Universités de Paris 6 & Paris 7 - CNRS (UMR 7599)

PRÉPUBLICATIONS DU LABORATOIRE

DE PROBABILITÉS & MODÈLES ALÉATOIRES

4, place Jussieu - Case 188 - 75 252 Paris cedex 05
http://www.proba.jussieu.fr

Genealogy of shocks in Burgers turbulence with white noise initial velocity

C. GIRAUD

JUILLET 2000

Prépublication n^o 605

Laboratoire de Probabilités et Modèles Aléatoires, CNRS-UMR 7599,
Université Paris VI & Université Paris VII,
4, place Jussieu, Tour 56, 3^o étage, F-75252 Paris Cedex 05.

Genealogy of shocks in Burgers turbulence with white noise initial velocity.

Christophe Giraud

Laboratoire de Probabilits et Modles Alatoires

Universit Pierre et Marie Curie, et C.N.R.S. UMR 7599

175, rue du Chevaleret

F-75013 PARIS, France.

Abstract : As time passes, the shocks of the solution of the inviscid Burgers equation aggregate. We characterize, in the case of white noise initial velocity, the stochastic fragmentation process obtained when time runs backwards. In other words, we specify the law of the genealogy of the shocks of the Burgers turbulence with white noise initial velocity.

Key words : Burgers turbulence, sticky particles, white noise initial velocity, Brownian excursion.

A.M.S. classification : 35Q53, 60J65.

e-mail : giraud@clipper.ens.fr

1 Introduction.

Burgers equation

$$\partial_t u + u.\nabla u = \mu \Delta u \tag{1}$$

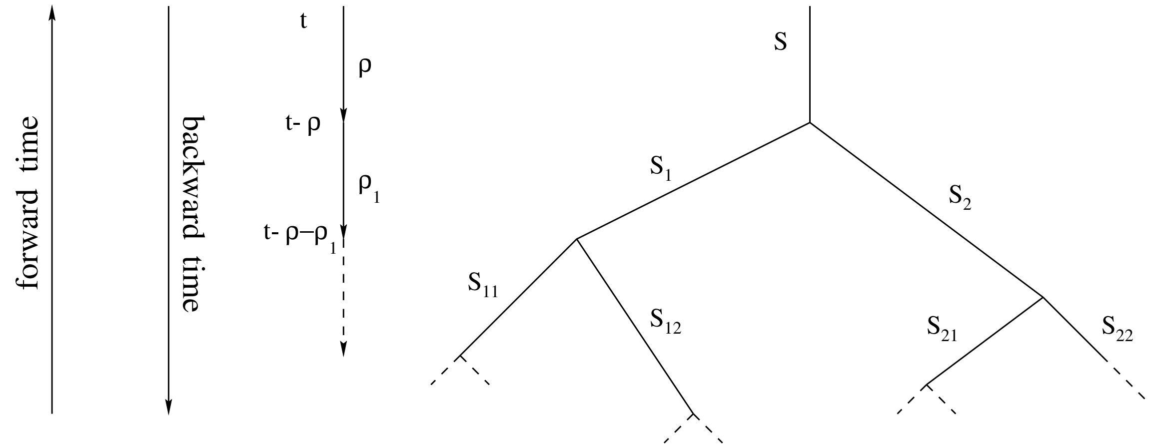
is a simplified version of the Navier-Stokes equation, in which the terms of pressure and force are neglected. First introduced by Burgers (see [6]) as a simple model for hydrodynamic turbulence, Burgers equation also appears in different fields such as the theory of the interfaces for ballistic aggregation or in cosmology for the formation of superstructures in the universe (see [19] and references therein).

We focus thenceforth on dimension one when the viscosity μ tends to 0. It is known that the solution u_μ of (1) converges when $\mu \to 0$ to $u_0 = u$, which is the (weak) entropy solution of the so-called inviscid Burgers equation

$$\partial_t u + u \partial_x u = 0. \tag{2}$$

Several studies concern statistics of the solution of (2) when the initial velocity $u(., 0)$ is a random process, see [6, 13, 19]. The solution $u(., t)$ at fixed time t of (2) is now well known in many cases, such as for example when $u(., 0)$ is a Brownian motion (see [18, 4])or a white noise (see [2, 3, 6, 11, 16, 17]). Recently, Bertoin has studied the evolution in time of the solution u when the initial velocity $u(., 0)$ is a Brownian motion, and proved a striking connection with the additive coalescent, see [5].

The aim of this paper is to describe the evolution in time of u when $u(.,0)$ is a white noise. The solution $u(.,t)$ is then a toothpath with a discrete sequence of discontinuities and segments of slope $1/t$. In particular, the solution $u(.,t)$ is determined by the location and amplitude of its discontinuities, which are usually called shocks. As time passes, when two shocks "collide", they form a single shock with amplitude the sum of the amplitudes of the two previous shocks. The evolution in time of u is thus governed by the deterministic dynamic of clustering of the shocks. This dynamic induces a loss of information, in the sense that if $t_1 < t_2$, the solution $u(.,t_1)$ cannot be recovered from $u(.,t_2)$. When time runs backwards, we thus obtain a random process of fragmentation : a shock S splits after a certain time ρ into two shocks S_1 and S_2, which in turn split after a time ρ_1 and ρ_2 into S_{11}, S_{12} and S_{21}, S_{22}, etc...



The purpose of the present work is to describe the law of this process of fragmentation, that is the genealogy of the shocks.

In this direction it is convenient to use the sticky particles interpretation of the inviscid Burgers equation. Consider at time $t = 0$, infinitesimal particles uniformly distributed on the line, with initial velocity $u(.,0)$. We suppose that they evolve with the dynamic of completely inelastic shocks. This means that the velocity of a particle only changes in case of collision, and when two (clusters) of particles collide, they form a heavier cluster with conservation of mass and momenta. The evolution of the system is completely described by the entropy solution of (2), and $u(x,t)$ then represents the velocity of the particle located at x at time t. This case of white noise initial velocity arises for example at the infinitesimal limit of a discrete sticky particles system, when particles are evenly spread with i.i.d. velocities (with zero mean and finite variance). A shock of $u(.,t)$ represents a cluster at time t with mass given by $t\times$the amplitude of the shock. The fragmentation of the shocks then corresponds to the fragmentation of the clusters obtained by reversing the dynamic of the sticky particles.

We prove here that,in the case of white noise initial data, conditionally on the state of the system at time t, masses split independently of their location, velocity and also of their environment. We compute then the law of the parameters which describe the splitting of a mass, and which also characterize the law of the fragmentation process.

We present in the second section some material for the study of the system. We specified the fragmentation in the second section. The third section is devoted to the proofs of the preliminary results. The Appendix contains the somewhat technical proofs of section 4.

2 Preliminaries.

2.1 Inviscid Burgers equation and sticky particles.

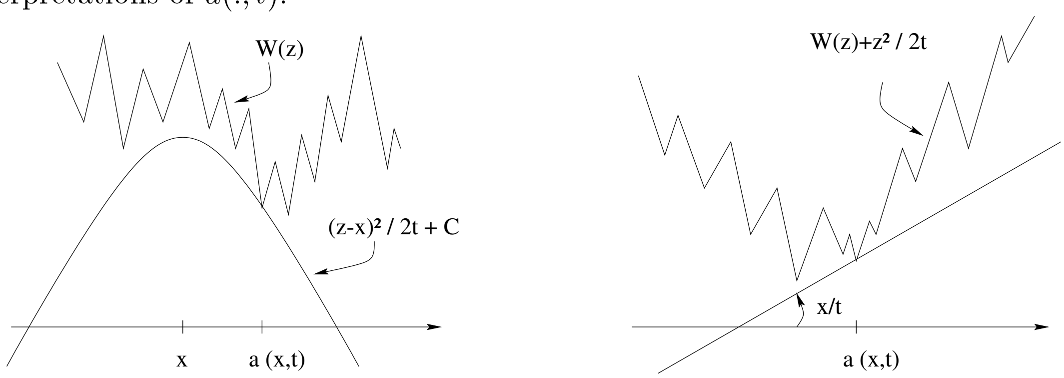
Let us consider the entropy solution u of the inviscid Burgers equation. This is the weak solution of (2) such that $x \mapsto u(x,t)$ has only discontinuities of the first kind and no positive jumps. We shall work here with the version satisfying

$$u(x,t) = \frac{u(x+,t) + u(x-,t)}{2},$$

where $u(x+,t)$ and $u(x-,t)$ refers to the right and left limit of $u(.,t)$ at x. We call initial potential the process

$$W(z) = \int_0^z u(x,0)dx, \ z \in \mathbb{R}.$$

When $W(z) = o(|z|^2)$ for $z \to \pm\infty$, we denote by $a(x,t)$ the largest location of the minimum of the function $z \to W(z) + \frac{1}{2t}(z-x)^2$. We have the following two geometrical interpretations of $a(.,t)$.



First, consider a parabola $z \mapsto -\frac{1}{2t}(z-x)^2 + C$, with C chosen such that this parabola is strictly below the path of W. Let C increase until this parabola touches the graph of

W. The largest abscissa of the contact points is then $a(x,t)$. Second, bring up a line of slope x/t, until it touches the graph of $z \mapsto W(z) + \frac{1}{2t}z^2$. The largest abscissa of contact is again $a(x,t)$. The function $a(.,t)$ may be thus described in terms of the convex minorant of the initial potential with a $\frac{1}{2t}$-parabolic drift, $z \mapsto W(z) + \frac{1}{2t}z^2$, or in terms of the "$\frac{1}{2t}$-parabolic" minorant of the initial potential W. One notices in particular that $a(.,t)$ is right continuous and non decreasing.

We can express the entropy solution of the inviscid Burgers equation in terms of $a(.,t)$. Yet, it is known (see Hopf [12] or Cole [7]) that

$$u(x,t) = \frac{x - a(x,t)}{t}$$

at every x where $a(.,t)$ is continuous, and

$$u(x,t) = \frac{u(x+,t) + u(x-,t)}{2},$$

elsewhere.

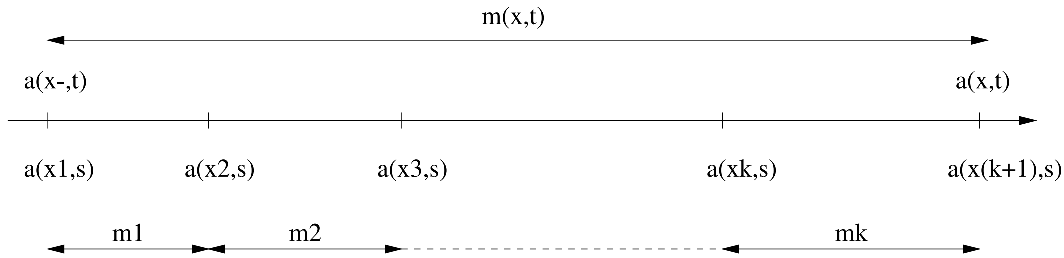
We have seen that the function $a(.,t)$ is right continuous and non decreasing. It possesses therefore a right continuous inverse $x(a,t) = \inf\{y \in \mathbb{R} : a(y,t) > a\}$, which is known as the *Lagrangian function*. In the sticky particles interpretation, the latter gives the location at time t of the particle started from $a \in \mathbb{R}$. In particular, the *Eulerian shock points* at time t, which are the abscissaes x of discontinuity of $a(.,t)$, are the locations of clusters in the system at time t. They have mass $m(x,t) = a(x+,t) - a(x-,t)$ and velocity

$u(x,t)$, since (conservation of momenta)

$$u(x,t) = \frac{u(x+,t)+u(x-,t)}{2} = \frac{1}{a(x+,t)-a(x-,t)} \int_{a(x-,t)}^{a(x+,t)} u(z,0)dz.$$

When the initial potential W is a Brownian motion (i.e. $u(.,0)$ is a white noise), it was shown (see [2, 11]) that $a(.,t)$ is a.s. a step function, which means that $u(.,t)$ is a toothpath. The particles are thus located at time $t>0$ on a discrete set of clusters. One says that the shock structure is discrete.

We are interested in the fragmentation of a cluster located at x at time t, when time runs backwards. We write $m_1(x,s,t),..,m_k(x,s,t)$ for the masses of the clusters at time s which form this cluster at time t. One may notice that $m_1(x,s,t),..,m_k(x,s,t)$ are exactly the length of the intervals of the partition of $[a(x-,t),a(x,t)]$ induced by the range of $a(.,s)$. We give therefore the following definition.



Definition 1 *For any Eulerian shock point x at time t and $0<s<t$, we set*

$$\mathcal{M}(x,s,t) = (m_1(x,s,t),..,m_k(x,s,t)),$$

where $m_1(x,s,t),..,m_k(x,s,t)$ are the length of the intervals appearing in the partition of $[a(x-,t),a(x,t)]$ by the range of $a(.,s)$ ranked according to the increasing order of their *location.* $\mathcal{M}(x,s,t)$ *takes values in the space of finite positive numerical sequence* $\mathbb{S} = \cup_{n\in\mathbb{N}^*}]0,\infty[^n$.

The process $(\mathcal{M}(x,t-r,t); 0 \le r \le t)$ *will be called the fragmentation process.*

We notice in the following lemma that the fragmentation of a cluster only depends of the "excursions" of the initial potential above the "parabolic minorant". We write in the sequel $\mathcal{E} = \cup_{m>0}\{m\} \times \mathrm{C}([0,m],\mathbb{R}^+)$ for the space of positive excursions, where m is meant to represent the duration of the excursion.

Lemma 1 *There exists a function* $F : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{E} \to \mathbb{S}$ *such that for any initial potential* W *which induces a discrete shock structure, and any Eulerian shock point* x *at time* t*, we have*

$$\mathcal{M}(x,s,t) = F(s,t,m(x,t),\varepsilon^{(x,t)}), \qquad \textit{for} \quad 0 < s < t,$$

where

$$\varepsilon^{(x,t)}(z) = W(a(x-,t)+z) - W(a(x-,t)) - \frac{1}{2t}z(m(x,t)-z) - zu(x,t), \ \textit{for } 0 \le z \le m(x,t).$$

Proof of Lemma 1.

We set

$$\overline{\varepsilon^{(x,t)}}(z) = W(a(x-,t)+z) - W(a(x-,t)) - \frac{1}{2t}z(m(x,t)-z) - zu(x,t), \ \text{for } z \in \mathbb{R}.$$

For any $y \in \mathbb{R}$ and $0 < s < t$, we have

$$\begin{aligned} a(y,s) &= a(x-,t) + \operatorname*{argmin}_{z \in R} \left(W(a(x-,t)+z) - W(a(x-,t)) + \frac{1}{2s}(z-y)^2 \right) \\ &= a(x-,t) + \operatorname*{argmin}_{z \in R} \left(\overline{\varepsilon^{(x,t)}}(z) + \frac{1}{2t} z(m(x,t)-z) + \frac{1}{2s}(z-y-su(x,t))^2 \right), \end{aligned}$$

where $\operatorname{argmin}_{z \in \mathbb{R}}(g(z))$ denotes the largest z that minimizes g. We deduce so that the range of $a(.,s) - a(x-,t)$ only depends on $s, t, m(x,t)$ and $\overline{\varepsilon^{(x,t)}}$. Now the intersection of the range of $a(.,s) - a(x-,t)$ with $[0, m(x,t)]$ depends on $s, t, m(x,t)$ and the restriction of $\overline{\varepsilon^{(x,t)}}$ to $[0, m(x,t)]$, which is $\varepsilon^{(x,t)}$. Since $M(x,s,t)$ is the finite sequence of the length of the intervals of the partition of $[0, m(x,t)]$ by the range of $a(.,s) - a(x-,t)$, it is therefore a function of $s, t, m(x,t)$ and $\varepsilon^{(x,t)}$. The proof is complete.

□

2.2 Laplace transform of the integral of a 3-d Bessel bridge (after Groeneboom).

We recall in this subsection the value of the Laplace transform of the integral of a 3-d Bessel bridge, which shall appear in further calculations. Let $(\alpha_n, n \in \mathbb{N}^*)$ denotes the negative decreasing sequence of the zeros of the Airy function Ai as defined on p. 446 of

[1]. We introduce

$$
\begin{aligned}
C(\lambda) =& \sqrt{2\pi}\lambda \sum_{n=1}^{\infty} \exp(2^{-1/3}\alpha_n \lambda^{2/3}),\\
\text{and } F^{(x,y)}(\lambda) =& 2^{-1/3}\lambda^{2/3} \sum_{n=1}^{\infty} \frac{\mathrm{Ai}(2^{1/3}\lambda^{1/3}y + \alpha_n)}{\mathrm{Ai}'(\alpha_n)} \exp(2^{-1/3}\lambda^{2/3}x\alpha_n).
\end{aligned}
$$

Groeneboom has computed the Laplace transform of the integral of a three dimensional Bessel bridge, see [11] Theorem 2-1, formulaes (4-9) and (4-13).

Lemma 2 *For $x, y \geq 0$, let $\beta_{0\to y}^{[x]}$ be a three dimensional Bessel bridge with duration x sarting at 0 and ending at y. We set*

$$
L^{(x,y)}(\lambda) = \mathbb{E}\left(\exp\left(\lambda \int_0^x u d\beta_{0\to y}^{[x]}(u)\right)\right).
$$

We have

$$
L^{(x,y)}(\lambda) = \begin{cases} \sqrt{2\pi\lambda^3}\, \exp\left(\lambda xy + y^2/2\lambda\right) F^{(x,y)}(\lambda)/\lambda & \text{for } y > 0 \\ \mathbb{E}\left(\exp\left(-\lambda \int_0^x e^{[x]}(s)\, ds\right)\right) = C\left(x^{3/2}\lambda\right) & \text{for } y = 0\ , \end{cases}
$$

where $e^{[x]}$ denotes a Brownian excursion with duration x.

2.3 Excursions above parabolaes.

We state here some useful results on excursions conditioned to stay above a parabola. The proof of these results are given in section 4. For any $a, m > 0$ we introduce the probability measure

$$\nu(a,m) = \frac{\exp\left(-a\int_0^m X_s ds\right)}{C(am^{3/2})}\,\mathbb{P}^{[m]}$$

which is absolutely continuous with respect to the law $\mathbb{P}^{[m]}$ of the Brownian excursion (or Bessel bridge) with duration m, where X_s denotes the canonical process and the normalizing factor C has been defined in the previous subsection. In the sequel, we will often write $e^{[m]}$ for a process with the law $\mathbb{P}^{[m]}$ of the Brownian excursion with duration m. We first connect $\nu(a,m)$ to the law of an excursion conditioned to stay above a parabola.

Lemma 3 *For any $a, m > 0$, we set*

$$p^{(a,m)}(z) = \frac{a}{2}z(m-z).$$

If $e^{[m]}$ is a Brownian excursion with duration m, then the law of $e^{[m]} - p^{(a,m)}$ conditionally on $e^{[m]} \geq p^{(a,m)}$ is $\nu(a,m)$.

We introduce now two variables

$$\begin{cases} \sigma(m) = \sup\{a \geq 0; e^{[m]} \geq p^{(a,m)}\} \\ \eta(m) = \text{ the largest abscissa } z_0 \in (0,1) \text{ such that } e^{[m]}(z_0) \geq p^{(\sigma,m)}(z_0). \end{cases}$$

We shall write in the sequel $e = e^{[1]}$, $\sigma = \sigma(1)$ and $\eta = \eta(1)$. Let us compute the law of $(\sigma(m), \eta(m))$.

Lemma 4 *Law of* $(\sigma(m), \eta(m))$.

i) *For any $m>0$ we have the scaling identity*

$$(\sigma(m),\eta(m)) \overset{law}{=} (m^{-3/2}\sigma, m\,\eta).$$

ii) *The law of (σ,η) is given by*

$$\mathbb{P}\left(\eta > x, \sigma \in da\right) = e^{-\frac{a^2}{24}}\left(\frac{a}{12}(1-x)^2(1-4x)C(a) - \partial_2 G^{(x)}(a,0)\right)da,$$

where

$$G^{(x)}(a,b) = \int_0^\infty y(y+\beta)e^{-y^2/2x(1-x)+(b(1-x)/2-a)y}\,L^{(x,y)}(a)\,L^{(1-x,y+\beta)}(a-b)\,\frac{dy}{\sqrt{2\pi x^3(1-x)^3}},$$

with $\beta = p^{(b,1)}(x) = \frac{b}{2}x(1-x)$ and $L^{(x,y)}$, C defined in the previous subsection.

In particular we have

$$\mathbb{P}(\sigma \geq a) = e^{-a^2/24}C(a).$$

We give in the last lemma the law of an excursion conditionally on $(\sigma = a, \eta = x)$. In this direction, it is convenient to define the concatenation of two processes $(\varepsilon_1(z); 0\leq z\leq m_1)$ and $(\varepsilon_2(z); 0 \leq z \leq m_2)$ as the process

$$\left(\varepsilon_1(z)\mathbf{1}_{0\leq z\leq m_1} + \varepsilon_2(z)\mathbf{1}_{m_1\leq z\leq m_1+m_2};\ 0\leq z\leq m_1+m_2\right).$$

Lemma 5 *Conditionally on* $(\sigma = a, \eta = x)$, $e - p^{(\sigma,1)}$ *has the law of the concatenation of two independent processes of law* $\nu(a,x)$ *and* $\nu(a, 1-x)$.

As a consequence, the law of $e^{[m]} - p^{(\sigma,m)}$ *under* $\nu(a,m)$ *conditionally on* $(\sigma(m) = b-a, \eta(m) = xm)$ *is the law of the concatenation of two independent processes of law* $\nu(xm, b)$ *and* $\nu((1-x)m, b)$.

Remark

The previous lemma ensures in particular that there exists a.s. a unique abscissa $z_0 \in (0,1)$ such that $e^{[m]}(z_0) = p^{(a,m)}(z_0)$. Indeed, an excursion is a.s. positive on (0,1), and this property still holds under $\nu(a,m)$. We can start now our investigations.

2.4 Conditional distribution of the initial data.

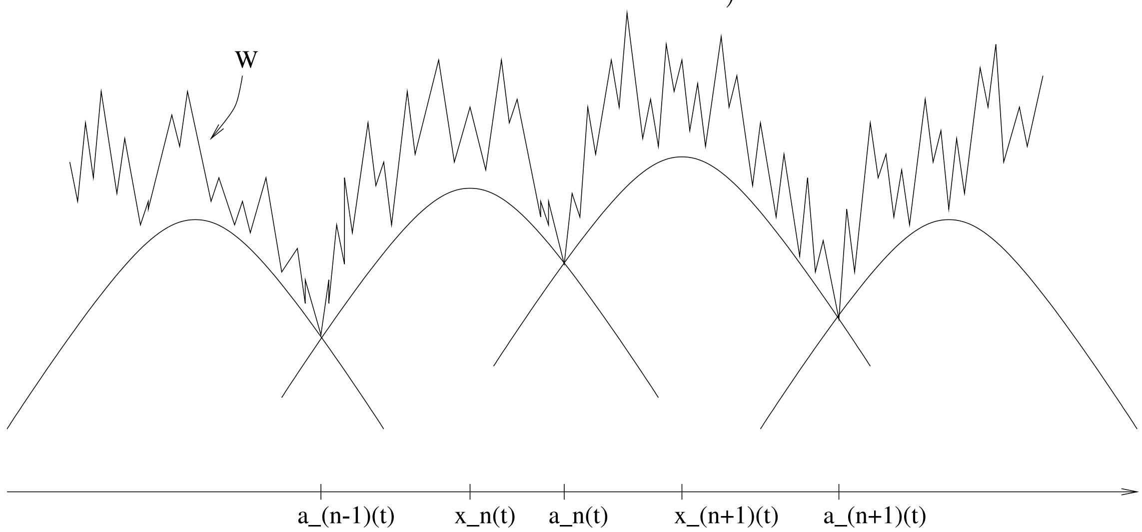
Clusters are ranked according to the increasing order of their location, with the convention that $x_1(t)$ is the location of the first cluster at the right of 0. From a physical point of view, the state of the system at time t is described by the sequence $((x_n(t), m_n(t), v_n(t)); n \in \mathbb{Z})$, where $m_n(t)$ and $v_n(t)$ are the mass and velocity of the n-th cluster. Yet, one may notice that the useful datum is the sequence $((x_n(t), a_n(t)); n \in \mathbb{Z})$, where $a_n(t) = a(x_n(t), t)$. Indeed, we have

$$m_n(t) = a_n(t) - a_{n-1}(t)$$

$$\text{and} \qquad v_n(t) = \frac{2x_n(t) - (a_n(t) + a_{n-1}(t))}{2t}.$$

We thus introduce $\mathcal{F}_t = \sigma\left(\left(x_n\left(t\right), a_n\left(t\right)\right); n \in \mathbb{Z}\right)$ which is the datum given by the state of the system at time t. The law of $\left(\left(x_n\left(t\right), m_n\left(t\right), v_n\left(t\right)\right); n \in \mathbb{Z}\right)$ is known see formulaes (99) and the following remark, (48),(52),(53),(64),(65) and (68) in [8].

We compute now the law of the initial potential W conditionally on the state of the turbulence at time t. We prove that the pieces of Brownian motion between the Lagrangian points $a_n(t)$ are independent conditionally on $\mathcal{F}_t$ and we connect their law to $\nu(a,m)$ (recall that this distribution has been introduced in section 2-3).



Lemma 6 *The "excursions" of the Brownian motion above the "$\frac{1}{2t}$-parabolic minorant"*

$$\varepsilon^{(x_n(t),t)}(z) = W(z + a_{n-1}(t)) - W(a_{n-1}(t)) - p^{(m_n(t),1/t)}(z) - zv_n(t),\ 0 \le z \le m_n(t),$$

are independent conditionally on $\mathcal{F}_t$ and their conditional law is $\nu(m_n(t), 1/t)$.

This lemma is the key of our analysis, since it connects the study of the fragmentation to the law $\nu(a,m)$. Let us sketch an explanation for the origin of this result.

Groeneboom [10] has studied the convex minorant of the Brownian motion (he actually focused on the concave majorant, which is its symmetric about the abscissaes line). He has shown it is a piecewise linear path, and conditionally on the edge points of this path, the Brownian motion realizes independent Brownian excursions above each segment of the convex minorant. By Girsanov Theorem, adding a parabolic drift $z \mapsto \frac{1}{2t}z^2$ to the Brownian motion amonts to work under the probability measure

$$\exp\left(\frac{1}{t}\int_0^T z dW_z - \frac{T^3}{6t^2}\right)\mathbb{P})_{\mathcal{G}_T},$$

where $\mathcal{G}_T = \sigma(W_s; 0 \leq s \leq T)$. In other words conditionally on the "$\frac{1}{2t}$−parabolic minorant" of the Brownian motion, the excursions $\varepsilon^{(x_n(t),t)}$ of the Brownian motion above the "$\frac{1}{2t}$−parabolic minorant" are independent and have the law $\nu(m_n(t), 1/t)$.

Nevertheless, we need to investigate the convex minorant of a Brownian motion on a finite interval in order to apply Girsanov Theorem, which cannot be deduced easily from the convex minorant of a Brownian motion on $[0, \infty)$. However, we can prove Lemma 6 from results in [11], and this somewhat technical proof is given in the Appendix.

3 Fragmentation statistics.

3.1 Statement of the main results.

We thenceforth turn our attention to the process of fragmentation generated by the dynamic of sticky particles as time runs backwards. Since the clustering dynamic of sticky particles is deterministic and induces a loss of information, the fragmentation process obtained by time reversal is a stochastic Markovian process. Our aim is to describe this process.

Recall that $(x_n(t), m_n(t), v_n(t))$ denotes the (location, mass, velocity) of the n-th cluster at time t and $\mathcal{F}_t = \sigma((x_n(t), m_n(t), v_n(t)); n \in \mathbb{Z})$. We point out that $(\mathcal{F}_t, t \geq 0)$ is a backwards filtration, since the evolution of the system is deterministic and induces a loss a information. The variable $\mathcal{M}(x, s, t)$ is defined in section 2-1 (definition 1). We first specify the dependence of the fragmentation of a cluster and its environment. In this direction, we write $\mu(t, m)$ for the law of $(\mathcal{M}(x_1(t), t-r, t); 0 \leq r \leq t)$ conditionally on $(m_1(t) = m)$.

Theorem 1 *Fix $t > 0$. Conditionally on the state $\mathcal{F}_t$ of the turbulence at time t, the fragmentation processes $(\mathcal{M}(x_n(t), t-r, t); 0 \leq r \leq t)$, $n \in \mathbb{Z}$, are independent, and their conditional laws only depend on $(t, m_n(t))$. More precisely, they are given by $\mu(t, m_n(t))$.*

Roughly, Theorem 1 claims that conditionally on the state of the system at time t, the masses of the clusters at time $t-r$ are obtained by breaking into pieces each cluster at time t independently of its location and velocity and of the other clusters. In particular, we deduce from Theorem 2 that for any Eulerian shock point x, the fragmentation process $(\mathcal{M}(x, t-r, t); 0 \leq r \leq t)$ is an inhomogeneous Markov process.

We want now to describe the law $\mu(t, m)$. We denote by $\rho(t, m)$ (or simply ρ) the time at which the cluster located at $x_1(t)$ at time t splits conditionally on $(m(x_1(t), t) = m)$. We will check that we have a binary splitting $\mathcal{M}(x_1(t), t-\rho, t) = (m_1, m_2)$ and we denote by $R(t, m)$ (or simply R) the ratio $R = m_1/(m_1 + m_2)$. The next theorem characterizes the law $\mu(t,m)$, in terms of the law of the variable (ρ, R). We introduce in this aim the operation $*$ on the space of finite numerical sequences : for any $M_1 = (m_1^1, .., m_{k_1}^1)$ and $M_2 = (m_1^2, .., m_{k_2}^2)$, we write

$$M_1 * M_2 = (m_1^1, .., m_{k_1}^1, m_1^2, .., m_{k_2}^2).$$

We can state now our result which is a splitting property at time ρ.

Theorem 2 *For any $t, m > 0$, let $M = (M(r); 0 \leq r \leq t)$ be a process of law $\mu(t,m)$. We have*

$$M(\rho + r) = M_1(r) * M_2(r) \qquad\qquad \textit{for} \;\; 0 \leq r \leq t - \rho,$$

where M_1 and M_2 are independent conditionally on (ρ, R) with conditional law $\mu(t-\rho, Rm)$ and $\mu(t-\rho, (1-R)m)$.

In other words, conditionally on $m(x_1(t), t)$ the cluster located at $x_1(t)$ at time t splits after a time $\rho_0 = \rho(t, m(x_1(t), t))$ into two clusters of mass $m_1 = R_0\, m(x_1(t), t)$ and $m_2 = (1 - R_0)\, m(x_1(t), t)$, where $R_0 = R(t, m(x_1(t), t))$. Moreover the fragmentation processes of these two clusters are independent conditionally on $(m(x_1(t), t), R_0, \rho_0)$ and their conditional law are $\mu(t - \rho_0, m_1)$ and $\mu(t - \rho_0, m_2)$. We can therefore iterate Theorem 3 in order to obtain that in turn each of these two clusters splits into two clusters of mass

$$m_{11} = R(t - \rho_0, m_1)\, m_1, \quad m_{12} = (1 - R(t - \rho_0, m_1))\, m_1$$

and

$$m_{21} = R(t-\rho_0, m_2)\, m_2, \quad m_{22} = (1 - R(t-\rho_0, m_2))\, m_2$$

at time $\rho_1 = \rho(t-\rho_0, m_1)$ and $\rho_2 = \rho(t-\rho_0, m_2)$, and so on.

The last theorem gives the joint law of $\rho(t,m)$ and $R(t,m)$ which completes the characterisation of the law $\mu(t,m)$.

Theorem 3 *For any $0 < r < t$, $m > 0$ and $0 \le \alpha \le 1$, the law of $(\rho(t,m), R(t,m))$ is given by*

$$\mu(t,m)\,(\rho \in dr, R > \alpha) = \left(\frac{m^{3/2}(1-\alpha)^2(1-4\alpha)}{12(t-r)} C\left(\frac{m^{3/2}}{t-r}\right) - \partial_2 G^{(\alpha)}\left(\frac{m^{3/2}}{t-r}, 0\right)\right) \\ \times \exp\left(-\frac{m^3}{24}\frac{(2t-r)r}{t^2(t-r)^2}\right) \frac{m^{3/2} dr}{C(m^{3/2}/t)(t-r)^2},$$

where $G^{(\alpha)}(a,b)$ is defined in Lemma 4 and C in section 2-2.

In particular, we have

$$\mu(t,m)(\rho \ge r) = \exp\left(-\frac{m^3}{24}\frac{(2t-r)r}{t^2(t-r)^2}\right) \frac{C(m^{3/2}/(t-r))}{C(m^{3/2}/t)}.$$

We deduce therefore the following asymptotics for the distribution of the splitting time ρ.

Corollary 3.1 *For any $t, m > 0$, we have*

$$\mu(t,m)(\rho \ge r) \sim_{r\to t} \frac{\exp\left(m^3/24t^2\right)}{C(m^{3/2}/t)} \frac{\sqrt{2\pi} m^{3/2}}{t-r} \exp\left(-\frac{m^3}{24(t-r)^2} + 2^{-1/3}\alpha_1 \frac{m}{(t-r)^{2/3}}\right),$$

with $\alpha_1 \approx -2.3381$, *and*

$$\mu(t,m)(\rho \le r) \sim_{r\to 0} \left(\frac{m^{3/2}}{12t} - \frac{C'(m^{3/2}/t)}{C(m^{3/2}/t)}\right) \times \frac{r}{t^2},$$

where C *is defined in section 2-2.*

Let us prove now these results.

3.2 Proofs.

Proof of Theorem 2.

Lemma 1 ensures that the fragmentation processes may be written

$$\mathcal{M}(x_n(t), t - ., t) = F(t - ., t, m_n(t), \varepsilon^{(x_n(t),t)}).$$

We thus deduce from Lemma 6 that they are independent conditionally on $\mathcal{F}_t$. Since the $\mathcal{F}_t$-conditional law of the processes $\varepsilon^{(x_n(t),t)}$, $n \in \mathbb{Z}$ is $\nu(t, m_n(t))$, the $\mathcal{F}_t$-conditional law of $\mathcal{M}(x_n(t), t - ., t)$, $n \in \mathbb{Z}$ only depends on $(t, m_n(t))$ and is $\mu(t, m_n(t))$.

Proof of Theorem 3.

The law of the excursion $\varepsilon^{(x_1(t),t)}$ of the Brownian motion above the "$\frac{1}{2t}$-parabolic minorant" is $\nu(m(x_1(t),t), 1/t)$. We write $\tau = t - \rho$,

$$\varepsilon_1(z) = \varepsilon^{(x_1(t),t)}(z) - p^{(m(x_1(t),t),1/\tau - 1/t)}(z), \qquad \text{for} \quad 0 \le z \le m_1 = Rm(x_1(t),t)$$

and

$$\varepsilon_2(z) = \varepsilon^{(x_1(t),t)}(m_1+z) - p^{(m(x_1(t),t),1/\tau-1/t)}(m_1+z), \text{ for } \ 0 \le z \le m_2 = m(x_1(t),t) - m_1.$$

Since

$$\frac{1}{\tau} = \sup\left\{a \ge 0; \varepsilon^{(x_1(t),t)}(z) \ge p^{(m(x_1(t),t),a-1/t)}, \text{ for } 0 \le z \le m(x_1(t),t)\right\},$$

Lemma 5 ensures that conditionally on $(m(x_1(t),t) = m, 1/\tau = 1/t_1 - 1/t, R = x)$ the law of ε_1 is $\nu(xm, 1/t_1)$ and the law of ε_2 is $\nu((1-x)m, 1/t_1)$.

Moreover, since the fragmentation of $[a(x_1(t)-,t), a(x_1(t),t)]$ after time ρ is obtained from the fragmentation of $[a(x_1(t)-,t), a(x_1(t)-,t) + Rm(x_1(t),t)]$ and $[a(x_1(t)-,t) + Rm(x_1(t),t), a(x_1(t),t)]$ we deduced from Lemma 1 that

$$\begin{aligned}\mathcal{M}(x_1(t),\tau-r,t) =& F(\tau-r,\tau,m(x_1(t),t),\varepsilon^{(x_1(t),\tau)})\\ =& F(\tau-r,\tau,m_1,\varepsilon_1) * F(\tau-r,\tau,m_2,\varepsilon_2).\end{aligned}$$

Putting pieces together, one obtains Theorem 3.

Proof of Theorem 4.

Conditionally on $m(x_1(t),t) = m$, we have :

$$\frac{1}{\tau} = \frac{1}{t-\rho} = \sup\left\{a \ge 0; \varepsilon^{(x_1(t),t)}(z) \ge p^{(m,a-1/t)}, \text{ for } 0 \le z \le m\right\}.$$

We thus have with Lemma 6 and the scaling property of Brownian excursions

$$\mu(t,m)(\rho \in dr, R < \alpha) = \mathbb{E}\left(\frac{\exp\left(-m^{3/2}t^{-1}\int_0^1 e_s ds\right)}{C\left(m^{3/2}/t\right)} ; \sigma \in d\left(\frac{m^{3/2}}{t-r} - \frac{m^{3/2}}{t}\right); \eta > \alpha \right),$$

where σ and η are defined in section 2-3. The Lemma 3 ensures now that

$$\begin{aligned}\mu(t,m)(\rho \in dr, R < \alpha) &= \mathbb{P}\left(\sigma \in d\left(\frac{m^{3/2}}{t-r}\right); \eta > \alpha | \sigma \geq \frac{m^{3/2}}{t}\right) \\ &= \mathbb{P}\left(\sigma \in d\left(\frac{m^{3/2}}{t-r}\right); \eta > \alpha\right) / \mathbb{P}\left(\sigma \geq \frac{m^{3/2}}{t}\right).\end{aligned}$$

We deduce so the joint law of (τ, R) from the joint law of (σ, η), which has been computed in Lemma 4.

4 Proof of the preliminary results.

4.1 Proof of Lemma 3.

Lemma 3 is mainly an application of Girsanov formula. Let W be a Brownian motion starting from $x > 0$ under $\mathbb{P}^x$. For any Borel bounded functional f

$$\mathbb{E}^x\left(f\left(W_s - p_s^{(a,1)}; 0 \leq s \leq 1\right)\right) = e^{-a^2/2}\mathbb{E}^x\left(f(W_s; 0 \leq s \leq 1)\exp\left(-\frac{a}{2}\int_0^1 (1-2s)dW_s\right)\right).$$

If we conditioned this identity by $(W_1 = y)$ with $y > 0$, we obtain after an integration by parts

$$\mathbb{E}\left(f\left(b^{[1]}_{x\to y}(s) - p^{(a,1)}(s); 0 \le s \le 1\right)\right) = \\ e^{a(x+y)/2-a^2/2}\mathbb{E}\left(f\left(b^{[1]}_{x\to y}(s); 0 \le s \le 1\right)\exp\left(-a\int_0^1 b^{[1]}_{x\to y}(s)ds\right)\right),$$

where $b^{[1]}_{x\to y}$ is a Brownian bridge of duration 1 from x to y. We have in particular

$$\mathbb{E}\left(f\left(b^{[1]}_{x\to y}(s) - p^{(a,1)}(s); 0 \le s \le 1\right); b^{[1]}_{x\to y} - p^{(a,1)} \ge 0\right) = \\ e^{a(x+y)/2-a^2/2}\mathbb{E}\left(f\left(b^{[1]}_{x\to y}(s); 0 \le s \le 1\right)\exp\left(-a\int_0^1 b^{[1]}_{x\to y}(s)ds\right); b^{[1]}_{x\to y} \ge 0\right).$$

We shall use now the fact that a Brownian bridge $b^{[1]}_{x\to y}$ conditioned to stay positive has the law of a Bessel(3) bridge $\beta^{[1]}_{x\to y}$. We divide the previous equality first by $\mathbb{P}\left(b^{[1]}_{x\to y} \ge 0\right)$

$$\mathbb{E}\left(f\left(\beta^{[1]}_{x\to y}(s) - p^{(a,1)}(s); 0 \le s \le 1\right); \beta^{[1]}_{x\to y} \ge p^{(a,1)}\right) = \\ e^{a(x+y)/2-a^2/2}\mathbb{E}\left(f\left(\beta^{[1]}_{x\to y}(s); 0 \le s \le 1\right)\exp\left(-a\int_0^1 \beta^{[1]}_{x\to y}(s)ds\right)\right)$$

and then by $\mathbb{P}\left(\beta^{[1]}_{x\to y} \ge p^{(a,1)}\right)$:

$$\mathbb{E}\left(f\left(\beta^{[1]}_{x\to y}(s) - p^{(a,1)}(s); 0 \le s \le 1\right) \mid \beta^{[1]}_{x\to y} \ge p^{(a,1)}\right) = \\ \mathbb{E}\left(\exp\left(-a\int_0^1 \beta^{[1]}_{x\to y}(s)ds\right)\right)^{-1} \times \mathbb{E}\left(f\left(\beta^{[1]}_{x\to y}(s); 0 \le s \le 1\right)\exp\left(-a\int_0^1 \beta^{[1]}_{x\to y}(s)ds\right)\right).$$

This means that for a Bessel(3) bridge $\beta^{[1]}_{x\to y}$ of duration 1 under $\mathbb{P}$, the law of $\beta^{[1]}_{x\to y} - p^{(a,1)}$ conditionally on $\left(\beta^{[1]}_{x\to y} \geq p^{(a,1)}\right)$ is

$$\mathbb{E}\left(\exp\left(-a\int_0^1 \beta^{[1]}_{x\to y}(s)ds\right)\right)^{-1} \times \exp\left(-a\int_0^1 \beta^{[1]}_{x\to y}(s)ds\right) \mathbb{P}.$$

Now, since the law of a Bessel(3) bridge $\beta^{[1]}_{x\to y}$ converges to the law of an excursion e when $x, y \to 0$, we claim that the law of $e - p^{(a,1)}$ conditionally on $(e \geq p^{(a,1)})$ is $\nu(a,1)$. Using the scaling property of Brownian excursions, one obtains Lemma 3.

4.2 Proof of Lemma 4.

We give first some relations between Bessel bridges.

Lemma 7 *Let $\beta^{[m]}_{x\to y}$ denotes a Bessel(3) bridge of duration m. For any $y, z, \alpha \geq 0$ and $x > 0$, the process*

$$\left(\beta^{[x]}_{z\to y+\alpha}(t) - \frac{\alpha}{x}t;\ 0 \leq t \leq x\right)$$

conditioned to stay positive has the same law as $\beta^{[x]}_{z\to y}$.

As a consequence, for $a, y \geq 0$, $x > 0$ and $\alpha = p^{(a,1)}(x) = \frac{a}{2}x(1-x)$ the law of the process

$$\beta^{[x]}_{0\to y+\alpha} - p^{(a,1)}$$

conditioned to stay positive is the same as the law of $\beta^{[x]}_{0\to y} - p^{(a,x)}$ conditioned to stay positive.

Proof of Lemma 7.

For any $y, z > 0$, let W^{z-y} be a Brownian motion starting from $z-y$. It is well known that the process

$$y + W^{z-y}(t) - \frac{t}{x} W^{z-y}(x), \qquad 0 \le t \le x,$$

is a Brownian bridge $b^{[x]}_{z\to y}$ which is independent of W^{z-y}_x. We can therefore conditioned W^{z-y} by $(W^{z-y}_x = \alpha)$ which gives

$$\left(y + b^{[x]}_{z-y\to\alpha}(t) - \frac{t}{x}\alpha;\; 0 \le t \le x \right) \stackrel{law}{=} \left(b^{[x]}_{z\to y}(t);\; 0 \le t \le x \right),$$

which may be written

$$\left(b^{[x]}_{z\to y+\alpha}(t) - \frac{\alpha}{x}t;\; 0 \le t \le x \right) \stackrel{law}{=} \left(b^{[x]}_{z\to y}(t);\; 0 \le t \le x \right).$$

Since a Brownian bridge conditioned to stay positive has the law of a Bessel(3) bridge, the process $\left(\beta^{[x]}_{z\to y+\alpha}(t) - \frac{\alpha}{x}t;\; 0 \le t \le x \right)$ conditioned to stay positive has same law as $\beta^{[x]}_{z\to y}$. The result still holds for $y = 0$ or $z = 0$ by taking the limits $y \to 0$, $z \to 0$.

The second part of the lemma follows from the first part and the equality

$$p^{(a,1)}(t) = p^{(a,x)}(t) + \frac{\alpha}{x}t. \tag{3}$$

The proof is complete.

□

We prove now Lemma 4. We set

$$\tau_a = \inf \left\{ u > 0, e(u) \leq p^{(a,1)}(u) \right\}.$$

We want to compute

$$\begin{aligned}\mathbb{P}(\eta > x, \sigma \in da) &= \mathbb{P}\left(\tau_a > x; \sigma \in da\right) \\ &= \lim_{b \downarrow 0} \frac{\mathbb{P}\left(\tau_a > x; \sigma \geq a-b\right) - \mathbb{P}\left(\tau_a > x; \sigma \geq a\right)}{b}.\end{aligned}$$

Conditionally on $(e_x = z)$, a Brownian excursion is the concatenation of two independent Bessel bridges $\beta^{[x]}_{0\to z}$ and $\beta^{[1-x]}_{z\to 0}$. This decomposition leads us to

$$\mathbb{P}\left(\tau_a > x; \sigma \geq a-b\right) = \int_0^\infty \mathbb{P}(e_x - \alpha \in dy) \mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha} \geq p^{(a,1)}\right) \mathbb{P}\left(\beta^{[1-x]}_{y+\alpha\to 0} \geq p^{(a-b,1)}(x+\cdot)\right),$$

where $\alpha = p^{(a,1)}(x)$. We deduce from Lemma 7 and (3) that

$$\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha} \geq p^{(a,1)}\right) = \mathbb{P}\left(\beta^{[x]}_{0\to y} \geq p^{(a,x)}\right) \mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha}(s) \geq \frac{\alpha}{x} s;\ 0 \leq s \leq x\right).$$

If one follows the same way as in proof of Lemma 3, one obtains

$$\begin{aligned}\mathbb{P}\left(\beta^{[x]}_{0\to y} \geq p^{(a,x)}\right) =& e^{-a^2x^3/24} \mathbb{E}\left(\exp\left(-\frac{a}{2}\int_0^x d\beta^{[x]}_{0\to y}(s)\right)\right) \\ =& e^{-a^2x^3/24} e^{-axy/2} L^{(x,y)}(a),\end{aligned}$$

where the second equality stems from Lemma 2. Let us compute $\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha}(s) \geq \frac{\alpha}{x}s;\ 0 \leq s \leq x\right)$. Let B denotes a three dimensional Bessel process starting from z under $\mathbb{P}^z$. Under $\mathbb{P}^0$, we have the identity in law

$$(sB_{1/s}; s \geq 0) \overset{law}{=} (B_s; s \geq 0),$$

which implies that

$$\begin{aligned}\mathbb{P}^0\left(B_s \geq \frac{\alpha}{x}s; 0 \leq s \leq x | B_x = y+\alpha\right) &= \mathbb{P}^0\left(B_{1/s} \geq \frac{\alpha}{x}; 0 \leq s \leq x | B_{1/x} = \frac{y+\alpha}{x}\right) \\ &= \mathbb{P}^{(y+\alpha)/x}\left(B_s \geq \frac{\alpha}{x}; s \geq 0\right),\end{aligned}$$

where the last equality follows from the Markov property of B. Moreover, it is known (see chap.VI Corollary (3-4) in [15]) that this last quantity is equal to $y/(y+\alpha)$. Putting the pieces together, we obtain

$$\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha} \geq p^{(a,1)}\right) = \frac{y}{y+\alpha} e^{-a^2x^3/24} e^{-axy/2} L^{(x,y)}(a).$$

Now, since (with $\beta = p^{(b,1)}(x)$)

$$\begin{aligned}\mathbb{P}\left(\beta^{[1-x]}_{y+\alpha\to 0} \geq p^{(a-b,1)}(x+\cdot)\right) &= \mathbb{P}\left(\beta^{[1-x]}_{0\to y+\alpha} \geq p^{(a-b,1)}\right) \\ &= \frac{y+\beta}{y+\alpha} e^{-(a-b)^2(1-x)^3/24} e^{-(a-b)(1-x)(y+\beta)/2} L^{(1-x,y+\beta)}(a-b),\end{aligned}$$

and

$$\mathbb{P}(e_x \in d(y+\alpha)) = \frac{2(y+\alpha)^2}{\sqrt{2\pi x^3(1-x)^3}} \exp\left(-\frac{(y+\alpha)^2}{2x(1-x)}\right)$$

we obtain

$$\mathbb{P}\left(\tau_a > x; \sigma \geq a-b\right) = e^{-a^2/24} e^{b(2a-b)(1-x)^3/24 - b(a-b)x(1-x)^2/4} G^{(x)}(a,b), \tag{4}$$

and at last

$$\mathbb{P}\left(\tau_a > x; \sigma \in da\right) = e^{-a^2/24} \left(\frac{a}{12}(1-x)^2(1-4x)G^{(x)}(a,0) - \partial_2 G^{(x)}(a,0)\right) da.$$

With Lemma 2 and the decomposition of an excursion into two Bessel bridges, one may check that $G^{(x)}(a,0) = C(a)$ for any $x \in [0,1]$. Finally, taking $x=0$ and $b=0$ in formula (4) gives

$$\mathbb{P}(\sigma \geq a) = e^{-a^2/24} C(a).$$

The proof of Lemma 4 is complete.

4.3 Proof of Lemma 5.

We shall mainly use here a path decomposition for Markov processes due to Millar [14]. For any $x > 0$ and $0 \leq z \leq 1$, let $\mathbb{P}^{(x,z)}$ be the law of $\{(\beta^{[1-z]}_{x\to 0}(s), z+s); 0 \leq s \leq 1-z\}$ where $\beta^{[m]}_{x\to y}$ is a Bessel(3) bridge of duration m from x to y. The canonical process X is

Markovian under $\mathbb{P}^{(x,z)}$ in its natural filtration $\{\mathcal{G}_s, s > 0\}$. We set

$$f(x,y) = \begin{cases} 2y(1-y)/x & \text{if } x > 0 \\ +\infty & \text{else.} \end{cases}$$

Under $\mathbb{P}^{(0,0)}$, we write $X(s) = (e(s), s)$ where e is a Brownian excursion. We thus have

$$\sigma = \inf_{s>0} f(X_s)$$

and η is the right most time where $f(X_s)$ reaches its overall minimum. The theorem of Millar in [14] allows us to decompose e at time η. Indeed, we apply this theorem to X under $\mathbb{P}^{(0,0)}$, and obtain that $(e(\eta+s); 0 \leq s \leq 1-\eta)$ is a Markov process independent of $(e(s); 0 \leq s \leq \eta)$ conditionally on (η, σ), with transitions

$$\mathbb{E}\left(g(e_{\eta+t}) \mid e_u; u \leq \eta + s\right) = \int g(y) H_{t-s}(e_{\eta+s}, \eta+s, \sigma; dy),$$

where

$$H_t(x, u, a; dy) = \mathbb{P}\left(\beta^{[1-u]}_{x\to 0}(t) \in dy \mid T_a = \infty\right)$$

and

$$T_a = \inf\left\{t > 0, \beta^{[1-u]}_{x\to 0}(t) < p^{(a,1)}(u+t)\right\}.$$

This means that conditionally on $(\sigma = a, \eta = x)$, the process $e_{\eta+\cdot}$ has the law of a Bessel(3) bridge $\beta^{[1-x]}_{p^{(a,1)}(x)\to 0}$ conditioned by $(T_a = \infty)$.

We have as in Lemma 7, that the process

$$\left(\beta_{\alpha\to 0}^{[1-x]}(t) - p^{(a,1)}(x+t);\ 0 \le t \le 1-x\right)$$

conditioned to stay positive has the law of

$$\left(e^{[1-x]}(t) - p^{(a,1-x)}(t);\ 0 \le t \le 1-x\right)$$

conditioned to stay positive, which is $\nu(a, 1-x)$. We have proved that the law of

$$\left(e(\eta+s) - p^{(\sigma,1)}(\eta+s); 0 \le s \le 1-\eta\right)$$

conditionally on $(\sigma = a, \eta = x)$ is $\nu(a, 1-x)$. Using the symmetry

$$(e(s); 0 \le s \le 1) \stackrel{law}{=} (e(1-s); 0 \le s \le 1)$$

of Brownian excursion, we obtain that the law of $\left(e(s) - p^{(\sigma,1)}(s); 0 \le s \le \eta\right)$ conditionally on $(\sigma = a, \eta = x)$ is $\nu(a, x)$. The first part of Lemma 5 is proved.

For any $0 \le a \le b$ and $0 \le x \le 1$, we write

$$e = e_2 + p^{(a,1)}$$

and

$$\sigma_2 = \sigma - a = \sup\left\{\alpha \geq -a, e_2(s) \geq p^{(\alpha,1)}(s),\ \text{for } 0 \leq s \leq 1\right\}.$$

Conditionally on $(\sigma = b, \eta = x)$, $e - p^{(\sigma,1)}$ has the law of the concatenation of two independent processes with law $\nu(b, x)$ and $\nu(b, 1-x)$ (first part of the lemma). Since

$$e - p^{(\sigma,1)} = e_2 - p^{(\sigma_2,1)}$$

we have that conditionally on $(\sigma = b, \eta = x)$, the law of e_2 is $\nu(a, 1)$ conditioned on $(\sigma_2 = b - a, \eta = x)$. The second part of Lemma 5 follows now from the scaling property of Brownian excursion.

5 Appendix.

The key of the proof of Lemma 6 is the following result.

Lemma 8 *Let us call $Y(z) = -W(z) - z^2$ and $Y^* = \sup_{z\in\mathbb{R}} Y(z)$. For $t_1 < t < w$ and $x < M$, we have*

$$\mathbb{P}\left(Y_t \in dx | Y^* = Y_{t_1} = Y_w = M\right) = \mathbb{E}\left(\frac{\exp\left(-2\int_0^{w-t_1} e_s^{[w-t_1]} ds\right)}{C\left(2\left(w-t_1\right)^{3/2}\right)}; e_{t-t_1}^{[w-t_1]} \in d\left(M - x\right)\right)$$

where C is defined in section 2-2.

Proof of Lemma 8.

The proof of this lemma is mainly based on results that can be found in [11]. We keep here the notation of Groeneboom, at the exception of the variables x and a, which have a different meaning. In particular, we refer to [11] for the definition of the functions which appear in the computations.

For any $t_1 < t < w$ and $x < y < M$, we deduce from (4-19), (4-20) in [11] and the Markov property of W that

$$\mathbb{P}\left(Y\left(t\right) \in dx|Y^* = Y(t_1) = M; Y\left(w\right) = y\right) = v\left(-t, x-M, -t_1\right) \frac{q^{\partial}\left(t, x, w, y\right)}{v\left(-w, y-M, -t_1\right)},$$

so

$$\lim_{y \uparrow M} \mathbb{P}\left(Y\left(t\right) \in dx|Y^* = Y(t_1) = M; Y\left(w\right) = y\right) = v\left(-t, x-M, -t_1\right) \frac{\partial_4 q^{\partial}\left(t, x, w, M\right)}{\partial_2 v\left(-w, 0, -t_1\right)}.$$

With the Lemma 2-2 in [11] we obtain that

$$\partial_4 q^{\partial}\left(t, x, w, M\right) = -2v(t, x-M, w),$$

so

$$\lim_{y \uparrow M} \mathbb{P}\left(Y\left(t\right) \in dx|Y^* = Y(t_1) = M; Y\left(w\right) = y\right) = 2\frac{v\left(-t, x-M, -t_1\right) v\left(t, x-M, w\right)}{v_2\left(-w, -t_1\right)}.$$

Moreover, we know (see [11] p.103) that

$$2v(-t, x-M, -t_1)\, v(t, x-M, w) = \mathbb{E}\left(\exp\left(-2\int_{t_1}^{w} e_s^{[w-t_1]} ds\right) | e_t^{[w-t_1]} = M - x\right)$$
$$\times f_{e_t^{[w-t_1]}}(M-x) \frac{1}{\sqrt{2\pi (w-t_1)^3}} \exp\left(-\frac{2}{3}\left(w^3 - t_1^3\right)\right),$$

where

$$f_{e_t^{[w-t_1]}}(y)dy = \mathbb{P}\left(e_t^{[w-t_1]} \in dy\right)$$

and $e^{[m]}$ is an excursion of duration m. Recall now from Lemma 4-1 in [11] that

$$v_2(-w, -t_1) = \frac{1}{\sqrt{2\pi (w-t_1)^3}} \exp\left(-\frac{2}{3}\left(w^3 - t_1^3\right)\right) \mathbb{E}\left(\exp\left(-2\int_{t_1}^{w} e_s^{[w-t_1]} ds\right)\right).$$

With the Lemma 4-2 in [11], we finally obtain that

$$\mathbb{P}(Y_t \in dx | Y^* = Y_{t_1} = Y_w = M) = \mathbb{E}\left(\frac{\exp\left(-2\int_0^{w-t_1} e_s^{[w-t_1]} ds\right)}{C\left(2(w-t_1)^{3/2}\right)} ; e_{t-t_1}^{[w-t_1]} \in d(M-x)\right)$$

since

$$C(\lambda) = \mathbb{E}\left(\exp\left(-\lambda \int_0^1 e_s ds\right)\right),$$

according to Lemma 2.

□

Proof of Lemma 6.

Let us define for $n \in \mathbb{Z}$ the processes

$$W_-^{a_n(t)} = (W(z); -\infty < z \leq a_n(t)),$$

$$W_+^{a_n(t)} = (W(z + a_n(t)) - W(a_n(t)); z \geq 0).$$

We know from the theory of splitting times (see [9] for a short introduction to splitting times) that $W_-^{a(0,t)}$ is independent of $W_+^{a(0,t)}$ conditionally on $a(0,t)$. We want to prove now that $W_-^{a_n(t)}$ is independent of $W_+^{a_n(t)}$ conditionally on $\mathcal{F}_t$.

For any real y, let us call $N_t(y)$ the index of the first cluster located at the right of y. By stationarity of W, the processes $(W(a_0(t) + z); z \in \mathbb{R})$ and $\left(W\left(a_{N_t(y)}(t) + z\right); z \in \mathbb{R}\right)$ have the same law. In particular, $W_-^{a_{N_t(y)}(t)}$ is independent of $W_+^{a_{N_t(y)}(t)}$ conditionally on $a_{N_t(y)}(t)$. Let $(q_k; k \in \mathbb{N})$ be an enumeration of the rational numbers, and define the event

$$\left\{q_k \in A_n^t\right\} = \{N_t(q_k) = n; N_t(q_l) \neq n, \ \forall l < k\}$$

which is $\mathcal{F}_t-$measurable. For any f, g Borel functional on path, we have

$$\begin{aligned}
&\mathbb{E}\left(f\left(W_-^{a_n(t)}\right) g\left(W_+^{a_n(t)}\right) |\mathcal{F}_t\right) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left(f\left(W_-^{a_{N_t(q_k)}(t)}\right) g\left(W_+^{a_{N_t(q_k)}(t)}\right); q_k \in A_n^t |\mathcal{F}_t\right) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left(\mathbb{E}\left(f\left(W_-^{a_{N_t(q_k)}(t)}\right) |\mathcal{F}_t\right) \mathbb{E}\left(g\left(W_+^{a_{N_t(q_k)}(t)}\right) |\mathcal{F}_t\right); q_k \in A_n^t\right) \\
&= \mathbb{E}\left(f\left(W_-^{a_n(t)}\right) |\mathcal{F}_t\right) \mathbb{E}\left(g\left(W_+^{a_n(t)}\right) |\mathcal{F}_t\right).
\end{aligned}$$

This ensures the independence of $W_-^{a_n(t)}$ and $W_+^{a_n(t)}$ conditionally on $\mathcal{F}_t$ and as a consequence the independence of the processes $\left(\varepsilon^{(x_n(t),t)}\ ;\ n\in\mathbb{Z}\right)$, conditionally on $\mathcal{F}_t$.

Let us give now the law of $\varepsilon^{(x_n(t),t)}$ conditionally on $\mathcal{F}_t$. Using the stationarity and the scaling property of W, we deduce from Lemma 8 that

$$\mathbb{P}\left(\varepsilon_u^{(x_n(t),t)}\in dz|x_n\left(t\right)=x,a_{n-1}\left(t\right)=\alpha,a_n\left(t\right)=\alpha+m\right)=\mathbb{E}\left(\frac{\exp\left(-\frac{1}{t}\int_0^m e_s^{[m]}\right)}{C\left(m^{3/2}/t\right)};e_u^{[m]}\in dz\right),$$

and

$$\mathbb{P}\left(\varepsilon_u^{(x_n(t),t)}\in dz|\mathcal{F}_t\right)=\mathbb{E}\left(\frac{\exp\left(-\frac{1}{t}\int_0^{m_n} e_s^{[m_n]}ds\right)}{C\left(m_n^{3/2}/t\right)};e_u^{[m_n(t)]}\in dz\right).$$

The multidimensional law may be deduced in the same way. The conditional law of $\varepsilon^{(x_n(t),t)}$ given $\mathcal{F}_t$ is then $\nu(m_n(t),1/t)$.

□

Acknowlegdment : I am very grateful to J. Bertoin for his guidance, without which this work would not have been possible.

References

[1] M. ABRAMOWITZ and I. A. STEGUN *Handbook of mathematical functions.* Nat. Bur. Stand., Washington, 1964.

[2] M. AVELLANEDA and W. E *Statistical properties of shocks in Burgers Turbulence.* Comm. Math. Phys. 172 (1995) p. 13-38.

[3] M. AVELLANEDA *Statistical properties of shocks in Burgers Turbulence II.* Comm. Math. Phys. 169 (1995) p. 45-59.

[4] J. BERTOIN *The inviscid Burgers equation with Brownian initial velocity.* Comm. Math. Phys. 193 (1998) p. 397-406.

[5] J. BERTOIN *Clustering statistics for sticky particles with Brownian initial velocity.* J. Math. Pures Appl. 79-2 (2000) p. 173-194.

[6] J. M. BURGERS *The nonlinear diffusion equation.* Dordrecht, Reidel (1974).

[7] J. D. COLE *On a quasi linear parabolic equation occuring in aerodynamics.* Quart. Appl. Math. 9 (1951) p. 225-236.

[8] L. FRACHEBOURG and P. A. MARTIN *Exact statistical properties of the Burgers equation.* To appear in J. Fluids Mech.

[9] R. K. GETOOR *Splitting times an shift functionals.* Z. Wahrscheinlichkeitstheorie Verw. Gebiete 47 (1979) p. 69-81.

[10] P. GROENEBOOM *The concave majorant of Brownian motion.* Ann. of Proba. 11-4 (1983) p. 1016-1027.

[11] P. GROENEBOOM *Brownian motion with a parabolic drift and Airy functions.* Probability Theory and Rel. Fields 81 (1989) p.79-109.

[12] E. HOPF, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. 3 (1950) p. 201-230.

[13] N. LEONENKO *Limit theorems for random fields with singular spectrum.* Math. and Appl. (1999) Kluwers Academic Publishers.

[14] P. W. MILLAR, *A path decomposition for Markov proecesses.* Ann. of Proba. 6 (1978) p. 345-348.

[15] D. REVUZ, M. YOR *Continuous martingales and Brownian motion.* Grundlehren Math. Wiss. 293 Springer-Verlag (1991).

[16] R. RYAN, *Large-deviation analysis of Burgers turbulence with white-noise initial data.* Comm. Pure Applied Math. 51 (1998) p. 47-75.

[17] Z. SHE, E. AURELL and U. FRISCH, *The inviscid Burgers equation with initial data of Brownian type.* Comm. Math. Phys. 148 (1992) p. 623-641.

[18] Y. SINAI, *Statistics of shocks in solutions of inviscid Burgers Equation.* Comm. Math. Phys. 148 (1992) p. 601-621.

[19] W. A. WOYCZYNSKI, *Burgers-KPZ Turbulence, Gttingen lectures* Lectures Notes in Math. 1700, Springer 1998.