# $p$-adic Hodge theory and the $p$-adic Langlands program

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## 1) Introduction

Let $p$ be a prime number, $F$ a number field and $n$ an integer $\geq 1$. Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$.

**Conjecture 1 (Langlands, Fontaine-Mazur)** *There is a unique bijection between the two sets:*

$\{$*isomorphisms classes of algebraic cuspidal automorphic representations* $\pi = \otimes' \pi_{\mathfrak{l}}$ *of* $\mathrm{GL}_n(\mathbb{A}_F)\}$

$$\updownarrow$$

$\{$*isomorphisms classes of continuous irreducible representations* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ *which are unramified at almost all places and potentially semi-stable (de Rham) at places dividing* $p\}$

*such that, for all* $\mathfrak{l} \mid \ell$, $\ell \neq p$*:*

$$\pi_{\mathfrak{l}} \longleftrightarrow \rho_{\mathfrak{l}} := \rho \mid_{\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/F_{\mathfrak{l}})}$$

*(local Langlands correspondence suitably normalized).*

An automorphic representation is algebraic if, for all infinite place, the restriction to $\mathbb{C}^\times$ of the Langlands parameter is a direct sum of characters:

$$z \mapsto z^{-a_1}\ \overline{z}^{-a_2}(z\overline{z})^{-\frac{n-1}{2}}$$

where $a_1, a_2 \in \mathbb{Z}$.

The correspondence:

$$\pi_{\mathfrak{l}} \longleftrightarrow \rho_{\mathfrak{l}}$$

actually factors as:

$$\pi_{\mathfrak{l}} \longleftrightarrow \mathsf{WD}(\rho_{\mathfrak{l}}) \longleftrightarrow \rho_{\mathfrak{l}}$$

where $\mathsf{WD}(\rho_{\mathfrak{l}})$ is the (conjecturally $F$-semi-simple) Weil-Deligne representation associated to the $p$-adic representation $\rho_{\mathfrak{l}}$.

If $\mathfrak{p} \mid p$, one should still have:

$$\pi_{\mathfrak{p}} \longleftrightarrow \mathsf{WD}(\rho_{\mathfrak{p}})$$

where $\mathsf{WD}(\rho_{\mathfrak{p}})$ is the Weil-Deligne representation associated by Fontaine to the potentially semi-stable $p$-adic representation $\rho_{\mathfrak{p}}$.

But this time, we DON'T have $\mathsf{WD}(\rho_{\mathfrak{p}}) \longleftrightarrow \rho_{\mathfrak{p}}$ (in general). For instance $\mathsf{WD}(\rho_{\mathfrak{p}})$ doesn't tell enough about the Hodge filtration of $D_{\mathsf{pst}}(V)$.

**Question 2** *Can one find a "natural $p$-adic" representation $\widehat{\pi} = \otimes' \widehat{\pi}_{\mathfrak{l}}$ of $\mathsf{GL}_n(\mathbb{A}_F)$ such that, for ALL finite places $\mathfrak{l}$:*

$$\widehat{\pi}_{\mathfrak{l}} \longleftrightarrow \rho_{\mathfrak{l}}?$$

If $\mathfrak{l} \nmid p$, one can take $\widehat{\pi}_{\mathfrak{l}} := \pi_{\mathfrak{l}}$. But what is $\widehat{\pi}_{\mathfrak{p}}$?

## 2) The $\mathbf{GL}_1$-case

$\pi$ is an algebraic Hecke character:

$$\pi : F^{\times}\backslash\mathbb{A}_F^{\times} \to \mathbb{C}^{\times}.$$

Let $I := \{\iota : F \hookrightarrow \mathbb{C}\}$ and $\pi_\infty$ the infinite part of $\pi$. One can write for $x \in F^{\times}$:

$$\pi_\infty(x) = \text{sign} \cdot \Pi_{\iota \in I}\iota(x)^{-a_\iota}$$

where sign $\in \{\pm 1\}$ and $(a_\iota)_{\iota \in I}$ are integers.

Let $\mathfrak{l}$ be any finite place, using the fixed embeddings and the fact that $\pi(\mathbb{A}_F^f)$ is contained in a finite extension of $\mathbb{Q}$, one can see $\pi_{\mathfrak{l}}$ as:

$$\pi_{\mathfrak{l}} : F_{\mathfrak{l}}^{\times} \to \overline{\mathbb{Q}}_p^{\times}.$$

For each $\mathfrak{p} \mid p$ and each $\sigma : F_\mathfrak{p} \hookrightarrow \overline{\mathbb{Q}}_p$, define $\iota(\mathfrak{p},\sigma) : F \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ such that the diagram commutes:

$$\begin{array}{ccc} F & \overset{\iota(\mathfrak{p},\sigma)}{\hookrightarrow} & \overline{\mathbb{Q}} \\ \downarrow & & \downarrow \\ F_\mathfrak{p} & \overset{\sigma}{\hookrightarrow} & \overline{\mathbb{Q}}_p. \end{array}$$

Define $\widehat{\pi} := \otimes' \widehat{\pi}_\mathfrak{l} : \mathbb{A}_F^\times \to \overline{\mathbb{Q}}_p^\times$ where:

$$\begin{array}{rcl} \widehat{\pi}_\mathfrak{l} & = & \text{sign in } \pi_\mathfrak{l} \text{ if } \mathfrak{l} \text{ is infinite} \\ \widehat{\pi}_\mathfrak{l} & = & \pi_\mathfrak{l} \ \text{ if } \mathfrak{l} \text{ is finite, } \mathfrak{l} \nmid p \\ \widehat{\pi}_\mathfrak{p} & = & \pi_\mathfrak{p} \displaystyle\prod_{\sigma : F_\mathfrak{p} \hookrightarrow \overline{\mathbb{Q}}_p} \sigma^{-a_{\iota(\mathfrak{p},\sigma)}} \text{ if } \mathfrak{l} = \mathfrak{p} \mid p. \end{array}$$

**Lemma 3** *The character* $\widehat{\pi} : \mathbb{A}_F^\times \to \overline{\mathbb{Q}}_p^\times$ *factors through* $\widehat{\pi} : \mathbb{A}_F^\times / \overline{(\mathbb{A}_F^\times)^0 F^\times} \to \overline{\mathbb{Q}}_p^\times$ *and :*

$$\rho := \widehat{\pi} \circ r_F : \mathrm{Gal}(\overline{\mathbb{Q}}/F)^{\mathrm{ab}} \to \overline{\mathbb{Q}}_p^\times = \mathrm{GL}_1(\overline{\mathbb{Q}}_p)$$

*is the corresponding Galois representation (where* $r_F : \mathrm{Gal}(\overline{\mathbb{Q}}/F)^{\mathrm{ab}} \overset{\sim}{\to} \mathbb{A}_F^\times / \overline{(\mathbb{A}_F^\times)^0 F^\times}$*).*

### 3) Locally algebraic representations of $\mathrm{GL}_n(F_{\mathfrak{p}})$

To go to $\mathrm{GL}_n$, the first thing is to generalize the construction of $\widehat{\pi}_{\mathfrak{p}} = \pi_{\mathfrak{p}}^{\mathrm{alg}}$. This is easy if we assume that $\pi$ is moreover regular.

For each $\iota : F \hookrightarrow \mathbb{C}$, one can associate a list of integers $(-a_{\iota,1}, \cdots, -a_{\iota,n})$ (the "weights" of $\pi$ at $\iota$). $\pi$ is said to be regular if, for any $\iota$, all the $a_{\iota,j}$ are distincts. We can assume $a_{\iota,j} < a_{\iota,j+1}$.

Let $L(\iota)$ be the algebraic representation of $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ (over $\overline{\mathbb{Q}}_p$) of highest weight:

$$\begin{aligned} -a_{\iota,n} \leq -a_{\iota,n-1} - 1 \leq -a_{\iota,n-2} - 2 \leq \cdots \\ \cdots \leq -a_{\iota,1} - n + 1 \end{aligned}$$

i.e. $L(\iota)$ is the algebraic parabolic induction of:

$$\begin{pmatrix} x_n & \cdots & \cdots & \cdot \\ 0 & x_{n-1} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_1 \end{pmatrix} \mapsto x_n^{-a_{\iota,n}} \cdots x_1^{-a_{\iota,1}-n+1}.$$

For $\mathfrak{p} \mid p$ and $\sigma : F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p$, denote by $\mathsf{alg}_{\mathfrak{p}}(\sigma)$ the representation of $\mathrm{GL}_n(F_{\mathfrak{p}})$ over $\overline{\mathbb{Q}}_p$:

$$\mathsf{alg}_{\mathfrak{p}}(\sigma) := L(\iota(\mathfrak{p},\sigma)) \circ \sigma$$

and set:

$$\begin{aligned} \mathsf{alg}_{\mathfrak{p}} &:= \bigotimes_{\sigma:F_{\mathfrak{p}}\hookrightarrow\overline{\mathbb{Q}}_p} \mathsf{alg}_{\mathfrak{p}}(\sigma) \\ \pi_{\mathfrak{p}}^{\mathsf{alg}} &:= \pi_{\mathfrak{p}} \otimes \mathsf{alg}_{\mathfrak{p}}. \end{aligned}$$

Contrary to what happens for $n = 1$, the representation $\pi_{\mathfrak{p}}^{\mathsf{alg}}$ is still not enough to "recover" $\rho_{\mathfrak{p}} = \rho\,|_{\mathsf{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})}$ in general.

Very rough hope: The missing data, at least if $\rho_{\mathfrak{p}}$ is irreducible, is a $p$-adic unitary completion $\widehat{\pi}_{\mathfrak{p}}$ of $\pi_{\mathfrak{p}}^{\mathsf{alg}}$. Equivalently, it is the data of an invariant norm on $\pi_{\mathfrak{p}}^{\mathsf{alg}}$ ($\|g(v)\| = \|v\|$).

This seems to hold for $\mathrm{GL}_n(F_{\mathfrak{p}}) = \mathrm{GL}_2(\mathbb{Q}_p)$.

From a local point of view, the very first question is thus:

**Question 4** *When is there an invariant norm on such representations as:*

$$\pi_{\mathfrak{p}} \otimes \bigotimes_{\sigma: F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p} \mathrm{alg}_{\mathfrak{p}}(\sigma)?$$

The second (more important) question is:

**Question 5** *To what extent do (some of) the invariant norms on* $\pi_{\mathfrak{p}} \otimes \bigotimes_{\sigma: F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p} \mathrm{alg}_{\mathfrak{p}}(\sigma)$ *"correspond" to irreducible* $n$*-diml de Rham representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})$ *with given Hodge-Tate weights and Weil-Deligne representation?*

In this talk, I give a conjectural answer to the first question, as well as the known cases so far. I then give examples concerning the second question for $\mathrm{GL}_n(F_{\mathfrak{p}}) = \mathrm{GL}_2(\mathbb{Q}_p)$.

**4) Local theory and the first question (joint with P. Schneider)**

Let $K$ be a finite extension of $\mathbb{Q}_p$ and $K_0$ its maximal unramified subfield. We normalize the reciprocity map:

$$\mathrm{rec} : W(\overline{\mathbb{Q}_p}/K)^{\mathrm{ab}} \xrightarrow{\sim} K^\times$$

(hence the local Langlands correspondence) so that arithmetic Frobeniuses go to uniformizers.

Fix $\pi$ a smooth irreducible representation of $\mathrm{GL}_n(K)$ over $\overline{\mathbb{Q}_p}$ and set ($|\cdot|_K := q_K^{-\mathrm{val}_K(\cdot)}$):

$$\mathrm{WD}(\pi) := \mathrm{LL}(\pi) \otimes |\mathrm{rec}|_K^{\frac{n-1}{2}}.$$

Let $K'$ be a finite Galois extension of $K$ such that $\mathrm{WD}(\pi)|_{W(\overline{\mathbb{Q}_p}/K')}$ is unramified and let $K_0'$ its maximal unramified subfield.

Both $\pi$ and $\mathrm{WD}(\pi)$ are defined over $E$-vector spaces for $E$ a sufficiently big finite extension of $\mathbb{Q}_p$. We assume $[K : \mathbb{Q}_p] = |\mathrm{Hom}(K, E)|$ and $[K_0' : \mathbb{Q}_p] = |\mathrm{Hom}(K_0', E)|$.

We call a $(\varphi, N, \mathsf{Gal}(K'/K))$-module any free $K'_0 \otimes_{\mathbb{Q}_p} E$-module $D$ of finite rank equipped with:

• $\varphi : D \to D$ bijective such that $\varphi(k \otimes e \cdot v) = \varphi(k) \otimes e \cdot v$ ($k \in K'_0$, $e \in E$)

• $N : D \to D$ linear such that $N\varphi = p\varphi N$

• an action of $\mathsf{Gal}(K'/K)$ commuting with $\varphi$ and $N$ such that $g(k \otimes e \cdot v) = g(k) \otimes e \cdot v$.

Fix an embedding $\sigma'_0 : K'_0 \hookrightarrow E$. Following Fontaine, attach a Weil-Deligne representation $\mathsf{WD}(D)$ to a $(\varphi, N, \mathsf{Gal}(K'/K))$-module $D$ as follows:

• $\mathsf{WD}(D) := D \otimes_{K'_0 \otimes E} E$ (via $\sigma'_0 \otimes \mathsf{Id}$)

• $w \in W(\overline{\mathbb{Q}_p}/K)$ acts via $\overline{w} \circ \varphi^{-\alpha(w)}$ where $w$ maps to $\overline{w} \in \mathsf{Gal}(K'/K)$ and to $\mathsf{Frob}^{\alpha(w)} \in \mathsf{Gal}(\mathbb{Q}_p^{\mathsf{nr}}/\mathbb{Q}_p)$ ($\mathsf{Frob}$ := arithmetic Frobenius on $\mathsf{Gal}(\mathbb{Q}_p^{\mathsf{nr}}/\mathbb{Q}_p)$)

• $N$ is the induced $N$.

Up to isomorphism, $\mathsf{WD}(D)$ doesn't depend on the embedding $\sigma'_0$.

For each $\sigma : K \hookrightarrow E$, fix $n$ integers $a_{\sigma,1} < a_{\sigma,2} < \cdots < a_{\sigma,n}$.

Let $L(\sigma)$ be the algebraic representation of $\mathrm{GL}_n(E)$ over $E$ of highest weight:

$$-a_{\sigma,n} \leq -a_{\sigma,n-1} - 1 \leq -a_{\sigma,n-2} - 2 \leq \cdots \\ \cdots \leq -a_{\sigma,1} - n + 1$$

i.e. $L(\sigma)$ is the algebraic parabolic induction of:

$$\begin{pmatrix} x_n & \cdots & \cdots & \cdot \\ 0 & x_{n-1} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_1 \end{pmatrix} \mapsto x_n^{-a_{\sigma,n}} \cdots x_1^{-a_{\sigma,1}-n+1}.$$

Set $\mathrm{alg}(\sigma) := L(\sigma) \circ \sigma$ and:

$$\mathrm{alg} := \bigotimes_{\sigma : K \hookrightarrow E} \mathrm{alg}(\sigma).$$

If $D$ is a $(\varphi, N, \mathrm{Gal}(K'/K))$-module, set for $\sigma : K \hookrightarrow E$:

$$D_{K',\sigma} := (K' \otimes_{K'_0} D) \otimes_{K' \otimes_{\mathbb{Q}_p} E} K' \otimes_{K,\sigma} E.$$

The following is the conjectural answer to the first question (for $\pi$ generic):

**Conjecture 6** *Assume (for simplicity) that $\pi$ is generic. The following conditions are equivalent:*

*(i) There is an invariant norm on $\pi \otimes_E \mathrm{Alg}$*

*(ii) There is a $(\varphi, N, \mathrm{Gal}(K'/K))$-module $D$ such that:*

$$\mathrm{WD}(D)^{F-ss} \simeq \mathrm{WD}(\pi)$$

*and a (weakly) admissible filtration preserved by $\mathrm{Gal}(K'/K)$ on:*

$$K' \otimes_{K'_0} D = \prod_{\sigma: K \hookrightarrow E} D_{K',\sigma}$$

*such that:*

$$\mathrm{Fil}^i D_{K',\sigma}/\mathrm{Fil}^{i+1} D_{K',\sigma} \neq 0 \Leftrightarrow i \in \{a_{\sigma,1}, \cdots, a_{\sigma,n}\}.$$

Remarks: • When $\pi$ is not generic, one has to replace $\pi$ in (i) by a reducible representation of $\mathrm{GL}_n(K)$ having $\pi$ as unique irreducible quotient. All local components $\pi_{\mathfrak{p}}$ as in the beginning are generic.

• One can make condition (ii) completely explicit in terms of Hodge and Newton polygons (see Fontaine and Rapoport's paper at Bull. S.M.F.).

• The functor WD induces an equivalence of categories between $(\varphi, N, \mathrm{Gal}(K'/K))$-modules and representations of the Weil-Deligne group of $K$ over $E$ that are unramified in restriction to $W(\overline{\mathbb{Q}_p}/K')$.

• One cannot replace $\mathrm{WD}(D)^{F-ss} \simeq \mathrm{WD}(\pi)$ by the stronger $\mathrm{WD}(D) \simeq \mathrm{WD}(\pi)$ in (ii).

• Via Fontaine's functor, (ii) is equivalent to the existence of an $n$-diml de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ over $E$ with given Hodge-Tate weights and Weil-Deligne representation.

The following statements sum up what is known of the above conjecture:

**Proposition 7** *The central character of* $\pi \otimes_E \mathrm{Alg}$ *in (i) is integral if and only if, for any* $(\varphi, N, \mathrm{Gal}(K'/K))$*-module* $D$ *as in (ii), we have* $t_H(K' \otimes_{K'_0} D) = t_N(D)$*.*

**Proposition 8** *The conjecture is true if* $\pi$ *is supercuspidal.*

**Theorem 9 (Schneider, Teitelbaum, B.)** *If* $\pi$ *is an unramified principal series and* $E$ *is sufficiently big, then (i) implies (ii) in the conjecture.*

**Theorem 10 (Berger, Colmez, B.)** *The conjecture is true if* $\mathrm{GL}_n(F_{\mathfrak{p}}) = \mathrm{GL}_2(\mathbb{Q}_p)$ *and* $E$ *is sufficiently big, except may-be if* $\mathrm{WD}(\pi)$ *is scalar.*

The first proposition is an exercise.

The second proposition follows because, if $\pi$ is supercuspidal, (i) is equivalent to the integrality of the central character and (ii) is equivalent to the equality $t_H(K' \otimes_{K'_0} D) = t_N(D)$ (as $D$ is an irreducible $(\varphi, N, \mathrm{Gal}(K'/K))$-module).

The first theorem can be proved using the theory of the $p$-adic Satake isomorphism of Schneider and Teitelbaum. See below.

The second theorem can be proved using $(\varphi, \Gamma)$-modules (the implication (ii) $\Rightarrow$ (i)). This sense seems much harder than the implication (i) $\Rightarrow$ (ii). Unfortunately, this method seems non-trivial to generalize to other groups than $\mathrm{GL}_2(\mathbb{Q}_p)$ so far.

Sketch of proof of first theorem:

Let $G := \mathrm{GL}_n(K)$ and $U := \mathrm{GL}_n(\mathcal{O}_K)$. Set:

$$\begin{aligned}\mathcal{H}_1 &:= \mathrm{End}_G(c-\mathrm{ind}_U^G 1_U)\\ \mathcal{H}_2 &:= \mathrm{End}_G(c-\mathrm{ind}_U^G \mathrm{Alg}|_U).\end{aligned}$$

One has an isomorphism of Hecke algebras:

$$i : \mathcal{H}_1 \xrightarrow{\sim} \mathcal{H}_2.$$

Assuming $E$ big enough, one can write:

$$\pi \otimes_E \mathrm{Alg} = E \otimes_{\mathcal{H}_2} (c-\mathrm{ind}_U^G \mathrm{Alg}|_U)$$

where $\mathcal{H}_2 \to E$ is given by:

$$\mathcal{H}_2 \xrightarrow{i^{-1}} \mathcal{H}_1 \xrightarrow{\text{Satake}} E[(K^\times/\mathcal{O}_K^\times)^n] \xrightarrow{\zeta} E$$

and $\zeta$ sends $(x_1, \cdots, x_n) \in (K^\times)^n$ to:

$$\zeta_1^{\mathrm{val}_K(x_1)} \cdot \zeta_2^{\mathrm{val}_K(x_2)}|x_2|_K \cdots \zeta_n^{\mathrm{val}_K(x_n)}|x_n|_K^{n-1}$$

for some $\zeta_i \in E^\times$.

One can assume $\mathrm{val}_K(\zeta_i) \leq \mathrm{val}_K(\zeta_{i+1})$.

The representation $c-\mathrm{ind}_U^G \mathrm{Alg}|_U$ has invariant lattices given by $c-\mathrm{ind}_U^G \mathrm{Alg}^0$ where $\mathrm{Alg}^0$ is an $U$-invariant lattice in Alg. Choose one and set $\mathcal{H}_2^0 := \mathrm{End}_G(c-\mathrm{ind}_U^G \mathrm{Alg}^0) \subset \mathcal{H}_2$.

Assume (i), then the image of $c-\mathrm{ind}_U^G \mathrm{Alg}^0$ in $E \otimes_{\mathcal{H}_2} (c-\mathrm{ind}_U^G \mathrm{Alg}|_U) = \pi \otimes_E \mathrm{Alg}$ is still a lattice. Equivalently $\zeta(\mathcal{H}_2^0)$ is bounded in $E$. An explicit computation gives this is equivalent to:

The polygon associated to:

$$\Big(\sum_\sigma a_{\sigma,1}, \sum_\sigma a_{\sigma,1} + \sum_\sigma a_{\sigma,2}, \cdots, \sum_{j=1}^{d+1} \sum_\sigma a_{\sigma,j}\Big)$$

is under the polygon associated to:

$$\Big(-\mathrm{val}_K(\zeta_{d+1}), -\mathrm{val}_K(\zeta_{d+1}) - \mathrm{val}_K(\zeta_d), \cdots, \\ -\sum_{j=1}^{d+1} \mathrm{val}_K(\zeta_j)\Big)$$

and both have the same endpoints. This is equivalent to (ii) by results “à la Fontaine-Rapoport”.

## 5) Local theory and the second question

The following statement sums up what is known of the second question so far. We keep the previous notations and recall that $GL_n(K) = GL_2(\mathbb{Q}_p)$.

**Theorem 11** *(i) (Berger, B.) Assume $\pi$ is a principal series (and* WD$(\pi)$ *is scalar), then there is at most one equivalence class of invariant norms on $\pi \otimes_E$* Alg.
*(ii) (Colmez) Assume $\pi$ is a special series (i.e. $\pi$ is the Steinberg representation up to twist), then there are at least as many equivalence classes of invariant norms on $\pi \otimes_E$* Alg *as there are irreducible admissible Hodge filtrations on $D$.*

In case (i), there is at most one irreducible admissible Hodge filtration on $D$. In case (ii), there are either none or infinitely many, parametrized by the so-called $\mathcal{L}$-invariant. Work on progress of Colmez should hopefully prove (ii) in the remaining cases ($\pi$ supercuspidal).

Sketch of proof of the theorem.

The proof of both (i) and (ii) is based on an idea of Colmez: use the $(\varphi, \Gamma)$-module theory.

Fix an irreducible admissible Hodge filtration on $D$ and let $V$ be the associated irreducible 2-diml de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Let $D(V)$ be the $(\varphi, \Gamma)$-module associated to $V$. It is a free $\mathcal{O}_E[[X]][1/X]^\wedge[1/p]$-module of finite type equipped with an "étale" Frobenius $\varphi : D(V) \to D(V)$ and a commuting action of $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$.

There is a unique surjection $\psi : D(V) \twoheadrightarrow D(V)$ such that, for $v_0, \cdots, v_{p-1} \in D(V)$:

$$\psi\Big(\sum_{i=0}^{p-1}(1+X)^i\varphi(v_i)\Big) := v_0.$$

Thus $\psi \circ \varphi = \mathrm{Id}$ and $\psi$ commutes with $\Gamma$.

Let $(\varprojlim_\psi D(V))^{\mathrm{b}}$ be the $E$-vector space of bounded $\psi$-compatible sequences in $D(V)$. It has infinite dimension.

The main result is that, in both (i) and (ii), one can put an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $(\varprojlim_\psi D(V))^{\mathrm{b}}$ in such a way that $(\varprojlim_\psi D(V))^{\mathrm{b}}$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-isomorphic to the topological dual of the completion of $\pi \otimes_E \mathrm{Alg}$ with respect to an invariant norm.

Moreover this completion is a topologically irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. This follows from the irreducibility of $V$, hence of $D(V)$.

In this identification, one has:
• the action of $\mathcal{O}_E[[X]]$ on $(\varprojlim_\psi D(V))^{\mathrm{b}}$ corresponds to the action of the Iwasawa algebra $\mathcal{O}_E[[\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)]]$ on the dual of the completion of $\pi \otimes_E \mathrm{Alg}$
• the action of $\Gamma \simeq \mathbb{Z}_p^\times$ on $(\varprojlim_\psi D(V))^{\mathrm{b}}$ corresponds to the action of $\left(\begin{smallmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{smallmatrix}\right)$
• the action of $\psi$ on $(\varprojlim_\psi D(V))^{\mathrm{b}}$ corresponds to the action of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

As $(\varprojlim_\psi D(V))^{\mathrm{b}} \simeq (\varprojlim_\psi (V'))^{\mathrm{b}}$ if and only if $V \simeq V'$, this proves that there are at least as many unitary completions of $\pi \otimes_E \mathsf{Alg}$ as irreducible $V$, i.e. as irreducible admissible filtrations on $D$.

In (i), one proves moreover that the resulting completion turns out to be the completion of $\pi \otimes_E \mathsf{Alg}$ with respect to an $\mathcal{O}_E$-lattice which is of finite type over $\mathcal{O}_E[\mathrm{GL}_2(\mathbb{Q}_p)]$ (this is false in (ii)). The topological irreducibility implies that this is then the only possible completion.

The difficulty for the general case is that it is easy to define an action of the upper Borel on such representations as $(\varprojlim_\psi D(V))^{\mathrm{b}}$ (see above), but the action of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ seems hard to see directly on $(\varprojlim_\psi D(V))^{\mathrm{b}}$.

This second question is completely open for other groups than $\mathrm{GL}_2(\mathbb{Q}_p)$. For instance, in order to generalize the $(\varphi, \Gamma)$-module method (at least in a naive way), one is immediately confronted with the following question:

**Question 12** *Let $N$ be the upper unipotent subgroup of $\mathrm{GL}_n(K)$. Can one find a "$(\varphi, \Gamma)$-module theory" (corresponding to some Galois representations (?)) where the role of:*

$$\mathcal{O}_E[[X]] = \mathcal{O}_E[[\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)]]$$

*is played by:*

$$\mathcal{O}_E[[N \cap \mathrm{GL}_n(\mathcal{O}_K)]]?$$