Eigenvarieties and the locally analytic Langlands program for GL_n

Christophe Breuil

Korea Institute for Advanced Study, April 2017

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Contents

1 Lecture 1: Global setting and main result

I first thank Chol Park for inviting me to the K.I.A.S. and giving me the opportunity to give these lectures.

In all the talks, p is a prime number, E is a finite extension of $\mathbb{Q}_p$ which is tacitly assumed "sufficiently large", $\mathcal{O}_E$ the ring of integers of E and k_E the residue field. I normalize local class field theory so that uniformizers correspond to geometric Frobeniuses. I denote by $\mathrm{unr}(a)$ the unramified character of $\mathbb{Q}_p^\times$ sending p to a and by ε the p-adic cyclotomic character (which has Hodge-Tate 1). For instance $|\ | = \mathrm{unr}(p^{-1})$ and $\varepsilon(x) = x|x|$.

The aim of these lectures is to explain a recent result, due to Eugen Hellmann, Benjamin Schraen and myself, unravelling a little bit the mysteries surrounding the (socle of the) locally $\mathbb{Q}_p$-analytic representations occuring in Hecke eigenspaces of the completed cohomology (for compact unitary groups).

1.1 Global setting

I fix a totally real number field F^+ and a quadratic totally imaginary extension F/F^+ where each place $v|p$ in F^+ splits in F. *For simplicity in these lectures* I will assume that p splits completely in F^+ (and thus in F) and denote by S_p the places of F^+ dividing p. I fix G a connected reductive algebraic group over F^+ which is an outer form of GL_n ($n \geq 2$) such that $G \times_{F^+} F \xrightarrow{\sim} \mathrm{GL}_{n/F}$ and $G \times_{F^+} F_v^+ \xrightarrow{\sim} U_n(\mathbb{R})$ for each infinite place v of F^+. I set $G_p := \prod_{v \in S_p} G(F_v^+) \cong \prod_{v \in S_p} \mathrm{GL}_n(\mathbb{Q}_p)$.

I fix a prime-to-p level $U^p = \prod_v U_v \subset G(\mathbb{A}_{F^+}^{p\infty})$ where U_v is a compact open subgroup of $G(F_v^+)$ and a finite extension E of $\mathbb{Q}_p$ (containing all Hecke eigenvalues I will consider). I denote by $\widehat{S}(U^p, E)$ the p-adic Banach space over E of continuous functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow E$ endowed with the left continuous action of G_p given by $(g'f)(g) := f(gg')$ (a special instance of Emerton's completed cohomology groups). The action of G_p preserves the unit ball given by the $\mathcal{O}_E$-submodule $\widehat{S}(U^p, \mathcal{O}_E)$ of continuous functions $G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow \mathcal{O}_E$, and $\widehat{S}(U^p, E)$ is called a unitary continuous representation of G_p. I denote by $\widehat{S}(U^p, E)^{\mathrm{an}} \subset \widehat{S}(U^p, E)$ the locally analytic representation of G_p defined as the E-subvector space of $\widehat{S}(U^p, E)$ of locally analytic vectors for the action of G_p, or equivalently the E-subvector space of locally analytic functions $G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^p \longrightarrow E$. This is an admissible locally analytic representation of G_p over E in the sense of Schneider and Teitelbaum.

Let $S = S(U^p)$ be the finite set of finite places of F^+ which is the union of

S_p and of the finite places $v \nmid p$ that split in F such that U_v is *not* maximal. For each $v \in S$, I choose $\tilde{v}$ one of the two places of F above v (this choice won't matter). One can define a commutative spherical Hecke $\mathcal{O}_E$-algebra $\mathbb{T} = \mathbb{T}(U^p)$ (a polynomial ring over $\mathcal{O}_E$ in the formal variables $T_w^{(j)}$ for $j \in \{1, \dots, n\}$ and w a finite place of F dividing a split place of F^+ not in S) which acts on $\widehat{S}(U^p, E)$ and $\widehat{S}(U^p, E)^{\mathrm{an}}$ and commutes with G_p. For any $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E)$ which is continuous and unramified outside the places above S one can associate a maximal ideal $\mathfrak{m}_\rho$ of $\mathbb{T}[1/p]$ by a standard recipee (looking at the characteristic polynomials of Frobenius at the above places w).

One far reaching aim of the locally analytic Langlands program is to describe the eigenspace $\widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ as a locally analytic representation of G_p (when it is non-zero). In particular we would wish to relate $\widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ to the local representations $\rho_{\tilde{v}} := \rho|_{\mathrm{Gal}(\overline{F}_{\tilde{v}}/F_{\tilde{v}})}$ for $v \in S_p$ (recall $\mathrm{Gal}(\overline{F}_{\tilde{v}}/F_{\tilde{v}}) = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ for $v \in S_p$). Note that the assumption $\widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho] \neq 0$ forces $\rho^c \cong \rho^\vee \otimes \varepsilon^{1-n}$ where $\rho^c(g) := \rho(cgc)$, c being the unique non-trivial element of $\mathrm{Gal}(F/F^+)$. This implies in particular $\rho_{\tilde{v}^c} \cong \rho_{\tilde{v}}^\vee \otimes \varepsilon^{1-n}$ if $\tilde{v}^c$ is the other place of F above $v \in S$, which is the reason why the choice of $\tilde{v}$ is harmless.

1.2 Socle conjecture

From now on, I assume that $\rho_{\tilde{v}}$ is crystalline for all $v \in S_p$. The G_p-representation $\widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ can be *a priori* quite complicated (e.g. it presumably can have even more constituents than Verma modules). As a first approximation, and by analogy with the mod p theory and the important works on Serre weight conjectures, I wish to understand its socle as a G_p-representation. More precisely, since the only irreducible admissible locally analytic representations of G_p we understand so far are irreducible locally algebraic representations and irreducible subquotients of locally analytic principal series[1], and since the locally algebraic constituents in the socle are already known by the classical theory, I wish to know for which irreducible subquotient C of a locally analytic principal series of G_p one has:

$$\mathrm{Hom}_{G_p}(C, \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]) \neq 0.$$

Let $\chi : T_p \longrightarrow E^\times$ be a locally analytic character of the diagonal torus T_p of G_p, that we see as a locally analytic character of $\overline{B}_p$ by inflation, where $\overline{B}_p \subset G_p$ is the subgroup of *lower* triangular matrices (this normalization turns out to be

[1]apart when $n = 2$ where other constituents can be constructed via (φ, Γ)-modules or via functions on the coverings of Drinfeld upper half plane by the work of Colmez and of Dospinescu-Le Bras on the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

convenient). Then the locally analytic principal series $(\mathrm{Ind}_{\overline{B}_p}^{G_p}\chi)^{\mathrm{an}}$ is:

$$\big(\mathrm{Ind}_{\overline{B}_p}^{G_p}\chi\big)^{\mathrm{an}}:=\{f:G_p\to E \text{ locally analytic, } f(bg)=\chi(b)f(g)\ \forall\, b\in\overline{B}_p, g\in G_p\}$$

with the left action of G_p given by $(g'f)(g):=f(gg')$. The irreducible constituents of $\big(\mathrm{Ind}_{\overline{B}_p}^{G_p}\chi\big)^{\mathrm{an}}$ have been described by Orlik and Strauch using the theory of Verma modules.

I now moreover assume that ρ is automorphic and absolutely irreducible. This implies in particular that each (crystalline) $\rho_{\tilde{v}}$ for $v\in S_p$ has distinct Hodge-Tate weights $h_{\tilde{v},1}<\cdots<h_{\tilde{v},n}$. I finally moreover assume that for each $v\in S_p$ the eigenvalues $\varphi_{\tilde{v},i}\in E$ of the crystalline Frobenius on $D_{\mathrm{cris}}(\rho_{\tilde{v}})=(\mathrm{B}_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}\rho_{\tilde{v}})^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ are all distinct and such that $\frac{\varphi_{\tilde{v},i}}{\varphi_{\tilde{v},j}}\neq p$ for all i,j.

For $v\in S_p$ I denote by $\mathcal{R}_v$ an ordering $\varphi_{\tilde{v},j_1},\ldots,\varphi_{\tilde{v},j_n}$ of the above eigenvalues and I set $\mathcal{R}:=(\mathcal{R}_v)_{v\in S_p}$ ($\mathcal{R}$ for refinement) and $\lambda_{\tilde{v},i}:=h_{\tilde{v},n+1-i}+i-1$ (so that $\lambda_{\tilde{v},1}\geq\cdots\geq\lambda_{\tilde{v},n}$). For $\mathcal{R}:=(\mathcal{R}_v)_{v\in S_p}$ I define the smooth character $\delta_{\mathcal{R}}=(\delta_{\mathcal{R}_v})_{v\in S_p}$ of T_p where $\delta_{\mathcal{R}_v}$ for each v is the following smooth character of the diagonal torus T_v of $G(F_v^+)\cong\mathrm{GL}_n(F_{\tilde{v}})=\mathrm{GL}_n(\mathbb{Q}_p)$:

$$\mathrm{unr}(\varphi_{\tilde{v},j_1})\otimes\mathrm{unr}(\varphi_{\tilde{v},j_2})|\ |\otimes\cdots\otimes\mathrm{unr}(\varphi_{\tilde{v},j_n})|\ |^{n-1}.$$

For $w=(w_v)_{v\in S_p}\in\mathcal{S}_n^{|S_p|}$ (the Weyl group of G_p) I define the algebraic character $ww_0\cdot\lambda=(w_vw_{v,0}\cdot\lambda_{\tilde{v}})_{v\in S_p}$ of T_p where $w_vw_{v,0}\cdot\lambda_{\tilde{v}}$ for each v is the following algebraic character of T_v:

$$\mathrm{diag}(t_1,\ldots,t_n)\longmapsto t_1^{h_{\tilde{v},w_v^{-1}(1)}}t_2^{h_{\tilde{v},w_v^{-1}(2)}+1}\cdots t_n^{h_{\tilde{v},w_v^{-1}(n)}+n-1}$$

(the notation comes from the fact that this is indeed $ww_0\cdot\lambda$ for $\cdot$ the dot action with respect to the upper Borel B_p and $w_0=(w_{v,0})_{v\in S_p}$ the longest element of $\mathcal{S}_n^{|S_p|}$). Then for each $\mathcal{R}$ and each $w\in\mathcal{S}_n^{|S_p|}$, I finally define the following locally analytic constituent:

$$C_{\mathcal{R},w}:=\mathrm{socle}\big(\mathrm{Ind}_{\overline{B}_p}^{G_p}(ww_0\cdot\lambda)\delta_{\mathcal{R}}\big)^{\mathrm{an}}$$

which turns out to be irreducible (using Orlik-Strauch's theory). For a fixed $\mathcal{R}$, the $C_{\mathcal{R},w}$ are all distinct. But for a fixed w, they are not all distinct, for instance the $C(\mathcal{R},w_0)$ are all isomorphic.

The refinement $\mathcal{R}$ defines for each $v\in S_p$ a unique Frobenius stable flag on $D_{\mathrm{cris}}(\rho_{\tilde{v}})$. Taking the induced Hodge filtration on this flag in turn determines a unique ordering $h_{w_{\mathcal{R}_v}^{-1}(1)},\ldots,h_{w_{\mathcal{R}_v}^{-1}(n)}$ of the Hodge-Tate weights of $\rho_{\tilde{v}}$ for some unique $w_{\mathcal{R}_v}\in\mathcal{S}_n$. I set $w_{\mathcal{R}}:=(w_{\mathcal{R}_v})_{v\in S_p}\in\mathcal{S}_n^{|S_p|}$. For instance, when the Hodge filtration is very generic for each $v\in S_p$, one finds $w_{\mathcal{R}}=w_0$.

Conjecture 1.2.1. *We have* $\mathrm{Hom}_{G_p}\left(C_{\mathcal{R},w}, \widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]\right) \neq 0$ *if and only if* $w_{\mathcal{R}} \preceq w$ *where* $\preceq$ *is the Bruhat order on* $\mathcal{S}_n^{|S_p|}$.

This statement is a quantitative version of the following qualitative statement: "the more the Hodge filtration is degenerate, the more constituents appear in the socle". For instance when the Hodge filtration is very generic for each $v \in S_p$, by which I now mean $w_{\mathcal{R}} = w_0$, Conjecture 1.2.1 only predicts the constituent $C(\mathcal{R}, w_0)$. This is a well-known theorem in that case: (i) the constituent $C(\mathcal{R}, w_0)$ is locally algebraic (the only such one among the $C(\mathcal{R}, w)$) and its existence easily follows from the automorphy of ρ whatever $w_{\mathcal{R}}$ is, (ii) the fact that no other $C(\mathcal{R}, w)$ occurs when $w_{\mathcal{R}} = w_0$ is a consequence of a result of Chenevier (or of more recent results of Kedlaya-Pottharst-Xiao/Liu).

Remark 1.2.2. I *do not* conjecture that the $C_{\mathcal{R},w}$ for $w_{\mathcal{R}} \preceq w$ exhaust the socle of $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$, as the latter could contain some unknown constituents of "supercuspidal nature" (even in this crystalline case). Note also that the socle is not known so far to be of finite length.

1.3 Main result and beginning of the proof

The main result is a proof of Conjecture 1.2.1 under several extra assumptions, which essentially come from the Taylor-Wiles method. Recall that $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E)$ is automorphic of prime-to-p level U^p (which *a fortiori* implies $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho] \neq 0$) and crystalline at $v \in S_p$ with $\frac{\varphi_{\tilde{v},i}}{\varphi_{\tilde{v},j}} \notin \{1, p\}$ for $i \neq j$.

Theorem 1.3.1. *Assume the following extra assumptions:*

(i) $p > 2$ *and* F/F^+ *is unramified;*

(ii) G *is quasi-split at all finite places of* F^+*;*

(iii) ρ *is residually absolutely irreducible and* $\overline{\rho}(\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1})))$ *is adequate;*

(iv) U^p *is small enough and* U_v *is maximal hyperspecial when* v *is inert in* F.

Then Conjecture 1.2.1 is true.

The case $n = 2$ of this theorem was already known and due to Yiwen Ding.

Remark 1.3.2. One can also prove the following strengthening of Theorem 1.3.1 (under the same assumptions): the irreducible constituents of locally analytic principal series of G_p which appear in the socle of $\widehat{S}(U^p,E)^{\mathrm{an}}[\mathfrak{m}_\rho]$ are the $C_{\mathcal{R},w}$

for all refinements $\mathcal{R}$ and all w such that $w_{\mathcal{R}} \preceq w$ (i.e. such constituents are automatically of the form $C_{\mathcal{R},w}$ for some $\mathcal{R}$ and some $w \in \mathcal{S}_n^{|S_p|}$). Due to limitation in the present status of Orlik-Strauch's theory, this strengthening so far requires that all local fields F_v^+ for $v \in S_p$ are equal to $\mathbb{Q}_p$ (which is our case, but here it is not just for simplicity anymore!), though it is undoubtedly true without this assumption.

I now start to sketch the proof of Theorem 1.3.1, that I have divided into several steps. This proof, and the material it requires, will occupy us until the end of these lectures. I adopt the following policy: technical complications which are not crucial to the proof but could obscure its understanding will be overlooked. I will explicitly mention in the course of the proof what is overlooked and from where it starts to be. In practice this means that several statements in what follows are (strictly speaking) wrong as stated, so be careful if you use them!

Step 1: The patched G_p-representation Π_∞.

Let $\mathfrak{m}_{\overline{\rho}}$ be the maximal ideal of $\mathbb{T}$ of residue field k_E associated to $\overline{\rho}$ (the irreducible mod p reduction of ρ) and $\widehat{S}(U^p, E)^{\mathrm{an}}_{\mathfrak{m}_{\overline{\rho}}}$ the locally analytic vectors of the p-adic completion of the corresponding localization. Let $R_{\overline{\rho},S}$ be the noetherian complete local $\mathcal{O}_E$-algebra of residue field k_E pro-representing the functor of (usual) deformations of $\overline{\rho}$ that are unramified outside S and conjugate self-dual (for instance ρ). Then $R_{\overline{\rho},S}$ naturally acts on $\widehat{S}(U^p, E)^{\mathrm{an}}_{\mathfrak{m}_{\overline{\rho}}}$ through a certain reduced quotient $R_{\overline{\rho},\mathcal{S}}$. Let R^{loc} be the reduced framed deformation ring at places $v \in S$, that is, $R^{\mathrm{loc}} := \widehat{\otimes}_{v\in S} R_{\overline{\rho}_{\tilde{v}}}$ where $R_{\overline{\rho}_{\tilde{v}}}$ is the reduced framed local deformation ring over $\mathcal{O}_E$ of $\overline{\rho}_{\tilde{v}} := \overline{\rho}|_{\mathrm{Gal}(\overline{F}_{\tilde{v}}/F_{\tilde{v}})}$. By the universal property of $R_{\overline{\rho}_{\tilde{v}}}$ there is a canonical map of local rings $R^{\mathrm{loc}} \to R_{\overline{\rho},\mathcal{S}}$.

Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin applied a generalization of Taylor-Wiles-Kisin's patching method (by patching also at places above p) to produce a patched locally analytic representation Π_∞^{an} of G_p over E which is a module over $R_\infty := R^{\mathrm{loc}}[[x_1, \ldots, x_g]]$ for some integer $g \geq 1$ such that $\Pi_\infty^{\mathrm{an}}[I] \cong \widehat{S}(U^p, E)^{\mathrm{an}}_{\mathfrak{m}_{\overline{\rho}}}$ as $R_\infty[G_p]$-modules where I is the kernel of a surjection $R_\infty[1/p] \twoheadrightarrow R_{\overline{\rho},\mathcal{S}}[1/p]$ compatible with $R^{\mathrm{loc}} \to R_{\overline{\rho},\mathcal{S}}$. This is essentially the place were all hypothesis (i) to (iv) in the main result above are used. Let $\mathfrak{m}_{\infty,\rho} := \ker(R_\infty[1/p] \twoheadrightarrow R_{\overline{\rho},\mathcal{S}}[1/p] \twoheadrightarrow E)$ where the last surjection is given by the deformation ρ, since:

$$\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_{\infty,\rho}] = \Pi_\infty^{\mathrm{an}}[I][\mathfrak{m}_{\infty,\rho}] \cong \widehat{S}(U^p, E)^{\mathrm{an}}_{\mathfrak{m}_{\overline{\rho}}}[\mathfrak{m}_\rho] \cong \widehat{S}(U^p, E)^{\mathrm{an}}[\mathfrak{m}_\rho]$$

it is enough to prove $\mathrm{Hom}_{G_p}\left(C_{\mathcal{R},w}, \Pi_\infty^{\mathrm{an}}[\mathfrak{m}_{\infty,\rho}]\right) \neq 0$ if and only if $w_{\mathcal{R}} \preceq w$.

I now write $\mathfrak{m}_\rho$ instead of $\mathfrak{m}_{\infty,\rho}$ (and forget the previous $\mathfrak{m}_\rho$). I set $\mathfrak{X}_\infty := (\mathrm{Spf}\ R_\infty)^{\mathrm{rig}}$ (generic fiber à la Raynaud of the formal scheme Spf R_∞) and denote by $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}$ the (underlying ring of the) completion of $\mathfrak{X}_\infty$ at the point defined by ρ. Then:

$$\mathcal{L}_{\mathcal{R},w} := \varprojlim_m \mathrm{Hom}_{G_p}\left(C_{\mathcal{R},w}, \Pi_\infty^{\mathrm{an}}[\mathfrak{m}_\rho^m]\right)^\vee$$

where $(-)^\vee$ is the dual (each $\mathrm{Hom}_{G_p}(C_{\mathcal{R},w}, \Pi_\infty^{\mathrm{an}}[\mathfrak{m}_\rho^m])$ is finite dimensional over E) is an $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}$-module via the action of R_∞ on each $\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_\rho^m]$. Since an eigenspace is non-zero if and only if the corresponding generalized eigenspace is non-zero, we see that it is equivalent to prove that the $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}$-module $\mathcal{L}_{\mathcal{R},w}$ is non-zero if and only if $w_{\mathcal{R}} \preceq w$.

2 Lecture 2: Eigenvarieties

I go on with the next steps in the (sketch of the) proof of Theorem 1.3.1, which crucially involve eigenvarieties.

Step 2: An equality of cycles.

Consider the locally analytic T_p-representation $J_{B_p}(\Pi_\infty^{\mathrm{an}})$ where J_{B_p} is Emerton's locally analytic Jacquet functor with respect to $B_p :=$ upper triangular matrices in G_p. It follows from the admissibility of Π_∞^{an} that the continuous dual of $J_{B_p}(\Pi_\infty^{\mathrm{an}})$ is the global section of a coherent sheaf $\mathcal{M}_\infty$ on the quasi-Stein reduced rigid analytic space $\mathfrak{X}_\infty \times \widehat{T}_p$ where $\widehat{T}_p$ is the rigid space over E parametrizing locally analytic characters of T_p. The schematic support of $\mathcal{M}_\infty$ turns out to be a reduced equidimensional Zariski-closed subvariety of $\mathfrak{X}_\infty \times \widehat{T}_p$ that I denote by $X_p(\overline{\rho})$. For any locally analytic character $\chi \in \widehat{T}_p$, I denote by $X_p(\overline{\rho})_\chi$ the fiber of $X_p(\overline{\rho})$ above χ. We have a closed immersion $X_p(\overline{\rho})_\chi \hookrightarrow \mathfrak{X}_\infty$.

For any locally analytic character $\chi : T_p \longrightarrow E^\times$, we have that:

$$\varprojlim_m \mathrm{Hom}_{T_p}\left(\chi, J_{B_p}(\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_\rho^m])\right)^\vee$$

(a projective limit of finite dimensional E-vector spaces with surjective transition maps) is non-zero if and only if $\mathrm{Hom}_{T_p}(\chi, J_{B_p}(\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_\rho]))$ is non-zero if and only if $(\rho, \chi) \in X_p(\overline{\rho})$, and is isomorphic to $\widehat{(\mathcal{M}_{\infty,\chi})}_{(\rho,\chi)}$:= completion at $(\rho, \chi) \in X_p(\overline{\rho})_\chi$ of the pull-back of $\mathcal{M}_\infty$ on $X_p(\overline{\rho})_\chi$ (this completion being zero when $(\rho, \chi) \notin X_p(\overline{\rho})$). Applying this to $\chi_{\mathcal{R},w} := (ww_0 \cdot \lambda)\delta_{\mathcal{R}}$ for any refinement $\mathcal{R}$ and any $w \in \mathcal{S}_n^{|S_p|}$ as in Lecture 1, I get finite type $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho,\chi_{\mathcal{R},w})}$-modules $\mathcal{M}_{\mathcal{R},w}$ where $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho,\chi_{\mathcal{R},w})}$ is the completed local ring of $X_p(\overline{\rho})_{\chi_{\mathcal{R},w}}$ at $(\rho, \chi_{\mathcal{R},w})$. Note that $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho} \twoheadrightarrow \widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho,\chi_{\mathcal{R},w})}$.

Using Orlik-Strauch's theory, one can define a finite length locally analytic representation (see e.g. my 2013 lectures at Beijing):

$$\mathcal{F}_{\overline{B}_p}^{G_p}\left(U(\mathfrak{g}_p) \otimes_{U(\overline{\mathfrak{b}}_p)} -ww_0 \cdot \lambda, \delta_{\mathcal{R}}\right)$$

(here $\mathfrak{g}_p$, $\overline{\mathfrak{b}}_p$ are the respective Lie algebras of G_p, $\overline{B}_p$ and $U(-)$ their enveloping algebra, so $U(\mathfrak{g}_p) \otimes_{U(\overline{\mathfrak{b}}_p)} -ww_0 \cdot \lambda$ is a Verma module with integral weights) which has exactly the same irreducible constituents as $(\mathrm{Ind}_{\overline{B}_p}^{G_p}(ww_0 \cdot \lambda)\delta_{\mathcal{R}})^{\mathrm{an}}$ but in the "reverse order". These constituents are precisely the $\mathcal{C}_{\mathcal{R},w'}$ for $w' \preceq w$ and from the theory of Verma modules one gets that $\mathcal{C}_{\mathcal{R},w'}$ occurs with multiplicity

$P_{w_0w,w_0w'}(1) \in \mathbb{Z}_{\geq 1}$ where the $P_{x,y} \in \mathbb{Z}_{\geq 0}[q]$ for $x \preceq y$ are the Kazhdan-Lusztig polynomials.

For $d \in \mathbb{Z}_{\geq 0}$ let $\mathrm{Z}^d(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho})$ be the free abelian group generated by the irreducible closed subschemes of codimension d in Spec $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}$. If $\mathcal{E}$ is any finite type $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}$-module such that its support has codimension $\geq d$, set:

$$[\mathcal{E}] := \sum_Z m(Z,\mathcal{E})[Z] \in \mathrm{Z}^d(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho})$$

where the sum runs over all irreducible subschemes Z of codimension d and $m(Z,\mathcal{E}) \in \mathbb{Z}_{\geq 0}$ is the length of the $(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho})_{\eta_Z}$-module $\mathcal{E}_{\eta_Z}$, η_Z being the generic point of Z. For instance we can apply this to $d = [F^+ : \mathbb{Q}]\frac{n(n+3)}{2}$ and $\mathcal{E} = \mathcal{M}_{\mathcal{R},w}$ (which can be proven to have support of codimension $\geq [F^+ : \mathbb{Q}]\frac{n(n+3)}{2}$).

There is an adjonction formula:

$$\mathrm{Hom}_{G_p}\left(\mathcal{F}_{\overline{B}_p}^{G_p}(U(\mathfrak{g}_p)\otimes_{U(\bar{\mathfrak{b}}_p)} -ww_0\cdot\lambda, \delta_{\mathcal{R}}), \Pi_\infty^{\mathrm{an}}[\mathfrak{m}_\rho^m]\right) \cong \mathrm{Hom}_{T_p}\left(\chi_{\mathcal{R},w}, J_{B_p}(\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_\rho^m])\right)$$

from which, taking duals, $\varprojlim_m$ and making dévissage, one can deduce that each $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}$-module $\mathcal{L}_{\mathcal{R},w'}$ (see Step 1) for $w' \preceq w$ is of finite type with support of codimension $\geq [F^+ : \mathbb{Q}]\frac{n(n+3)}{2}$ and that:

$$[\mathcal{M}_{\mathcal{R},w}] = \sum_{w' \preceq w} P_{w_0w,w_0w'}(1)[\mathcal{L}_{\mathcal{R},w'}] \ \text{ in } \ \mathrm{Z}^{[F^+:\mathbb{Q}]\frac{n(n+3)}{2}}(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}).$$

In fact, this is not quite true. To have exact sequences in the dévissage, and thus the above formula, one also needs to take everywhere the generalized eigenspace on the smooth part $\delta_{\mathcal{R}}$. However this technical point is not crucial in understanding the proof and I forget it here (as I explained in Lecture 1).

This finishes Step 2. Recall I want to prove $\mathcal{L}_{\mathcal{R},w'} \neq 0$ if and only if $w_{\mathcal{R}} \preceq w'$. But there is no direct way to see which $\mathcal{L}_{\mathcal{R},w'}$ are non-zero. For this, I will first need to introduce a new tool: a more tractable purely local version of the patched eigenvariety $X_p(\overline{\rho})$ (which doesn't involve any patching).

Step 3: Link with the trianguline variety.

For $v \in S_p$ I set $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}} := (\mathrm{Spf}\ R_{\overline{\rho}_{\tilde{v}}})^{\mathrm{rig}}$ and $\widehat{T}_v$ the rigid space parametrizing locally analytic characters on T_v. I set:

$$\widehat{T}_{v,\mathrm{reg}} := \{\delta = \delta_1 \otimes \cdots \otimes \delta_n,\ \tfrac{\delta_i}{\delta_j} \notin \{x \mapsto x^{-m}, x \mapsto x^m\varepsilon(x),\ m \in \mathbb{Z}_{\geq 0}\} \text{ for } i \neq j\}$$

(a Zariski open subset of $\widehat{T}_v$) and define $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}}) \subset \mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_v$ as the reduced Zariski-closure of the points $(r,\delta) \in \mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_{v,\mathrm{reg}}$ such that r is trianguline (in the sense of Colmez) and the locally analytic character δ comes from a triangulation on $D_{\mathrm{rig}}(r)$. I call these points on $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ *saturated points*. Here, recall that $D_{\mathrm{rig}}(r)$ is the étale (φ,Γ)-module over the Robba ring associated to r by the work of Fontaine, Cherbonnier, Colmez, ... and that a triangulation - or equivalently a flag by not-necessarily étale sub-(φ,Γ)-modules that are direct summands as modules - gives rise to a locally analytic character δ (essentially) by local class field theory. It turns out that $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ is equidimensional, and that many new non-saturated points (r,δ) arise in this Zariski-closure. More precisely it follows from work of Kedlaya-Pottharst-Xiao or Liu that any r that appears is still trianguline, but that δ need not come from a triangulation on $D_{\mathrm{rig}}(r)$.

I set $R_{\overline{\rho}^p} := \widehat{\otimes}_{v\in S\backslash S_p} R_{\overline{\rho}_{\tilde{v}}}$, $\mathfrak{X}_{\overline{\rho}^p} := (\mathrm{Spf}\ R_{\overline{\rho}^p})^{\mathrm{rig}}$, $\mathbb{U}^g := (\mathrm{Spf}\ \mathcal{O}_E[[x_1,\dots,x_g]])^{\mathrm{rig}}$ and I recall that there is a closed immersion:

$$X_p(\overline{\rho}) \hookrightarrow \mathfrak{X}_\infty \times \widehat{T}_p \cong \Big(\prod_{v\in S_p} \mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_v\Big) \times \mathfrak{X}_{\overline{\rho}^p} \times \mathbb{U}^g.$$

Using that crystalline points with a very generic Hodge filtration (in the sense of Lecture 1) are Zariski-dense in $X_p(\overline{\rho})$ and that the image of those points on the right hand side lie in $(\prod_{v\in S_p} X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})) \times \mathfrak{X}_{\overline{\rho}^p} \times \mathbb{U}^g$, one can deduce by density that the above closed immersion factors as a closed immersion:

$$X_p(\overline{\rho}) \hookrightarrow \Big(\prod_{v\in S_p} X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})\Big) \times \mathfrak{X}_{\overline{\rho}^p} \times \mathbb{U}^g.$$

There is a small technical point here: in order to see points of $X_p(\overline{\rho})$ as points of the right hand side, one actually needs to make the following shift on $\widehat{T}_v$ for each $v \in S_p$:

$$\chi_1 \otimes \cdots \otimes \chi_n \longmapsto \chi_1 \otimes \chi_2\varepsilon^{-1} \otimes \cdots \otimes \chi_n\varepsilon^{-(n-1)}.$$

Set $X_{\mathrm{tri}}(\overline{\rho}_p) := \prod_{v\in S_p} X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$, it can be checked that the two equidimensional reduced rigid spaces $X_p(\overline{\rho})$ and $X_{\mathrm{tri}}(\overline{\rho}_p) \times \mathfrak{X}_{\overline{\rho}^p} \times \mathbb{U}^g$ have the same dimension, and thus $X_p(\overline{\rho})$ is a union of irreducible components of $X_{\mathrm{tri}}(\overline{\rho}_p) \times \mathfrak{X}_{\overline{\rho}^p} \times \mathbb{U}^g$. Since $\mathfrak{X}_{\overline{\rho}^p} \times \mathbb{U}^g$ never plays a key role in the rest of the proof, I will forget it from now on and do as if $X_p(\overline{\rho})$ is a union of irreducible components of $X_{\mathrm{tri}}(\overline{\rho}_p)$. It can be checked that the (usual) automorphy conjectures in all Hodge-Tate weights imply that $X_p(\overline{\rho})$ should be exactly the union of those irreducible components of $X_{\mathrm{tri}}(\overline{\rho}_p)$ containing a saturated crystalline point, but fortunately we won't need this in the sequel.

I would like to emphasize that the two definitions of $X_p(\overline{\rho})$ and $X_{\mathrm{tri}}(\overline{\rho}_p)$ are very different: the first one is the support of the coherent sheaf defined by applying the Jacquet-Emerton functor to the locally analytic representation Π_∞^{an} of G_p, whereas the second one is the Zariski-closure of triangulations on (φ,Γ)-modules of n-dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. There is so far (unfortunately!) no purely local analogue of the locally analytic representation Π_∞^{an} from which one could try to obtain $X_{\mathrm{tri}}(\overline{\rho}_p)$ by the Jacquet-Emerton functor method (except maybe for $\mathrm{GL}_2(\mathbb{Q}_p)$).

Since I decided to forget $\mathfrak{X}_{\overline{\rho}^p} \times \mathbb{U}^g$, I have now to replace $\mathfrak{X}_\infty$ by $\mathfrak{X}_{\overline{\rho}_p} := \prod_{v\in S_p} \mathfrak{X}_{\overline{\rho}_{\tilde{v}}}$ and $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty,\rho}$ by $\widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_p},\rho_p}$ where $\rho_p := (\rho_{\tilde{v}})_{v\in S_p}$. One idea in order to understand the $\widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_p},\rho_p}$-modules $\mathcal{L}_{\mathcal{R},w'}$ of Step 1 is to look for a purely *local* Galois-theoretic definition of $\widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_p},\rho_p}$-modules (for instance coming from $X_{\mathrm{tri}}(\overline{\rho}_p)$) which would be more tractable than the $\mathcal{L}_{\mathcal{R},w'}$ and hopefully ultimately equal (the hope being that all these constructions from the patched Π_∞^{an} should anyway be essentially local at p). What I will do is close to that: I will give a purely local definition of *cycles* $\mathcal{C}_{\mathcal{R},w'} \in \mathrm{Z}^{[F^+:\mathbb{Q}]\frac{n(n+3)}{2}}(\mathrm{Spec}\, \widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_p},\rho_p})$ and prove that $[\mathcal{L}_{\mathcal{R},w'}]$ and $\mathcal{C}_{\mathcal{R},w'}$ are closely related. But before going into this, I need material concerning (i) companion points on $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ and (ii) the local description of $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ around companion points.

Step 4: Local companion points.

Fix $v \in S_p$. For any ordering $\mathcal{R}_v$ as in Lecture 1 and any $w_v \in \mathcal{S}_n$, let $\delta_{\mathcal{R}_v,w_v} \in \widehat{T}_v$ be the shift of $(w_v w_{v,0}\cdot\lambda_{\tilde{v}})\delta_{\mathcal{R}_v}$ defined in Step 3 and set:

$$x_{\mathcal{R}_v,w_v} := (\rho_{\tilde{v}}, \delta_{\mathcal{R}_v,w_v}) \in \mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_v$$

(see Lecture 1 for the notation). We know that $x_{\mathcal{R}_v,w_{\mathcal{R}_v}} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}}) \subset \mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_v$ since by definition of $w_{\mathcal{R}_v}$ it is a saturated point (the Frobenius stable flag on $D_{\mathrm{cris}}(\rho_{\tilde{v}})$ defined by $\mathcal{R}_v$ gives rise to a triangulation on $D_{\mathrm{rig}}(\rho_{\tilde{v}})$ which, unravelling the definitions, gives rise to $\delta_{\mathcal{R}_v,w_v}$). But it is not clear which other $x_{\mathcal{R}_v,w_v}$ for $w_v \in \mathcal{S}_n$ belong to $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$.

Let $R_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}}$ be the reduced quotient of $R_{\overline{\rho}_{\tilde{v}}}$ parametrizing (framed) cristalline deformations of $\overline{\rho}_{\tilde{v}}$ with Hodge-Tate weights $h_{\tilde{v},i}$. Set $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}} := (\mathrm{Spf}\ R_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}})^{\mathrm{rig}}$ and:

$$\widetilde{\mathfrak{X}}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}} := \mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}} \times_{T^{\mathrm{rig}}/\mathcal{S}_n} T^{\mathrm{rig}}$$

where $T^{\mathrm{rig}} \cong (\mathbb{G}_{\mathrm{m}}^{\mathrm{rig}})^n$ is the rigid diagonal torus (over E) and the morphism $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}} \to T^{\mathrm{rig}}/\mathcal{S}_n \cong \mathbb{A}_E^{n,\mathrm{rig}}$ sends a crystalline deformation to the coefficients of the characteristic polynomial of its crystalline Frobenius. One can check that $\widetilde{\mathfrak{X}}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}}$ is a reduced (equidimensional) rigid space.

There is a morphism $\widetilde{\mathfrak{X}}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}} \to (\mathrm{GL}_n/B)^{\mathrm{rig}}$ given by the Hodge filtration (in fact, this morphism is only defined locally, but this is enough for what follows and I overlook that). For $w_v \in \mathcal{S}_n$, the inverse image of the Bruhat cell $(Bw_vB/B)^{\mathrm{rig}}$ in $\widetilde{\mathfrak{X}}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}}$ via this morphism can be embedded into the *saturated* locus of $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$

(this embedding depends on w_v). Since $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ is closed in $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_v$, the Zariski-closure in $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_v$ of the inverse image of $(Bw_vB/B)^{\mathrm{rig}}$ still lies in $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$, but now contains new non-saturated points. Using that the Zariski-closure of $(Bw_vB/B)^{\mathrm{rig}}$ in $(\mathrm{GL}_n/B)^{\mathrm{rig}}$ is $\cup_{w'_v \preceq w_v}(Bw'_vB/B)^{\mathrm{rig}}$, one can explicitely describe these non-saturated points (the new positions of the Hodge filtration is given by the w'_v, and it is not saturated when $w'_v \neq w_v$). We deduce in particular that the points $x_{\mathcal{R}_v,w_v}$ for $w_{\mathcal{R}_v} \preceq w_v$ are in $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ ($w_{\mathcal{R}_v}$ is one of the w'_v).

Fixing $\mathcal{R}_v$, I call *companion point* (implicitly of $x_{\mathcal{R}_v,w_{v,0}}$) any point $x_{\mathcal{R}_v,w_v} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$. From what I just proved we have the companion points $x_{\mathcal{R}_v,w_v}$ for $w_{\mathcal{R}_v} \preceq w_v$ but we do not know so far that there can't be some other $x_{\mathcal{R}_v,w_v} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ for some other w_v not satisfying $w_{\mathcal{R}_v} \preceq w_v$. This is true, but the proof of this fact is surprisingly difficult and will follow from the next lecture (no direct proof is known when $n > 3$).

To make a link with Theorem 1.3.1, note that the companion points $x_{\mathcal{R}_v,w_v}$ for $w_{\mathcal{R}_v} \preceq w_v$ (for each $v \in S_p$) can be seen as a "sign" that the constituents $C_{\mathcal{R},w}$ for $w_{\mathcal{R}} \preceq w$ are in $\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_{\rho_p}]$. Indeed, assuming this is the case, then $J_{B_p}(C_{\mathcal{R},w}) \subseteq J_{B_p}(\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_{\rho_p}])$ and since $\chi_{\mathcal{R},w} = (ww_0 \cdot \lambda)\delta_{\mathcal{R}} \subseteq J_{B_p}(C_{\mathcal{R},w})$, we deduce $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$, and from the embedding $X_p(\overline{\rho}) \hookrightarrow X_{\mathrm{tri}}(\overline{\rho}_p)$ of Step 3 that all the above $x_{\mathcal{R}_v,w_v}$ should indeed be there. Unfortunately, it is not possible to go back in general because (i) we don't know *a priori* that the points $(x_{\mathcal{R}_v,w_v})_{v\in S_p} \in X_{\mathrm{tri}}(\overline{\rho}_p)$ lie in the subspace $X_p(\overline{\rho})$ and (ii) even if they do, this wouldn't imply that they come from embeddings $C_{\mathcal{R},w} \hookrightarrow \Pi_\infty^{\mathrm{an}}[\mathfrak{m}_{\rho_p}]$.

3 Lecture 3: A formal local model

I continue the proof of Theorem 1.3.1. One key result is that it turns out one can describe the local geometry of $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ at a companion point $x_{\mathcal{R}_v,w_v}$, more precisely one can describe the formal completion $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}}$ of $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ at $x_{\mathcal{R}_v,w_v}$. But in order to do so, I need to recall material from geometric representation theory.

Step 5: Grothendieck's simultaneous resolution.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ the respective Lie algebras of GL_n, B (upper Borel) and T (diagonal torus) that we see as affine schemes over E and recall that GL_n/B is the moduli space of complete flags $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$. I define the following subscheme of $\mathrm{GL}_n/B \times \mathfrak{g}$:

$$\widetilde{\mathfrak{g}} := \{(\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n, \mathfrak{u}) \in \mathrm{GL}_n/B \times \mathfrak{g},\ \mathfrak{u}(\mathcal{F}_i) \subseteq \mathcal{F}_i\}$$

or equivalently:

$$\widetilde{\mathfrak{g}} = \{(gB, \mathfrak{u}) \in \mathrm{GL}_n/B \times \mathfrak{g},\ g^{-1}\mathfrak{u}g \in \mathfrak{b}\}.$$

This is a smooth irreducible scheme over E and the natural projection $\widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is called *Grothendieck's simultaneous resolution* (though $\mathfrak{g}$ is already smooth).

More interesting is the fiber product $X := \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ over the natural projection. Its points are triples $(g_1B, g_2B, \mathfrak{u})$ where $\mathfrak{u}$ respects the two flags associated to g_1B and g_2B (i.e. $g_1^{-1}\mathfrak{u}g_1 \in \mathfrak{b}$ and $g_2^{-1}\mathfrak{u}g_2 \in \mathfrak{b}$). It is reduced but not irreducible anymore: one can check that $X = \amalg_{w\in\mathcal{S}_n} V_w = \cup_{w\in\mathcal{S}_n} X_w$ where X_w is the Zariski-closure of the locus $V_w := \{(gB, gwB, \mathfrak{u}) \in X\}$ where the two flags are in relative position w. The X_w have the same dimension $(= n^2)$ and are known to be Cohen-Macaulay (a result of Bezrukavnikov-Riche). One can moreover prove that they are normal (by proving that their singularities have codimension ≥ 2 and applying Serre's criterion since they are Cohen-Macaulay). Hence they are also locally irreducible. This local irreducibility will be a key result.

The natural projection $X \longrightarrow \mathrm{GL}_n/B \times \mathrm{GL}_n/B$ (coming from the projections $\widetilde{\mathfrak{g}} \to \mathrm{GL}_n/B$ of the two copies of $\widetilde{\mathfrak{g}}$) sends surjectively V_w to the Bruhat cell $U_w := \{(gB, gwB) \in \mathrm{GL}_n/B \times \mathrm{GL}_n/B\}$ and X_w to the Zariski-closure of U_w in $\mathrm{GL}_n/B \times \mathrm{GL}_n/B$ which is known to be:

$$\bigcup_{w' \preceq w} \{(gB, gw'B) \in \mathrm{GL}_n/B \times \mathrm{GL}_n/B\}.$$

In particular if a point in X_w also belongs to some $V_{w'}$ then this implies $w' \preceq w$. This, again, will be quite important.

Step 6: Another equality of cycles.

There is a surjection $\widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{t}$ given by $(gB, \mathfrak{u}) \mapsto \overline{g^{-1}\mathfrak{u}g}$ where $\overline{g^{-1}\mathfrak{u}g}$ is the image of $g^{-1}\mathfrak{u}g \in \mathfrak{b}$ in the quotient $\mathfrak{t}$, and thus there are two distinct surjections $X = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{t}$ depending on which copy of $\widetilde{\mathfrak{g}}$ one first projects to. If $(g_1B, g_2B, \mathfrak{u}) \in X$ the two elements $\overline{g_1^{-1}\mathfrak{u}g_1}$, $\overline{g_2^{-1}\mathfrak{u}g_2}$ are not necessarily equal in $\mathfrak{t}$, but they are equal in $\mathfrak{t}/\mathcal{S}_n \xrightarrow{\sim} \mathbb{A}_E^n$ (the morphism here being given by the coefficients of the characteristic polynomial), hence there is a well-defined unique surjection $\kappa : X \twoheadrightarrow \mathfrak{t}/\mathcal{S}_n$.

Denote by $Z := \kappa^{-1}(0)^{\mathrm{red}}$ the reduced fiber of κ over $0 \in \mathfrak{t}/\mathcal{S}_n$. It is again equidimensional with irreducible components Z_w parametrized by $\mathcal{S}_n$. Denote by $\mathrm{Z}^0(Z) := \oplus_{w\in\mathcal{S}_n}\mathbb{Z}[Z_w]$ the free abelian group on the irreducible components Z_w. For each closed subscheme Y of the (non-reduced) fiber $\kappa^{-1}(0)$, define (compare Step 2):

$$[Y] := \sum_{w\in\mathcal{S}_n} m(Z_w, Y)[Z_w] \in \mathrm{Z}^0(Z)$$

where $m(Z_w, Y) \in \mathbb{Z}_{\geq 0}$ is the length of the localized ring $\mathcal{O}_{Y,\eta_{Z_w}}$ as a module over itself (η_{Z_w} being the generic point of Z_w).

It turns out that the fibers $\overline{X_w} := X_w \times_{\mathfrak{t}/\mathcal{S}_n} \{0\}$ for $w \in \mathcal{S}_n$ are far from being reduced in general (even if X_w is) and so it is interesting to consider $[\overline{X_w}] \in \mathrm{Z}^0(Z)$. Using Beilinson-Bernstein's theory of localization of Verma modules as $\mathcal{D}$-modules on the flag variety GL_n/B, one has the following remarkable formula, which is "well-known" to specialists:

$$[\overline{X_w}] = \sum_{w'\preceq w} P_{w_0w,w_0w'}(1)\mathcal{C}_{w'} \quad \text{in} \quad \mathrm{Z}^0(Z)$$

where $\mathcal{C}_{w'} \in \mathrm{Z}^0(Z)$ are certain non-zero cycles which only depend on w' (not on w). The cycle $\mathcal{C}_{w'}$ is close to $[Z_{w'}]$, e.g. $[Z_{w'}]$ is always in its support with multiplicity 1 (one even has $\mathcal{C}_{w'} = [Z_{w'}]$ if $n \leq 7$), but $\mathcal{C}_{w'}$ can be non-irreducible in general (if $n \geq 8$) though its support is always contained in $\{[Z_{w''}],\ w'' \preceq w'\}$.

Step 7: A formal local isomorphism I.

I now fix $v \in S_p$, $\mathcal{R}_v$ an ordering of the $\varphi_{\tilde{v},i}$ and $w_v \in \mathcal{S}_n$ such that $x_{\mathcal{R}_v,w_v} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$. The formal completion of $\mathfrak{X}_{\overline{\rho}_{\tilde{v}}}$ at the point $\rho_{\tilde{v}}$ is easily checked to be isomorphic to the formal scheme $X_{\rho_{\tilde{v}}}$ pro-representing the fonctor of (equal characteristic 0) framed deformations of $\rho_{\tilde{v}}$ on local artinian E-algebras. The projection $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}}) \longrightarrow \mathfrak{X}_{\overline{\rho}_{\tilde{v}}}$ induces a morphism of formal schemes:

$$\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}} \longrightarrow X_{\rho_{\tilde{v}}}.$$

As briefly seen in Step 4, the ordering $\mathcal{R}_v$ defines a unique triangulation on the (φ, Γ)-module $D_{\mathrm{rig}}(\rho_{\tilde{v}})$ and I denote by $\mathcal{M}_\bullet = \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_n$ the associated flag on $D_{\mathrm{rig}}(\rho_{\tilde{v}})[\frac{1}{t}]$ where t is Fontaine's p-adic $2i\pi$ (this only depends on $\mathcal{R}_v$, not on any w_v). For any local artinian E-algebra A, let $\mathcal{R}_A$ be the Robba ring with A-coefficients. I denote by $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ the formal scheme pro-representing the functor of framed deformations of $\rho_{\tilde{v}}$ *and of the flag* $\mathcal{M}_\bullet$ *on* $D_{\mathrm{rig}}(\rho_{\tilde{v}})[\frac{1}{t}]$, that is, the functor sending a local artinian E-algebra A to the set of $(\rho_{\tilde{v},A}, \mathcal{M}_{A,\bullet})$ where $\rho_{\tilde{v},A}$ is a framed deformation of $\rho_{\tilde{v}}$ and $\mathcal{M}_{A,\bullet}$ is a triangulation on $D_{\mathrm{rig}}(\rho_{\tilde{v},A})[\frac{1}{t}]$ deforming $\mathcal{M}_\bullet$. It is still pro-representable because of the framing on $\rho_{\tilde{v},A}$ (which makes all automorphisms trivial). Using the genericity assumptions on the $\varphi_{\tilde{v},i}$, one can check that the obvious forgetful functor $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet} \longrightarrow X_{\rho_{\tilde{v}}}$ is a closed immersion of formal schemes (one shows that there is a unique way to deform $\mathcal{M}_\bullet$ in $D_{\mathrm{rig}}(\rho_{\tilde{v},A})[\frac{1}{t}]$).

From the work of Kedlaya-Pottharst-Xiao or of Liu, one can deduce a global triangulation in a neighbourhood of $x_{\mathcal{R}_v,w_v} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ *provided one inverts* t (without inverting t, this would be true only when $w_v = w_{\mathcal{R}_v}$, i.e. only when $x_{\mathcal{R}_v,w_v}$ is saturated). This implies that the morphism $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}} \to X_{\rho_{\tilde{v}}}$ factors as:

$$\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}} \longrightarrow X_{\rho_{\tilde{v}},\mathcal{M}_\bullet} \hookrightarrow X_{\rho_{\tilde{v}}}.$$

Moreover one can check that $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}} \longrightarrow X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ is also a closed immersion (thus so is $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}} \to X_{\rho_{\tilde{v}}}$) and that the two formal schemes $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}}$ and $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ have the same dimension.

Step 8: A formal local isomorphism II.

Now I relate $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ and $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}}$ to the variety X of Step 5.

I start with $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$. From the (decreasing) Hodge filtration $\mathrm{Fil}^\bullet$ on the n-dimensional E-vector space $D_{\mathrm{dR}}(\rho_{\tilde{v}})$, I define the flag:

$$\mathrm{Fil}_\bullet = \mathrm{Fil}_1 := \mathrm{Fil}^{-h_{\tilde{v},1}} \subset \mathrm{Fil}_2 := \mathrm{Fil}^{-h_{\tilde{v},2}} \subset \cdots \subset \mathrm{Fil}_n := \mathrm{Fil}^{-h_{\tilde{v},n}}.$$

I denote by $\mathcal{F}_\bullet$ the Frobenius stable flag on $D_{\mathrm{dR}}(\rho_{\tilde{v}}) = D_{\mathrm{cris}}(\rho_{\tilde{v}})$ coming from the ordering $\mathcal{R}_v$. To these two flags I associate the point:

$$y_{\mathcal{R}_v} := (\mathcal{F}_\bullet, \mathrm{Fil}_\bullet, 0) \in X(E)$$

i.e. the endomorphism $\mathfrak{u}$ is here 0. Going back to the definition of the variety X in Step 5, one can see that, strictly speaking, $y_{\mathcal{R}_v}$ is not well-defined as one should also fix an E-basis of $D_{\mathrm{dR}}(\rho_{\tilde{v}})$ so that one can identify flags on $D_{\mathrm{dR}}(\rho_{\tilde{v}})$ with GL_n/B and endomorphisms of $D_{\mathrm{dR}}(\rho_{\tilde{v}})$ with $\mathfrak{g}$, that is, one should fix a framing on $D_{\mathrm{dR}}(\rho_{\tilde{v}})$. However, this is a non-crucial technical complication that I overlook everywhere in the sequel.

Let A be a local artinian E-algebra and $(\rho_{\tilde{v},A},\mathcal{M}_{A,\bullet})\in X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}(A)$. I would like to define a morphism of formal schemes $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}\to\widehat{X}_{y_{\mathcal{R}_v}}$, and one could think of considering $D_{\mathrm{dR}}(\rho_{\tilde{v},A})$ for that (since we considered $D_{\mathrm{dR}}(\rho_{\tilde{v}})$ above). However, in general, $\rho_{\tilde{v},A}$ is not a de Rham deformation of $\rho_{\tilde{v}}$. To see this, consider the baby case $n=1$, $\rho_{\tilde{v}}=1$ (trivial representation) and $A=E[x]/(x^2)$ (dual numbers). Then in that case the deformation of $\rho_{\tilde{v}}=1$ over A given by the p-adic logarithm certainly appears on (the analogue of) $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$, and it is clearly not de Rham (not even Hodge-Tate). However, Fontaine has defined a slightly more general notion than de Rham representations which turns out to be exactly what is needed here.

Replace B_{dR} by the larger ring $\mathrm{B}_{\mathrm{pdR}}:=\mathrm{B}_{\mathrm{dR}}[\log(t)]$ with an obvious action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on $\log(t)$, and say r is an *almost de Rham* representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ if $\dim_{\mathbb{Q}_p}D_{\mathrm{pdR}}(r)=\dim_{\mathbb{Q}_p}r$ where $D_{\mathrm{pdR}}(r):=(\mathrm{B}_{\mathrm{pdR}}\otimes_{\mathbb{Q}_p}r)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$. The B_{dR}-linear derivation with respect to $\log(t)$ commutes with $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and yields a nilpotent endomorphism ν on $D_{\mathrm{pdR}}(r)$ which is zero if and only if r is de Rham if and only if $D_{\mathrm{dR}}(r)\xrightarrow{\sim}D_{\mathrm{pdR}}(r)$. Moreover the decreasing filtration $(t^i\mathrm{B}_{\mathrm{dR}}^+[\log(t)])$ on $\mathrm{B}_{\mathrm{pdR}}$ induces a filtration $\mathrm{Fil}^\bullet$ on $D_{\mathrm{pdR}}(r)$ that can be rescaled as a flag $\mathrm{Fil}_\bullet$ on $D_{\mathrm{pdR}}(r)$ as I did above. Finally, any extension of almost de Rham representations remains almost de Rham (e.g. any trianguline representation with integral Sen weights is almost de Rham), which implies that the previous problem with Galois deformations now disappears.

Berger has defined a covariant exact functor from (φ,Γ)-modules over the Robba ring to semi-linear representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over free $\mathrm{B}_{\mathrm{dR}}^+$-modules. Inverting t, one can show that his functor sends $D_{\mathrm{rig}}(\rho_{\tilde{v},A})[\frac{1}{t}]$ (for $\rho_{\tilde{v},A}$ as above) to $\mathrm{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\rho_{\tilde{v},A}$. I can now define a morphism of formal schemes $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}\longrightarrow\widehat{X}_{y_{\mathcal{R}_v}}$ by sending $(\rho_{\tilde{v},A},\mathcal{M}_{A,\bullet})$ to $(\mathcal{F}_{A,\bullet},\mathrm{Fil}_{A,\bullet},\nu_A)$ where:

(i) $\mathcal{F}_{A,\bullet}$ is the flag on the free A-module $D_{\mathrm{pdR}}(\rho_{\tilde{v},A})=(\mathrm{B}_{\mathrm{pdR}}\otimes_{\mathrm{B}_{\mathrm{dR}}}(\mathrm{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}\rho_{\tilde{v},A}))^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}$ functorially induced by the flag $\mathcal{M}_{A,\bullet}$ on $D_{\mathrm{rig}}(\rho_{\tilde{v},A})[\frac{1}{t}]$;
(ii) $\mathrm{Fil}_{A,\bullet}$ is the flag on $D_{\mathrm{pdR}}(\rho_{\tilde{v},A})$ induced by the Hodge filtration on $D_{\mathrm{pdR}}(\rho_{\tilde{v},A})$ rescaled as at the beginning of Step 8;
(iii) ν_A is the nilpotent A-linear endomorphism on $D_{\mathrm{pdR}}(\rho_{\tilde{v},A})$.

Using (again) the genericity of the $\varphi_{\tilde{v},i}$ together with Berger's description of (φ,Γ)-modules in terms of B-pairs, one can prove that the above morphism $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}\longrightarrow\widehat{X}_{y_{\mathcal{R}_v}}$ is *formally smooth.* It follows by formal smoothness that the irreducible components of $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ are the inverse image of those of $\widehat{X}_{y_{\mathcal{R}_v}}$ (or rather of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{X,y_{\mathcal{R}_v}}$ where $\widehat{\mathcal{O}}_{X,y_{\mathcal{R}_v}}$ is the underlying completed local ring). But the irreducible components of $\widehat{X}_{y_{\mathcal{R}_v}}$ are the $\widehat{(X_{w'_v})}_{y_{\mathcal{R}_v}}$ for $w'_v\in\mathcal{S}_n$ such that $y_{\mathcal{R}_v}\in X_{w'_v}$

(we crucially use here that the normality of the $X_{w'_v}$ implies that they remain locally irreducible after completion at any point). I denote by $X^{w'_v}_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ the inverse image of $\widehat{(X_{w'_v})}_{y_{\mathcal{R}_v}}$. In particular $X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ is equidimensional.

Now recall from Step 7 that we have a closed immersion of equidimensional formal schemes of the same dimension $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}} \hookrightarrow X_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ (here $x_{\mathcal{R}_v,w_v} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ as in Step 7). Hence $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}}$ is a union of some $X^{w'_v}_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$. Which are these w'_v? One can characterize $X_{w'_v}$ by looking at its image via $X_{w'_v} \hookrightarrow X \longrightarrow \mathfrak{t} \times_{\mathfrak{t}/\mathcal{S}_n} \mathfrak{t}$ (see beginning of Step 6). Using this characterization and unravelling the definition of w_v, it is then not too hard to check that there can only be one w'_v as above which is in fact w_v. I thus finally have my formal local isomorphism (or formal local model):

$$\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}} \xrightarrow{\sim} X^{w_v}_{\rho_{\tilde{v}},\mathcal{M}_\bullet} \longrightarrow \widehat{(X_{w_v})}_{y_{\mathcal{R}_v}}$$

where the last morphism is formally smooth. Note that this implies in particular $y_{\mathcal{R}_v} \in X_{w_v}(E) \subset X(E)$ for any $w_v \in \mathcal{S}_n$ such that $x_{\mathcal{R}_v,w_v} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$.

Steps 4 to 8 have several crucial outcomes that I will state in the next and last lecture.

4 Lecture 4: End of proof

I am now in a position to put everything together to deduce Theorem 1.3.1. I fix $\rho_p := (\rho_{\tilde{v}})_{v\in S_p} \in \prod_{v\in S_p} \mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}}$ with each $\rho_{\tilde{v}}$ satisfying the assumptions of Lecture 1, so that I can define and use all the previous material. Note that I don't need to assume that ρ_p comes from a global ρ.

I first state direct consequences of the formal local isomorphism of Lecture 3.

Step 9: First consequences of the local isomorphism.

Since X_{w_v} is normal, so is $\widehat{(X_{w_v})}_{y_{\mathcal{R}_v}}$ (normality is preserved by completion), hence also $X^{w_v}_{\rho_{\tilde{v}},\mathcal{M}_\bullet}$ (as it is formally smooth over $\widehat{(X_{w_v})}_{y_{\mathcal{R}_v}}$), and thus also $\widehat{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})}_{x_{\mathcal{R}_v,w_v}}$. We then have the following first consequence:

Theorem 4.1. *The rigid analytic variety $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ is irreducible in the neighbourhood of any companion point $x_{\mathcal{R}_v,w_v}$. In particular, if $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$, then we have $X_p(\overline{\rho}) \xrightarrow{\sim} X_{\mathrm{tri}}(\overline{\rho}_p)$ in the neighbourhood of $(\rho_p, \chi_{\mathcal{R},w})$.*

Note that $(\rho_p, \chi_{\mathcal{R},w})$ is sent to $x_{\mathcal{R},w} := (x_{\mathcal{R}_v,w_v})_{v\in S_p} = ((\rho_{\tilde{v}}, \delta_{\mathcal{R}_v,w_v}))_{v\in S_p}$ under the shift of Step 3.

Let $w_v \in \mathcal{S}_n$ such that $x_{\mathcal{R}_v,w_v} \in X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$. From the definition of $w_{\mathcal{R}_v}$, we have $y_{\mathcal{R}_v} \in V_{w_{\mathcal{R}_v}}$ (recall $w_{\mathcal{R}_v}$ measures the relative position of the two flags $\mathcal{F}_\bullet$ and $\mathrm{Fil}_\bullet$ on $D_{\mathrm{dR}}(\rho_{\tilde{v}})$). From the end of Step 5 we have $y_{\mathcal{R}_v} \in X_{w'_v}(E) \Rightarrow w_{\mathcal{R}_v} \preceq w'_v$ (using that $\mathfrak{u} = 0$ on $y_{\mathcal{R}_v}$ one can check that the converse also holds: $y_{\mathcal{R}_v} \in X_{w'_v}(E) \Leftrightarrow w_{\mathcal{R}_v} \preceq w'_v$). Since $y_{\mathcal{R}_v} \in X_{w_v}(E)$ (see the end of Step 8, recall this statement comes from the study of deformations) we deduce $w_{\mathcal{R}_v} \preceq w_v$, thus completing the description of local companion points in Step 4. Using the argument at the end of Step 4 together with Step 1, we then have the second consequence, which is one implication in Theorem 1.3.1:

Theorem 4.2. *If $\mathrm{Hom}_{G_p}\left(C_{\mathcal{R},w}, \Pi_\infty^{\mathrm{an}}[\mathfrak{m}_{\rho_p}]\right) \neq 0$, then we have $w_{\mathcal{R}} \preceq w$.*

Step 10: Yet another equality of cycles.

For any $\delta \in \widehat{T}_v$ denote by $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})_\delta$ the fiber of $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})$ above δ via $X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})_\delta \hookrightarrow \mathfrak{X}_{\overline{\rho}_{\tilde{v}}} \times \widehat{T}_v \twoheadrightarrow \widehat{T}_v$. We have $y_{\mathcal{R}_v} \in \overline{X_{w_v}}(E) \subseteq X_{w_v}(E)$ for w_v as before (as $\mathfrak{u} = 0$ on $y_{\mathcal{R}_v}$) and from the end of Step 8 we deduce by base change a formally smooth morphism:

$$(X_{\mathrm{tri}}\widehat{(\overline{\rho}_{\tilde{v}})_{\delta_{\mathcal{R}_v,w_v}}})_{x_{\mathcal{R}_v,w_v}} \longrightarrow \widehat{(\overline{X_{w_v}})}_{y_{\mathcal{R}_v}}.$$

Consider the cycle $\mathcal{C}_{w'_v} \in \mathrm{Z}^0(Z)$ in Step 6 for some $w'_v \in \mathcal{S}_n$. Recall that the support of $\mathcal{C}_{w'_v}$ consists of some $Z_{w''_v}$ with $w''_v \preceq w'_v$ containing $Z_{w'_v}$ and that $y_{\mathcal{R}_v} \in Z_{w''_v}(E) \subseteq \overline{X_{w''_v}}(E) \subset X_{w''_v}(E)$ if and only if $w_{\mathcal{R}_v} \preceq w''_v$. So $y_{\mathcal{R}_v}$ is in the support of $\mathcal{C}_{w'_v}$ if and only if $w_{\mathcal{R}_v} \preceq w'_v$. When $w'_v \preceq w_v$, the cycle $\mathcal{C}_{w'_v}$ appears in $\overline{X_{w_v}}$ (see Step 6) and I denote by $\mathcal{C}_{\mathcal{R}_v,w'_v}$ the pull-back of its formal completion at $y_{\mathcal{R}_v}$ along the above formally smooth morphism (the formal completion of $\mathcal{C}_{w'_v}$ at $y_{\mathcal{R}_v}$ is the cycle obtained by taking the sum of the completion of each irreducible component[2] at $y_{\mathcal{R}_v}$ keeping the same multiplicities). Then $\mathcal{C}_{\mathcal{R}_v,w'_v}$ is a cycle in $\mathrm{Z}^{\frac{n(n+3)}{2}}(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_{\tilde{v}}},\rho_v})$ (see Step 2 for the notation) which only depends on $\mathcal{R}_v$ and w'_v (if we change w_v such that $w'_v \preceq w_v$ and do the same procedure with this new w_v, we still get the same cycle $\mathcal{C}_{\mathcal{R}_v,w'_v}$ in $\mathrm{Z}^{\frac{n(n+3)}{2}}(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_{\tilde{v}}},\rho_v})$).

Now, the formal local isomorphism at the end of Step 8 together with the equality of cycles in Step 6 and $\mathcal{C}_{\mathcal{R}_v,w'_v} \neq 0 \Leftrightarrow w_{\mathcal{R}_v} \preceq w'_v$ imply an equality of cycles in $\mathrm{Z}^{\frac{n(n+3)}{2}}(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_{\tilde{v}}},\rho_v})$:

$$[\widehat{\mathcal{O}}_{X_{\mathrm{tri}}(\overline{\rho}_{\tilde{v}})_{\delta_{\mathcal{R}_v,w_v}},x_{\mathcal{R}_v,w_v}}] = \sum_{w_{\mathcal{R}_v} \preceq w'_v \preceq w_v} P_{w_{v,0}w_v,w_{v,0}w'_v}(1)[\mathcal{C}_{\mathcal{R}_v,w'_v}]$$

where all terms in the sum are non-zero. Finally, taking the product over $v \in S_p$ we get an equality of cycles for any refinement $\mathcal{R}$ and any w such that $w_{\mathcal{R}} \preceq w$:

$$[\widehat{\mathcal{O}}_{X_{\mathrm{tri}}(\overline{\rho}_p)_{\delta_{\mathcal{R},w}},x_{\mathcal{R},w}}] = \sum_{w_{\mathcal{R}} \preceq w' \preceq w} P_{w_0w,w_0w'}(1)[\mathcal{C}_{\mathcal{R},w'}] \in \mathrm{Z}^{[F^+:\mathbb{Q}]\frac{n(n+3)}{2}}(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_p},\rho_p})$$

where all the terms in the sum are non-zero.

Step 11: End of proof of Theorem 1.3.1 I.

I now give the argument of the proof of Theorem 1.3.1 but *assuming that the* $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})}$*-module* $\mathcal{M}_{\mathcal{R},w}$ *in Step 2 is free of rank* 1 *when it is non-zero*, i.e. when $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$ or equivalently $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})} \neq 0$ (recall that $\chi_{\mathcal{R},w}$ is $\delta_{\mathcal{R},w}$ "unshifted", see Step 3). In general, this is presumably not the case (i.e. $\mathcal{M}_{\mathcal{R},w}$ might not even be free), however the proof is easier in that case, and the modifications in order to bypass this appear more natural (they will be given in the next and last step).

From the end of Step 1 and the theorem at the end of Step 9 we have $\mathcal{L}_{\mathcal{R},w'} = 0$ if $w_{\mathcal{R}} \not\preceq w'$. From the end of Step 2 the above assumption on $\mathcal{M}_{\mathcal{R},w}$ then implies

[2] note that we do not know the normality of the irreducible components $Z_{w''}$, thus their completion might not remain irreducible

an equality of cycles for any refinement $\mathcal{R}$ and any w such that $w_{\mathcal{R}} \preceq w$:

$$[\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})}] = \sum_{w_{\mathcal{R}} \preceq w' \preceq w} P_{w_0 w, w_0 w'}(1)[\mathcal{L}_{\mathcal{R},w'}] \in \mathrm{Z}^{[F^+:\mathbb{Q}]\frac{n(n+3)}{2}}(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathfrak{X}_{\overline{\rho}_p},\rho_p}).$$

Note that we don't know so far which terms in this sum are non-zero, or even when $[\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})}]$ is non-zero. By the first consequence in Step 9, if $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$ then we have $[\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})}] = [\widehat{\mathcal{O}}_{X_{\mathrm{tri}}(\overline{\rho}_p)_{\delta_{\mathcal{R},w}},x_{\mathcal{R},w}}]$. From the end of Step 10 we thus deduce the equalities for any refinement $\mathcal{R}$ and any w such that $w_{\mathcal{R}} \preceq w$ and $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$:

$$\sum_{w_{\mathcal{R}} \preceq w' \preceq w} P_{w_0 w, w_0 w'}(1)[\mathcal{C}_{\mathcal{R},w'}] \quad = \sum_{w_{\mathcal{R}} \preceq w' \preceq w} P_{w_0 w, w_0 w'}(1)[\mathcal{L}_{\mathcal{R},w'}].$$

Assume first that all $(\rho_p, \chi_{\mathcal{R},w})$ are in $X_p(\overline{\rho})$ for $w_{\mathcal{R}} \preceq w$. Then I claim that the above equalities imply $\mathcal{L}_{\mathcal{R},w'} \neq 0$ for $w_{\mathcal{R}} \preceq w'$. Indeed, by this assumption we have for $w_{\mathcal{R}} \preceq w$:

$$\sum_{w_{\mathcal{R}} \preceq w' \preceq w} P_{w_0 w, w_0 w'}(1)\big([\mathcal{C}_{\mathcal{R},w'}] - [\mathcal{L}_{\mathcal{R},w'}]\big) = 0.$$

Since $P_{w_0 w, w_0 w}(1) = 1$ for all w, this is a system of "upper triangular" linear equations (the "unknowns" being the $[\mathcal{C}_{\mathcal{R},w'}] - [\mathcal{L}_{\mathcal{R},w'}]$ for $w_{\mathcal{R}} \preceq w'$) with coefficients in $\mathbb{Z}$ which are 1 on the diagonal. It is thus invertible and we deduce $[\mathcal{C}_{\mathcal{R},w'}] - [\mathcal{L}_{\mathcal{R},w'}] = 0$ for $w_{\mathcal{R}} \preceq w'$, which implies $[\mathcal{L}_{\mathcal{R},w'}] = [\mathcal{C}_{\mathcal{R},w'}] \neq 0$ and thus $\mathcal{L}_{\mathcal{R},w'} \neq 0$ for $w_{\mathcal{R}} \preceq w'$.

I now assume that $(\rho_p, \chi_{\mathcal{R},w_0}) \in X_p(\overline{\rho})$ (this assumption is satisfied in the global situation we started with, see Lecture 1) and I prove by induction that $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$ for $w_{\mathcal{R}} \preceq w$.

I denote by lg be the length function on $\mathcal{S}_n^{|S_p|}$ and I consider the following induction hypothesis $(\mathcal{H}_\ell)$ for $\ell \leq \lg(w_0)$ and all ρ_p, $\mathcal{R}$ as at the beginning of this lecture *such that* $(\rho_p, \chi_{\mathcal{R},w_0}) \in X_p(\overline{\rho})$:

$(\mathcal{H}_\ell)$ When $\lg(w_{\mathcal{R}}) \geq \ell$ then $(\rho_p, \chi_{\mathcal{R},w'}) \in X_p(\overline{\rho})$ for all w' such that $w_{\mathcal{R}} \preceq w'$.

The hypothesis $(\mathcal{H}_{\ell(w_0)})$ is satisfied by assumption since $\lg(w_{\mathcal{R}}) \geq \ell(w_0)$ implies $w_{\mathcal{R}} = w_0$. Assuming $(\mathcal{H}_\ell)$, I now prove $(\mathcal{H}_{\ell-1})$.

I first deduce from $(\mathcal{H}_\ell)$ that we have $(\rho_p, \chi_{\mathcal{R},w'}) \in X_p(\overline{\rho})$ for $w_{\mathcal{R}} \preceq w'$ and $\lg(w') \geq \ell$ whatever $w_{\mathcal{R}}$ is. The argument is *mutatis mutandis* the same Zariski-closure argument as that of Step 4 in order to show the existence of local companion points. Let w' such that $\lg(w') \geq \ell$. The hypothesis $(\mathcal{H}_\ell)$ implies that the inverse image of the Bruhat cell $(Bw'B/B)^{\mathrm{rig}}$ in $\prod_{v \in S_p} \widetilde{\mathfrak{X}}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}}$ (with the notation of Step 4) can be embedded into $X_p(\overline{\rho})_{\chi_{\mathcal{R},w'}} \subset X_p(\overline{\rho}) \subset \mathfrak{X}_{\overline{\rho}_p} \times \widehat{T}_p$ (note that the corresponding points are saturated, i.e. we are in the case $w_{\mathcal{R}} = w'$ to which we apply $(\mathcal{H}_\ell)$). Since $X_p(\overline{\rho})_{\chi_{\mathcal{R},w'}}$ is closed in $\mathfrak{X}_{\overline{\rho}_p} \times \widehat{T}_p$, the Zariski-closure of this inverse image remains in $X_p(\overline{\rho})_{\chi_{\mathcal{R},w'}}$, and as in Step 4 this Zariski-closure now contains the points $(\rho_p, \chi_{\mathcal{R},w'})$ for $w_{\mathcal{R}} \preceq w'$. Applying this to all w' such that $\lg(w') \geq \ell$, we get the result.

Now I prove $(\mathcal{H}_{\ell-1})$. I can assume $\lg(w_{\mathcal{R}}) = \ell - 1$ and from what was just proven, it only remains to check $(\rho_p, \chi_{\mathcal{R},w_{\mathcal{R}}}) \in X_p(\overline{\rho})$. For that it is enough to prove $[\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] \neq 0$. Since $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$ for $w_{\mathcal{R}} \preceq w$ and $w_{\mathcal{R}} \neq w$ we have for such w:

$$\sum_{w_{\mathcal{R}} \preceq w' \preceq w} P_{w_0 w, w_0 w'}(1)\big([\mathcal{C}_{\mathcal{R},w'}] - [\mathcal{L}_{\mathcal{R},w'}]\big) = 0.$$

Assume first $\ell = \lg(w_0)$, then $w = w_0$ is the only such w and this gives:

$$[\mathcal{C}_{\mathcal{R},w_{\mathcal{R}}}] - [\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] + [\mathcal{C}_{\mathcal{R},w_0}] - [\mathcal{L}_{\mathcal{R},w_0}] = 0.$$

But it is not difficult to check that the cycle $[\mathcal{C}_{\mathcal{R},w_0}]$ is irreducible and closed in $\prod_{v \in S_p} \mathfrak{X}_{\overline{\rho}_{\tilde{v}}}^{h_{\tilde{v}}-\mathrm{cr}} \subset \mathfrak{X}_{\overline{\rho}_p}$. Since $[\mathcal{L}_{\mathcal{R},w_0}]$ corresponds to the locally algebraic constituent $C_{\mathcal{R},w_0}$, results of Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin then imply that either $[\mathcal{L}_{\mathcal{R},w_0}] = 0$ or $[\mathcal{L}_{\mathcal{R},w_0}] = [\mathcal{C}_{\mathcal{R},w_0}]$. Assume now $[\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] = 0$, then we have either $[\mathcal{C}_{\mathcal{R},w_{\mathcal{R}}}] + [\mathcal{C}_{\mathcal{R},w_0}] = 0$ or $[\mathcal{C}_{\mathcal{R},w_{\mathcal{R}}}] = 0$, both being impossible. Hence $[\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] \neq 0$.

Assume now $\ell \leq \lg(w_0) - 1$ so that $\lg(w_{\mathcal{R}}) \leq \lg(w_0) - 2$. Then by standard facts one can find 3 elements w_1, w_2, w_3 in $\mathcal{S}_n^{|S_p|}$ such that $w_{\mathcal{R}} \preceq w_i \preceq w_3$ $(i = 1, 2)$ with $\lg(w_1) = \lg(w_2) = \lg(w_{\mathcal{R}}) + 1$ and $\lg(w_3) = \lg(w_{\mathcal{R}}) + 2$. Moreover w_1, w_2 are the only elements strictly between $w_{\mathcal{R}}$ and w_3. We can apply the above equality with $w \in \{w_1, w_2, w_3\}$. Since all relevant values $P_{w_0 w, w_0 w'}(1)$ are 1 in that case this gives the equalities:

$$\begin{cases} [\mathcal{C}_{\mathcal{R},w_{\mathcal{R}}}] - [\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] + [\mathcal{C}_{\mathcal{R},w_i}] - [\mathcal{L}_{\mathcal{R},w_i}] = 0, \quad i = 1, 2 \\ [\mathcal{C}_{\mathcal{R},w_{\mathcal{R}}}] - [\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] + \sum_{i=1}^{3}([\mathcal{C}_{\mathcal{R},w_i}] - [\mathcal{L}_{\mathcal{R},w_i}]) = 0. \end{cases}$$

Assuming $[\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] = 0$ this implies $[\mathcal{C}_{\mathcal{R},w_3}] = [\mathcal{L}_{\mathcal{R},w_3}] + [\mathcal{C}_{\mathcal{R},w_{\mathcal{R}}}]$. But properties of the cycles $\mathcal{C}_{w'}$ make it impossible for $\mathcal{C}_{w''}$ to appear in $\mathcal{C}_{w'}$ when $w'' \preceq w'$ *and* $\lg(w'') = \lg(w') - 2$, hence this equality can't hold no matter what $[\mathcal{L}_{\mathcal{R},w_3}]$ is. This again implies $[\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] \neq 0$.

This finishes the proof of Theorem 1.3.1 under the assumption on $\mathcal{M}_{\mathcal{R},w}$.

Step 12: End of proof of Theorem 1.3.1 II.

I indicate now how one can modify the previous arguments in order to dispense with the assumption on $\mathcal{M}_{\mathcal{R},w}$. This modification will also make the proof slightly more direct.

One can prove (using input from the patching process) that the coherent module $\mathcal{M}_\infty$ in Step 2 is Cohen-Macaulay, hence if $X_p(\overline{\rho})$ is smooth at the point $(\rho_p, \chi_{\mathcal{R},w})$ (assumed to be in $X_p(\overline{\rho})$) then $\mathcal{M}_\infty$ is locally free at $(\rho_p, \chi_{\mathcal{R},w})$, and thus (by taking fibers and completing) $\mathcal{M}_{\mathcal{R},w}$ is a free $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})}$-module of finite rank ≥ 1. Unfortunately, it turns out that $X_{\mathrm{tri}}(\overline{\rho}_p)$ is in general *not* smooth at $x_{\mathcal{R},w}$, hence from Step 9 $X_p(\overline{\rho})$ usually won't be smooth at $(\rho_p, \chi_{\mathcal{R},w})$. *But* $X_{\mathrm{tri}}(\overline{\rho}_p)$ is smooth at $x_{\mathcal{R},w}$ when $\lg(w)-\lg(w_{\mathcal{R}}) \in \{0,1,2\}$ (and $w_{\mathcal{R}} \preceq w$) and this gives sufficiently many smooth points so that a modification of the previous argument can be carried through, as I indicate now.

However, I will still make a simplifying assumption (but much weaker than the previous one) and finish the lecture by indicating how one can dispense also with this one. This simplifying assumption is: *if* $\lg(w) - \lg(w_{\mathcal{R}}) \in \{0,1,2\}$*, then the free* $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})}$*-module* $\mathcal{M}_{\mathcal{R},w}$ *has rank exactly* 1 *if non-zero.* The modification consists in considering now the stronger induction hypothesis:

$(\mathcal{H}_\ell^{\mathrm{strong}})$ When $\lg(w_{\mathcal{R}}) \geq \ell$ then $[\mathcal{L}_{\mathcal{R},w'}] \neq 0$ for all w' such that $w_{\mathcal{R}} \preceq w'$.

The hypothesis $(\mathcal{H}_{\lg(w_0)}^{\mathrm{strong}})$ is satisfied since $(\rho_p, \chi_{\mathcal{R},w_0}) \in X_p(\overline{\rho})$ is smooth and thus one has the equality $[\mathcal{C}_{\mathcal{R},w_0}] - [\mathcal{L}_{\mathcal{R},w_0}] = 0$ so that $[\mathcal{L}_{\mathcal{R},w_0}] \neq 0$.

Let w' such that $\lg(w') \geq \ell$ and recall from Step 2 that the fiber $X_p(\overline{\rho})_{\chi_{\mathcal{R},w'}}$ is the support of the continuous dual of $\mathrm{Hom}_{G_p}\big(\mathcal{F}_{\overline{B}_p}^{G_p}(U(\mathfrak{g}_p)\otimes_{U(\overline{\mathfrak{b}}_p)} -w'w_0\cdot\lambda, \delta_{\mathcal{R}}), \Pi_\infty^{\mathrm{an}}\big)$. Assuming $(\mathcal{H}_\ell^{\mathrm{strong}})$, an analogous Zariski-density argument replacing $X_p(\overline{\rho})_{\chi_{\mathcal{R},w'}}$ by the support of the continuous dual of $\mathrm{Hom}_{G_p}(C_{\mathcal{R},w'}, \Pi_\infty^{\mathrm{an}})$ (Zariski-closed in $X_p(\overline{\rho})_{\chi_{\mathcal{R},w'}}$) gives that $[\mathcal{L}_{\mathcal{R},w'}] \neq 0$ for all w' such that $w_{\mathcal{R}} \preceq w'$ and $\lg(w') \geq \ell$, no matter what $w_{\mathcal{R}}$ is.

Then in order to prove $(\mathcal{H}_{\ell-1}^{\mathrm{strong}})$ we can as before assume $\lg(w_{\mathcal{R}}) = \ell - 1$ and prove $[\mathcal{L}_{\mathcal{R},w_{\mathcal{R}}}] \neq 0$. But this statement is in fact already what we proved in order to deduce $(\mathcal{H}_{\ell-1})$! Note that the argument goes through *verbatim* because the points $x_{\mathcal{R},w_i}$, $i = 1,2,3$ we used are still smooth on $X_{\mathrm{tri}}(\overline{\rho}_p)$.

Now, to finish, we have to deal with the fact that, for w such that $\lg(w) - \lg(w_{\mathcal{R}}) \in \{0,1,2\}$, the free $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{\chi_{\mathcal{R},w}},(\rho_p,\chi_{\mathcal{R},w})}$-modules $\mathcal{M}_{\mathcal{R},w}$ may have various ranks ≥ 1. Even when $(\rho_p, \chi_{\mathcal{R},w}) \in X_p(\overline{\rho})$ is not smooth, there is an open and irreducible neighbourhood of $(\rho_p, \chi_{\mathcal{R},w})$ in $X_p(\overline{\rho})$ (here, we use again Step 9) consisting of smooth points on which $\mathcal{M}_\infty$ is locally free of constant rank $d_{\mathcal{R},w}$. One problem is that $d_{\mathcal{R},w}$ may vary with w. However, one can then consider the even stronger induction hypothesis:

($\mathcal{H}_\ell^{\text{super strong}}$) When $\lg(w_{\mathcal{R}}) \geq \ell$ then $[\mathcal{L}_{\mathcal{R},w'}] \neq 0$ for all w' such that $w_{\mathcal{R}} \preceq w'$ and one has $d_{\mathcal{R},w'} = d_{\mathcal{R},w_0}$.

Then all the previous arguments can go through, proving finally Theorem 1.3.1.