$-module;
(ii) $M|_{U(\mathfrak{l}_P)}$ is a direct sum of irreducible algebraic $U(\mathfrak{l}_P) \otimes_{\mathbb{Q}_p} E$-modules;
(iii) for all $v \in M$, the E-vector space $U(\mathfrak{n}_P) \otimes_{\mathbb{Q}_p} E \cdot v$ is finite dimensional.

If $P = G$, the category $\mathcal{O}^{\mathfrak{g}}_{\mathrm{alg}}$ is just the semi-simple category of finite dimensional algebraic representations of $\mathfrak{g}$ over E. In general however, an object of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ is far from being semi-simple.

Exercice 2.1.3. If $P \subseteq Q$ are two parabolic subgroups containing B, prove that $\mathcal{O}^{\mathfrak{q}}_{\mathrm{alg}}$ is a full subcategory of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$.

Proposition 2.1.4. *The generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ is in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$.*

Proof. (sketch) Condition (i) is obviously satisfied. Let us sketch the proof for (ii) and (iii) in the simpler case where $P = B$, which is the most important case in these lectures. We then have $W = Ev$ and $hv = \lambda(h)v$ for $h \in \mathfrak{t}$ where $\lambda : \mathfrak{t} \to E$ is the *weight* of v. We denote by $M(\lambda)$ the Verma module in that case. We let $\overline{B}$ be the opposite Borel subgroup, $\overline{N}$ its unipotent radical and $\overline{\mathfrak{n}}$ the Lie algebra of $\overline{N}(\mathbb{Q}_p)$. We have $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{t} \oplus \mathfrak{n} = \overline{\mathfrak{n}} \oplus \mathfrak{b}$ and decompositions $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$, $\overline{\mathfrak{n}} = \oplus_{\alpha>0}\mathfrak{g}_{-\alpha}$ where α runs among the set of positive roots of G and where $\mathfrak{g}_{\pm\alpha} := \{x \in \mathfrak{g}, [h,x] = \pm\alpha(h)x \ \forall \ h \in \mathfrak{t}\}$ has dimension 1 over E. Then $U(\mathfrak{g}) \cong U(\overline{\mathfrak{n}}) \otimes_{\mathbb{Q}_p} U(\mathfrak{b})$ and thus $M(\lambda) \cong U(\overline{\mathfrak{n}}) \otimes_{\mathbb{Q}_p} Ev$. Let $\alpha_1, \cdots, \alpha_m$ be the positive roots, then $U(\overline{\mathfrak{n}}) \otimes_{\mathbb{Q}_p} Ev \cong \oplus_{\underline{d}} Ey_1^{d_1} \cdots y_m^{d_m} \otimes v$ where $\mathfrak{g}_{-\alpha_i} = Ey_i$ and $\underline{d} = (d_1, \cdots, d_n) \in \mathbb{Z}_{\geq 0}^m$. Since each vector $y_1^{d_1} \cdots y_m^{d_m} \otimes v$ has weight $\lambda - \sum_{i=1}^m d_i \alpha_i$ (use inductively $[h, y_i] = -\alpha_i(h) y_i$ for $h \in \mathfrak{t}$), we see that $M(\lambda)|_{U(\mathfrak{t})}$ is semi-simple (condition (ii) in Definition 2.1.2) and that v is the unique *highest weight vector* of $M(\lambda)$. Moreover, the action of an element of $\mathfrak{g}_\alpha$ on an eigenvector for $U(\mathfrak{t})$ is either zero or increases the corresponding weight by α. Since all the weights are bounded by λ, this action has to be nilpotent, which easily implies condition (iii). □

One can prove that the category $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ is abelian, closed under submodules, quotients and finite direct sums.

Proposition 2.1.5. *The generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ has a unique irreducible quotient.*

Proof. Let λ be the highest weight of W, since $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ is obviously a quotient of $M(\lambda) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$ (see the proof of Proposition 2.1.4 for $M(\lambda)$), it is enough to prove the statement for $M(\lambda)$. Let $v \in M(\lambda)$ be the unique (up to scalar) nonzero vector of weight λ. The image of v in any irreducible quotient of $M(\lambda)$ is nonzero since v generates $M(\lambda)$. Thus *any* irreducible quotient has a nonzero weight subspace for the weight λ. But this weight space has dimension 1 in $M(\lambda)$ (see the proof of Proposition 2.1.4) and thus there can be only one irreducible quotient. □

We denote by $L(\lambda)$ the unique irreducible quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ (where λ is the highest weight of W), or equivalently the unique irreducible quotient of $M(\lambda)$.

Proposition 2.1.6. *Any irreducible object of $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ is isomorphic to $L(\lambda)$ for some weight λ.*

Proof. It is enough to prove it for the category $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{b}}$ by Exercise 2.1.3. Any object M of $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{b}}$ has at least one nonzero weight vector v (= eigenvector for $\mathfrak{t}$) such that $\mathfrak{n} \cdot v = 0$ (start from any nonzero weight vector w in M, let v be a weight vector in the finite dimensional $U(\mathfrak{n}) \cdot w$ which has a maximal weight and use the fact that the weight can only increase under $\mathfrak{n}$). Let λ be the weight of v, then we have a nonzero morphism of $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$-modules $M(\lambda) \to M$. If moreover M is irreducible, then M is necessarily the unique quotient of $M(\lambda)$ by Proposition 2.1.5, that is $L(\lambda)$. □

Exercice 2.1.7. Let M be an irreducible object of $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$. Prove that there is a unique maximal parabolic subgroup P of G containing B such that $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. Hint: write $M = L(\lambda)$ and recall that parabolic subgroups of G containing B correspond to subsets of the set of simple positive roots of G. Then take P corresponding to the set of simple roots α such that λ is dominant for α, i.e. $\langle \lambda, \alpha^\vee \rangle \geq 0$. We call P the maximal parabolic subgroup of M.

It is true that each Verma module is of *finite length*, but we won't prove it here (the proof is more involved). From this, it is easy to deduce that any object M of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ has also finite length, and thus that the category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ is artinian. Indeed, it is again enough to prove it for $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$. Let $M \in \mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$ and $v_1, \cdots, v_m \in M$ be weight vectors which generate M under $U(\mathfrak{g})$ (conditions (i) and (ii)) such that $V := \sum_i U(\mathfrak{n}) \cdot v_i$ has a dimension as small as possible (condition (iii)). Then argue by induction on $\dim_E V$ as follows. Let $v \in V$ be a weight vector which has a maximal weight among weight vectors of V, then $M_v := U(\mathfrak{g}) \cdot v \subseteq M$ is a quotient of $M(\lambda)$ where λ is the weight of v and thus M_v is of finite length. The image of V in M/M_v has strictly smaller dimension and one can repeat the argument with M/M_v. By induction, we see that M has finite length.

By Proposition 2.1.6, the list of constituents of $M(\lambda)$ (up to multiplicity) are some $L(\mu)$ for certain weights μ. We already know that $L(\lambda)$ appears in this list (and we know it appears with multiplicity one because λ is the unique highest weight in $M(\lambda)|_{\mathfrak{t}}$). Although strictly speaking, we do not really need it in these lectures, it is good to know what this precise list is, at least in a special case.

Let $\mathcal{W}$ be the Weyl group of G, then $\mathcal{W}$ acts on the $\mathbb{Z}$-module of weights: $(w, \lambda) \mapsto w\lambda$. Let $\rho := \frac{1}{2}\sum_{\alpha>0} \alpha$ be half the sum of the positive roots (because of the $\frac{1}{2}$, ρ is not always a weight, but this won't matter), we define the *dot action* ($w \in \mathcal{W}$, λ a weight):

$$w \cdot \lambda := w(\lambda + \rho) - \rho \tag{1}$$

which is always a weight (even if ρ is not). Then one has the following important result, which is a special case of a more general deep theorem due to Bernstein, Gelfand and Gelfand:

Theorem 2.1.8. *Assume that λ is dominant, i.e. λ is the highest weight of an irreducible algebraic representation of $\mathfrak{g}$ over E. Then the constituents of $M(\lambda)$ are, up to multiplicity, exactly the $L(w \cdot \lambda)$ for $w \in \mathcal{W}$.*

Remark 2.1.9. The multiplicities of the $L(w \cdot \lambda)$ are not always 1 and are quite subtle to understand (this is the subject of Kazhdan-Lusztig polynomials).

This is all we need to know on the categories $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$.

2.2 The representations $\mathcal{F}_P^G(M, \pi_P)$ (after Orlik & Strauch)

We now explain important results due to Orlik and Strauch which allow to understand the topological constituents of some locally analytic parabolic inductions.

Let G, P be as in §2.1 and let V_P be a locally analytic representation of $P(\mathbb{Q}_p)$ over E as in Example 1.2.5. For any $f \in C^{\mathrm{an}}(G(\mathbb{Q}_p), V_P)$ and $\mathfrak{x} \in \mathfrak{g}$, define $\mathfrak{x} \cdot f \in C^{\mathrm{an}}(G(\mathbb{Q}_p), V_P)$ by:

$$(\mathfrak{x} \cdot f)(g) := \frac{d}{dt} f\big(\exp(-t\mathfrak{x})g\big)|_{t=0} \in V_P. \tag{2}$$

The endomorphism $f \mapsto \mathfrak{x} \cdot f$ of $C^{\mathrm{an}}(G(\mathbb{Q}_p), V_P)$ is continuous. By composition, one obtains a left action of $\mathfrak{g}$ and of its enveloping algebra $U(\mathfrak{g})$ on $C^{\mathrm{an}}(G(\mathbb{Q}_p), V_P)$ by continuous endomorphisms that we still write $f \mapsto \mathfrak{x} \cdot f$.

Now let M be any object of the category $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ of §2.1, and let $W \subseteq M$ be a finite dimensional algebraic representation of $\mathfrak{p}$ over E (such a W exists because of the properties defining $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$). Since $W|_{\mathfrak{l}_P}$ is a direct sum of irreducible algebraic representations of $\mathfrak{l}_P$ over E, we know that it comes from an algebraic representation of $L_P(\mathbb{Q}_p)$ over E. Hence we have an action of $L_P(\mathbb{Q}_p)$ on W. Writing any element of $N_P(\mathbb{Q}_p)$ as the exponential of an element in $\mathfrak{n}_P$ and using the fact that the action of each element of $\mathfrak{n}_P$ on W is nilpotent (either as W is finite dimensional or as $\mathfrak{n}_P$ acts nilpotently on all elements of M by definition), we see that the action of $L_P(\mathbb{Q}_p)$ canonically extends to $L_P(\mathbb{Q}_p)N_P(\mathbb{Q}_p) = P(\mathbb{Q}_p)$. We thus have a finite dimensional algebraic representation of $P(\mathbb{Q}_p)$ that we still denote W. Let $W' := \mathrm{Hom}_E(W, E)$ with $(p \cdot f)(v) := f(p^{-1}v)$ ($p \in P(\mathbb{Q}_p)$, $v \in W$).

Let π_P be a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E. Using (2) applied to $V_P = W' \otimes_E \pi_P$, for $\mathfrak{x} \in U(\mathfrak{g})$ and $v \in W$ we define $(\mathfrak{x} \otimes v) \cdot f \in C^{\mathrm{an}}(G(\mathbb{Q}_p), \pi_P)$ by:

$$(\mathfrak{x} \otimes v) \cdot f := \big(g \mapsto (\mathfrak{x} \cdot f)(g)(v)\big). \tag{3}$$

By linearity, each element of $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} W$ induces a continuous morphism $C^{\mathrm{an}}(G(\mathbb{Q}_p), W' \otimes_E \pi_P) \longrightarrow C^{\mathrm{an}}(G(\mathbb{Q}_p), \pi_P)$. The following lemma is straightforward:

Lemma 2.2.1. *Let $\mathfrak{x} \in U(\mathfrak{p})$, $v \in W$ and $f \in \big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}} \subset C^{\mathrm{an}}(G(\mathbb{Q}_p), W' \otimes_E \pi_P)$, then $(\mathfrak{x} \otimes v) \cdot f = (1 \otimes \mathfrak{x}v) \cdot f \in C^{\mathrm{an}}(G(\mathbb{Q}_p), \pi_P)$.*

By Lemma 2.2.1 we can define $\mathfrak{d} \cdot f \in C^{\mathrm{an}}(G(\mathbb{Q}_p), \pi_P)$ for $\mathfrak{d} \in U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ and $f \in \big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$. Recall that $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ is a generalized Verma module if W is irreducible (which will be the most important case for us), see Definition 2.1.1. In general, it is a successive extension in $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ of Verma modules.

Now assume that $W \subseteq M$ generates M over $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$ so that we have an exact sequence of $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$-modules:

$$0 \longrightarrow \ker(\phi) \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \xrightarrow{\phi} M \longrightarrow 0. \tag{4}$$

Following Orlik and Strauch, we then define:

$$\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\ker(\phi)} := \Big\{f \in \big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}, \mathfrak{d}\cdot f = 0 \ \forall\ \mathfrak{d} \in \ker(\phi)\Big\}.$$

Since the action of $G(\mathbb{Q}_p)$ on $\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$ is by *right* translation, this is obviously a closed invariant subspace of $\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$. Moreover it is an admissible locally analytic representation of $G(\mathbb{Q}_p)$, being a subrepresentation of the admissible $\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$ (see §1.2).

Theorem 2.2.2 (Orlik-Strauch). *Let π_P be a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E, M an object of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and $W \subseteq M$ an $U(\mathfrak{p}) \otimes_{\mathbb{Q}_p} E$-submodule which is finite dimensional over E and which generates M over $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$. Define $\ker(\phi)$ as in (4).*
(i) The $G(\mathbb{Q}_p)$-representation $\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\ker(\phi)}$ is nonzero if and only if both M and π_P are nonzero and it only depends on π_P and M and not on the choice of W as above.
(ii) Denote by $\mathcal{F}_P^G(M, \pi_P)$ the representation in (i), the functor $(M, \pi_P) \mapsto \mathcal{F}_P^G(M, \pi_P)$ is exact in both arguments.
(iii) Let $Q \supseteq P$ be another parabolic subgroup and assume M lies in $\mathcal{O}^{\mathfrak{q}}_{\mathrm{alg}} \subseteq \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$, then $\mathcal{F}_P^G(M, \pi_P) \cong \mathcal{F}_Q^G(M, \operatorname{Ind}_{P(\mathbb{Q}_p)\cap L_Q(\mathbb{Q}_p)}^{L_Q(\mathbb{Q}_p)} \pi_P)$ where $\operatorname{Ind}_{P(\mathbb{Q}_p)\cap L_Q(\mathbb{Q}_p)}^{L_Q(\mathbb{Q}_p)} \pi_P$ is the usual smooth parabolic induction.
(iv) If M and π_P are irreducible and P is the maximal parabolic subgroup of M (Exercise 2.1.7), then $\mathcal{F}_P^G(M, \pi_P)$ is an irreducible representation of $G(\mathbb{Q}_p)$.

Remark 2.2.3. The proof given by Orlik and Strauch of part (iv) in Theorem 2.2.2 sometimes requires $p \geq 5$ depending on the form of G. However, when $G = \mathrm{GL}_n$, which will be soon our case, there is no assumption on p.

By combining parts (ii), (iii) and (iv), we get that $\mathcal{F}_P^G(M, \pi_P)$ is irreducible if and only if M is irreducible and the $L_Q(\mathbb{Q}_p)$-representation $\operatorname{Ind}_{P(\mathbb{Q}_p)\cap L_Q(\mathbb{Q}_p)}^{L_Q(\mathbb{Q}_p)} \pi_P$ is irreducible where $Q \supsetneq P$ is the maximal parabolic subgroup of M (Exercise 2.1.7).

In the proof of the above theorem, the first step is to compute and study the continuous dual of $\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\ker(\phi)}$. In the case where π_P is the trivial representation (which is the main case as far as proofs are concerned), one finds:

$$\Big(\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W'\big)^{\ker(\phi)}\Big)' \cong \big(D(G(\mathbb{Q}_p), E) \otimes_{D(P(\mathbb{Q}_p),E)} W\big)/D(G(\mathbb{Q}_p), E)\ker(\phi). \tag{5}$$

Moreover, there is a canonical map:

$$M \cong \big(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W\big)/\ker(\phi) \longrightarrow \\ \big(D(G(\mathbb{Q}_p), E) \otimes_{D(P(\mathbb{Q}_p),E)} W\big)/D(G(\mathbb{Q}_p), E)\ker(\phi). \quad (6)$$

Let $D\big(\mathfrak{g}, P(\mathbb{Q}_p), E\big) \subseteq D(G(\mathbb{Q}_p), E)$ be the subring generated by $U(\mathfrak{g})$ and $D(P(\mathbb{Q}_p), E)$, Orlik and Strauch prove that (6) extends to an isomorphism of $D(G(\mathbb{Q}_p), E)$-modules:

$$D(G(\mathbb{Q}_p), E) \otimes_{D(\mathfrak{g},P(\mathbb{Q}_p),E)} M \xrightarrow{\sim} \\ \big(D(G(\mathbb{Q}_p), E) \otimes_{D(P(\mathbb{Q}_p),E)} W\big)/D(G(\mathbb{Q}_p), E)\ker(\phi).$$

In particular, using (5) we get that $\big(\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W'\big)^{\ker(\phi)}\big)'$, and thus also $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W'\big)^{\ker(\phi)}$, doesn't depend on the choice of W and is nonzero if and only if M is nonzero (= statement (i) for π_P trivial). We do not sketch the proof of the other statements in Theorem 2.2.2. The hardest part is the irreducibility in (iv), which, very roughly, comes from the irreducibility of M using (6).

Example 2.2.4. (i) When $\ker(\phi) = 0$, that is $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$, one obviously has $\mathcal{F}_P^G(M, \pi_P) = \big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$.
(ii) When G is the maximal parabolic subgroup of M, i.e. $M \in \mathcal{O}^{\mathfrak{g}}_{\mathrm{alg}}$, then $\mathcal{F}_P^G(M, \pi_P) = M' \otimes_E \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \pi_P$, in particular $\mathcal{F}_P^G(M, \pi_P)$ is then a locally algebraic representation of $G(\mathbb{Q}_p)$.

We will give concrete examples of $\mathcal{F}_P^G(M, \pi_P)$ in the next lecture.

3 Lecture 3

3.1 More on the representations $\mathcal{F}_P^G(M, \pi_P)$

Before giving explicit examples of representations $\mathcal{F}_P^G(M, \pi_P)$, let us first state (and sometimes prove) some other useful statements about these representations.

Recall that G is a split connected reductive algebraic group over $\mathbb{Q}_p$ with a split torus T and a Borel subgroup B containing T. Let M be an irreducible object of $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$, by Propositions 2.1.5 and 2.1.6 there is a unique highest weight λ in $M|_{\mathfrak{t}}$ and M is the unique irreducible quotient $L(\lambda)$ of $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$. Note that, if $P = L_P N_P$ is the maximal parabolic subgroup of G such that $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ (Exercise 2.1.7), then by *loc.cit.* λ is the highest weight of an irreducible finite dimensional algebraic representation of $L_P(\mathbb{Q}_p)$ over E.

The following useful proposition is an easy generalization of a result due to Orlik and Schraen.

Proposition 3.1.1. *Let $M = L(\lambda)$ be an irreducible object of $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}}$, P be the maximal parabolic subgroup of G such that $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ (Exercise 2.1.7) and W be the irreducible (finite dimensional) algebraic representation of $L_P(\mathbb{Q}_p)$ over E of highest weight λ. Let π_P be a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E. Then there is an $L_P(\mathbb{Q}_p)$-equivariant isomorphism:*

$$H^0\big(N_P(\mathbb{Q}_p), \mathcal{F}^G_P(M, \pi_P)'\big) = W \otimes_E \pi'_P$$

where $\mathcal{F}^G_P(M, \pi_P)'$ (resp. π'_P) is the continuous (resp. algebraic) dual of $\mathcal{F}^G_P(M, \pi_P)$ (resp. π_P).

Let us give a few explanations on its proof. Passing to continuous duals, the injection $\mathcal{F}^G_P(M, \pi_P) \hookrightarrow \big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W'\otimes_E \pi_P\big)^{\mathrm{an}}$ yields a surjection of $D(G(\mathbb{Q}_p), E)$-modules:

$$\begin{aligned}\Big(\big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}\Big)' \cong D(G(\mathbb{Q}_p), E) \otimes_{D(P(\mathbb{Q}_p),E)} (W \otimes_E \pi'_P)& \\ \twoheadrightarrow \mathcal{F}^G_P(M, \pi_P)'&\end{aligned}$$

where the first isomorphism is similar to (5) but with $\ker(\phi) = 0$ and π_P not necessarily trivial. Since $W \otimes_E \pi'_P$ is obviously invariant under $N_P(\mathbb{Q}_p)$, there is nonzero $L_P(\mathbb{Q}_p)$-equivariant map $W \otimes_E \pi'_P \to H^0\big(N_P(\mathbb{Q}_p), \mathcal{F}^G_P(M, \pi_P)'\big)$ and it is enough to prove that the composition:

$$W \otimes_E \pi'_P \to H^0\big(N_P(\mathbb{Q}_p), \mathcal{F}^G_P(M, \pi_P)'\big) \hookrightarrow H^0\big(\mathfrak{n}_P, \mathcal{F}^G_P(M, \pi_P)'\big) \qquad (7)$$

is an isomorphism (recall that $H^0(\mathfrak{n}_P, *)$ is the subspace of $*$ cancelled by $\mathfrak{n}_P$ and thus contains $H^0(N_P(\mathbb{Q}_p), *)$). It is not difficult to reduce to the case π_P trivial, i.e. $\pi_P = 1$. We have $M \hookrightarrow \mathcal{F}^G_P(M, 1)'$ by (6) and (5), and $H^0\big(\mathfrak{n}_P, M\big) = W$ by the irreducibility of M, thus $W \cong H^0\big(\mathfrak{n}_P, M\big) \hookrightarrow H^0\big(\mathfrak{n}_P, \mathcal{F}^G_P(M, 1)'\big)$. Following Orlik and Schraen, we then prove that this injection is actually an isomorphism. The proof relies on the same kind of techniques that are used in the proof of statement (iv) of Theorem 2.2.2.

Proposition 3.1.1 has two useful consequences. We define the *socle* of a locally analytic representation of $G(\mathbb{Q}_p)$ over E as the closure of the direct sum of all its irreducible closed subrepresentations. Notation: $\mathrm{soc}_{G(\mathbb{Q}_p)}$.

Corollary 3.1.2. *Let $P \subseteq G$ be a parabolic subgroup containing B, W an irreducible algebraic representation of $L_P(\mathbb{Q}_p)$ over E, M the irreducible quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ (Proposition 2.1.5) and π_P a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E. Then we have:*

$$\mathrm{soc}_{G(\mathbb{Q}_p)} \mathcal{F}^G_P(M, \pi_P) = \mathrm{soc}_{G(\mathbb{Q}_p)} \big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}.$$

In particular, if $\mathrm{Ind}_{P(\mathbb{Q}_p)\cap L_Q(\mathbb{Q}_p)}^{L_Q(\mathbb{Q}_p)}\pi_P$ *is irreducible (where* Q *is the maximal parabolic subgroup of* M*), then:*

$$\mathcal{F}_P^G(M,\pi_P)=\mathrm{soc}_{G(\mathbb{Q}_p)}\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}W'\otimes_E\pi_P\big)^{\mathrm{an}}.$$

Proof. The second part follows from the first together with statements (iii) and (iv) of Theorem 2.2.2. Note that thanks to this theorem, we already know that the list of all the irreducible constituents of $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}W'\otimes_E\pi_P\big)^{\mathrm{an}}$ is $\{\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})\}$ where $\widetilde{M}$ is a constituent of $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}W$, $\widetilde{Q}$ its maximal parabolic subgroup and $\pi_{\widetilde{Q}}$ a constituent of $\mathrm{Ind}_{P(\mathbb{Q}_p)\cap L_{\widetilde{Q}}(\mathbb{Q}_p)}^{L_{\widetilde{Q}}(\mathbb{Q}_p)}\pi_P$. Since the socle of $\mathcal{F}_P^G(M,\pi_P)$ is clearly contained in the socle of $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}W'\otimes_E\pi_P\big)^{\mathrm{an}}$, it is therefore enough to prove the following statement:
Let $\widetilde{M}$ *be any irreducible constituent of* $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}W$ *distinct from* M*,* $\widetilde{Q}$ *its maximal parabolic subgroup and let* $\pi_{\widetilde{Q}}$ *be any irreducible smooth representation of* $L_{\widetilde{Q}}(\mathbb{Q}_p)$ *over* E*. Then the irreducible representation* $\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})$ *is not a* $G(\mathbb{Q}_p)$*-subrepresentation of* $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}W'\otimes_E\pi_P\big)^{\mathrm{an}}$*.*
Indeed, any constituent of $\mathrm{soc}_{G(\mathbb{Q}_p)}\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}W'\otimes_E\pi_P\big)^{\mathrm{an}}$ must then be a constituent of the subrepresentation $\mathcal{F}_Q^G(M,\mathrm{Ind}_{P(\mathbb{Q}_p)\cap L_Q(\mathbb{Q}_p)}^{L_Q(\mathbb{Q}_p)}\pi_P)=\mathcal{F}_P^G(M,\pi_P)$ (recall that M occurs only once in $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}W$, namely as its irreducible quotient) and hence must lie in $\mathrm{soc}_{G(\mathbb{Q}_p)}\mathcal{F}_P^G(M,\pi_P)$.
So let $\widetilde{M}$, $\widetilde{Q}$ and $\pi_{\widetilde{Q}}$ be as above and assume that $\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})$ is a subrepresentation of $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}W'\otimes_E\pi_P\big)^{\mathrm{an}}$. Passing to continuous duals we have as above a surjection of $D(G(\mathbb{Q}_p),E)$-modules:

$$\Big(\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}W'\otimes_E\pi_P\big)^{\mathrm{an}}\Big)'\cong D(G(\mathbb{Q}_p),E)\otimes_{D(P(\mathbb{Q}_p),E)}(W\otimes_E\pi_P')$$
$$\twoheadrightarrow\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})'$$

and thus obtain a nonzero $L_P(\mathbb{Q}_p)$-equivariant morphism :

$$W\otimes_E\pi_P'\longrightarrow H^0\big(N_P(\mathbb{Q}_p),\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})'\big).\tag{8}$$

Proposition 3.1.1 applied to $\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})$ implies:

$$\begin{aligned}H^0\big(N_P(\mathbb{Q}_p),\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})'\big)&=H^0\big(N_P(\mathbb{Q}_p)\cap L_{\widetilde{Q}}(\mathbb{Q}_p),H^0\big(N_{\widetilde{Q}}(\mathbb{Q}_p),\mathcal{F}_{\widetilde{Q}}^G(\widetilde{M},\pi_{\widetilde{Q}})'\big)\big)\\&=H^0\big(N_P(\mathbb{Q}_p)\cap L_{\widetilde{Q}}(\mathbb{Q}_p),\widetilde{W}\otimes_E\pi_{\widetilde{Q}}'\big)\end{aligned}$$

where $\widetilde{W}$ is the unique irreducible algebraic representation of $L_{\widetilde{Q}}(\mathbb{Q}_p)$ over E such that $\widetilde{M}$ is the irreducible quotient of $U(\mathfrak{g})\otimes_{U(\widetilde{\mathfrak{q}})}\widetilde{W}$ (Proposition 2.1.5).

Let $\mathfrak{u}_P$ (resp. $\mathfrak{u}_{\widetilde{Q}}$, resp. $\mathfrak{u}_{P,\widetilde{Q}}$) be the Lie algebra of the p-adic analytic group $N(\mathbb{Q}_p) \cap L_P(\mathbb{Q}_p)$ (resp. $N(\mathbb{Q}_p) \cap L_{\widetilde{Q}}(\mathbb{Q}_p)$, resp. $N_P(\mathbb{Q}_p) \cap L_{\widetilde{Q}}(\mathbb{Q}_p)$) and let v^+ (resp. $\widetilde{v}^+$) be a highest weight vector in W (resp. $\widetilde{W}$). We deduce from (8) that $Ev^+ \otimes_E \pi'_P \cong H^0\big(\mathfrak{u}_P, W \otimes_E \pi'_P\big)$ has a nonzero image in:

$$\begin{aligned} H^0\big(\mathfrak{u}_P, H^0\big(N_P(\mathbb{Q}_p), \mathcal{F}^G_{\widetilde{Q}}(\widetilde{M}, \pi_{\widetilde{Q}})'\big) &\cong H^0\big(\mathfrak{u}_P, H^0\big(N_P(\mathbb{Q}_p) \cap L_{\widetilde{Q}}(\mathbb{Q}_p), \widetilde{W} \otimes_E \pi'_{\widetilde{Q}}\big)\big) \\ &\hookrightarrow H^0\big(\mathfrak{u}_{\widetilde{Q}}, \widetilde{W} \otimes_E \pi'_{\widetilde{Q}}\big) \cong E\widetilde{v}^+ \otimes_E \pi'_{\widetilde{Q}} \end{aligned}$$

where the second injection follows from:

$$H^0\big(N_P(\mathbb{Q}_p) \cap L_{\widetilde{Q}}(\mathbb{Q}_p), \widetilde{W} \otimes_E \pi'_{\widetilde{Q}}\big) \hookrightarrow H^0\big(\mathfrak{u}_{P,\widetilde{Q}}, \widetilde{W} \otimes_E \pi'_{\widetilde{Q}}\big)$$

and $\mathfrak{u}_{\widetilde{Q}} \cong \mathfrak{u}_P \oplus \mathfrak{u}_{P,\widetilde{Q}}$. Looking at the action of $\mathfrak{t}$, the Lie algebra of $T(\mathbb{Q}_p)$ acting trivially on π'_P and $\pi'_{\widetilde{Q}}$, we get that v^+ and $\widetilde{v}^+$ must have the same weight, which is impossible since $1 \otimes v^+ \in U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ is the unique (up to scalar) highest weight vector and $M \not\cong \widetilde{M}$. $\square$

If, for all constituents $\widetilde{M}$ of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$, the representations $\mathrm{Ind}^{L_{\widetilde{Q}}(\mathbb{Q}_p)}_{P(\mathbb{Q}_p) \cap L_{\widetilde{Q}}(\mathbb{Q}_p)} \pi_P$ are irreducible, then by Corollary 3.1.2 the socle filtration of $\big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$ completely reflects the cosocle filtration of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$.

We also deduce another even more useful consequence.

Corollary 3.1.3. *Let $P \subsetneq G$ be a parabolic subgroup containing B, M_1, M_2 two irreducible objects of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$, Q_1 and Q_2 their respective maximal parabolic subgroups, and π_{Q_1}, π_{Q_2} two finite length smooth admissible representations of respectively $L_{Q_1}(\mathbb{Q}_p)$ and $L_{Q_2}(\mathbb{Q}_p)$ over E. We have $\mathcal{F}^G_{Q_1}(M_1, \pi_{Q_1}) \cong \mathcal{F}^G_{Q_2}(M_2, \pi_{Q_2})$ if and only if $M_1 \cong M_2$ and $\pi_{Q_1} \cong \pi_{Q_2}$.*

Proof. Let us assume $\mathcal{F}^G_{Q_1}(M_1, \pi_{Q_1}) \cong \mathcal{F}^G_{Q_2}(M_2, \pi_{Q_2})$. For $i \in \{1, 2\}$, let v_i^+ be a highest weight vector in M_i. By a result due to Orlik and Schraen which is a variant of Proposition 3.1.1 above (more precisely of the isomorphism (7)), we have an isomorphism $H^0\big(\mathfrak{n}, \mathcal{F}^G_{Q_i}(M_i, \pi_{Q_i})'\big) \cong Ev_i^+ \otimes_E \pi'_{Q_i}$ compatible with the action of $\mathfrak{t}$ (acting trivially on π'_{Q_i}). We deduce that the highest weights of M_1 and M_2 are the same, and hence that $M_1 = M_2$ and $Q_1 = Q_2$. Then Proposition 3.1.1 implies $W \otimes_E \pi'_{Q_1} \cong W \otimes_E \pi'_{Q_2}$ for the same irreducible algebraic representation W of $L_{Q_1}(\mathbb{Q}_p) = L_{Q_2}(\mathbb{Q}_p)$. As $\mathrm{End}_{\mathfrak{l}_{Q_i}}(W) = E$ and $\mathfrak{l}_{Q_i}$ acts trivially on π'_{Q_i}, we have $L_{Q_i}(\mathbb{Q}_p)$-equivariant morphisms $\pi'_{Q_1} = \mathrm{Hom}_{\mathfrak{l}_{Q_1}}(W, W \otimes_E \pi'_{Q_1}) = \mathrm{Hom}_{\mathfrak{l}_{Q_2}}(W, W \otimes_E \pi'_{Q_2}) = \pi'_{Q_2}$. $\square$

3.2 Examples for $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_3(\mathbb{Q}_p)$

We give the explicit socle filtration of some locally analytic principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_3(\mathbb{Q}_p)$.

Let us start with $G = \mathrm{GL}_2$, T the diagonal matrices and B the *lower* triangular matrices (this is our convention). Recall from the proof of Proposition 2.1.4 that $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{b} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \mathfrak{g}_\alpha$ where α is the unique positive root of $\mathfrak{g}$ (with respect to $\mathfrak{b}$) and $\mathfrak{g}_{\pm\alpha} = \{x \in \mathfrak{g}, [h,x] = \pm\alpha(h)x \ \forall \ h \in \mathfrak{t}\}$. Concretely, we have $\mathfrak{g}_{-\alpha} = \mathbb{Q}_p y$, $\mathfrak{g}_\alpha = \mathbb{Q}_p x$ and $\mathfrak{t} = \mathbb{Q}_p h \oplus \mathbb{Q}_p z$ with $y := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $x := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $z := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Moreover $\alpha(h) = 2$, $\alpha(z) = 0$ and $[x,y] = h$ (and $[h,y] = -2y$, $[h,x] = 2x$ as seen above). Iterating, we get in $U(\mathfrak{g})$ the relations $[x,y^n] = ny^{n-1}h - n(n-1)y^{n-1}$ and $[h,y^n] = -2ny^n$ for $n \in \mathbb{Z}_{>0}$.

Let $\lambda : \mathfrak{t} \to E$ be an (integral) weight and v^+ a highest weight vector of $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda \cong U(\mathfrak{g}_{-\alpha}) \otimes_{\mathbb{Q}_p} Ev^+ \cong \oplus_{n\in\mathbb{Z}_{\geq 0}} Ey^n \otimes v^+$. Note that $\lambda(h) = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$. For $n \in \mathbb{Z}_{>0}$ we have $h(y^n \otimes v^+) = (\lambda(h) - 2n)y^n \otimes v^+$ and, since $xv^+ = 0$, $x(y^n \otimes v^+) = [x,y^n]v^+ = (n\lambda(h) - n(n-1))y^{n-1} \otimes v^+$. Thus $x(y^n \otimes v^+) = 0$ if and only if $\lambda(h) = n - 1 \in \mathbb{Z}_{\geq 0}$ (i.e. if and only if λ is dominant) and we deduce easily from all this that $M(\lambda)$ is irreducible if and only $\lambda(h) \in \mathbb{Z}_{<0}$ and otherwise is a nonsplit extension of $L(\lambda) \cong \oplus_{n=0}^{\lambda(h)} Ey^n \otimes v^+$ by $L(s_\alpha \cdot \lambda) = M(s_\alpha \cdot \lambda) \cong \oplus_{n=\lambda(h)+1}^{+\infty} Ey^n \otimes v^+$ (see (1), s_α is the unique non trivial element of the Weyl group). Note that we recover in this simple case Theorem 2.1.8, and that $\ker(\phi) = L(s_\alpha \cdot \lambda) = y^{\lambda(h)+1}M(\lambda)$ if $W = \lambda$ and $M = L(\lambda)$.

Denote by $\chi_{-\mu}$ the algebraic character of $T(\mathbb{Q}_p)$ corresponding to the *dual* $-\mu$ of an integral weight $\mu : \mathfrak{t} \to E$. For instance one has $\chi_{-\lambda}\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) = x^{\lambda(h)}$ and $\chi_{-\lambda}\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) = x^{-\lambda(z)}$. Now let $\pi_B : T(\mathbb{Q}_p) \to E^\times$ be any smooth character, we have $\mathcal{F}_B^{\mathrm{GL}_2}(M(\lambda), \pi_B) = \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{-\lambda}\pi_B\right)^{\mathrm{an}}$ and, if $\lambda(h) \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} \mathcal{F}_B^{\mathrm{GL}_2}(L(\lambda), \pi_B) &= \mathcal{F}_{\mathrm{GL}_2}^{\mathrm{GL}_2}\left(L(\lambda), \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \pi_B\right) = L(\lambda)' \otimes_E \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \pi_B \\ \mathcal{F}_B^{\mathrm{GL}_2}(L(s_\alpha \cdot \lambda), \pi_B) &= \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{-s_\alpha \cdot \lambda}\pi_B\right)^{\mathrm{an}} \end{aligned}$$

where $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \pi_B$ is the smooth induction. By the previous theory, we deduce that we have a nonsplit extension of admissible locally analytic representations when $\lambda(h) \geq 0$:

$$0 \to L(\lambda)' \otimes_E \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \pi_B \to \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{-\lambda}\pi_B\right)^{\mathrm{an}} \to \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{-s_\alpha \cdot \lambda}\pi_B\right)^{\mathrm{an}} \to 0$$

where the right hand side is always irreducible and the left hand side is irreducible if and only if $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \pi_B$ is. Since $\ker(\phi)$ is generated by $y^{\lambda(h)+1}$, the map on the right is explicitly given by:

$$f \mapsto \Big[g \mapsto \Big(\frac{d^{\lambda(h)+1}}{dt^{\lambda(h)+1}} f\big(\left(\begin{smallmatrix} 1 & -t \\ 0 & 1 \end{smallmatrix}\right) g\big)\Big)|_{t=0}\Big].$$

We rewrite the above nonsplit exact sequence simply as:

$$L(\lambda)' \otimes_E \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \pi_B \;\text{——}\; \big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_{-s_\alpha\cdot\lambda}\pi_B\big)^{\mathrm{an}}. \tag{9}$$

Let us now switch to $G = \mathrm{GL}_3$. Here $\mathfrak{g} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{-\gamma} \oplus \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma$ where α, β are the two simple roots and $\gamma = \alpha + \beta$. We write as before $\mathfrak{g}_{-*} = \mathbb{Q}_p y_*$ where $* \in \{\alpha, \beta, \gamma\}$ and $y_\alpha := \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $y_\beta := \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $y_\gamma := \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

Let $\lambda : \mathfrak{t} \to E$ be an (integral) weight and let us assume that λ is dominant, that is, $\langle \lambda, \alpha^\vee \rangle \geq 0$ and $\langle \lambda, \beta^\vee \rangle \geq 0$ (this is the most interesting case and all other cases are simpler than this one). Let $n_\alpha := \langle \lambda, \alpha^\vee \rangle + 1 > 0$, $n_\beta := \langle \lambda, \beta^\vee \rangle + 1 > 0$ and $n_\gamma := n_\alpha + n_\beta = \langle \lambda, \gamma^\vee \rangle + 2 = \langle s_\alpha \cdot \lambda, \beta^\vee \rangle + 1 = \langle s_\beta \cdot \lambda, \alpha^\vee \rangle + 1 > 0$. Let v^+ a highest weight vector in $M(\lambda)$, then a computation analogous to the one for GL_2 shows that the weight vectors of $M(\lambda)$ killed by $\mathfrak{n}$ are exactly the following 6 weight vectors (up to scalar): v^+, $y_\alpha^{n_\alpha} v^+$, $y_\beta^{n_\beta} v^+$, $y_\beta^{n_\gamma} y_\alpha^{n_\alpha} v^+$, $y_\alpha^{n_\gamma} y_\beta^{n_\beta} v^+$ and $y_\alpha^{n_\beta} y_\beta^{n_\gamma} y_\alpha^{n_\alpha} v^+ = y_\beta^{n_\alpha} y_\alpha^{n_\gamma} y_\beta^{n_\beta} v^+$ of respective weights λ, $\lambda - n_\alpha \alpha = s_\alpha \cdot \lambda$, $\lambda - n_\beta \beta = s_\beta \cdot \lambda$, $\lambda - n_\alpha \alpha - n_\gamma \beta = (s_\beta s_\alpha) \cdot \lambda$, $\lambda - n_\beta \beta - n_\gamma \alpha = (s_\alpha s_\beta) \cdot \lambda$ and $\lambda - n_\gamma \gamma = (s_\alpha s_\beta s_\alpha) \cdot \lambda = (s_\beta s_\alpha s_\beta) \cdot \lambda = s_\gamma \cdot \lambda$. Any weight vector $w \in M(\lambda)$ killed by $\mathfrak{n}$ of weight μ gives rise to a nonzero morphism $M(\mu) \to M(\lambda)$ in $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{b}}$ (and conversely) obtained by sending a highest weight vector of $M(\mu)$ to w. Morever, one can prove that any such morphism is necessarily injective (this comes from the fact that the multiplication in $U(\overline{\mathfrak{n}})$ by any nonzero element of $U(\overline{\mathfrak{n}})$ is an injective map). Therefore the $M(\mu)$ contained in $M(\lambda)$ are exactly:

$$\begin{array}{lclclcl}
y_\alpha^{n_\alpha} M(\lambda) & \cong & M(s_\alpha \cdot \lambda) & & y_\beta^{n_\beta} M(\lambda) & \cong & M(s_\beta \cdot \beta) \\
y_\beta^{n_\gamma} y_\alpha^{n_\alpha} M(\lambda) & \cong & M((s_\beta s_\alpha) \cdot \lambda) & & y_\alpha^{n_\gamma} y_\beta^{n_\beta} M(\lambda) & \cong & M((s_\alpha s_\beta) \cdot \lambda) \\
y_\alpha^{n_\beta} y_\beta^{n_\gamma} y_\alpha^{n_\alpha} M(\lambda) & = & y_\beta^{n_\alpha} y_\alpha^{n_\gamma} y_\beta^{n_\beta} M(\lambda) & \cong & M((s_\beta s_\alpha s_\beta) \cdot \lambda) & = & M(s_\gamma \cdot \lambda).
\end{array}$$

Moreover, a direct computation using (inductively) $[y_\alpha, y_\beta] = y_\gamma$ and $[y_\alpha, y_\gamma] = [y_\beta, y_\gamma] = 0$ shows that $y_\beta^{n_\gamma} y_\alpha^{n_\alpha} = y_\beta^{n_\beta + n_\alpha} y_\alpha^{n_\alpha} \in U(\overline{\mathfrak{n}}) y_\beta^{n_\beta}$ and $y_\alpha^{n_\gamma} y_\beta^{n_\beta} = y_\alpha^{n_\beta + n_\alpha} y_\beta^{n_\beta} \in U(\overline{\mathfrak{n}}) y_\alpha^{n_\alpha}$. We thus have chains of inclusions:

$$\begin{tikzcd}
& M\big((s_\beta s_\alpha) \cdot \lambda\big) \arrow[r, hook] \arrow[drr, hook] & M\big(s_\alpha \cdot \lambda\big) \arrow[dr, hook] & \\
M\big(s_\gamma \cdot \lambda\big) \arrow[ur, hook] \arrow[dr, hook] & & & M\big(\lambda\big). \\
& M\big((s_\alpha s_\beta) \cdot \lambda\big) \arrow[r, hook] \arrow[urr, hook] & M\big(s_\beta \cdot \lambda\big) \arrow[ur, hook] &
\end{tikzcd}$$

With further work, one can deduce from these inclusions the complete (co)socle filtration of $M(\lambda)$ (and also of $M\big(s_\alpha\cdot\lambda\big)$, etc.):

$$\begin{tikzcd}
 & L\big((s_\beta s_\alpha)\cdot\lambda\big) \arrow[r, dash] \arrow[ddr, dash] & L\big(s_\alpha\cdot\lambda\big) \arrow[ddl, dash] \arrow[dr, dash] & \\
L\big(s_\gamma\cdot\lambda\big) \arrow[ur, dash] \arrow[dr, dash] & & & L\big(\lambda\big) \\
 & L\big((s_\alpha s_\beta)\cdot\lambda\big) \arrow[r, dash] & L\big(s_\beta\cdot\lambda\big) \arrow[ur, dash] &
\end{tikzcd}$$

where (following our convention in (9)) a line between two constituents means a nonsplit extension as subquotient. Note that $L(s_\gamma\cdot\lambda) = M(s_\gamma\cdot\lambda)$.

For $\pi_B : T(\mathbb{Q}_p)\to E^\times$ a smooth character, we thus deduce as for GL_2 that we have an analogous decomposition of $\big(\operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\chi_{-\lambda}\pi_B\big)^{\mathrm{an}} = \mathcal{F}_B^{\mathrm{GL}_3}(M(\lambda),\pi_B)$:

$$\begin{tikzcd}
 & \mathcal{F}_B^{\mathrm{GL}_3}(L(s_\alpha\cdot\lambda),\pi_B) \arrow[r, dash] \arrow[ddr, dash] & \mathcal{F}_B^{\mathrm{GL}_3}(L((s_\beta s_\alpha)\cdot\lambda),\pi_B) \arrow[ddl, dash] \arrow[dr, dash] & \\
\mathcal{F}_B^{\mathrm{GL}_3}(L(\lambda),\pi_B) \arrow[ur, dash] \arrow[dr, dash] & & & \mathcal{F}_B^{\mathrm{GL}_3}(L(s_\gamma\cdot\lambda),\pi_B). \\
 & \mathcal{F}_B^{\mathrm{GL}_3}(L(s_\beta\cdot\lambda),\pi_B) \arrow[r, dash] & \mathcal{F}_B^{\mathrm{GL}_3}(L((s_\alpha s_\beta)\cdot\lambda),\pi_B) \arrow[ur, dash] &
\end{tikzcd}$$

Note that $\mathcal{F}_B^{\mathrm{GL}_3}(L(\lambda),\pi_B)\cong L(\lambda)'\otimes_E \operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\pi_B$ and $\mathcal{F}_B^{\mathrm{GL}_3}(L(s_\gamma\cdot\lambda),\pi_B)\cong \big(\operatorname{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_3(\mathbb{Q}_p)}\chi_{-s_\gamma\cdot\lambda}\pi_B\big)^{\mathrm{an}}$.

Exercice 3.2.1. (i) Check that $\mathcal{F}_B^{\mathrm{GL}_3}(L((s_\beta s_\alpha)\cdot\lambda),\pi_B)$ is the locally anaytic parabolic induction from $P_1(\mathbb{Q}_p)$ to $\mathrm{GL}_3(\mathbb{Q}_p)$ of a locally algebraic representation of $P_1(\mathbb{Q}_p)$ where $P_1\subset\mathrm{GL}_3$ is the lower parabolic subgroup of Levi $\mathrm{GL}_2\times\mathrm{GL}_1$. Likewise with $\mathcal{F}_B^{\mathrm{GL}_3}(L((s_\alpha s_\beta)\cdot\lambda),\pi_B)$ and the lower parabolic of Levi $\mathrm{GL}_1\times\mathrm{GL}_2$.
(ii) Show that there is an analogous description for the extensions $\mathcal{F}_B^{\mathrm{GL}_3}(L(s_\alpha\cdot\lambda),\pi_B) \frac{\quad\quad}{} \mathcal{F}_B^{\mathrm{GL}_3}(L((s_\alpha s_\beta)\cdot\lambda),\pi_B)$ and $\mathcal{F}_B^{\mathrm{GL}_3}(L(s_\beta\cdot\lambda),\pi_B) \frac{\quad\quad}{} \mathcal{F}_B^{\mathrm{GL}_3}(L((s_\beta s_\alpha)\cdot\lambda),\pi_B)$.

4 Lecture 4

4.1 Necessary conditions for integrality

We finish our general treatment of locally analytic representations by giving necessary conditions for the representations $\mathcal{F}_P^G(M,\pi_P)$ to admit an invariant $\mathcal{O}_E$-lattice. In the case of locally analytic parabolic inductions, these conditions are due to Emerton.

We keep the previous notation: G, T, B, etc. As we have seen in the introduction, the most interesting locally analytic representations of $G(\mathbb{Q}_p)$ over E are those which arise inside some unitary Banach space representations of $G(\mathbb{Q}_p)$, for instance completed étale cohomology groups. Such a locally analytic representation automatically has the induced invariant norm of the Banach space. It is therefore important to understand which conditions are satisfied by locally analytic representations of $G(\mathbb{Q}_p)$ which possess invariant norms, or equivalently invariant lattices (see below). In the case of the representations $\mathcal{F}_P^G(M,\pi_P)$, we do not know in general whether they admit invariant norms or not (and this question is presumably hard). However, using a result of Emerton and the previous theory, we can at least give some *necessary* conditions on M and π_P for $\mathcal{F}_P^G(M,\pi_P)$ to possess an invariant norm (or lattice). What makes these necessary conditions interesting is that they will be related to Fontaine's weakly admissible conditions.

Definition 4.1.1. *An invariant lattice in a locally analytic representation of $G(\mathbb{Q}_p)$ on a Hausdorff locally convex E-vector space of compact type V is an open $\mathcal{O}_E$-submodule $V^0\subset V$ which is preserved by $G(\mathbb{Q}_p)$, which doesn't contain any E-line and such that $V^0\otimes_{\mathcal{O}_E}E\xrightarrow{\sim}V$.*

Set $|E|:=\{|x|,x\in E\}\subset\mathbb{Q}$ where $|x|:=p^{-\mathrm{val}(x)}$ with $\mathrm{val}(p)=1$ (we endow $|E|$ with the topology induced by the usual transcendental topology of $\mathbb{Q}$). Equivalently, an invariant lattice on V is the same thing as a continuous invariant norm $\|\cdot\|:V\to|E|$, where V^0 is sent to $\|v\|:=\inf_{v\in\lambda V^0}|\lambda|$ and $\|\cdot\|$ is sent to $V^0:=\{v\in V,\|v\|\leq 1\}$.

Let $P=L_PN_P\subseteq G$ be a parabolic subgroup containing B, $\overline{P}=L_{\overline{P}}N_{\overline{P}}=L_PN_{\overline{P}}$ its opposite parabolic subgroup (containing $\overline{B}$) and fix an open compact subgroup $N_{\overline{P}}^0$ of $N_{\overline{P}}(\mathbb{Q}_p)$. Define:

$$L_P(\mathbb{Q}_p)^+:=\{g\in L_P(\mathbb{Q}_p),gN_{\overline{P}}^0g^{-1}\subseteq N_{\overline{P}}^0\}\subseteq L_P(\mathbb{Q}_p)$$

and let $Z_{L_P}(\mathbb{Q}_p)^+:=Z_{L_P}(\mathbb{Q}_p)\cap L_P(\mathbb{Q}_p)^+$ where Z_{L_P} is the center of L_P. Then $L_P(\mathbb{Q}_p)^+$ (resp. $Z_{L_P}(\mathbb{Q}_p)^+$) is a submonoid of $L_P(\mathbb{Q}_p)$ (resp. $Z_{L_P}(\mathbb{Q}_p)$) which contains an open compact subgroup of $L_P(\mathbb{Q}_p)$ (resp. $Z_{L_P}(\mathbb{Q}_p)$) as well as $Z_G(\mathbb{Q}_p)$ where Z_G is the center of G.

Exercice 4.1.2. If $G=\mathrm{GL}_n$, $P=B$ is the subgroup of lower triangular matrices, $\overline{N}$ the subgroup of upper triangular unipotent matrices and $N_{\overline{P}}^0:=\overline{N}(\mathbb{Z}_p)\subset\overline{N}(\mathbb{Q}_p)$, check that:

$$L_P(\mathbb{Q}_p)^+=Z_{L_P}(\mathbb{Q}_p)^+=\{\mathrm{diag}(a_1,a_2,\cdots,a_n),\ a_i\in\mathbb{Q}_p^\times,\ \mathrm{val}(a_i)\geq\mathrm{val}(a_{i+1})\ \forall\ i\}.$$

For any locally analytic representation of $G(\mathbb{Q}_p)$ on a Hausdorff locally convex E-vector space of compact type V, we can endow the E-vector space $V^{N_{\overline{P}}^0}$ with a continuous Hecke action of $L_P(\mathbb{Q}_p)^+$ as follows:

$$\pi_g v := \sum_{n \in N_{\overline{P}}^0 / g N_{\overline{P}}^0 g^{-1}} (ng)v \tag{10}$$

where $g \in L_P(\mathbb{Q}_p)^+$ and $v \in V^{N_{\overline{P}}^0}$ ($V^{N_{\overline{P}}^0}$ being a closed subspace of V remains a Hausdorff locally convex E-vector space of compact type). Note that $\pi_g v = gv$ if $g N_{\overline{P}}^0 g^{-1} = N_{\overline{P}}^0$ which holds for g in an open compact subgroup of $L_P(\mathbb{Q}_p)^+$ or for $g \in Z_G(\mathbb{Q}_p)$.

Exercice 4.1.3. If $g_1, g_2 \in L_P(\mathbb{Q}_p)^+$, check that $\pi_{g_1 g_2} v = \pi_{g_1}(\pi_{g_2} v)$.

Proposition 4.1.4 (Emerton)**.** *Let W be a finite dimensional algebraic representation of $L_P(\mathbb{Q}_p)$ over E and π_P a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E. We have a continuous $L_P(\mathbb{Q}_p)^+$-equivariant injection:*

$$(W' \otimes_E \pi_P)|_{L_P(\mathbb{Q}_p)^+} \hookrightarrow \Big(\big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}\Big)^{N_{\overline{P}}^0}.$$

where $L_P(\mathbb{Q}_p)^+$ acts on the right hand side via the Hecke action (10).

Proof. Continuity is automatic since $W' \otimes_E \pi_P$ is equipped with the finest locally convex topology (see Example 1.2.1). Let $C_c^\infty(N_{\overline{P}}(\mathbb{Q}_p), W' \otimes_E \pi_P)$ be the $N_{\overline{P}}(\mathbb{Q}_p)$-representation of smooth functions with compact support $N_{\overline{P}}(\mathbb{Q}_p) \to W' \otimes_E \pi_P$ where $N_{\overline{P}}(\mathbb{Q}_p)$ acts by right translation. We extend this action to $\overline{P}(\mathbb{Q}_p) = L_P(\mathbb{Q}_p) N_{\overline{P}}(\mathbb{Q}_p)$ by the formula:

$$\big((mn) \cdot f\big)(n') := m\big(f(m^{-1} n' m n)\big)$$

where $f \in C_c^\infty(N_{\overline{P}}(\mathbb{Q}_p), W' \otimes_E \pi_P)$, $m \in L_P(\mathbb{Q}_p)$, $n, n' \in N_{\overline{P}}(\mathbb{Q}_p)$ (it is formal to check that this indeed yields an action of $\overline{P}(\mathbb{Q}_p)$). There is a $\overline{P}(\mathbb{Q}_p)$-equivariant injection $C_c^\infty(N_{\overline{P}}(\mathbb{Q}_p), W' \otimes_E \pi_P) \hookrightarrow \big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$ defined as follows: $f \in C_c^\infty(N_{\overline{P}}(\mathbb{Q}_p), W' \otimes_E \pi_P)$ is sent to the unique function $F \in \big(\operatorname{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$ such that $F(g) = 0$ if $g \in G(\mathbb{Q}) \backslash P(\mathbb{Q}_p) N_{\overline{P}}(\mathbb{Q}_p)$ and $F(pn) := p(f(n))$ if $p \in P(\mathbb{Q}_p)$ and $n \in N_{\overline{P}}(\mathbb{Q}_p)$ (the function F is locally analytic on the whole of $G(\mathbb{Q}_p)$ since f has compact support on $N_{\overline{P}}(\mathbb{Q}_p)$ and the $\overline{P}(\mathbb{Q}_p)$-equivariance is formal to check). Now all that remains to be checked is that there is an $L_P(\mathbb{Q}_p)^+$-equivariant injection $(W' \otimes_E \pi_P)|_{L_P(\mathbb{Q}_p)^+} \hookrightarrow C_c^\infty(N_{\overline{P}}(\mathbb{Q}_p), W' \otimes_E \pi_P)^{N_{\overline{P}}^0}$ where the right hand side is endowed with the Hecke action (10). For $v \in W' \otimes_E \pi_P$, let f_v be the function on $N_{\overline{P}}(\mathbb{Q}_p)$ which is constant on $N_{\overline{P}}^0$ with value v and which is 0 on $N_{\overline{P}}(\mathbb{Q}_p) \backslash N_{\overline{P}}^0$. A computation shows that $f_{mv} = \pi_m f_v$ for $m \in L_P(\mathbb{Q}_p)^+$, thus $v \mapsto f_v$ is the required injection. $\square$

Exercice 4.1.5. Check the last equality of the proof, that is:

$$f_{mv}(n') = \sum_{n \in N^0_{\overline{P}}/mN^0_{\overline{P}}m^{-1}} \big((nm) \cdot f_v\big)(n') \qquad \forall\, n' \in N_{\overline{P}}(\mathbb{Q}_p).$$

We now turn to the representations $\mathcal{F}^G_P(M, \pi_P)$.

Proposition 4.1.6. *Let W be an irreducible algebraic representation of $L_P(\mathbb{Q}_p)$ over E, M the irreducible quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ (Proposition 2.1.5) and π_P a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E. The $G(\mathbb{Q}_p)$-equivariant injection $\mathcal{F}^G_P(M, \pi_P) \hookrightarrow \big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$ induces an isomorphism compatible with the Hecke action of $L_P(\mathbb{Q}_p)^+$ in (10):*

$$\mathcal{F}^G_P(M, \pi_P)^{N^0_{\overline{P}}} \xrightarrow{\ \sim\ } \Big(\big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}\Big)^{N^0_{\overline{P}}}.$$

Proof. It is enough to prove that any $f \in \big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$ such that $f|_{N_{\overline{P}}(\mathbb{Q}_p)}$ is a locally constant function on $N_{\overline{P}}(\mathbb{Q}_p)$ is in $\mathcal{F}^G_P(M, \pi_P)$. Indeed, any $f \in \big(\big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}\big)^{N^0_{\overline{P}}}$ is *a fortiori* such that $f|_{N_{\overline{P}}(\mathbb{Q}_p)}$ is locally constant and thus belongs to $\mathcal{F}^G_P(M, \pi_P)^{N^0_{\overline{P}}}$ (the inclusion in the other sense being obvious). For any $0 \neq f \in \big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} W' \otimes_E \pi_P\big)^{\mathrm{an}}$, we have a nonzero morphism of left $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$-modules thanks to Lemma 2.2.1:

$$\Delta_f : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \longrightarrow C^{\mathrm{an}}(G(\mathbb{Q}_p), \pi_P), \quad \mathfrak{d} \mapsto \mathfrak{d} \cdot f$$

where the action of $U(\mathfrak{g})$ on the right hand side is given by (2). By definition of $\mathcal{F}^G_P(M, \pi_P)$, we have to prove that $\Delta_f(\ker(\phi)) = 0$ if $f|_{N_{\overline{P}}(\mathbb{Q}_p)}$ is locally constant (where $\ker(\phi)$ is as in (4)), or equivalently $\mathrm{im}(\Delta_f) \cong M$. Note that we always have surjections $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \twoheadrightarrow \mathrm{im}(\Delta_f) \twoheadrightarrow M$ in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and that by definition $\mathrm{im}(\Delta_f)$ is the smallest quotient M' of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ such that $f \in \mathcal{F}^G_P(M', \pi_P)$. Let us assume that $\ker(\mathrm{im}(\Delta_f) \twoheadrightarrow M)$ contains an irreducible object (of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$), which is the quotient of another generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$ by Proposition 2.1.6. So we have a diagram:

$$\begin{tikzcd} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \arrow[r, two heads, "\Delta_f"] & \mathrm{im}(\Delta_f) \arrow[r, two heads] & M. \\ & U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V \arrow[u, "\psi"] & \end{tikzcd} \tag{11}$$

Letting $\widetilde{M} \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be the cokernel of ψ, we have $f \notin \mathcal{F}^G_P(\widetilde{M}, \pi_P)$ and an exact sequence of $G(\mathbb{Q}_p)$-representations by (ii) of Theorem 2.2.2:

$$0 \to \mathcal{F}^G_P(\widetilde{M}, \pi_P) \to \mathcal{F}^G_P(\mathrm{im}(\Delta_f), \pi_P) \to \big(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} V' \otimes_E \pi_P\big)^{\mathrm{an}}. \tag{12}$$

Therefore it is enough to prove that, if $f|_{N_{\overline{P}}(\mathbb{Q}_p)}$ is locally constant, its image in $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V' \otimes_E \pi_P\big)^{\mathrm{an}}$ is zero, because then we would have $f \in \mathcal{F}_P^G(\widetilde{M}, \pi_P)$ which is a contradiction. So let h be the image of f in $\big(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} V' \otimes_E \pi_P\big)^{\mathrm{an}}$, since h is locally analytic, it is easy to see that $h \equiv 0$ if and only if $h(\cdot)(v)|_{N_{\overline{P}}(\mathbb{Q}_p)} \equiv 0$ in $C^{\mathrm{an}}(G(\mathbb{Q}_p), \pi_P)$ for all $v \in V$. For each $v \in V$, there is $\mathfrak{d}_v \in U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ such that $\psi(1 \otimes v) = \Delta_f(\mathfrak{d}_v)$ in $\mathrm{im}(\Delta_f)$ (see (11)). Moreover it is formal to check that $h(\cdot)(v) = \mathfrak{d}_v \cdot f \in C^{\mathrm{an}}(G(\mathbb{Q}_p), \pi_P)$. The fact that $\Delta_f(\mathfrak{d}_v) \mapsto 0$ in M implies that $\mathfrak{d}_v \in U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \cong U(\mathfrak{n}_{\overline{P}}) \otimes_{\mathbb{Q}_p} W$ must involve only non constant elements of $U(\mathfrak{n}_{\overline{P}})$ (elements like $1 \otimes w$ all remain nonzero in M). But then this implies $(\mathfrak{d}_v \cdot f)|_{N_{\overline{P}}(\mathbb{Q}_p)} \equiv 0$ when $f|_{N_{\overline{P}}(\mathbb{Q}_p)}$ is locally constant, i.e. $h(\cdot)(v)|_{N_{\overline{P}}(\mathbb{Q}_p)} \equiv 0$. □

The proof of the corollary below is due to Emerton.

Corollary 4.1.7. *Let $P \subseteq G$ be a parabolic subgroup containing B, M an irreducible object of $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ and π_P a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over E which has a central character $\chi_{\pi_P} : Z_{L_P}(\mathbb{Q}_p) \to E^\times$. Let λ be the highest weight of $M|_{\mathfrak{t}}$ (Proposition 2.1.6) and $\chi_{-\lambda} : T(\mathbb{Q}_p) \to E^\times$ the algebraic character associated to its dual $-\lambda$. If $\mathcal{F}_P^G(M, \pi_P)$ has an invariant lattice, then $\chi_{-\lambda}(z)\chi_{\pi_P}(z) \in \mathcal{O}_E$ for all $z \in Z_{L_P}(\mathbb{Q}_p)^+$.*

Proof. Let W be the unique irreducible algebraic representation of $L_P(\mathbb{Q}_p)$ (or of $\mathfrak{l}_P$) over E such that M is the unique irreducible quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$. By Propositions 4.1.6 and 4.1.4, we have a continuous $L_P(\mathbb{Q}_p)^+$-equivariant injection:

$$(W' \otimes_E \pi_P)|_{L_P(\mathbb{Q}_p)^+} \hookrightarrow \mathcal{F}_P^G(M, \pi_P)^{N_P^0}.$$

In particular there is a nonzero $v \in \mathcal{F}_P^G(M, \pi_P)^{N_P^0}$ such that $\pi_z v = \chi_{-\lambda}(z)\chi_{\pi_P}(z)v$ for all $z \in Z_{L_P}(\mathbb{Q}_p)^+$ (note that $\chi_{-\lambda}|_{Z_{L_P}(\mathbb{Q}_p)}\chi_{\pi_P}$ is the central character of $W' \otimes_E \pi_P$). If $\mathcal{F}_P^G(M, \pi_P)$ has an invariant lattice, or equivalently an invariant norm, we have for all $z \in Z_{L_P}(\mathbb{Q}_p)^+$:

$$\begin{aligned}\|\chi_{-\lambda}(z)\chi_{\pi_P}(z)v\| = |\chi_{-\lambda}(z)\chi_{\pi_P}(z)|\|v\| = \|\pi_z v\| &= \|\sum_{n \in N_P^0/zN_P^0z^{-1}} (nz)v\| \\ &\leq \max\{\|(nz)v\|,\ n \in N_P^0/zN_P^0z^{-1}\} = \|v\|.\end{aligned}$$

Hence $|\chi_{-\lambda}(z)\chi_{\pi_P}(z)| \leq 1$ since $\|v\| \neq 0$. □

4.2 Definition of the representations $\pi(\underline{D}, \underline{h})$

We define some locally algebraic representations $\pi(\underline{D}, \underline{h})$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E and recall a (special case of a) conjecture of Schneider and myself predicting when they should admit an invariant lattice.

From now on and till the end of these lectures, we set $G = \mathrm{GL}_n$, T the torus of diagonal matrices and B the Borel subgroup of lower triangular matrices (and we keep the previous notation: N, $\overline{B}$, etc.).

We fix a Galois extension K of $\mathbb{Q}_p$ and let $K_0 \subseteq K$ be its maximal unramified extension. We always assume $|\mathrm{Hom}(K, E)| = [K : \mathbb{Q}_p]$. We let φ_0 be the absolute arithmetic Frobenius on K_0 (= raising to the p mod p).

Definition 4.2.1. *A Deligne-Fontaine module is a 4-tuple $(\varphi, N, \mathrm{Gal}(K/\mathbb{Q}_p), D)$ where D is a free $K_0 \otimes_{\mathbb{Q}_p} E$-module of finite rank equipped with a bijective application (the Frobenius) $\varphi : D \to D$ such that $\varphi((k_0 \otimes e)d) = (\varphi_0(k_0) \otimes e)\varphi(d)$, with a $K_0 \otimes_{\mathbb{Q}_p} E$-linear endomorphism $N : D \to D$ (the monodromy operator) such that $N\varphi = p\varphi N$ and with an action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ (the descent data) which commutes with φ, N and such that $g((k_0 \otimes e)d) = (g(k_0) \otimes e)g(d)$ ($k_0 \in K_0$, $e \in E$, $d \in D$, $g \in \mathrm{Gal}(K/\mathbb{Q}_p)$).*

Remark 4.2.2. The operator N (which will disappear soon) shouldn't be confused with the lower unipotent radical $N \subset \mathrm{GL}_n$.

Deligne-Fontaine modules form an abelian category with morphisms being $K_0 \otimes_{\mathbb{Q}_p} E$-linear maps which commute with φ, N and $\mathrm{Gal}(K/\mathbb{Q}_p)$. Replacing the semi-linear actions of φ and $\mathrm{Gal}(K/\mathbb{Q}_p)$ by the linear action of the elements $\overline{g} \circ \varphi^{-\alpha(g)}$ where g is in the Weil group of $\mathbb{Q}_p$, $\overline{g}$ is its image in $\mathrm{Gal}(K/\mathbb{Q}_p)$ and $\alpha(g) \in \mathbb{Z}$ is such that $g \mapsto (x \mapsto x^{p^{\alpha(g)}}) \in \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$, we obtain the following well-known result of Fontaine that we leave as an exercise (and which justifies the terminology):

Exercice 4.2.3. The category of Deligne-Fontaine modules is equivalent to the category of representations of the Weil-Deligne group of $\mathbb{Q}_p$ over E which become unramified after restriction to the Weil group of K.

We now fix a rank n Deligne-Fontaine module $\underline{D} := (\varphi, N, \mathrm{Gal}(K/\mathbb{Q}_p), D)$. Enlarging E if necessary, we can assume that its irreducible constituents are all absolutely irreducible (use e.g. Exercise 4.2.3). We assume:

Hypothesis 4.2.4. The irreducible constituents of $\underline{D}$ are distinct.

Then we can write:

$$\underline{D} = \bigoplus_{i=1}^{r} \Big[\bigoplus_{\ell=1}^{\ell_i} \big(p^{-(\ell_i - \ell)}\varphi_i, \mathrm{Gal}(K/\mathbb{Q}_p), D_i\big) \Big] \tag{13}$$

where $r \in \mathbb{Z}_{>0}$, $\ell_i \in \mathbb{Z}_{>0}$, $(\varphi_i, 0, \mathrm{Gal}(K/\mathbb{Q}_p), D_i)$ is an absolutely irreducible Deligne-Fontaine module, N is zero on $\big(p^{-(\ell_i-1)}\varphi_i, \mathrm{Gal}(K/\mathbb{Q}_p), D_i\big)$ and sends $\big(p^{-(\ell_i-\ell)}\varphi_i, \mathrm{Gal}(K/\mathbb{Q}_p), D_i\big)$ to $\big(p^{-(\ell_i-\ell+1)}\varphi_i, \mathrm{Gal}(K/\mathbb{Q}_p), D_i\big)$ via the identity on D_i if $1 < \ell \leq \ell_i$. We assume:

Hypothesis 4.2.5. For all $i, j \in \{1, \cdots, r\}$, we have $\big(\varphi_i, 0, \mathrm{Gal}(K/\mathbb{Q}_p), D_i\big) \not\cong \big(p^{-\ell_j}\varphi_j, 0, \mathrm{Gal}(K/\mathbb{Q}_p), D_j\big)$.

Remark 4.2.6. For those who are familiar with this terminology, the Deligne-Fontaine modules $\bigoplus_{\ell=1}^{\ell_i} \big(p^{-(\ell_i-\ell)}\varphi_i, \mathrm{Gal}(K/\mathbb{Q}_p), D_i\big)$ are *segments* and Hypothesis 4.2.4 and 4.2.5 mean that the segments of $\underline{D}$ are *distinct* and *not linked*.

Let $\Pi^{\mathrm{Langlands}}$ be the smooth irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\overline{\mathbb{Q}_p}$ corresponding to the Weil-Deligne representation of $\underline{D}$ (Exercise 4.2.3) by the classical local Langlands correspondence (for our normalization of local class field theory). It is a well-known result that $\Pi^{\mathrm{Langlands}}|\det|^{\frac{1-n}{2}}$ admits a canonical model Π defined over E, that is, $\Pi^{\mathrm{Langlands}}|\det|^{\frac{1-n}{2}} \cong \Pi \otimes_E \overline{\mathbb{Q}_p}$.

Let us fix a list of distinct integers $\underline{h} := (h_1, \cdots, h_n)$ in $\mathbb{Z}$ such that $h_1 < h_2 < \cdots < h_n$. Define $\lambda_i := -h_i - (n-i)$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We let $\overline{L}(\lambda)$ be the irreducible algebraic representation of GL_n over E of highest weight $\chi_\lambda : T \to \mathbb{G}_{\mathrm{m}}$, $\mathrm{diag}(x_1, \cdots, x_n) \mapsto x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ with respect to the roots of $\overline{B}$, that is, the algebraic induction:

$$\overline{L}(\lambda) := \big(\mathrm{Ind}_B^{\mathrm{GL}_n} \chi_\lambda\big)^{\mathrm{alg}}.$$

We still denote by $\overline{L}(\lambda)$ its restriction to $\mathrm{GL}_n(\mathbb{Q}_p)$ which is still (absolutely) irreducible. We define the following locally algebraic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E:

$$\pi(\underline{D}, \underline{h}) \;:=\; \overline{L}(\lambda) \otimes_E \Pi \;\cong\; \mathcal{F}_{\mathrm{GL}_n}^{\mathrm{GL}_n}\big(\overline{L}(\lambda)', \Pi\big) = \mathcal{F}_{\mathrm{GL}_n}^{\mathrm{GL}_n}\big(L(-\lambda), \Pi\big) \qquad (14)$$

where $\overline{L}(\lambda)'$ is the dual representation of $\overline{L}(\lambda)$ seen as an object of $\mathcal{O}^{\mathfrak{g}}_{\mathrm{alg}}$, that is $L(-\lambda)$ with the notation of §2.1 (note that $-\lambda$ is dominant with respect to the roots of $\mathfrak{b}$).

Exercice 4.2.7. Prove directly that $\pi(\underline{D}, \underline{h})$ is absolutely irreducible without using (iv) of Theorem 2.2.2.

We conjecturally know when $\pi(\underline{D}, \underline{h})$ should admit an invariant lattice:

Conjecture 4.2.8 (B.-Schneider)**.** *The* $\mathrm{GL}_n(\mathbb{Q}_p)$*-representation* $\pi(\underline{D}, \underline{h})$ *admits an invariant lattice if and only if* $\underline{D}$ *admits a weakly admissible filtration of Hodge-Tate weights* $\underline{h}$*.*

We recall the definitions of a Hodge filtration of fixed Hodge-Tate weights and of weak admissibility in the next lecture. The direction $\Rightarrow$ is completely known thanks to work of Schneider, Teitelbaum, Hu and myself. The direction $\Leftarrow$ is much more difficult and still open in general, but many cases are now known (e.g. the case $n = 2$) thanks to work of Berger, Colmez, Vignéras, de Shalit, Kazhdan, Sorensen, de Ieso, Assaf, myself, and especially the ongoing recent work of Emerton, Gee, Paskunas and al.

5 Lecture 5

5.1 Some preliminaries

We give several preliminaries which will be used afterwards to define some semi-simple locally analytic representations $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}})$ containing $\pi(\underline{D}, \underline{h})$.

We fix an arbitrary Deligne-Fontaine module $\underline{D}$ of rank n over $K_0 \otimes_{\mathbb{Q}_p} E$ and a list of distinct integers $\underline{h} := (h_1 < \cdots < h_n)$. We extend the action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ to $D_K := D \otimes_{K_0} K$ by $g((k \otimes e) \cdot d) = (g(k) \otimes e) \cdot g(d)$.

Definition 5.1.1. *A Hodge filtration of Hodge-Tate weights $\underline{h}$ on $\underline{D}$ is the data of $K \otimes_{\mathbb{Q}_p} E$-submodules $(\mathrm{Fil}^i D_K)_{i \in \mathbb{Z}}$ of D_K such that:*
(i) $\mathrm{Fil}^{i+1} D_K \subseteq \mathrm{Fil}^i D_K$ for all i, $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$, $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$;
(ii) $\mathrm{Fil}^i D_K$ is stable under the action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ for all i;
(ii) $\mathrm{Fil}^i D_K / \mathrm{Fil}^{i+1} D_K \neq 0$ if and only if $i \in \{h_1, \cdots, h_n\}$.

We denote by $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ a Hodge filtration of Hodge-Tate weights $\underline{h}$ on $\underline{D}$. By Hilbert Thm. 90, we have $K \otimes_{\mathbb{Q}_p} (\mathrm{Fil}^i D_K)^{\mathrm{Gal}(K/\mathbb{Q}_p)} \xrightarrow{\sim} \mathrm{Fil}^i D_K$ for all i. Hence a $K \otimes_{\mathbb{Q}_p} E$-submodule $\mathrm{Fil}^i D_K$ of D_K stable by $\mathrm{Gal}(K/\mathbb{Q}_p)$ is just equivalent to an E-subvector space of $D_K^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ (which is $(\mathrm{Fil}^i D_K)^{\mathrm{Gal}(K/\mathbb{Q}_p)}$). In particular each $\mathrm{Fil}^i D_K$ and each $\mathrm{Fil}^i D_K / \mathrm{Fil}^{i+1} D_K$ is free (of finite rank) over $K \otimes_{\mathbb{Q}_p} E$.

Let $f := [K_0 : \mathbb{Q}_p]$ and consider the following integers:

$$t_N(\underline{D}) := \frac{1}{f} \mathrm{val}\big(\det_{K_0}(\varphi^f)\big), \quad t_H(\underline{D}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}) := \sum_{i \in \mathbb{Z}} i \dim_K\big(\mathrm{Fil}^i D_K / \mathrm{Fil}^{i+1} D_K\big)$$

(recall that φ^f acts K_0-linearly on the K_0-vector space D). If $\underline{D}' \subseteq \underline{D}$ is a Deligne-Fontaine submodule, we endow it with the induced Hodge filtration $\mathrm{Fil}^i D'_K := D'_K \cap \mathrm{Fil}^i D_K$ ($i \in \mathbb{Z}$). Recall the following definition:

Definition 5.1.2 (Fontaine)**.** *The Hodge filtration $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ is weakly admissible if, for every Deligne-Fontaine submodule $\underline{D}' \subseteq \underline{D}$, we have $t_H(\underline{D}', \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}) \leq t_N(\underline{D}')$ and if moreover $t_H(\underline{D}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}) = t_N(\underline{D})$.*

The main "raison d'être" of this definition is the following theorem:

Theorem 5.1.3 (Colmez-Fontaine)**.** *Assume that $\underline{D}$ has rank n over $K_0 \otimes_{\mathbb{Q}_p} E$. If $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ is a weakly admissible filtration on $\underline{D}$, then the $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-representation $\mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}} \otimes_K D_K) \cap (\mathrm{B}_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1, N=0}$, where the action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is induced by that on B_{dR} and B_{st} and by the action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ on D, has dimension n over E and becomes semi-stable in restriction to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$.*

I don't recall the definitions of B_{dR} and B_{st} here (we won't use them).

From now on, we assume that $\underline{D}$ satisfies Hypothesis 4.2.4 and 4.2.5 and also, in order to avoid too many technicalities, that $\underline{D}$ is of the form:

$$\underline{D} = \oplus_{i=1}^{n} \underline{D}_i \tag{15}$$

where $\underline{D}_i := (\varphi_i, 0, \mathrm{Gal}(K/\mathbb{Q}_p), K_0 \otimes_{\mathbb{Q}_p} E \cdot e_i)$ with the notation of (13). All the results that follow can be extended to the general case (13) (that is, only assuming Hypothesis 4.2.4 and 4.2.5). In the case (15), the representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ given by Theorem 5.1.3 are called *crystabelline*.

Exercice 5.1.4. Prove that the crystabelline representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ become crystalline over an *abelian* extension of $\mathbb{Q}_p$ contained in K, whence their name (use the Weil representations associated to the $\underline{D}_i$).

Let χ_i be the character of the Weil group of $\mathbb{Q}_p$ associated to $\underline{D}_i$. Explicitly $\chi_i(g)$ is just the linear action of $\overline{g} \circ \varphi^{-\alpha(g)}$ on $K_0 \otimes_{\mathbb{Q}_p} E \cdot e_i$ (see §4.2) followed by $K_0 \otimes_{\mathbb{Q}_p} E \cdot e_i \twoheadrightarrow E e_i \cong E$ for any embedding $K_0 \hookrightarrow E$. Seeing the χ_i as characters of $\mathbb{Q}_p^\times$ by local class field theory, we then have:

$$\Pi = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \chi_1 |\cdot|^{-(n-1)} \otimes \chi_2 |\cdot|^{-(n-2)} \otimes \cdots \otimes \chi_n$$

(smooth induction). Note that Π is irreducible because of Hypothesis 4.2.5. By (iii) of Theorem 2.2.2 and Corollary 3.1.2, we deduce from (14):

$$\begin{aligned}\pi(\underline{D}, \underline{h}) &= \mathcal{F}_B^{\mathrm{GL}_n}\big(L(-\lambda), \chi_1 |\cdot|^{-(n-1)} \otimes \chi_2 |\cdot|^{-(n-2)} \otimes \cdots \otimes \chi_n\big) \\ &= \mathrm{soc}_{\mathrm{GL}_n(\mathbb{Q}_p)}\Big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \chi_\lambda \big(\chi_1 |\cdot|^{-(n-1)} \otimes \chi_2 |\cdot|^{-(n-2)} \otimes \cdots \otimes \chi_n\big)\Big)^{\mathrm{an}}\end{aligned}$$

where $\chi_\lambda : B(\mathbb{Q}_p) \twoheadrightarrow T(\mathbb{Q}_p) \to \mathbb{Q}_p^\times \subseteq E^\times$, $\mathrm{diag}(x_1, \cdots, x_n) \mapsto x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.

Remark 5.1.5. Let $\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{Z}_p^\times \hookrightarrow E^\times$ be the p-adic cyclotomic character. Seen as a character of $\mathbb{Q}_p^\times$, we have $\varepsilon(x) = x|x|$. Then one checks that:

$$\pi(\underline{D}, \underline{h}) = \mathrm{soc}_{\mathrm{GL}_n(\mathbb{Q}_p)}\Big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} x_1^{-h_1} \chi_1 \varepsilon^{-(n-1)} \otimes x_2^{-h_2} \chi_2 \varepsilon^{-(n-2)} \otimes \cdots \otimes x_n^{-h_n} \chi_n\Big)^{\mathrm{an}}.$$

Recall that the Weyl group of GL_n is $\mathcal{S}_n$, the set of permutations on n elements. There is a length function $\ell : \mathcal{S}_n \to \mathbb{Z}_{\geq 0}$ defined as the smallest integer $\ell(w)$ such that w is a product of $\ell(w)$ simple reflections (i.e. adjacent transpositions). Also $\mathcal{S}_n$ is endowed with a partial order called the Bruhat order: $w \leq w'$ if and only if w is a subexpression of an expression of w' as a product of $\ell(w')$ simple reflections. There is a unique maximal element w_0 in $\mathcal{S}_n$ (i.e. $w \leq w_0$ for all $w \in \mathcal{S}_n$) given by $w_0(i) = n + 1 - i$, $i \in \{1, \cdots, n\}$ and one has $ww_0 \leq w'w_0$ if and only if $w' \leq w$.

For $w \in \mathcal{S}_n$, we define $w\bar{\cdot}\lambda$ as in (1) but replacing ρ (= half the sum of the roots of B, writing the weights additively as in §2.1) by $\overline{\rho} = -\rho$ = half the sum of the roots of $\overline{B}$. For instance one has:

$$w\bar{\cdot}\lambda = -(w \cdot (-\lambda)). \tag{16}$$

For $(w^{\mathrm{alg}}, w) \in \mathcal{S}_n \times \mathcal{S}_n$, define:

$$\pi_{B,w} := \chi_{w^{-1}(1)}|\cdot|^{-(n-1)} \otimes \chi_{w^{-1}(2)}|\cdot|^{-(n-2)} \otimes \cdots \otimes \chi_{w^{-1}(n)}, \tag{17}$$

a smooth (1-dimensional) representation of $T(\mathbb{Q}_p)$ over E, and set:

$$C(w^{\mathrm{alg}}, w) := \mathcal{F}_B^{\mathrm{GL}_n}\big(L(w^{\mathrm{alg}} \cdot (-\lambda)), \pi_{B,w}\big)$$

with $L(w^{\mathrm{alg}} \cdot (-\lambda))$ as in §2.1. From Corollary 3.1.2 and (16), we have explicitly:

$$C(w^{\mathrm{alg}}, w) = \mathrm{soc}_{\mathrm{GL}_n(\mathbb{Q}_p)}\Big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} \chi_{w^{\mathrm{alg}}\bar{\cdot}\lambda}\big(\chi_{w^{-1}(1)}|\cdot|^{-(n-1)} \otimes \cdots \otimes \chi_{w^{-1}(n)}\big)\Big)^{\mathrm{an}} \tag{18}$$

(see §3.2 for the notation $\chi_{w^{\mathrm{alg}}\bar{\cdot}\lambda} = \chi_{-w^{\mathrm{alg}}\cdot(-\lambda)}$). The $C(w^{\mathrm{alg}}, w)$ are irreducible admissible locally analytic representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E (the irreducibility follows from Theorem 2.2.2 together with Hypothesis 4.2.5) and $C(1, w) \cong \pi(\underline{D}, \underline{h})$ for all $w \in \mathcal{S}_n$. Even for $w^{\mathrm{alg}} \neq 1$, the $C(w^{\mathrm{alg}}, w)$ are not all distinct when w varies and we have the following (important) proposition:

Proposition 5.1.6. *For $w^{\mathrm{alg}} \in \mathcal{S}_n$, let $P(w^{\mathrm{alg}}) \subseteq \mathrm{GL}_n$ be the maximal parabolic subgroup of $L(w^{\mathrm{alg}} \cdot (-\lambda))$, that is, the maximal parabolic subgroup of GL_n containing B such that $\langle w^{\mathrm{alg}} \cdot (-\lambda), \alpha^\vee\rangle \geq 0$ for all its simple roots α, and let $W_{P(w^{\mathrm{alg}})} \subseteq \mathcal{S}_n$ be its Weyl group. Then the following are equivalent (where $w_i^{\mathrm{alg}}, w_i \in \mathcal{S}_n$):*
(i) $C(w_1^{\mathrm{alg}}, w_1) \cong C(w_2^{\mathrm{alg}}, w_2)$;
(ii) $w_1^{\mathrm{alg}} = w_2^{\mathrm{alg}}$ and $w_2 w_1^{-1} \in W_{P(w_1^{\mathrm{alg}})} = W_{P(w_2^{\mathrm{alg}})}$;
(iii) $w_1^{-1}\overline{\overline{B}w_1^{\mathrm{alg}}w_0\overline{B}}/\overline{B} = w_2^{-1}\overline{\overline{B}w_2^{\mathrm{alg}}w_0\overline{B}}/\overline{B}$ (Zariski closures).

Proof. By Corollary 3.1.3, we have (i) if and only if $L(w_1^{\mathrm{alg}}\cdot(-\lambda)) \cong L(w_2^{\mathrm{alg}}\cdot(-\lambda))$ and:

$$\mathrm{Ind}_{B(\mathbb{Q}_p)\cap L_{P(w_1^{\mathrm{alg}})}(\mathbb{Q}_p)}^{L_{P(w_1^{\mathrm{alg}})}(\mathbb{Q}_p)} \pi_{B,w_1} \cong \mathrm{Ind}_{B(\mathbb{Q}_p)\cap L_{P(w_2^{\mathrm{alg}})}(\mathbb{Q}_p)}^{L_{P(w_2^{\mathrm{alg}})}(\mathbb{Q}_p)} \pi_{B,w_2}$$

(smooth inductions). As $-\lambda$ is dominant with respect to the roots of B, the first isomorphism is equivalent to $w_1^{\mathrm{alg}} = w_2^{\mathrm{alg}}$ (which implies $P(w_1^{\mathrm{alg}}) = P(w_2^{\mathrm{alg}})$) and it is a classical fact on intertwinings between irreducible smooth parabolic inductions that, granting Hypothesis 4.2.4 and 4.2.5, the second is equivalent to $w_2 w_1^{-1} \in W_{P(w_1^{\mathrm{alg}})} = W_{P(w_2^{\mathrm{alg}})}$. Hence (i) is equivalent to (ii). Let us prove that

(ii) implies (iii). Since $W_{P(w^{\mathrm{alg}})}$ is generated by those reflections s_α such that $s_\alpha \in W_{P(w^{\mathrm{alg}})}$ (where α is a simple root of B), it is enough to prove it for $w_2w_1^{-1} = s_\alpha$. It is a classical fact that $s_\alpha \in W_{P(w^{\mathrm{alg}})}$ is equivalent to $s_\alpha w^{\mathrm{alg}}w_0 \leq w^{\mathrm{alg}}w_0$ which in turn is equivalent to the statement:

$$\big[w' \leq w^{\mathrm{alg}}w_0 \Longleftrightarrow s_\alpha w' \leq w^{\mathrm{alg}}w_0\big]. \tag{19}$$

Now using the well-known result of Chevalley:

$$\overline{\overline{B}w^{\mathrm{alg}}w_0\overline{B}/\overline{B}} = \amalg_{w' \leq w^{\mathrm{alg}}w_0} \overline{B}w'\overline{B}/\overline{B} \tag{20}$$

together with:

$$s_\alpha \overline{B}w'\overline{B} \subseteq \overline{B}s_\alpha w'\overline{B} \ \amalg\ \overline{B}w'\overline{B}$$

for any $w' \in \mathcal{S}_n$ (another classical fact), we easily see that (19) is equivalent to $s_\alpha\overline{\overline{B}w^{\mathrm{alg}}w_0\overline{B}/\overline{B}} = \overline{\overline{B}w^{\mathrm{alg}}w_0\overline{B}/\overline{B}}$ which is (iii) for $w_2w_1^{-1} = s_\alpha$. To sum up: $s_\alpha \in W_{P(w^{\mathrm{alg}})}$ is equivalent to $s_\alpha\overline{\overline{B}w^{\mathrm{alg}}w_0\overline{B}/\overline{B}} = \overline{\overline{B}w^{\mathrm{alg}}w_0\overline{B}/\overline{B}}$. Finally, let us prove that (iii) implies (ii) (the following proof is due to Sasha Orlik). The elements of GL_n stabilizing the Schubert variety $\overline{\overline{B}w^{\mathrm{alg}}w_0\overline{B}/\overline{B}}$ form a closed algebraic subgroup of GL_n containing $\overline{B}$, hence a parabolic subgroup containing $\overline{B}$. Such a parabolic subgroup is determined by its simple roots $-\alpha$ (with α a simple root of B), and we see from what we have just proven that these simple roots are precisely the opposite of those of $L_{P(w^{\mathrm{alg}})}$. So this parabolic subgroup is $\overline{P(w^{\mathrm{alg}})}$. Now (iii) implies $w_1^{-1}\overline{P(w_1^{\mathrm{alg}})}w_1 = w_2^{-1}\overline{P(w_2^{\mathrm{alg}})}w_2$ or equivalently $\overline{P(w_2^{\mathrm{alg}})} = w_2w_1^{-1}\overline{P(w_1^{\mathrm{alg}})}w_1w_2^{-1}$. Since $\overline{P(w_1^{\mathrm{alg}})}$ and $\overline{P(w_2^{\mathrm{alg}})}$ are both *standard* (i.e. containing $\overline{B}$) and *conjugate*, they must be equal and $w_2w_1^{-1}$ must be in their commun Weyl group (again, classical facts). Hence in particular $w_2w_1^{-1}$ fixes $\overline{\overline{B}w_1^{\mathrm{alg}}w_0\overline{B}/\overline{B}}$ and (iii) is $\overline{\overline{B}w_1^{\mathrm{alg}}w_0\overline{B}/\overline{B}} = \overline{\overline{B}w_2^{\mathrm{alg}}w_0\overline{B}/\overline{B}}$ from which it is straightforward to deduce $w_1^{\mathrm{alg}} = w_2^{\mathrm{alg}}$ (use (20)). We have all of (ii). $\square$

Exercice 5.1.7. For $w^{\mathrm{alg}} \in \mathcal{S}_n$, α a simple root of B and μ a dominant weight for $\overline{B}$, prove the following statements: $\langle w^{\mathrm{alg}}\cdot\mu, -\alpha^\vee\rangle \geq 0 \Leftrightarrow s_\alpha \in W_{P(w^{\mathrm{alg}})} \Leftrightarrow s_\alpha w^{\mathrm{alg}}w_0 \leq w^{\mathrm{alg}}w_0 \Leftrightarrow [s_\alpha w' \leq w^{\mathrm{alg}}w_0$ if and only if $w' \leq w^{\mathrm{alg}}w_0]$.

We define an equivalence relation on $\mathcal{S}_n \times \mathcal{S}_n$ as follows: $(w_1^{\mathrm{alg}}, w_1) \sim (w_2^{\mathrm{alg}}, w_2)$ if and only if $w_1^{\mathrm{alg}} = w_2^{\mathrm{alg}}$ and $w_2w_1^{-1} \in W_{P(w_1^{\mathrm{alg}})} = W_{P(w_2^{\mathrm{alg}})}$ (that is, if and only if condition (ii) in Proposition 5.1.6 holds). Set $\mathcal{C} := (\mathcal{S}_n \times \mathcal{S}_n)/\sim$, then by Proposition 5.1.6 the map $(w^{\mathrm{alg}}, w) \mapsto C(w^{\mathrm{alg}}, w)$ induces a bijection between $\mathcal{C}$ and the set of isomorphism classes of the representations $C(w^{\mathrm{alg}}, w)$.

5.2 Definition of the representations $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^\cdot)$

To any Hodge filtration $\underline{\mathrm{Fil}}^\cdot$ of Hodge-Tate weights $\underline{h}$ on a Deligne-Fontaine module $\underline{D}$ as in (15), we associate a finite length semi-simple locally analytic representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^\bullet)$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E.

We keep the previous notation and recall that the assumption (15) is there only to avoid technicalities and ease understanding. For $i \in \{1, \cdots, n\}$, we fix a basis $e_{i,K}$ of the 1-dimensional E-vector space $D_{i,K}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$. Then, in the basis $(e_{i,K})_{1\leq i\leq n}$, the upper triangular matrices $\overline{B}(E)$ stabilize the flag $Ee_{1,K} \subsetneq Ee_{1,K} \oplus Ee_{2,K} \subsetneq \cdots \subsetneq \oplus_{i=1}^n Ee_{i,K}$. Let $\underline{\mathrm{Fil}}^\bullet$ be a Hodge filtration of Hodge-Tate weights $\underline{h}$ on $\underline{D}$, then the corresponding flag:

$$\begin{aligned}(\mathrm{Fil}^{h_n} D_K)^{\mathrm{Gal}(K/\mathbb{Q}_p)} \subsetneq (\mathrm{Fil}^{h_{n-1}} D_K)^{\mathrm{Gal}(K/\mathbb{Q}_p)} \subsetneq \cdots \\ \subsetneq (\mathrm{Fil}^{h_1} D_K)^{\mathrm{Gal}(K/\mathbb{Q}_p)} = D_K^{\mathrm{Gal}(K/\mathbb{Q}_p)}\end{aligned}$$

expressed in the basis $(e_{i,K})_{1\leq i\leq n}$ of $D_K^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ defines an E-point of the flag variety $\mathrm{GL}_n/\overline{B}$ (that is, an element of $\mathrm{GL}_n(E)/\overline{B}(E)$) that we still denote by $\underline{\mathrm{Fil}}^\bullet$. We define the following semi-simple locally analytic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over E:

$$\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^\bullet) \quad := \quad \bigoplus_{\overline{(w^{\mathrm{alg}}, w)}} C(w^{\mathrm{alg}}, w) \tag{21}$$

where the direct sum is over the equivalence classes $\overline{(w^{\mathrm{alg}}, w)} \in \mathcal{C}$ such that:

$$\underline{\mathrm{Fil}}^\bullet \in w^{-1}\overline{\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)} \subset \mathrm{GL}_n(E)/\overline{B}(E). \tag{22}$$

By Proposition 5.1.6, we see that it is well defined.

Exercice 5.2.1. (i) Prove that the representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^\bullet)$ doesn't depend on the choice of $e_{i,K}$ (i.e. one can replace $e_{i,K}$ by $\lambda_i e_{i,K}$ for any $\lambda_i \in E^\times$).
(ii) Prove that the representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^\bullet)$ doesn't depend on the order of the $\underline{D}_i$ (or equivalently of the $e_{i,K}$). Hint: the constituent $C(w^{\mathrm{alg}}, w)$ does depend on this order, as well as the set of $\overline{(w^{\mathrm{alg}}, w)} \in \mathcal{C}$ for which (22) holds. These two dependences cancel each other.

Since $w^{-1}\overline{\overline{B}(E)w_0\overline{B}(E)/\overline{B}(E)} = \mathrm{GL}_n(E)/\overline{B}(E)$ for all $w \in \mathcal{S}_n$, we see that $C(1, w) \cong C(1, 1) = \pi(\underline{D}, \underline{h})$ is always a constituent of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^\bullet)$.

Remark 5.2.2. The reader may ask where (21) and (22) are coming from. Theorem 6.1.4 below is a big motivation, as well as compatibility with Proposition 5.1.6 (handling all intertwinings between the $C(w^{\mathrm{alg}}, w)$), with some aspects of the mod p Langlands program such as Serre weights (see Remark 6.2.1 below), and also with what can be reasonnably expected on "overconvergent companion eigenforms".

Example 5.2.3. It is a good place to describe explicitly the case $n = 2$. Write $(\mathrm{Fil}^{h_2} D_K)^{\mathrm{Gal}(K/\mathbb{Q}_p)} := E(a_1 e_{1,K} \oplus a_2 e_{2,K})$ for some $(a_1, a_2) \in E^2 \backslash \{(0,0)\}$. If $a_1 \neq 0$, then $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ corresponds to the point $\overline{\left(\begin{smallmatrix} a_1 & 0 \\ a_2 & 1 \end{smallmatrix}\right)} \in \mathrm{GL}_2(E)/\overline{B}(E)$ and if $a_2 \neq 0$, $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ corresponds to the point $\overline{\left(\begin{smallmatrix} a_1 & 1 \\ a_2 & 0 \end{smallmatrix}\right)} \in \mathrm{GL}_2(E)/\overline{B}(E)$. An easy computation shows that we have the following (recall that $w_0 = s_\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the unique non trivial element of $\mathcal{S}_2$):

$$\begin{cases} a_1 \neq 0 & \text{if and only if} \quad \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}} \in w_0\big(\overline{B}(E) w_0 \overline{B}(E)/\overline{B}(E)\big) \\ a_2 \neq 0 & \text{if and only if} \quad \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}} \in \overline{B}(E) w_0 \overline{B}(E)/\overline{B}(E) \\ a_1 = 0 & \text{if and only if} \quad \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}} = \overline{w_0} = w_0\big(\overline{B}(E) w_0^2 \overline{B}(E)/\overline{B}(E)\big) \\ a_2 = 0 & \text{if and only if} \quad \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}} = \overline{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \overline{B}(E) w_0^2 \overline{B}(E)/\overline{B}(E). \end{cases}$$

This implies:

$$\begin{cases} \text{if } \mathrm{a}_1 \neq 0 \text{ and } \mathrm{a}_2 \neq 0, & \text{then} \quad \pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}) = C(1,1) = \pi(\underline{D}, \underline{h}) \\ \text{if } \mathrm{a}_1 \neq 0 \text{ and } \mathrm{a}_2 = 0, & \text{then} \quad \pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}) = C(1,1) \oplus C(w_0, 1) \\ \text{if } \mathrm{a}_2 \neq 0 \text{ and } \mathrm{a}_1 = 0, & \text{then} \quad \pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}) = C(1,1) \oplus C(w_0, w_0). \end{cases}$$

Then one has the following theorem (already mentioned in the introduction):

Theorem 5.2.4 (Colmez, R. Liu)**.** *Assume that* $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ *is weakly admissible and let* ρ *be the associated* 2*-dimensional crystabelline representation of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *over* E *(see Theorem 5.1.3). Then* $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}})$ *is the* $\mathrm{GL}_2(\mathbb{Q}_p)$*-socle of the locally analytic representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *over* E *associated to* ρ *by the* p*-adic Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$*.*

Note that when $a_1 a_2 = 0$, the two constituents of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}})$ are the two constituents of a locally analytic principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ as is easily seen from (9) and $w_0 = s_\alpha$ (however, this fact doesn't generalize to $n > 2$).

Example 5.2.3 seems however a bit too simple to provide a good intuition. For instance, in general, contrary to what happens for $n = 2$, the representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}})$ is of course *not sufficient* to recover the Hodge filtration $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}$ (up to isomorphism), as is clear from the next proposition.

Proposition 5.2.5. *We have* $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\boldsymbol{\cdot}}) = \pi(\underline{D}, \underline{h})$ *if and only if* $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}} \in w^{-1}\big(\overline{B}(E) w_0 \overline{B}(E)/\overline{B}(E)\big)$ *for all* $w \in \mathcal{S}_n$*.*

Proof. Recall first the Bruhat decomposition:

$$\mathrm{GL}_n(E)/\overline{B}(E) = \amalg_{w^{\mathrm{alg}} \in \mathcal{S}_n} \overline{B}(E) w^{\mathrm{alg}} w_0 \overline{B}(E)/\overline{B}(E). \tag{23}$$

Assume $\underline{\mathrm{Fil}}^{\boldsymbol{\cdot}} \in w^{-1}\big(\overline{B}(E) w_0 \overline{B}(E)/\overline{B}(E)\big)$ for all $w \in \mathcal{S}_n$. Then by (20) we have:

$$w^{-1}\big(\overline{B}(E) w_0 \overline{B}(E)/\overline{B}(E)\big) \bigcap w^{-1}\overline{\overline{B}(E) w^{\mathrm{alg}} w_0 \overline{B}(E)/\overline{B}(E)} = \emptyset$$

as soon as $w^{\mathrm{alg}} \neq 1$ (since then $w^{\mathrm{alg}}w_0 < w_0$) which implies that $C(1,1) = \pi(\underline{D}, \underline{h})$ is the only constituent of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$. Conversely, assume that $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot}) = \pi(\underline{D}, \underline{h})$. Let $w \in \mathcal{S}_n$, then by (23) there is a unique $w^{\mathrm{alg}} \in \mathcal{S}_n$ such that $\underline{\mathrm{Fil}}^{\cdot} \in w^{-1}\big(\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)\big)$. But if $w^{\mathrm{alg}} \neq 1$, we know by assumption (and the definition of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$) that $\underline{\mathrm{Fil}}^{\cdot} \notin w^{-1}\overline{\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)}$. Hence we must have $w^{\mathrm{alg}} = 1$, i.e. $\underline{\mathrm{Fil}}^{\cdot} \in w^{-1}\big(\overline{B}(E)w_0\overline{B}(E)/\overline{B}(E)\big)$. $\square$

6 Lecture 6

6.1 The link with weak admissibility

We prove that if $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ has an invariant lattice, then the Hodge filtration $\underline{\mathrm{Fil}}^{\cdot}$ is weakly admissible.

Our aim here is to prove that, although $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ is not sufficient to determine $\underline{\mathrm{Fil}}^{\cdot}$, it is sufficient to determine whether $\underline{\mathrm{Fil}}^{\cdot}$ is weakly admissible or not. We keep the previous notation (in particular we fix a basis $e_{i,K}$ of $D_{i,K}^{\mathrm{Gal}(K/\mathbb{Q}_p)}$ as before) and let $T(\mathbb{Q}_p)^+$ be the submonoid of $T(\mathbb{Q}_p)$ denoted by $L_P(\mathbb{Q}_p)^+$ in Exercise 4.1.2. If $w \in \mathcal{S}_n$ and $z \in T(\mathbb{Q}_p)$, recall that $\pi_{B,w}(z) \in E^\times$ is defined in (17).

Proposition 6.1.1. *Let $(w^{\mathrm{alg}}, w) \in \mathcal{S}_n \times \mathcal{S}_n$ and $\underline{\mathrm{Fil}}^{\cdot}$ a Hodge filtration of Hodge-Tate weights $\underline{h}$ on $\underline{D}$ such that $\underline{\mathrm{Fil}}^{\cdot} \in w^{-1}\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)$. The conditions:*

$$\begin{array}{rcll} t_H\big(\oplus_{j=1}^{i}\, \underline{D}_{w^{-1}(j)}, \underline{\mathrm{Fil}}^{\cdot}\big) & \leq & t_N\big(\oplus_{j=1}^{i}\, \underline{D}_{w^{-1}(j)}\big) & 1 \leq i \leq n-1 \\ t_H\big(\oplus_{j=1}^{n}\underline{D}_{w^{-1}(j)}, \underline{\mathrm{Fil}}^{\cdot}\big) & = & t_N\big(\oplus_{j=1}^{n}\underline{D}_{w^{-1}(j)}\big) & \end{array} \qquad (24)$$

are equivalent to the conditions :

$$\chi_{w^{\mathrm{alg}}\cdot\lambda}(z)\pi_{B,w}(z) \in \mathcal{O}_E \quad \forall\ z \in T(\mathbb{Q}_p)^+. \qquad (25)$$

Proof. Let us first assume $w = 1$, and thus $\underline{\mathrm{Fil}}^{\cdot} \in \overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)$. Then a straightforward computation shows that the graded pieces of such a filtration are as follows for $j \in \{1, \cdots, n\}$:

$$\begin{aligned} &\mathrm{Fil}^{h_{n+1-j}}\, D_K/\,\mathrm{Fil}^{h_{n+2-j}}\, D_K = \\ &\qquad (K \otimes_{\mathbb{Q}_p} E) \otimes_E E\big(*\, e_{1,K} \oplus *e_{2,K} \oplus \cdots \oplus *e_{(w^{\mathrm{alg}}w_0)(j)-1,K} \oplus e_{(w^{\mathrm{alg}}w_0)(j),K}\big) \end{aligned}$$

for some $* \in E$. Replacing j by $(w^{\mathrm{alg}}w_0)^{-1}(j)$, we get:

$$\begin{aligned} &\mathrm{Fil}^{h_{n+1-(w^{\mathrm{alg}}w_0)^{-1}(j)}}\, D_K/\,\mathrm{Fil}^{h_{n+2-(w^{\mathrm{alg}}w_0)^{-1}(j)}}\, D_K = \\ &\qquad\qquad (K \otimes_{\mathbb{Q}_p} E) \otimes_E E\big(*\, e_{1,K} \oplus *e_{2,K} \oplus \cdots \oplus *e_{j-1,K} \oplus e_{j,K}\big) \end{aligned}$$

for all j, from which we deduce for $i \in \{1, \cdots, n\}$:

$$\begin{aligned}
\frac{1}{[E:\mathbb{Q}_p]} t_H\big(\oplus_{j=1}^{i} \underline{D}_j, \underline{\mathrm{Fil}^{\cdot}}\big) &= \sum_{j=1}^{i} h_{n+1-(w^{\mathrm{alg}}w_0)^{-1}(j)} \\
&= \sum_{j=1}^{i} h_{(w^{\mathrm{alg}})^{-1}(j)}
\end{aligned}$$

where we have used $w_0^{-1}((w^{\mathrm{alg}})^{-1}(j)) = w_0((w^{\mathrm{alg}})^{-1}(j)) = n+1-(w^{\mathrm{alg}})^{-1}(j)$ for the second equality. For $j \in \{1, \cdots, n\}$, we have:

$$\frac{1}{[E:\mathbb{Q}_p]} t_N(\underline{D}_j) = \mathrm{val}\big(\chi_j(p)\big)$$

so that the inequalities (24) (for $w = 1$) are equivalent to the inequalities for $i \in \{1, \cdots, n\}$:

$$\sum_{j=1}^{i} h_{(w^{\mathrm{alg}})^{-1}(j)} \leq \sum_{j=1}^{i} \mathrm{val}\big(\chi_j(p)\big) \tag{26}$$

(with an equality for $i = n$). From the other hand, we have:

$$\begin{aligned}
w^{\mathrm{alg}} \cdot \lambda &= w^{\mathrm{alg}}\big((\lambda_j + n - j)_{1 \leq j \leq n}\big) - (n-j)_{1 \leq j \leq n} \\
&= \big(\lambda_{(w^{\mathrm{alg}})^{-1}(j)} + n - (w^{\mathrm{alg}})^{-1}(j)\big)_{1 \leq j \leq n} - (n-j)_{1 \leq j \leq n} \\
&= \big(\lambda_{(w^{\mathrm{alg}})^{-1}(j)} + j - (w^{\mathrm{alg}})^{-1}(j)\big)_{1 \leq j \leq n} = \\
&= \big(-h_{(w^{\mathrm{alg}})^{-1}(j)} - (n-j)\big)_{1 \leq j \leq n}.
\end{aligned}$$

Hence the conditions (25) are equivalent to the following inequalities for $i \in \{1, \cdots, n\}$ (using the description of $T(\mathbb{Q}_p)^+$ given in Exercise 4.1.2 together with (17)):

$$\sum_{j=1}^{i} \big(-h_{(w^{\mathrm{alg}})^{-1}(j)} - (n-j)\big) + \sum_{j=1}^{i} \Big(\mathrm{val}\big(\chi_j(p)\big) + \big(n-j\big)\Big) \geq 0$$

with $= 0$ (instead of ≥ 0) when $i = n$ (= the case when $z \in Z_G(\mathbb{Q}_p) \subseteq T(\mathbb{Q}_p)^+$). That is to say:

$$\sum_{j=1}^{i} h_{(w^{\mathrm{alg}})^{-1}(j)} \leq \sum_{j=1}^{i} \mathrm{val}\big(\chi_j(p)\big) \tag{27}$$

for $i \in \{1, \cdots, n\}$ (with an equality for $i = n$): we exactly recover (26).
The proof in the case $w \neq 1$ is completely similar replacing everywhere $\underline{D}_j$ (resp. $e_{j,K}$, resp. χ_j) by $\underline{D}_{w^{-1}(j)}$ (resp. $e_{w^{-1}(j),K}$, resp. $\chi_{w^{-1}(j)}$). $\square$

Proposition 6.1.2. *Let $(w^{\mathrm{alg}}, w) \in \mathcal{S}_n \times \mathcal{S}_n$ and $\underline{\mathrm{Fil}}^\cdot$ a Hodge filtration of Hodge-Tate weights $\underline{h}$ on $\underline{D}$ such that $\underline{\mathrm{Fil}}^\cdot \in w^{-1}\overline{\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)}$. The conditions (24) imply the conditions (25).*

Proof. Here again, we can assume $w = 1$. By (20), we have $\underline{\mathrm{Fil}}^\cdot \in \overline{B}(E)w_1^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)$ for some unique $w_1^{\mathrm{alg}} \in \mathcal{S}_n$ such that $w_1^{\mathrm{alg}} \geq w^{\mathrm{alg}}$ and by Proposition 6.1.1, (24) for $\underline{\mathrm{Fil}}^\cdot$ is equivalent to (25) for w_1^{alg}. It is thus enough to prove that (25) for w_1^{alg} implies (25) for w^{alg}. By induction, we can assume $w_1^{\mathrm{alg}} = s_\alpha w^{\mathrm{alg}}$ with $\ell(w_1^{\mathrm{alg}}) > \ell(w^{\mathrm{alg}})$ where α is a root of B (this is a property of the Bruhat order, note that α is not simple in general). Let $j_0 < j_1 \in \{1, \cdots, n\}$ such that s_α is the transposition on $\{1, \cdots, n\}$ which swaps j_0 et j_1. The condition $\ell(s_\alpha w^{\mathrm{alg}}) > \ell(w^{\mathrm{alg}})$ is then equivalent to $(w^{\mathrm{alg}})^{-1}(j_0) < (w^{\mathrm{alg}})^{-1}(j_1)$ (again a classical property), i.e. $h_{(w^{\mathrm{alg}})^{-1}(j_0)} < h_{(w^{\mathrm{alg}})^{-1}(j_1)}$. So we have $(w_1^{\mathrm{alg}})^{-1}(j) = (w^{\mathrm{alg}})^{-1}(j)$ if $j \notin \{j_0, j_1\}$, $(w_1^{\mathrm{alg}})^{-1}(j_0) = (w^{\mathrm{alg}})^{-1}(j_1) > (w^{\mathrm{alg}})^{-1}(j_0)$ and $(w_1^{\mathrm{alg}})^{-1}(j_1) = (w^{\mathrm{alg}})^{-1}(j_0)$ so that for all $i \in \{1, \cdots, n\}$:

$$\sum_{j=1}^{i} h_{(w^{\mathrm{alg}})^{-1}(j)} \leq \sum_{j=1}^{i} h_{(w_1^{\mathrm{alg}})^{-1}(j)}$$

with an equality for $i = n$. Thus we see that if (27) holds for w_1^{alg}, then it holds for w^{alg}. □

Proposition 6.1.3. *Let $\underline{\mathrm{Fil}}^\cdot$ be a Hodge filtration of Hodge-Tate weights $\underline{h}$ on $\underline{D}$. Then $\underline{\mathrm{Fil}}^\cdot$ is weakly admissible if and only if the conditions (25) hold for all $(w^{\mathrm{alg}}, w) \in \mathcal{S}_n \times \mathcal{S}_n$ such that $\underline{\mathrm{Fil}}^\cdot \in w^{-1}\overline{\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)}$.*

Proof. Assume that $\underline{\mathrm{Fil}}^\cdot$ is weakly admissible and let (w^{alg}, w) such that $\underline{\mathrm{Fil}}^\cdot \in w^{-1}\overline{\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)}$. Then by Proposition 6.1.2 the conditions (25) are satisfied for (w^{alg}, w). Assume that the conditions (25) hold for all (w^{alg}, w) as in the statement and let $\underline{D}' \subseteq \underline{D}$ be a Deligne-Fontaine submodule. By Hypothesis 4.2.4, there is $w \in \mathcal{S}_n$ (non unique) and $i \in \{1, \cdots, n\}$ such that $\underline{D}' = \oplus_{j=1}^{i} \underline{D}_{w^{-1}(j)}$. By (23) there is $w^{\mathrm{alg}} \in \mathcal{S}_n$ (unique) such that $\underline{\mathrm{Fil}}^\cdot \in w^{-1}\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)$. From Proposition 6.1.1 (and the assumption), we get $t_H(\underline{D}', \underline{\mathrm{Fil}}^\cdot) \leq t_N(\underline{D}')$ with an equality if $\underline{D}' = \underline{D}$, i.e. $\underline{\mathrm{Fil}}^\cdot$ is weakly admissible. □

We can now prove our main theorem:

Theorem 6.1.4. *Let $\underline{D}$, $\underline{h}$ as above and let $\underline{\mathrm{Fil}}^\cdot$ be a Hodge filtration of Hodge-Tate weights $\underline{h}$ on $\underline{D}$. If the locally analytic representation $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^\cdot)$ has an invariant lattice (Definition 4.1.1) then $\underline{\mathrm{Fil}}^\cdot$ is weakly admissible.*

Proof. From the definition of $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)$ (cf. (21) and (22)), we get that for all $(w^{\mathrm{alg}},w)\in\mathcal{S}_n\times\mathcal{S}_n$ such that $\underline{\mathrm{Fil}}^\bullet\in w^{-1}\overline{\overline{B}(E)w^{\mathrm{alg}}w_0\overline{B}(E)/\overline{B}(E)}$ the constituent $C(w^{\mathrm{alg}},w)=\mathcal{F}_B^{\mathrm{GL}_n}\big(L(w^{\mathrm{alg}}\cdot(-\lambda)),\pi_{B,w}\big)$ has an invariant lattice. Then Corollary 4.1.7 applied to $G=\mathrm{GL}_n$, $M=L(w^{\mathrm{alg}}\cdot(-\lambda))$, $P=B$ and $\pi_P=\pi_{B,w}$ gives $\chi_{-w^{\mathrm{alg}}\cdot(-\lambda)}(z)\pi_{B,w}(z)\in\mathcal{O}_E$ if $z\in T(\mathbb{Q}_p)^+$. Since $\chi_{-w^{\mathrm{alg}}\cdot(-\lambda)}=\chi_{w^{\mathrm{alg}}\cdot\lambda}$ (cf. (16)), the result follows from Proposition 6.1.3. □

6.2 Examples for $\mathrm{GL}_3(\mathbb{Q}_p)$ and open questions

We finish these lectures with the description of a few representations $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)$ (with $\underline{\mathrm{Fil}}^\bullet$ weakly admissible) in the case of $\mathrm{GL}_3(\mathbb{Q}_p)$, and with a few questions for possible future developments.

We assume $n=3$ and wish to give explicitly $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)$ when $\underline{\mathrm{Fil}}^\bullet$ is weakly admissible and the associated crystabelline 3-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E (see Theorem 5.1.3) is upper triangular. We thus assume $h_i=\mathrm{val}(\chi_i(p))$ for $i\in\{1,2,3\}$ and $\underline{\mathrm{Fil}}^\bullet$ of the form:

$$\left\{\begin{array}{rcl}\mathrm{Fil}^{h_3}D_K&=&(K\otimes_{\mathbb{Q}_p}E)\otimes_E E\big(a_3e_{1,K}\oplus a_2e_{2,K}\oplus e_{3,K}\big)\\ \mathrm{Fil}^{h_2}D_K/\mathrm{Fil}^{h_3}D_K&=&(K\otimes_{\mathbb{Q}_p}E)\otimes_E E\big(a_1e_{1,K}\oplus e_{2,K}\big)\\ \mathrm{Fil}^{h_1}D_K/\mathrm{Fil}^{h_2}D_K&=&(K\otimes_{\mathbb{Q}_p}E)\otimes_E Ee_{1,K}\end{array}\right.\tag{28}$$

for some $a_i\in E$. Then the representation $\rho:=\mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}\otimes_K D_K)\cap(\mathrm{B}_{\mathrm{cris}}\otimes_{K_0}D)^{\varphi=1}$ has the form:

$$\rho\cong\begin{pmatrix}\eta_1\varepsilon^{-h_1}&*_1&*_3\\0&\eta_2\varepsilon^{-h_2}&*_2\\0&0&\eta_3\varepsilon^{-h_3}\end{pmatrix}\tag{29}$$

where ε is the p-adic cyclotomic character and η_i is the locally constant integral character $\chi_i|\cdot|^{h_i}$ seen as a character of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ by local class field theory.

The computations that follow are easy from the definition (21) and we just give the result (leaving the details as a last exercise). We use the notation s_α and s_β of §3.2 for the two simple reflections of $\mathcal{S}_3$.

If $a_1a_2a_3\neq0$ and $a_3\neq a_1a_2$ in (28), then we are in the situation of Proposition 5.2.5, and $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)=C(1,1)=\pi(\underline{D},\underline{h})$.

If $a_1a_2a_3\neq0$ and $a_3=a_1a_2$, then $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)=C(1,1)\oplus C(s_\alpha,s_\alpha s_\beta)$.

If $a_1a_2\neq0$ and $a_3=0$, then $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)=C(1,1)\oplus C(s_\beta,s_\beta s_\alpha)$.

If $a_1a_3\neq0$ and $a_2=0$, then $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)=C(1,1)\oplus C(s_\beta,s_\beta)$.

If $a_2a_3\neq0$ and $a_1=0$, then $\pi(\underline{D},\underline{h},\underline{\mathrm{Fil}}^\bullet)=C(1,1)\oplus C(s_\alpha,s_\alpha)$.

If $a_1 \neq 0$ and $a_2 = a_3 = 0$, then:

$$\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot}) = \big(C(1,1) \oplus C(s_\beta, s_\beta) \oplus C(s_\alpha s_\beta, s_\alpha s_\beta)\big) \oplus \big(C(s_\alpha, s_\alpha s_\beta) \oplus C(s_\beta, s_\beta s_\alpha)\big).$$

If $a_2 \neq 0$ and $a_1 = a_3 = 0$, then:

$$\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot}) = \big(C(1,1) \oplus C(s_\alpha, s_\alpha) \oplus C(s_\beta s_\alpha, s_\beta s_\alpha)\big) \oplus \big(C(s_\alpha, s_\alpha s_\beta) \oplus C(s_\beta, s_\beta s_\alpha)\big).$$

If $a_3 \neq 0$ and $a_1 = a_2 = 0$, then:

$$\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot}) = C(1,1) \oplus C(s_\alpha, s_\alpha) \oplus C(s_\beta, s_\beta).$$

Finally if $a_1 = a_2 = a_3 = 0$, then:

$$\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot}) = \big(\oplus_{w \in \mathcal{S}_3} C(w,w)\big) \oplus \big(C(s_\alpha, s_\alpha s_\beta) \oplus C(s_\beta, s_\beta s_\alpha)\big).$$

Remark 6.2.1. (i) One can prove that the six $C(w,w)$ for $w \in \mathcal{S}_3$ are the $\mathrm{GL}_3(\mathbb{Q}_p)$-socles of the locally analytic vectors of six distinct continuous unitary principal series of $\mathrm{GL}_3(\mathbb{Q}_p)$ over E (check that $\chi_{w \cdot \lambda}\pi_{B,w}$ takes values in $\mathcal{O}_E^\times$, see also Remark 5.1.5 for $w = 1$). These constituents $C(w,w)$ in each case fit with analogous constituents mod p which are *known* to occur in Hecke eigenspaces of $\overline{S}(U^p) := \varinjlim\{f : G(\mathbb{Q})\backslash G(\mathbb{A}^{\infty,p})/U^pU_p \to \mathcal{O}_E/\varpi_E\}$ for G as in the introduction. (ii) It is quite tantalizing to think of the "extra" constituents $C(s_\alpha, s_\alpha s_\beta)$ and $C(s_\beta, s_\beta s_\alpha)$ as being contained in the $\mathrm{GL}_3(\mathbb{Q}_p)$-socle of the locally analytic vectors of a mysterious continuous unitary Banach space representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ of "supercuspidal nature". Again, this fits with a similar observation on Serre weights in an analogous situation mod p.

Let us close these lectures with a few natural open questions.

(i) Can we extend these definitions and results to include more p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$? For instance some trianguline representations, or some representations for which Hypothesis 4.2.4 or 4.2.5 are not satisfied? Relaxing Hypothesis 4.2.4 on $\underline{D}$ means that there can exist lots of Deligne-Fontaine submodules $\underline{D}'$ inside $\underline{D}$ (eg. think about the case when φ has several equal eigenvalues). The weak admissibility conditions thus become more involved, and probably more delicate to "capture" on the $\mathrm{GL}_n(\mathbb{Q}_p)$-side.

(ii) Can we extend these constructions to more general reductive groups that GL_n? The answer is surely yes, however some complications will occur. For instance, the definition of $\pi(\underline{D}, \underline{h})$ in §4.2 uses the local Langlands correspondence for GL_n, which is a $1-1$ bijection. We know it is not the case in general for other groups as there are packets. So do we have to consider "packets" of $\pi(\underline{D}, \underline{h})$ (not to speak of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot}))$? Also, the definition of Deligne-Fontaine modules and Hodge filtrations should be adapted so that those which are weakly admissible correspond to Galois representations with values in the dual group.

(iii) In a global situation when $\underline{\mathrm{Fil}}^{\cdot}$ is weakly admissible, can we find all constituents of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ inside the correct Hecke or Galois isotopic subspace of some completed étale cohomology group? This should be related to the results on existence of overconvergent companion modular forms. Though we didn't define $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ for fields $L \neq \mathbb{Q}_p$ in these lectures, let us mention that Y. Ding has (ongoing) partial results along these lines in the case of $\mathrm{GL}_2(L)$ with L unramified over $\mathbb{Q}_p$ and a unitary Shimura curve, extending the method of $\mathrm{GL}_2(\mathbb{Q}_p)$ using p-adic comparison theorems. Once $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ will be defined for $\mathrm{GSp}_4(\mathbb{Q}_p)$ (see (ii)), one may also be able to extend this method to prove some partial results in that case (see e.g. work of Tilouine in the mod p case).

(iv) Conversely, still when we are in a global situation with $\underline{\mathrm{Fil}}^{\cdot}$ weakly admissible, can we prove that the $C(w^{\mathrm{alg}}, w)$ which are *not* in $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ do *not* either occur inside the corresponding Hecke/Galois isotopic subspace of the completed cohomology? This part is related to the classical statement that the existence of a companion form implies a splitting of the local Galois representation (or rather its contrapositive), which is usually easier to prove than the existence of a companion form when there is a splitting. For instance, if $\widehat{S}(U^p)[\pi]^{\mathrm{an}}$ is as in the introduction and if $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ corresponds to a $\underline{\mathrm{Fil}}^{\cdot}$ such that $\underline{\mathrm{Fil}}^{\cdot} \in w^{-1}\overline{B}(E)w_0\overline{B}(E)/\overline{B}(E)$ for a fixed $w \in \mathcal{S}_n$, then using results of Chenevier (plus some representation theory) one should be able to prove, at least under some regularity assumption on the eigenvalues of φ, that $C(w^{\mathrm{alg}}, w)$ is *never* in the socle of $\widehat{S}(U^p)[\pi]^{\mathrm{an}}$ unless $w^{\mathrm{alg}} = 1$ (see the proof of Proposition 5.2.5).

(v) Is the converse of Theorem 6.1.4 true? (This would be a generalization of Conjecture 4.2.8.) Certainly, if we want the constituents of $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ to occur in some completed étale cohomology group as soon as $\underline{D}$, $\underline{h}$ and $\underline{\mathrm{Fil}}^{\cdot}$ have a global origin (which in particular implies $\underline{\mathrm{Fil}}^{\cdot}$ weakly admissible), then $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ has an invariant lattice (as already mentioned): namely the one induced by the completed integral étale cohomology.

(vi) Finally, let me mention once again the following recurring issue: can we construct an explicit locally analytic representation of $\mathrm{GL}_n(\mathbb{Q})$ which contains $\pi(\underline{D}, \underline{h}, \underline{\mathrm{Fil}}^{\cdot})$ and which completely determines $\underline{\mathrm{Fil}}^{\cdot}$? This is presumably a (very) hard question, at least from a local point of view.