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FÉVRIER 2001

Prépublication n^o 639

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Statistics of a Flux in Burgers Turbulence with One-Sided Brownian Initial Data

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Summary. We study the statistics of the flux of particles crossing the origin, which is induced by the dynamics of ballistic aggregation in dimension 1, under certain random initial conditions for the system. More precisely, we consider the cases when particles are uniformly distributed on $\mathbb{R}$ at the initial time, and if $u(x,t)$ denotes the velocity of the particle located at x at time t, then $u(x,0)=0$ for $x<0$ and $(u(x,0), x \geq 0)$ is either a white noise or a Brownian motion.

Key words. Burgers turbulence, ballistic aggregation, sticky particles, shocks, Brownian data.

A.M.S. Classification. 35 Q 53, 60 H 15, 60 J 65.

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1 Introduction

This work is motivated by a question in the so-called Burgers turbulence that is perhaps more easily formulated in terms of the model of ballistic aggregation (i.e. system of free sticky particles). Specifically, consider infinitesimal particles that are uniformly spread in the one-dimensional space $\mathbb{R}$, and suppose that at the initial time, particles receive some impulse, inducing a velocity field. Then let the system evolve according to the dynamics of completely inelastic shocks. As time passes, this induces the formation of clusters (i.e. point masses); more precisely, particles clump in case of collision to form a single cluster whose mass and velocity are determined by the law of conservation of mass and momentum. Of course, the velocity of particles is unchanged as long as they are not involved in shocks.

Suppose now that the particles in $(-\infty, 0]$ are at rest at the initial time and that sufficiently many particles in $[0,\infty)$ have a negative velocity. We may observe as time passes a flux of clusters that cross the point 0 from the right to the left. Typically, if we denote by $a(0,t)$ the total mass of particles that crossed 0 up to time t, then t is a jump time of $a(0,\cdot)$ if and only if there is a cluster crossing 0 at time t, and the mass of this cluster is just the size of the jump, $a(0,t) - a(0,t-)$. Moreover, it can be seen that the velocity of this cluster is then a function of t, $a(0,t-)$ and $a(0,t)$, so that the flux is completely determined by the process $(a(0,t), t \geq 0)$.

The purpose of this work is to investigate the statistics of this flux when the initial velocities on $[0,\infty)$ are random. Specifically, we will be interested in the cases when they are given by either a white noise or a Brownian motion, which have been considered first by Burgers [5], and by Sinai [15] and She et al. [14]. We also refer to Tribe and Zaboronski [16] and Frachebourg et al. [7] for recent works on Burgers turbulence where the initial impulse is given by a Gaussian white noise supported by a finite or semi-infinite interval. Roughly, in both cases $a(0,\cdot)$ is pure jump and satisfies a time-inhomogeneous Markov property [1]; more precisely jumps occur on a discrete set of times in the case of white noise initial velocity, whereas the jump times are everywhere dense for Brownian initial velocity. Our main results specify the statistics of the jumps of $a(0,\cdot)$ in the white noise case, and its transition probabilities in the Brownian case.

The rest of this note is organized as follows. In the next section we recall some standard features on the evolution of sticky particles and its connection with the inviscid Burgers equation. Section 3 and 4 are devoted to the statements and proofs of our results in the case of white noise and Brownian initial velocities, respectively.

2 Some background on Burgers equation

Provided that at the initial time the mass distribution of particles is given by the Lebesgue measure on $\mathbb{R}$, it is well-known (see [5], [11], [10], [6], [12], ...) that the state of the system at time $t > 0$ for the dynamics of ballistic aggregation can be completely described in terms of the initial velocity field $u(\cdot, t=0)$ as follows.

First, we introduce the so-called initial potential $U(\cdot) = \int_0^{\cdot} u(y,0)dy$, so for $x \geq 0$, $U(x)$ represents the initial momentum of particles located in $[0,x]$. We assume that the initial data ensure that $U(x) = o(x^2)$ as $x \to \pm\infty$. For every $x \in \mathbb{R}$ and $t > 0$, we denote by $a(x,t)$ the right-most location of the overall minimum of the function

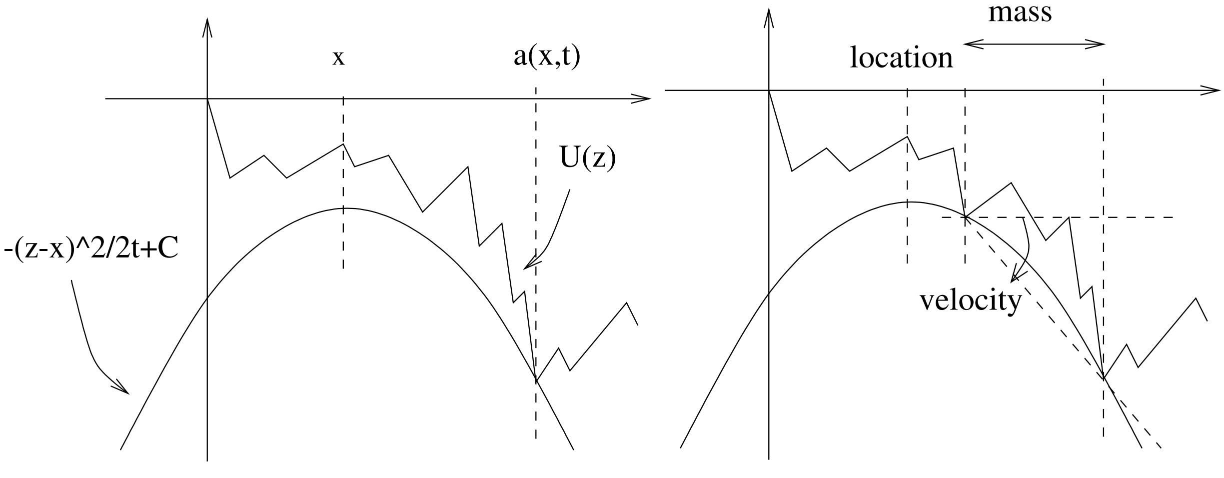
$$z \to U(z) + \frac{(z-x)^2}{2t}\,;$$

see the first figure below. Alternatively, this quantity can be viewed as the right-most location of the overall minimum of

$$z \to \int_0^z (tu(y,0) + y - x)\,dy\,, \qquad z \in \mathbb{R}\,.$$

We shall refer to $a(x,t)$ as the *inverse Lagrangian function* evaluated at location x and time t. It corresponds to the right-most initial location of the particles that lie in $(-\infty, x]$ at time t.

[1] The Markov property relies crucially on the hypothesis that at the initial time, the particles in $(-\infty,0)$ have velocity zero, and would fail if we considered instead e.g. a two-sided Brownian motion as initial data. Indeed, it may happen in that case that at time $t > 0$, there is a cluster with mass m that passes across 0 from the left to the right. The dynamics of ballistic aggregation impose that, if the next cluster that will cross 0 arrives from the right, then it mass will be greater than m, which clearly impedes the Markov property.



geometrical interpretation of a(x,t) geometrical interpretation of a shock

For every fixed $t > 0$, $x \to a(x,t)$ is a right-continuous and increasing function, and its jumps are related to the clusters in the system. More precisely, there is a cluster located at $x \in \mathbb{R}$ at time t if and only if $a(x,t) > a(x-,t)$. By the conservation of mass and momentum, the mass of this cluster is given by the size of the jump $a(x,t) - a(x-,t)$ and its velocity by

$$u(x,t) = \frac{1}{a(x,t) - a(x-,t)} \int_{a(x-,t)}^{a(x,t)} u(y,0)dy = \frac{1}{2t}(2x - a(x,t) - a(x-,t)). \tag{1}$$

See the second figure above.

When the initial velocities are identically zero on $(-\infty, 0]$ (which is assumed in this work), $a(0,t)$ represents the total mass of the particles that crossed 0 up to time t. In particular the map $t \to a(0,t)$ is also right-continuous and increasing. On the other hand, it is easily seen that its left-limit $a(0,t-)$ at $t > 0$ coincides with $a(0-,t)$. So the velocity of the cluster crossing 0 at time t (if any) can also be expressed as

$$u(0,t) = -\frac{a(0,t) + a(0-,t)}{2t} = -\frac{a(0,t) + a(0,t-)}{2t}, \tag{2}$$

which shows that the process $(a(0,t), t \geq 0)$ completely describes the flux of clusters crossing 0.

Let us turn our attention to the connection with the inviscid Burgers equation. If we define

$$u(x,t) = \frac{x - a(x,t)}{t} \tag{3}$$

for every point $x \in \mathbb{R}$ at which $a(\cdot,t)$ is continuous and by (1) otherwise, then u is the well-known entropy solution to the inviscid Burgers equation

$$\partial_t u + u \partial_x u = 0. \tag{4}$$

In particular, describing the solution at the fixed location 0 as time varies, $(u(0,t), t \geq 0)$, is the same as describing the flux of particles crossing 0 in the model of ballistic aggregation. An interesting consequence of (3) and (4) for the study of the flux is that it solves

$$\partial_t a + u \partial_x a = 0 \tag{5}$$

in the weak sense (in (4) and (5), it is of course crucial to define properly the value of $u(x,t)$ at its discontinuity points, which is the purpose of (1)). The physical interpretation of this PDE is clear : $\partial_x a$ gives the mass field and u the velocity field of the particles, so (5) is just the transport equation. Observe also that taking derivatives with respect to the spatial variable x in (5) yields the equation of conservation of masses, see e.g. [6].

3 White noise initial velocity

3.1 Main results

In this section, we focus on white noise initial velocity, i.e.

$$u(x,0) \;=\; 0 \text{ for } x \leq 0 \text{ and } u(x,0) \;=\; \frac{dW_x}{dx} \text{ for } x > 0$$

where $(W_x, x \geq 0)$ is a standard Brownian motion which thus coincides with the initial potential U. So, strictly speaking, the initial velocities $u(\cdot,0)$ are not given by a classical function, but rather by a generalized function which is the derivative of a random continuous path. However it is well known that the analysis of the inviscid Burgers equation can be extended to such singular initial data. More precisely the singularity immediately disappears, in the sense that the solution $u(\cdot,s)$ to Burgers equation is a classical function for any $s > 0$.

An important guide for our analysis is the fact that the shock structure is discrete (see [2]). This means that at time $s > 0$, all the particles are concentrated on clusters whose locations form a discrete set in space. As a consequence, $(a(0,s), s \geq 0)$ is a.s. a step process, in the sense that it increases only by jumps and the number of the jumps (i.e. of clusters of particles that cross 0) during the time interval $[\varepsilon, 1/\varepsilon]$ is finite for every $\varepsilon > 0$.

We shall show that $a(0,\cdot)$ is a time-inhomogeneous Markov process. Its one-dimensional distributions have been determined by Groeneboom [9], see also Section 3 in Frachebourg et al. [7]. To give an explicit expression, it is convenient to first introduce the following notation. We write $\mathtt{Ai}$ for the Airy function (see [1] on page 446), and $-\omega_1 > -\omega_2 > \ldots$ for its zeros ranked in the decreasing order. Following Groeneboom [9], we introduce the function $g : \mathbb{R} \to \mathbb{R}_+$ which has Fourier transform

$$\hat{g}(\xi) \;=\; \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} g(x) dx \;=\; \frac{2^{1/3}}{\mathtt{Ai}(\mathrm{i}2^{-1/3}\xi)}\,, \qquad \xi \in \mathbb{R}\,,$$

and the function $h(w,.) : \mathbb{R}_+ \to \mathbb{R}_+$ that is determined via its Laplace transform

$$\int_0^{\infty} e^{-\lambda x} h(w,x)\, dx = \frac{\mathtt{Ai}(2^{2/3}w + 2^{-1/3}\lambda)}{\mathtt{Ai}(2^{-1/3}\lambda)}\,.$$

There is also a representation of $h(w,m)$ as a series :

$$h(w,m) = 2^{1/3} \sum_{n=1}^{\infty} \frac{\mathtt{Ai}(2^{2/3}w - \omega_n)}{\mathtt{Ai}'(-\omega_n)} \exp\left(-2^{1/3} m\omega_n\right)\,.$$

Then according to Corollary 3.1 in [9], we have

$$\mathbb{P}(a(0,s) \in dm) = (2s)^{-2/3} g\left((2s)^{-2/3}m\right) \left(\int_0^\infty h\left(w,(2s)^{-2/3}m\right)dw\right) dm.$$

By the Markov property, since the one-dimensional distributions of the process $a(0,\cdot)$ are known, the statistics of its evolution are completely determined by the family of conditional distribution of the pair $(T(s),M(s))$ given $a(0,s)$, where

$$T(s) = \inf\{t > s : a(0,t) > a(0,s)\} \quad , \quad M(s) = a(0,T(s)) - a(0,s)$$

denote the first instant after time s at which a cluster crosses 0, and the mass of this cluster. The calculations of these conditional laws rely again crucially on the work of Groeneboom [9]; more precisely, the fact that the shock structure is discrete will enable us to relate the rates of jump of the map $s \to a(0,s)$ to that of $x \to a(x,s)$ at $x = 0$ (which has been computed by Groeneboom). In order to state the result, we need to introduce the function $p : \mathbb{R}_+ \to \mathbb{R}_+$

$$p(x) = 2\sum_{k\geq 1} \exp\{-2^{1/3}\omega_k x\}.$$

Alternatively, p is determined by the Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x}\left(p(x) - (2\pi x^3)^{-1/2}\right)dx = 2^{2/3}\mathtt{Ai}'(2^{-1/3}\lambda)/\mathtt{Ai}(2^{-1/3}\lambda) + \sqrt{2\lambda}.$$

Theorem 1 *For white noise initial velocity, $a(0,\cdot)$ is a time-inhomogeneous Markov step process. For any $m,s > 0$, the conditional distribution of the instant and the size of the first jump of $a(0,\cdot)$ after time s given $a(0,s) = m$ is*

$$\begin{aligned}&\mathbb{P}\left(T(s) \in dt, M(s) \in dy \mid a(0,s) = m\right)\\ &= \left(\frac{s}{t}\right)^{1/3} \exp\left(-\frac{m^3}{6}\left(\frac{1}{s^2} - \frac{1}{t^2}\right)\right) \frac{y(2m+y)}{4t^3} p\left((2t)^{-2/3}y\right) \frac{g\left((2t)^{-2/3}(m+y)\right)}{g\left((2s)^{-2/3}m\right)}\, dy\, dt,\end{aligned}$$

where $y > 0$ and $t > s$.

Specifying the above formula for $t = s$, we obtain the rates of jump of $s \to a(0,s)$:

$$\begin{aligned}&\lim_{t\to s} \frac{1}{t-s}\mathbb{P}\left(a(0,t) - a(0,s) \in dy \mid a(0,s) = m\right)\\ &= \frac{y(2m+y)}{4s^3} p\left((2s)^{-2/3}y\right) \frac{g\left((2s)^{-2/3}(m+y)\right)}{g\left((2s)^{-2/3}m\right)}\, dy.\end{aligned}$$

Before starting to prove Theorem 1, recall from Corollary 3.4(ii) in [9] that

$$g(x) \sim 4x\exp\left\{-\frac{2}{3}x^3\right\} \qquad \text{as } x \to \infty,$$

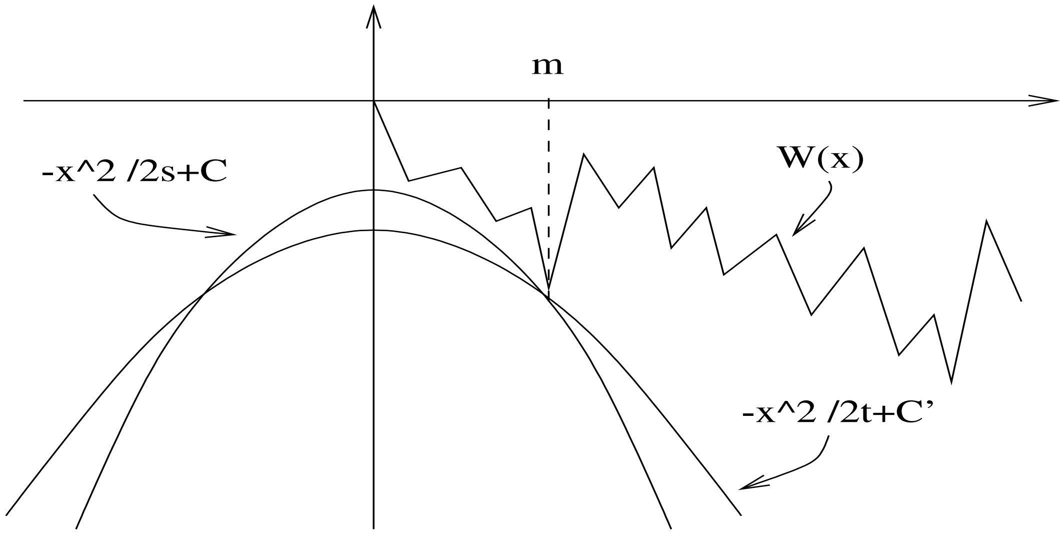
which enables us to estimate the decay of the rates of jump e.g. when the size $y \to \infty$. For instance one gets for $s = 1/2$ and $a(0,1/2) = m$ that the rate of jump of size y is of order

$$\asymp y^3 \exp\left(-\frac{2}{3}(y+m)^3 - 2^{1/3}\omega_1 y\right)$$

when y is large (the first root of the Airy function is $-\omega_1 \approx -2.3381$).

3.2 Proof of Theorem 1

We have already explained why $a(0,\cdot)$ must be a step process. An argument closed to that in Avellaneda and E [2] shows that the (time-inhomogeneous) Markov property follows from a general path decomposition for Markov processes due to Millar [13]. More precisely, Millar's result entails that conditionally on $a(0,s) = m$, the processes $(W_x, x \le a(0,s))$ and $\left(W_{a(0,s)+x} - W_{a(0,s)}, x \ge 0\right)$ are independent. For $r < s < t$, $a(0,r)$ can be viewed as the location of the minimum of $x \to W_x + x^2/2r$ for $x \le m$, whereas $a(0,t) - a(0,s)$ coincides with the location of the minimum of $x \to W_{m+x} - W_m + (x+m)^2/2t$ for $x \ge 0$; see the figure below. The Markov property follows.



We now turn our attention to the conditional distribution of the pair $(T(s), M(s))$ given $a(0,s) = m$, that is we have to calculate the asymptotic of

$$\mathbb{P}\left(T(s) \in [t, t+h], M(s) \in [y, y+\eta] \mid a(0,s) = m\right)$$

as $h, \eta \to 0+$. As a first step, we express the latter quantity in the form $A \times B$ where

$$\begin{aligned} A &= \mathbb{P}\left(T(s) \ge t \mid a(0,s) = m\right) \\ B &= \mathbb{P}\left(T(s) \in [t, t+h], M(s) \in [y, y+\eta] \mid a(0,s) = m, T(s) \ge t\right) . \end{aligned}$$

We start with the computation of the conditional probability A. For every $z > 0$, let $\mathbb{P}^z$ denote the law of a Brownian motion $(W_x, x \ge 0)$ starting from z. On the one-hand, it is known from the work of Millar [13] that the conditional law

$$\mathbb{P}^z(\cdot \mid W_x \ge -x(x+2m)/2s) \text{ for all } x \ge 0)$$

has a weak limit when z decreases to 0, which serves as the conditional distribution of the process $(W_{a(0,s)+x} - W_{a(0,s)}, x \ge 0)$ given $a(0,s) = m$. On the other hand, the analysis of the dynamics of ballistic aggregation sketched in Section 2 shows that the first instant $T(s)$ after time s at which a cluster crosses 0, occurs after time t if and only if $W_{a(0,s)+x} - W_{a(0,s)} \ge -\frac{1}{2t}x(x+2a(0,s))$ for all $x \ge 0$. As a consequence

$$
\begin{aligned}
A &= \mathbb{P}\left(W_{a(0,s)+x} - W_{a(0,s)} \geq -\frac{1}{2t} x(x+2a(0,s)) \text{ for all } x \geq 0 \mid a(0,s)=m \right) \\
&= \lim_{z \downarrow 0} \frac{\mathbb{P}^z\left(W_x \geq -\frac{1}{2t}x(x+2m) \text{ for all } x \geq 0\right)}{\mathbb{P}^z\left(W_x \geq -\frac{1}{2s}x(x+2m) \text{ for all } x \geq 0\right)}
\end{aligned}
$$

According to Corollary 3.1 in [9], we have for every $r > 0$ that

$$
\lim_{z \downarrow 0} \frac{\partial}{\partial z} \mathbb{P}^z \left(W_x \geq -\frac{1}{2r} x(x+2m) \text{ for all } x \geq 0 \right) = (2r)^{-1/3} \exp\left(\frac{m^3}{6r^2} \right) g\left((2r)^{-2/3} m \right) ,
$$

where g has been defined before Theorem 1. By l'Hospital's rule, we conclude that the first conditional probability is

$$
A = \left(\frac{s}{t}\right)^{1/3} \exp\left(-\frac{m^3}{6}\left(\frac{1}{s^2} - \frac{1}{t^2} \right) \right) \frac{g\left((2t)^{-2/3}m\right)}{g\left((2s)^{-2/3}m\right)}. \tag{6}
$$

We next turn our attention to the conditional probability B. The Markov property of $s \to a(0,s)$ applied at time t shows that

$$
B = \mathbb{P}\left(T(t) \in [t, t+h], M(t) \in [y, y+\eta] \mid a(0,t) = m\right) .
$$

We shall first estimate this quantity when $h \to 0+$ and $\eta \to 0+$, in terms of the jump rate of $x \to a(0,x)$.

Lemma 1 *We have the following estimate when $h, \eta \to 0+$:*

$$
\begin{aligned}
& \mathbb{P}\left(T(t) \in [t, t+h], M(t) \in [y, y+\eta] \mid a(0,t) = m\right) \\
\sim \;& \mathbb{P}\left(a((m+y/2)h/t, t) - a(0,t) \in [y, y+\eta] \mid a(0,t) = m\right) .
\end{aligned}
$$

Let us sketch a heuristic argument for Lemma 1 (the rigorous proof is postponed to the end of this section). It is convenient to consider the left-most cluster located in $[0,\infty)$ at time t, so the location of this cluster is $\ell = \inf\{x \geq 0 : a(x,t) > a(0,t)\}$ and its mass $\mu = a(\ell,t) - a(0,t)$. On the one hand, as the shock structure is discrete, we may neglect the effects of collisions during small time-intervals. Therefore, a cluster of mass $\approx y$ and velocity $\approx v$, crosses 0 during the time-interval $[t, t+h]$ "if and only if" the left-most cluster in $[0,\infty)$ at time t is located at $\ell \in [0, -vh]$ and has mass $\mu \approx y$ and velocity $\approx v$ (roughly, this is a loose interpretation of the transport equation (5)). Recall from (2) that given $a(0,t) = m$, we must have $v \approx -(m+y/2)/t$. On the other hand, because the process $x \to a(x,t)$ is an inhomogeneous Markov step process (see [9] or [2]), we have as $h \to 0+$

$$
\begin{aligned}
& \mathbb{P}\left(\ell \in [0, (m+y/2)h/t] \text{ and } \mu \in [y, y+\eta] \mid a(0,t) = m\right) \\
\sim \;& \mathbb{P}\left(a((m+y/2)h/t, t) - a(0,t) \in [y, y+\eta] \mid a(0,t) = m\right) .
\end{aligned} \tag{7}
$$

We thus arrive at the stated estimate.

In order to evaluate the quantities appearing in Lemma 1, we can invoke Theorem 4.1 of Groeneboom [2]. If we denote by $\nu_t(m,y)$ the rate of jump of size y for $x \to a(x,t)$ at $x=0$, conditionally on $a(0,t)=m$, then we have for $t=1/2$

$$\nu_{1/2}(m,y) \;=\; 2yp(y)\,\frac{g\,(m+y)}{g\,(m)}\,.$$

This means that for every $y' > y$, we have the following estimate as $h \to 0+$

$$\mathbb{P}\left(a(h,1/2)-a(0,1/2)\in[y,y'] \mid a(0,1/2)=m\right) \;\sim\; h\int_y^{y'} \nu_{1/2}(m,z)\,dz\,.$$

It is an easy matter to deduce from this the expression of $\nu_t(m,y)$ for an arbitrary $t>0$. Indeed, the scaling property of Brownian motion propagates to the inverse Lagrangian function, in the sense that for every $t>0$

$$(a(x,t), x\geq 0) \;\overset{\text{law}}{=}\; \left((2t)^{2/3}a(x(2t)^{2/3},1/2), x\geq 0\right)\,,$$

see e.g. Equation (21) in She et al. [14]. It follows readily that the conditional rates of jump at time t are given by

$$\begin{aligned}\nu_t(m,y) \;&=\; (2t)^{-4/3}\nu_{1/2}\left((2t)^{-2/3}m,(2t)^{-2/3}y\right)\\ &=\; \frac{y}{2t^2}\,p((2t)^{-2/3}y)\,\frac{g\left((2t)^{-2/3}(m+y)\right)}{g\left((2t)^{-2/3}m\right)}\,.\end{aligned}$$

By Lemma 1, we get as $h,\eta \to 0+$ that

$$\begin{aligned}&\mathbb{P}\left(T(t)\in[t,t+h], M(t)\in[y,y+\eta] \mid a(0,t)=m\right)\\ \sim\;\; & h\eta\,\frac{y(2m+y)}{4t^3}\,p((2t)^{-2/3}y)\,\frac{g\left((2t)^{-2/3}(m+y)\right)}{g\left((2t)^{-2/3}m\right)}\,.\end{aligned}$$

Combining this with (6) yields the formula stated in Theorem 1.

So all that we need to complete the proof of Theorem 1 is to establish Lemma 1.

Proof of Lemma 1 : We shall follow the heuristic approach sketched above; the main technical problem is to check that the effects of shocks during a small time interval can be neglected. Let us work conditionally on $a(0,t)=m$ and fix $\varepsilon>0$. Denote by $\rho(\ell,\mu,m,\varepsilon)$ the conditional probability given the location ℓ and the mass μ of the first cluster at the right of 0 at time t, that during the time interval $(t,t+\varepsilon]$ there is at least one cluster that crosses the location ℓ. By the same argument based on Corollary 3.1 in [9] as that we used to obtain (6), we see that

$$\rho(\ell,\mu,m,\varepsilon) = 1-\frac{k(t+\varepsilon,m+\mu-\ell)}{k(t,m+\mu-\ell)}\,,$$

[2]In fact, Groeneboom considered a two-sided white noise as initial velocity, which induces spatial stationarity for the inviscid Burgers turbulence. Replacing the two-sided white noise by a one-sided white noise modifies the one-dimensional distributions of the process $a(\cdot,1/2)$, but not its transition probabilities nor its infinitesimal generator (this is easily seen from a perusal of Groeneboom's argument).

where

$$k(r,m) \;=\; (2r)^{-1/3}\exp\left(\frac{m^3}{6r^2}\right) g\left((2r)^{-2/3}m\right) .$$

Hence, for every $c>0$, if we set

$$\rho^\star(\varepsilon,m,c) := \sup_{0\le \ell,\mu\le c} \rho(\ell,\mu,m,\varepsilon)\,,$$

then

$$\rho^\star(\varepsilon,m,c) \le \varepsilon \times \sup_{\substack{t\le\tau\le t+\varepsilon \\ 0\le\mu,\ell\le c}} \frac{|\partial_1 k(\tau, m+\mu-\ell)|}{k(t,m+\mu-\ell)}\,, \tag{8}$$

and this uniform upper bound tends to 0 as ε decreases to 0.

Next, we write for simplicity T for $T(t)$ and M for $M(t)$, i.e. T is the first instant after time t at which a cluster crosses 0, and M is the mass of this cluster. We thus have $a(0,T-)=m$ and $a(0,T)=m+M$, and by (2), the velocity of this cluster is $v=-(m+M/2)/T$. By the conservation of masses and momenta, the center of the mass of the particles that constitute this cluster is located at $-v(T-t)$ at time t. As a consequence, the location ℓ of the left-most cluster in $[0,\infty)$ at time t must fulfill

$$\ell \;\le\; (m+M/2)(T-t)/T \;\sim\; (m+M/2)(T-t)/t \qquad \text{as } T\to t+\ .$$

Combining these two observations, we deduce the following for every fixed $0<dt\ll\varepsilon$. On the one hand, the event, say Λ, that $T\in[t,t+dt]$ and $M\in[y,y']$, is implied by the event that $\ell\in[0,(m+y/2)dt/t]$, $\mu\in[y,y']$, and that no cluster crosses ℓ during the time interval $(t,t+\varepsilon]$. This yields the lower-bound

$$\begin{aligned} &\mathbb{P}\left(T\in[t,t+dt], M\in[y,y'] \mid a(0,t)=m\right)\\ \ge\;& \mathbb{P}\left(\ell\in[0,(m+y/2)dt/t]\,,\mu\in[y,y'] \mid a(0,t)=m\right)(1-\rho^\star(\varepsilon,m,y'))\ .\end{aligned}$$

On the other hand, this same event Λ forces that $\ell\in[0,(m+y'/2)dt/t]$, and either that $\mu\in[y,y']$, or that $\mu<y$ and there is at least one cluster that crosses the location ℓ during the time interval $(t,t+\varepsilon]$. This yields the upper-bound

$$\begin{aligned} &\mathbb{P}\left(T\in[t,t+dt], M\in[y,y'] \mid a(0,t)=m\right)\\ \le\;& \mathbb{P}\left(\ell\in[0,(m+y'/2)dt/t]\,,\mu\in[y,y'] \mid a(0,t)=m)\right)\\ &+\mathbb{P}\left(\ell\in[0,(m+y'/2)dt/t]\,,\mu<y \mid a(0,t)=m)\right)\rho^\star(\varepsilon,m,y)\,;\end{aligned}$$

Finally, letting $\varepsilon\to0+$ and then $y'\to y$, we obtain from (8) that

$$\begin{aligned} &\frac{\mathbb{P}\left(T\in[t,t+dt], M\in[y,y+dy]\ \mid a(0,t)=m\right)}{dt\,dy}\\ &=\frac{\mathbb{P}\left(\ell\in[0,(m+y/2)dt/t], \mu\in[y,y+dy]\ \mid a(0,t)=m\right)}{dt\,dy}\,.\end{aligned}$$

By (7), the proof of Lemma 1 is complete. ■

4 Brownian initial velocity

4.1 Main results

Throughout this section, we assume Brownian initial velocities, i.e.

$$u(x,0) \;=\; 0 \text{ for } x<0 \text{ and } (u(x,0), x\geq 0) \text{ is a standard Brownian motion.}$$

One major difference with the white noise case is that now there are no rarefaction intervals, i.e. at any time $s>0$, the locations of clusters form an everywhere dense subset on $[0,\infty)$ with probability 1 (cf. Sinai [15]). We now state the main result of this work in this setting.

Theorem 2 *For Brownian initial velocities, $a(0,\cdot)$ is a time-inhomogeneous Markov process that increases only by jumps. The transition probabilities can be described as follows. For every $q,m>0$ and $0<s<t$, we have*

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{-q\left(a(0,t)-a(0,s)\right)\right\} \mid a(0,s)=m\right)\\
=\;& \sqrt{\frac{s}{t}+\frac{t-s}{t\sqrt{2t^2q+1}}}\,\exp\left\{-\frac{m(t-s)}{st^2}\left(\sqrt{2t^2q+1}-1\right)\right\}.
\end{aligned}$$

It is interesting to observe that

$$\lim_{q\to\infty}\mathbb{E}\left(\exp\left\{-q\left(a(0,t)-a(0,s)\right)\right\} \mid a(0,s)=m\right) \;=\; 0\,,$$

which entails that on any non-void time-interval $(s,t]$, there are clusters of particles that cross 0. This is certainly not surprising due to the absence of rarefaction intervals. We also point out that it is easy to derive from the formula in Theorem 2 the jump rates of $a(0,\cdot)$ (that is the rate of clusters with a given mass crossing zero) conditional on its current value. Specifically, we see that when t decreases to s,

$$\begin{aligned}
&\frac{1}{t-s}\mathbb{E}\left(1-\exp\left\{-q\left(a(0,t)-a(0,s)\right)\right\} \mid a(0,s)=m\right)\\
\sim\;& \frac{m}{s^3}\left(\sqrt{2s^2q+1}-1\right)+\frac{1}{2s}\left(1-1/\sqrt{2s^2q+1}\right).
\end{aligned}$$

Inverting the Laplace transform, we obtain the following formula for the rate of jump of size y for $a(0,\cdot)$ at time s conditionally on $a(0,s)=m$

$$\begin{aligned}
&\frac{1}{t-s}\mathbb{P}\left(a(0,t)-a(0,s)\in[y,y+dy] \mid a(0,s)=m\right)\\
\sim\;& \left(\frac{2m+y}{2s}\right)\frac{1}{s\sqrt{2\pi y^3}}\exp\left(-\frac{y}{2s^2}\right)dy\,. \qquad (9)
\end{aligned}$$

As a check, it may be interesting to recover (9) by the following informal argument, inspired by our approach via Lemma 1 in the white noise case. If we neglect the effects of collisions during a small time-interval, then we should have

$$\begin{aligned}
&\mathbb{P}\left(a(0,t)-a(0,t-)\in[y,y']\text{ for some }t\in[s,s+ds]\mid a(0,s)=m\right)\\
\sim\quad&\mathbb{P}\left(a(x,s)-a(x-,s)\in[y,y']\text{ for some }x\in[0,\frac{2m+y}{2s}ds]\mid a(0,s)=m\right). \qquad (10)
\end{aligned}$$

On the other hand, we know from Theorem 1 in [3] that the point process of the jumps of the inverse Lagrangian function, $(a(x,s)-a(x-,s),x\geq 0)$ (that is the process of the masses of clusters at time s as a function of their location) is independent of $a(0,s)$ and has the distribution of a Poisson point process on $(0,\infty)$ with characteristic measure

$$\nu_s(dy) = \frac{1}{s\sqrt{2\pi y^3}}\exp\left(-\frac{y}{2s^2}\right)dy\,.$$

So the probability of the right-hand side in (10) equals

$$\left(\frac{2m+y}{2s}\right)ds\,\nu_s(dy)\,,$$

which is the formula found in (9). Unfortunately, it does not seem easy to make this heuristic argument rigorous, because it does not take into account the collisions during the time interval $[s,s+ds]$ (recall that there are no rarefaction intervals a.s.).

The rest of this section is devoted to the proof of Theorem 2.

4.2 Proof of Theorem 2

Let us first establish the easiest parts of the statement. To start with, recall that with probability one, at any positive time almost every particle belongs to some clusters. See Sinai [15] or E et al. [6]. We may therefore identify $a(0,s)$ with the total mass of the clusters that crossed 0 by time s, and the latter, viewed as a process depending on s, is obviously pure jump. Next, we lift from Lemma 1 in [3] the key fact that $a(0,s)$ is a splitting time for the Brownian motion $u(\cdot,0)$. Specifically, write $\mathcal{F}_s$ for the sigma-field generated by $(u(x,0),x\leq a(0,s))$. Because $a(0,s)$ increases with s, $(\mathcal{F}_s)_{s\geq 0}$ is a filtration and the process $a(0,\cdot)$ is clearly $(\mathcal{F}_s)$-adapted. Then for each $s>0$

$$(u(x+a(0,s),0),x\geq 0)\text{ is independent of }\mathcal{F}_s. \qquad (11)$$

On the other hand, the increment $a(0,t)-a(0,s)$ can be identified as the (right-most) location of the minimum of the function

$$z\to\int_0^z\left(tu(y+a(0,s),0)+y+a(0,s)\right)dy\,,\qquad z\geq 0\,.$$

We thus see from (11) that the conditional distribution of $a(0,t)$ given $\mathcal{F}_s$ only depends on $a(0,s)$, which establishes the Markov property. So all that we need now is to calculate the (time-inhomogeneous) transition probabilities.

In this direction, we consider the solution v to the inviscid Burgers equation with initial velocity $v(\cdot,0)$ given for $x \le 0$ by $v(x,0) = 0$ and for $x \ge 0$ by

$$v(x,0) \;=\; u(x,s) - u(0,s) \;=\; u(x,s) + a(0,s)/s\,.$$

We write

$$(x,r) \;\to\; \alpha(x,r) \;=\; x - rv(x,r)$$

for the associated inverse Lagrangian function.

Lemma 2 *We have :*

$$\frac{t}{t-s}\left(\alpha(x,t-s) - \alpha(0,t-s)\right) \;=\; \frac{sx}{t-s} + a(x,t) - a(0,t)\,, \tag{12}$$

and

$$a(0,t) \;=\; \frac{t}{t-s}\alpha\left(\frac{t-s}{s}\,a(0,s), t-s\right)\,. \tag{13}$$

Proof : It is convenient to consider first the so-called delayed solution (cf. Section 4.3 in [4]), that is for every $x \in \mathbb{R}$ and $r \ge 0$, we set $\tilde{u}(x,r) = u(x,s+r)$. Then $\tilde{u}$ is the entropy solution to the inviscid Burgers equation $\partial_t \tilde{u} + \tilde{u}\partial_x \tilde{u} = 0$ with initial velocity $\tilde{u}(\cdot,0) = u(\cdot,s)$. We write $\tilde{a}(x,r)$ for the inverse Lagrangian function evaluated at location x and at time r for the delayed solution, that is for the right-most location of the minimum of the function

$$z \to \int_0^z \left(ru(y,s) + y - x\right)dy\,, \qquad z \in \mathbb{R}\,.$$

Of course, we have

$$\frac{x - \tilde{a}(x,t-s)}{t-s} \;=\; \tilde{u}(x,t-s) \;=\; u(x,t) \;=\; \frac{x - a(x,t)}{t}\,,$$

which yields the identity

$$a(0,t) \;=\; \frac{t}{t-s}\tilde{a}(0,t-s)\,. \tag{14}$$

Let us now compare the two sticky particle systems with respective initial velocities $v(\cdot,0)$ and $\tilde{u}(\cdot,0)$. On the one hand, because the initial relative velocities $v(x,0) - v(y,0) = \tilde{u}(x,0) - \tilde{u}(y,0)$ are the same for $x, y \ge 0$ and the initial velocities $\tilde{u}(\cdot,0)$ and $v(\cdot,0)$ both are non-positive on $(-\infty,0]$, it should be plain from the dynamics of the system and the interpretation of v and $\tilde{u}$ as a velocity fields that relative velocities still coincide on $[0,\infty)$ at any time $r > 0$, i.e. that we still have $v(x,r) - v(y,r) = \tilde{u}(x,r) - \tilde{u}(y,r)$ for $x, y \ge 0$. We refer to [4] on page 186 for an elementary mathematical check of a closely related heuristic argument. In particular, we have

$$\begin{aligned}
\frac{x - \alpha(x,t-s) + \alpha(0,t-s)}{t-s} \;&=\; v(x,t-s) - v(0,t-s) \\
&=\; \tilde{u}(x,t-s) - \tilde{u}(0,t-s) \\
&=\; u(x,t) - u(0,t) \\
&=\; \frac{x - a(x,t) + a(0,t)}{t}\,,
\end{aligned}$$

which yields (12).

On the other hand, as $u(y,s) = \tilde{u}(y,0) \le 0$ for every $y \le 0$, we may identify $\tilde{a}(0,r)$ as the right-most location of

$$z \to \int_0^z (rv(y,0) + y - ra(0,s)/s)\, dy\,, \qquad z \in \mathbb{R}\,.$$

Specifying this for $r = t - s$ gives

$$\alpha((t-s)a(0,s)/s, t-s) \;=\; \tilde{a}(0, t-s)\,.$$

We arrive at (13) by combination with (14). ■

We may now use results in [3] to analyze the distribution of variables arising in Lemma 2. In this direction, (13) leads us to apply (12) for $x = \frac{t-s}{s}a(0,s)$. This yields the identity

$$a(0,t) \;=\; a(0,s) + \tau_{s,t}(a(0,s)) + \frac{t}{t-s}\,\alpha(0,t-s)\,, \tag{15}$$

where

$$\tau_{s,t}(y) \;=\; a((t-s)y/s, t) - a(0,t)\,, \qquad y \ge 0\,.$$

Theorem 2 now follows from (15) and an appeal to the following lemma.

Lemma 3 (i) *The process $(\tau_{s,t}(x), x \ge 0)$ is independent of the variables $a(0,s)$ and $\alpha(0,t-s)$. It is a subordinator (increasing process with stationary and independent increments) with Laplace transform*

$$\mathbb{E}\left(\exp\left\{-q\tau_{s,t}(x)\right\}\right) \;=\; \exp\left\{-\frac{x(t-s)}{st^2}\left(\sqrt{2t^2q+1}-1\right)\right\}\,.$$

(ii) *The Laplace transform of the random variable $t\alpha(0,t-s)/(t-s)$ is given by*

$$\mathbb{E}\left(\exp\left\{-\frac{qt}{t-s}\alpha(0,t-s)\right\}\right) \;=\; \sqrt{\frac{s}{t} + \frac{t-s}{t\sqrt{2t^2q+1}}}\,.$$

Proof : (i) According to Theorem 1 in [3], the process $v(\cdot,0)$ is a Lévy process with no positive jumps which is independent of $a(0,s)$. Because the inverse Lagrangian function $\alpha(\cdot,t-s)$ is measurable with respect to $v(\cdot,0)$, it is also independent of $a(0,s)$. Then Theorem 2 in [3] applies to the initial velocity $v(\cdot,0)$, and we get that the increments of $\alpha(\cdot,t-s)$ are independent of $\alpha(0,t-s)$. Since the increments of $a(\cdot,t)$ are related to those of $\alpha(\cdot,t-s)$ by the identity (12), we conclude that they are independent of $a(0,s)$ and $\alpha(0,t-s)$. Finally, the expression for their Laplace transform is given (up to the scaling property) in Theorem 1 in [3] :

$$\mathbb{E}\left(\exp\left\{-q\left(a(x,t)-a(0,t)\right)\right\}\right) \;=\; \exp\left\{-\frac{x}{t^2}\left(\sqrt{2t^2q+1}-1\right)\right\}\,,$$

and the formula for the Laplace transform of $\tau_{s,t}(x)$ follows.

(ii) We can use the identity (15) and the independence stated in part (i) to determine the Laplace transform of $t\alpha(0,t-s)/(t-s)$ in terms of the other variables. Specifically, we get

$$\mathbb{E}\left(\exp\{-qt\alpha(0,t-s)/(t-s)\}\right) = \frac{\mathbb{E}\left(\exp\{-qa(0,t)\}\right)}{\mathbb{E}\left(\exp\{-qa(0,s)-q\tau_{s,t}(a(0,s))\}\right)}.$$

On the one hand, the Laplace exponent of the subordinator $\tau_{s,t}$ has been given in part (i). On the other hand, we know from Theorem 1 in [3] (and the scaling property of Brownian motion) that for every $r>0$, $a(0,r)$ has a Gamma$(1/4, 1/2r^2)$ distribution, i.e. its Laplace transform is given by

$$\mathbb{E}\left(\exp\{-qa(0,r)\}\right) = \left(2r^2q+1\right)^{-1/4}.$$

Straightforward (but lengthy) calculations then yield the stated formula.

We mention that there is also an alternative proof which we now sketch. Because $v(\cdot,0)$ is a Lévy process with no positive jumps, we can also determine the distribution of $\alpha(0,t-r)$ by adapting the argument of [3] on pages 402-403, which deals with Brownian initial velocities. Specifically, one gets that if $\tilde{\alpha}(0,t-r)$ is an independent copy of $\alpha(0,t-r)$, then the sum $\alpha(0,t-r)+\tilde{\alpha}(0,t-r)$ has the same distribution as the last passage time

$$\ell_{s,t} := \sup\{x\geq 0 : (t-s)v(x,0)+x=0\}.$$

So all that is needed is to calculate the Laplace transform of $\ell_{s,t}$, which is an easy task since the distribution of the Lévy process $v(\cdot,0)$ is known explicitly. One finds

$$\mathbb{E}\left(\exp\{-q\ell_{s,t}\}\right) = \frac{s}{t}+\frac{t-s}{t\sqrt{2qt(t-s)+1}},$$

so

$$\mathbb{E}\left(\exp\{-q\alpha(0,t-r)\}\right) = \sqrt{\frac{s}{t}+\frac{t-s}{t\sqrt{2qt(t-s)+1}}},$$

and we recover the formula (ii). ■

Acknowledgment. We should like to thank Ph. A. Martin as the question about the statistics of the flux that motivates this work was raised by him during a visit of J.B. to the Institut de Physique Théorique at the École Polytechnique Fédérale in Lausanne.

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