Gravitational clustering and additive coalescence

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Abstract: we investigate a gravitational system in dimension one, started from some "uniform" random initial data. In Section 2, a connection is established with the additive coalescence. An hydrodynamic limit is obtained in Section 3 and it suggests a new construction of the standard additive coalescent. The latter is given in Section 3.2. An infinite system is considered in Section 4, and is shown to be closely related to the Smoluchowski additive and multiplicative equations.

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1 Introduction

1.1 The models under investigation

The distribution of the mass in the universe is commonly believed to reflect a small density fluctuation around the uniform density at the time of baryon-photon decoupling, see e.g. the survey article [18]. Hence arises the natural question: what kind of mass distribution appears when the dynamic starts from some "uniform" random configuration? In this note, we investigate the mass distribution of a one-dimensional system of particles in newtonian interaction started from a random setting. We refer to [12, 9, 11] for some recent studies on the subject. In the case we will focus on, many features are known, such as the statistics of the last time of collision or the size of the largest cluster, see [9, 11]. Our intention in this work is to emphasize a close link between the gravitational dynamic and the additive coalescence.

In dimension one, two masses m and m' attract each other according to the force $F = \mathcal{G}mm'$, where $\mathcal{G}$ denotes the gravitational constant. In particular, the strength F does not depend on the distances between the two masses. When two (or more) particles collide, we will assume that they stick and merge into a new particle (we call cluster) with conservation of the mass and the momentum. We fix henceforth $\mathcal{G} = 1$ and we will focus in Section 2 and 3 on a system of N particles with initial masses $m_1, \ldots, m_N$, initial locations $x_1, \ldots, x_N$ and initial velocities $v_1, \ldots, v_N$. We will assume that the m_i's are deterministic, and we will focus on the special case where the v_i's are coupled with the m_i's via the relation

$$v_{i+1} - v_i = -\frac{\lambda}{2}(m_i + m_{i+1}), \qquad \lambda \geq 0, \tag{1}$$

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which is crucial for the following analysis. The x_i's will be random, and a first idea to mimic a uniform configuration is to consider as initial locations N independent uniform variables on $[0,1]$. This will be our first assumption:

Assumption 1 : *the initial locations $x_1, \ldots, x_N$ are given by N ordered independent uniform variables on* $[0,1]$.

In fact, it will be often more convenient to replace the previous assumption by the following:

Assumption 2 : *x_1 and the $d_i = x_{i+1} - x_i$, $i \in \{1, \ldots, N-1\}$ are N independent exponential variables with parameter $c > 0$.*

These two assumptions are actually closely related since if $d_0, \ldots, d_N$ are $N+1$ independent exponential variables with parameter $c > 0$, then the

$$\tilde{x}_i := \frac{d_0 + \ldots + d_{i-1}}{d_0 + \ldots + d_N}, \quad i \in \{1, \ldots, N\} \tag{2}$$

are distributed as N ordered independent uniform variables on $[0,1]$.

The purpose of this work is to compare this model of gravitational clustering with the additive coalescent, which is a mathematical object that describes the evolution of the masses in a system of clusters, where each pair of clusters, say with mass m_i and m_j, merges as a single cluster with mass $m_i + m_j$, at rate $m_i + m_j$, independently of the other pairs, see Section 2.2 for a more detailed description. Besides, we shall also consider in the last section, an infinite system, we will define there and connect to the Smoluchowski coagulation equation.

1.2 Organization of the paper

Next subsection is devoted to some remarks on the gravitational dynamic in dimension one. We link in Section 2 the gravitational clustering (under Assumption 2) with the additive coalescence. We consider in Section 3 the case $m_i = 1/N$, and we show the existence of an hydrodynamic limit as $N \to \infty$, which can be expressed in terms of the convex hull of some "Brownian" path. This limit strongly suggests a construction of the so-called standard additive coalescent in terms of the convex hull of a special "Brownian" process, which is done in Section 3.2. In Section 4, an infinite system is investigated. Some initial conditions called "Poisson clouds" are shown to be stable under the gravitational dynamic and their evolution appears to be connected to the Smoluchowki additive equation, see Theorem 2 there. It is interesting to mention that this equation has already been proposed as a model for gravitational clustering, see [15]. Finally, some technicalities are postponed to the Appendix.

1.3 Analyzing the dynamic

Consider two sets $\mathcal{I}_i$ and $\mathcal{I}_{i+1}$ of consecutive indices, i.e. of the form $\mathcal{I}_i = \{k, k+1, \ldots, k+j\}$ and $\mathcal{I}_{i+1} = \{k+j+1, k+j+2, \ldots, k+q\}$. Assuming condition (1), one can check that the velocities of the barycenters of the particles with label in $\mathcal{I}_i$ and $\mathcal{I}_{i+1}$ also fulfills relation (1), viz

$$V_{\mathcal{I}_{i+1}} - V_{\mathcal{I}_i} = -\frac{\lambda}{2}(M_{\mathcal{I}_i} + M_{\mathcal{I}_{i+1}}), \tag{3}$$

where

$$M_{\mathcal{I}_i} = \sum_{j \in \mathcal{I}_i} m_j \quad \text{and} \quad V_{\mathcal{I}_i} = \frac{1}{M_{\mathcal{I}_i}} \sum_{j \in \mathcal{I}_i} m_j v_j .$$

Let the system evolve according to the dynamic described above. Since the mass and momentum are conserved during collisions, as long as there is no interaction between the particles with initial label in the set $\mathcal{I}_i$ and the others, the location $X_{\mathcal{I}_i}(s)$ of their barycenter at time s is given by

$$X_{\mathcal{I}_i}(s) = X_{\mathcal{I}_i} + sV_{\mathcal{I}_i} + \frac{s^2}{2}\left(M_{\mathcal{I}_i}^{(R)} - M_{\mathcal{I}_i}^{(L)}\right)$$

where $M_{\mathcal{I}_i}^{(R)}$ (respectively $M_{\mathcal{I}_i}^{(L)}$) represents the mass of the particles at the right (resp. left) of the particles with indices in $\mathcal{I}_i$. In particular, if the particles with initial label in $\mathcal{I}_i$ and $\mathcal{I}_{i+1}$ do not have interacted with the exterior up to time s then

$$X_{\mathcal{I}_{i+1}}(s) - X_{\mathcal{I}_i}(s) = X_{\mathcal{I}_{i+1}} - X_{\mathcal{I}_i} + s(V_{\mathcal{I}_{i+1}} - V_{\mathcal{I}_i}) - \frac{s^2}{2}\left(M_{\mathcal{I}_i} + M_{\mathcal{I}_{i+1}}\right)$$

In view of (3), if we set $t = 2\lambda s + s^2$, inversed by $s = \tau(t)$, we obtain

$$X_{\mathcal{I}_{i+1}}(\tau(t)) - X_{\mathcal{I}_i}(\tau(t)) = X_{\mathcal{I}_{i+1}} - X_{\mathcal{I}_i} - \frac{t}{2}\left(M_{\mathcal{I}_i} + M_{\mathcal{I}_{i+1}}\right) . \tag{4}$$

In the sequel, we will *always* consider the system at time $\tau(t)$, and refer to

$$t = 2\lambda s + s^2$$

as the time, i.e. we completely forget the time s. We index the n_t clusters present at time t from left to right; cluster number 1 will correspond to the left most cluster, whereas cluster number n_t will correspond to the right most cluster. For $i \in \{1, \ldots, n_t\}$, we write $\mathcal{I}(i,t)$ for the set of the labels of the initial particles (the ones present at time 0) making up the i'th cluster at time t. Applying formula (4) to the $\mathcal{I}(i,t)$'s, one notices that the dynamic of the system between the shocks can be described as follows. The distance d between two consecutive masses m and m' decreases according to

$$\dot{d}\,(t) = -\frac{m + m'}{2} . \tag{5}$$

This point of view will be especially appropriate to investigate the clustering dynamic of the system.

Before starting this analysis, we mention that in view of (2) and (4) the statistics of the system at time t under Assumption 1 are the same as the statistics of the system at time $T_N\, t$ under Assumption 2, after a space rescaling $x \mapsto x/T_N$, with $T_N = d_0 + \ldots + d_N$.

2 Clustering dynamic

2.1 Description

We are interested in this section in the clustering dynamic of the system, viz on the evolution of the masses $m_1(t), \ldots, m_{n_t}(t)$ of the n_t clusters present at time t, ranked in their spatial order. We will work under Assumption 2, i.e. with a system which is initially made of N particles of mass $m_1, \ldots, m_N$, with interdistances $d_i = x_{i+1} - x_i$ distributed as independent exponential variables with parameter $c > 0$. Next proposition describes the evolution of $M(t) = (m_1(t), \ldots, m_{n_t}(t))$.

Proposition 1 *Under Assumption 2, each pair of neighbouring clusters, say with masses m_i and m_{i+1}, merges as a single cluster with mass $m_i + m_{i+1}$ at rate $\frac{c}{2}(m_i + m_{i+1})$ independently of the other pairs.*

More precisely, M is a Markov step process, and for $i = 1$ to $n-1$ its jump rate from the state $(m_1, \ldots, m_n)$ to the state $(m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_n)$ is

$$c\,\frac{m_i + m_{i+1}}{2}\,.$$

All the other jump rates are zero.

Proof: Call t_1 the time of the first aggregation. Since at time $t < t_1$, the distance between the cluster i and $i+1$ is given by $d_i(t) = d_i - \frac{t}{2}(m_i + m_{i+1})$, the first aggregation time corresponds to

$$t_1 = \min\left(\frac{2d_1}{m_1 + m_2}, \ldots, \frac{2d_{N-1}}{m_{N-1} + m_N}\right).$$

In particular, t_1 is distributed as an exponential variable of parameter

$$c\left(\frac{m_1}{2} + m_2 + \ldots + m_{N-1} + \frac{m_N}{2}\right),$$

and the probability that at time t_1 the aggregation occurs between the particles i and $i+1$ is

$$\frac{m_i + m_{i+1}}{2\left(\frac{m_1}{2} + m_2 + \ldots + m_{N-1} + \frac{m_N}{2}\right)}\,.$$

Assume that the collision at time t_1 involves particles i and $i+1$, i.e. $t_1 = 2d_i/(m_i + m_{i+1})$. Then, for $j \leq i-1$, the distance between the cluster j and the cluster $j+1$ at time t_1+ is $d_j(t_1) = d_j - t_1(m_j + m_{j+1})/2$, whereas for $j \geq i$, it is $d_j(t_1) = d_{j+1} - t_1(m_{j+1} + m_{j+2})/2$. Conditionally on $t_1 = 2d_i/(m_i + m_{i+1})$, the $(d_j, j \neq i)$ are independent exponential variables conditioned to be larger than $t_1(m_j + m_{j+1})/2$, so due to the absence of memory of the exponential law, the $d_j(t_1)$ are distributed as independent exponential variables of parameter c. Iterating the argument, it should be plain that M is a Markov process with the jump rates specified in the proposition. ■

2.2 Connection with the additive coalescence

We notice in this subsection, that if we transform the system in a periodic (or equivalently cyclic) system, then the clustering dynamic is, up to a time change, nothing but the one of an additive coalescent. An additive coalescent $(C^{\downarrow}(t), t \geq 0)$ started from N clusters with total mass $m = m_1 + \ldots + m_N$ is a step Markov process in the space $\mathcal{S}_m^{\downarrow} := \{(s_1, s_2, \ldots) : s_1 \geq s_2 \geq \cdots \geq 0,\ s_1 + s_2 + \cdots = m\}$ with $N-1$ jump times $\tau_1 < \cdots < \tau_{N-1}$. The times between two consecutive jumps, $\tau_1, \tau_2 - \tau_1, \ldots, \tau_{N-1} - \tau_{N-2}$ are independent exponential variables with parameters $m(N-1), m(N-2), \ldots, m$, which are independent of the Markov chain $(C^{\downarrow}(\tau_k), k = 0 \ldots N-1)$ (with the convention $\tau_0 = 0$). The transitions of the latter chain can be described as follows. The probability given $C^{\downarrow}(\tau_k) = (m_1, \ldots, m_{N-k})$ that $C^{\downarrow}(\tau_{k+1})$ is obtained from the coalescence of the clusters with mass m_i and m_j, $1 \leq i < j \leq N-k$, is

$$\frac{m_i + m_j}{m(N-k-1)}\,.$$

We make the gravitational system cyclic, in considering the same dynamic but on a circle (of variable size). We introduce a new variable d_N independent of the others with an exponential distribution of parameter c. The particles are put on a circle of perimeter $d_1 + \cdots + d_N$, so that the distance between the particle i and $i+1$ is given by d_i for $i = 1, \ldots, N-1$ and the distance between the particle N and 1 is d_N. Then, between shocks, we let the distance between two adjacent clusters of mass m_i and m_{i+1} evolve according to

$$\dot{d_i}\,(t) = -\frac{m_i + m_{i+1}}{2}\,.$$

Notice that at time t, the perimeter $d_1(t) + \cdots + d_{n_t}(t)$ of the circle equals $d_1 + \cdots + d_N - mt$. In terms of the process M described in Proposition 1, this system also allows the jump from the state $(m_1, \ldots, m_n)$ to the state $(m_1 + m_n, m_2, \ldots, m_{n-1})$ at rate $c(m_1 + m_n)/2$.

Write $(m_1^{\downarrow}(t), \ldots, m_{n_t}^{\downarrow}(t))$ for the sequence of the masses of the clusters present at time t ranked in the decreasing order. Next result states that $M^{\downarrow}(t) = (m_1^{\downarrow}(t), \ldots, m_{n_t}^{\downarrow}(t))$ evolves after a time change as an additive coalescent. It rephrases in our setting and up to a time change Proposition 2 in [4].

Fact 1 *The evolution of the ranked sequences $M^{\downarrow}(t) = (m_1^{\downarrow}(t), \ldots, m_{n_t}^{\downarrow}(t))$ of the masses of the cyclic system at time t can be depicted as follows. Each pair of cluster, say with mass $m_i^{\downarrow}$ and $m_j^{\downarrow}$ merges, independently of the others, into a single cluster with mass $m_i^{\downarrow} + m_j^{\downarrow}$ at rate*

$$c\,\frac{m_i^{\downarrow} + m_j^{\downarrow}}{n_t - 1}\,,$$

where n_t denotes the number of the clusters present in the system at time t.

More precisely, the process $M^{\downarrow}$ is a step Markov process, with $N-1$ jump times $T_1 < \cdots < T_{N-1}$. The times between two consecutive jumps, $T_1, T_2 - T_1, \ldots, T_{N-1} - T_{N-2}$ are independent exponential variables with parameter cm, which are independent of the Markov chain $(M^{\downarrow}(T_k), k = 0 \ldots N-1)$ (with the convention $T_0 = 0$). The latter chain has the same distribution as the chain $(C^{\downarrow}(\tau_k), k = 0 \ldots N-1)$. In other word, $(M^{\downarrow}(t), 0 \leq t \leq T_{N-1})$ is distributed as the additive coalescent $(C^{\downarrow}(\sigma_t), \sigma_t \leq \tau_{N-1})$, with the time change

$$\int_0^{\sigma_t} (k_s - 1)\, ds = ct\,, \tag{6}$$

where k_t denotes the total number of clusters in $C^{\downarrow}(t)$.

3 An hydrodynamic limit

We focus in this section on the case where the masses of the initial particles are $m_i = 1/N$, $i = 1, \ldots, N$ and we are interested in the statistical behaviour of the system when N becomes large.

3.1 Statistical behaviour

As a sketch of the dynamic, we recall what happens when at the initial time the distances are given by $d_i = 1/N$, $i = 1, \ldots, N-1$. Then $d_i(t) = (1-t)/N$, which means that every particles collide simultaneously at time $t = 1$ and merge into a single cluster of mass 1. According to the law of large numbers, the same kind of behaviour is expected for large N when one works under the Assumption 1 or 2. Indeed, in the first case it has been shown that the last collision occurs between two macroscopic clusters (i.e. clusters whose mass does not vanish as $N \to \infty$) at a time that differs from 1 at order $1/\sqrt{N}$, see Theorem 3.1 in [9]. Moreover, at a time $t < 1$ the largest cluster has a size of order $\log(N)$, whereas a typical cluster is of bounded size (see [9, 11] for precise statements).

Assume then, that at the initial time the masses of the particles are $1/N$ and consider Assumption 2 with $c = N$ (then $d_1 + \cdots + d_{N-1} \to 1$ in probability). Since a typical cluster is only involved in a finite number of collision before time 1, the original system and the cyclic system have a similar statistical behaviour for large N when $t < 1$. Now, the statistics of an additive coalescent $C^{\downarrow}$ started from N masses $1/N$ are well-known, so we can compute many statistical properties of the cyclic model. For example, for $t < 1$ the law of large numbers ensures that the number n_t of particles at time t is equivalent to $N(1-t)$ as N goes to infinity and we can lift from [2] that the sequence $M^{\downarrow}(T_{N-k})$ is distributed as

$$\frac{1}{N}(B_1^{\downarrow}, \ldots, B_k^{\downarrow} \mid B_1^{\downarrow} + \ldots + B_k^{\downarrow} = N)$$

where $B_1^{\downarrow}, \ldots, B_k^{\downarrow}$ are k ordered independent Borel variables[1]. In particular, via a large deviation analysis, one can show that conditionally on $B_1^{\downarrow} + \ldots + B_{N(1-t)}^{\downarrow} = N$,

$$B_1^{\downarrow} = \frac{\log N - \frac{3}{2}\log\log N}{t - 1 - \log t} + o(\log\log N).$$

This permits for example to recover the result of Lifshits and Shi, which states that in the original system the size of the largest cluster at time $t < 1$ is equivalent to

$$\frac{\log N}{t - 1 - \log t}$$

as N tends to infinity, see Theorem 3.3 in [11].

An interesting question when one considers the limit $N \to \infty$ is the question of existence of an hydrodynamic limit. The additive coalescent is known to admit such a limit. More precisely, if $C_N^{\downarrow}(t)$ denotes an additive coalescent started from N particles of mass $1/N$, then the process $\left(C_N^{\downarrow}(\frac{1}{2}\log N + t), -\frac{1}{2}\log N \le t < \infty\right)$ is known to converge when $N \to \infty$ towards the so-called standard additive coalescent $\Pi^{\downarrow}$, see Evans and Pitman [8]. Now, one can check from (6) that $\sigma_t^{(N)}$ converges in probability to $-\log(1-t)$, so putting pieces together this suggests an hydrodynamical limit for the original system in a time scale of order $1/\sqrt{N}$ around 1. We emphasize yet that in this time scale the statistics of the original system differ perceptibly from the statistics of the cyclic system and the hydrodynamic limit will not correspond to the additive coalescent. We thus have to lead a different analysis, which is the topic of the next subsection.

3.2 An hydrodynamic limit

A glance to the relation (4) tells us a bit more than (5). Indeed, the system behaves has a system of free sticky particles (which means that between shocks particles move at constant

[1] remind that the law of a Borel variable B is given by $\mathbb{P}(B = k) = e^{-k}k^{k-1}/k!$.

speed and the shocks are completely inelastic) with initial velocities $(u_i)_{1,N}$ fulfilling

$$u_2 - u_1 = -\frac{1}{2}(m_1 + m_2), \dots, u_N - u_{N-1} = -\frac{1}{2}(m_{N-1} + m_N)\,.$$

Recall that $\mathcal{I}(1,t), \dots, \mathcal{I}(n_t,t)$ stands for the set of the labels of the particles making up the n_t clusters present at time t. An integer k will be called a right-label at time t, if there exist i and $l \le k$ such that $\mathcal{I}(i,t) = \{l, \dots, k\}$. It should be plain that an integer $k \le N$ is a right-label if and only if

$$\sup_{1\le l\le k} \frac{\sum_{j=l}^k m_j(x_j + tu_j)}{\sum_{j=l}^k m_j} < \inf_{k+1\le r\le N} \frac{\sum_{j=k+1}^r m_j(x_j + tu_j)}{\sum_{j=k+1}^r m_j}\,.$$

In a geometrical point of view, the right labels correspond to the labels of the contact points between the set

$$\mathcal{P}_N = \left\{ \left(\sum_{j=1}^k m_j, \sum_{j=1}^k m_j(x_j + tu_j) \right), k = 1, \dots, N \right\} \bigcup \{(0,0)\}$$

and its convex hull $\mathcal{C}_N$, the point $\{(0,0)\}$ excepted.

Let us come back to the special case $m_i = 1/N$. For the sake of simplicity, we will assume that $\lambda = 0$ in condition (1) and $v_1 = 0$, so that $v_i = 0$ for $i = 1, \dots, N$, and $u_i = 1/2 - (2i-1)/2N$, for $i = 1, \dots, N$. The quantity $\sum_{j=1}^k m_j(x_j + tu_j)$ then equals

$$\frac{1}{N}\sum_{j=1}^k x_j - \frac{t}{2}\left(\frac{k}{N}\right)^2 + \frac{kt}{2N}.$$

We define X_N by

$$X_N(y) = x_k \quad \text{for} \quad \frac{k-1}{N} < y \le \frac{k}{N} \quad \text{and} \quad X_N(0) = 0,$$

so that the set $\mathcal{P}_N$ can be expressed as

$$\mathcal{P}_N = \left\{ \left(\frac{k}{N}, \int_0^{k/N} X_N(y) - ty + \frac{t}{2}\, dy \right), k = 0, \dots, N \right\}.$$

Under Assumption 1 or Assumption 2 with $c = N$, it is appropriate to introduce the process $b_N(y) := \sqrt{N}(X_N(y) - y)$, which is known to converge in law towards a brownian bridge in the first case, and towards a brownian motion in the second case, see e.g. [16] Theorem 1. At time $t = 1 + s/\sqrt{N}$, with $s \ge -\sqrt{N}$, the set $\mathcal{P}_N$ then corresponds to

$$\mathcal{P}_N = \left\{ \left(\frac{k}{N}, \frac{1}{\sqrt{N}} \int_0^{k/N} b_N(y) - sy + \frac{\sqrt{N}+s}{2}\, dy \right), k = 0, \dots, N \right\}.$$

Moreover, if k and m are two consecutive right-labels, it should be plain from the conservation of momentum, that the slope of $\mathcal{C}_N$ between k/N and m/N corresponds to the location at time

t of the cluster made up of the particles with label $k+1,\dots,m$. This suggests a connection in time scale $s=\sqrt{N}\,(t-1)$ and space scale $x'=\sqrt{N}\,(x-t/2)$ between the density of mass

$$\rho_N(t,dx)=\sum_i M_{\mathcal{I}(i,t)}\delta_{X_{\mathcal{I}(i,t)}(\tau(t))}(dx)$$

and the convex hull of the integral of a brownian motion or bridge with drift.

To make the statement rigorous we associate to $x,t\in\mathbb{R}$ and a càdlàg[2] path $b:[0,1]\to\mathbb{R}$ the largest abscissa $a_b(t,x)$ in $[0,1]$ where

$$z\mapsto\int_0^z b-\frac{t}{2}z^2-xz$$

reaches its minimum. A moment of thought shows that $\{a_b(t,x),x\in\mathbb{R}\}$ coincides with the abscissaes where $z\mapsto\int_0^z b-\frac{t}{2}z^2$ touches its convex hull. For any $t\in\mathbb{R}$, the function $x\mapsto a_b(t,x)$ is càdlàg, non decreasing from 0 to 1, and its weak derivative ρ_b defined by

$$\rho_b(t,]x,y])=a_b(t,y)-a_b(t,x) \tag{7}$$

is a probability measure on $\mathbb{R}$. We endow henceforth the space $\mathcal{P}(\mathbb{R})$ of probability measure on $\mathbb{R}$ with the topology of weak convergence, i.e. a sequence of probability measure μ_N converges towards μ if and only if $\langle f,\mu_N\rangle\to\langle f,\mu\rangle$ for any f continuous and bounded on $\mathbb{R}$. The next result exhibits an hydrodynamic limit for the system.

Proposition 2 *Consider a system started from N particles with mass $1/N$, velocity 0 and assume that either Assumption 1 or Assumption 2 with $c=N$ holds. Then the density*

$$\left(\rho_N\left(1+\frac{t}{\sqrt{N}},d\left(\frac{1}{2}+\frac{x+t/2}{\sqrt{N}}\right)\right),t\geq-\sqrt{N}\right)$$

converges in the sense of finite dimensional distribution towards $\rho_b(t,dx)$ (defined by (7)), with b distributed as a Brownian bridge in the first case and as a Brownian motion in the second case.

Remark: the previous proposition fits in the first case with the convergence of the last time of collision $T_N^{l.c.}$ stated in Theorem 3.1 of [9]:

$$\sqrt{N}\left(T_N^{l.c.}-1\right)\overset{\text{law}}{\to}\sup_{x\in[0,1]}\left(\frac{1}{1-x}\int_x^1 b(t)\,dt-\frac{1}{x}\int_0^x b(t)\,dt\right),$$

with b a Brownian bridge.

Proof: We focus for the sake of simplicity on the one-dimensional distribution, the general case can be treated in a similar way. The convergence is based on the following elementary lemma.

Lemma 1 *The function $b\to\rho_b(t,\cdot)$ is continuous from the space $(\mathbf{D},||\,||)$ of càdlàg functions from $[0,1]$ to $\mathbb{R}$ endowed with the uniform metric to $\mathcal{P}(\mathbb{R})$ endowed with the topology of weak convergence.*

[2]right continuous with left limits ("continu à droite avec limites à gauche" in french)

Proof of lemma 1: Consider a sequence b_N in $\mathbf{D}$ that converges uniformly to b and $x \in \mathbb{R}$ such that $a_b(t,x) = a_b(t,x-)$, where $f(x-)$ stands for the left limit of f at x. If $a_{b_N}(t,x)$ does not converge to $a_b(t,x)$, then up to an extraction we can assume that e.g.

$$a_{b_N}(t,x) \to a > a_b(t,x).$$

Then taking the limit in the inequality

$$\int_0^{a_{b_N}(t,x)} b_N - \frac{t}{2}(a_{b_N}(t,x))^2 - x a_{b_N}(t,x) \le \int_0^{a_b(t,x)} b_N - \frac{t}{2}(a_b(t,x))^2 - x a_b(t,x)\,,$$

leads to (using the uniform convergence of $\int_0^z b_N$ to $\int_0^z b$ on $[0,1]$)

$$\int_0^a b - \frac{t}{2}a^2 - xa \le \int_0^{a_b(t,x)} b - \frac{t}{2}(a_b(t,x))^2 - x a_b(t,x)$$

with $a > a_b(t,x)$, which contradicts the definition of $a_b(t,x)$. The convergence of $a_{b_N}(t,x)$ to $a_b(t,x)$ for every x where $a_b(t,\cdot)$ is continuous enforces the weak convergence of $\rho_{b_N}(t,\cdot)$ to $\rho_b(t,\cdot)$. ■

The convergence in distribution of b_N to b distributed as a brownian bridge under the first assumption and as a brownian motion under the second assumption combined with Lemma 1 ensures the convergence in distribution of $\rho_{b_N}(t,\cdot)$ to $\rho_b(t,\cdot)$. Proposition 2 then follows from the fact that

$$\rho_N\left(1 + \frac{t}{\sqrt{N}}, d\left(\frac{1}{2} + \frac{x + t/2}{\sqrt{N}}\right)\right) = \rho_{\tilde{b}_N}(t, dx)$$

where

$$\tilde{b}_N(y) := b_N(y) + \frac{t}{2}\left(y - \frac{k}{N}\right) \quad \text{for} \quad \frac{k}{N} \le y < \frac{k+1}{N}$$

fulfills $||\tilde{b}_N - b_N|| \to 0$ when $N \to \infty$. ■

Let us discuss shortly some of the properties of the hydrodynamic limit. Adapting the arguments of Sinaï [17], one can show that $x \mapsto a_b(t,x)$ is a pure jump process, which means that all the mass is contained into the macroscopic clusters. Moreover, a moment of thought shows that the location of the left-most (macroscopic) cluster at time t corresponds to

$$x_-(t) = \inf_{z\in[0,1]} \frac{1}{z}\int_0^z b^{(t)},$$

with $b_s^{(t)} = b_s - ts$ and the location of the right-most cluster corresponds to

$$x_+(t) = \sup_{z\in[0,1]} \frac{1}{1-z}\int_z^1 b^{(t)}.$$

These locations are in particular finite with probability one.

3.3 Standard additive coalescent

The standard additive coalescent $\Pi^{\downarrow}$ is a Markov process in the simplex

$$\mathcal{S}_1^{\downarrow} = \{(s_1, s_2, \dots) : \ s_1 \geq s_2 \geq \cdots \geq 0 \ \text{ and } \ s_1 + \cdots = 1\}$$

endowed with the uniform metric. It arises in the limit as $N \to \infty$ of the coalescent

$$\left(C_N^{\downarrow} \left(\frac{1}{2} \log N + t \right), -\frac{1}{2} \log N \leq t < \infty \right)$$

started from N masses $1/N$ and its statistics are well-known, see [2]. For instance, the distribution of $\Pi_t^{\downarrow}$ is given by that of the ranked sequence $\pi_1 \geq \pi_2 \geq \cdots$ of the atoms of a Poisson measure on $]0, \infty[$ with intensity $e^{-t}\, dx/\sqrt{2\pi x^3}$ and conditioned by $\pi_1 + \cdots = 1$. In view of the previous hydrodynamic limit and the close relationship between the system of particles and the additive coalescent, one expects a connection between the standard additive coalescent and the convex hull of some "brownian" process. Next theorem states such a connection.

As before, to $(x, t) \in \mathbb{R} \times \mathbb{R}^-$ and a 1-periodic càdlàg path b we associate the largest abscissa $\tilde{a}_b(t, x)$ in $\mathbb{R}$ where

$$z \mapsto \int_0^z b - \frac{t}{2} z^2 - xz$$

reaches its minimum. Note that the path $x \mapsto \tilde{a}_b(t, x) - x$ is then 1-periodic and $x \mapsto \tilde{a}_b(t, x)$ is again non-decreasing. We consider the jumps of $\tilde{a}_b(t, x)$ on an interval of size 1 and write $\Gamma_b^{\downarrow}(t)$ for the sequence of these jumps ranked in the decreasing order.

Theorem 1 *Assume that b is 1-periodic and is distributed as a brownian bridge on $[0, 1]$. Then $\left(\Gamma_b^{\downarrow}(-e^{-t}), t \in \mathbb{R}\right)$ is a standard additive coalescent.*

Remark: in [3], Bertoin has shown a connection between the standard additive coalescent and the solution of Burgers equation

$$\partial_t u + u \partial_x u = 0 \tag{8}$$

with an initial velocity $u(0, \cdot)$ equal to 0 on $\mathbb{R}^-$ and distributed as a brownian motion on $\mathbb{R}^+$. Theorem 1 provides a new (simplest) connection between Burgers equation and the standard additive coalescent. Indeed, it is standard (see [10, 6]) that

$$u(t, x) = \frac{x - \tilde{a}_b\left(-t^{-1}, x\right)}{t}$$

is solution of (8) with initial condition $u(0, \cdot) = b$. Notice that the additive coalescent appears here naturally from the dynamic, whereas its appearence in [3] follows from some statistical properties of the brownian path.

Proof: Consider Assumption 2 and recall that $M^{\downarrow}(t)$ denotes the ranked sequence of the masses in the cyclic model started from N particles of mass $1/N$. We set

$$t(N) = T_N \left(1 - \frac{e^{-t}}{\sqrt{N}} \right),$$

where T_k refers to the k-th time of jump of $M^\downarrow$ and we get from [4] (see Theorem 1 there) that when N goes to infinity $\big(M^\downarrow(t(N)), t \geq -\log N\big)$ converges in law to a standard additive coalescent $\Pi^\downarrow$. Now the distribution of

$$\left(\left(\frac{1}{T_N} x_i(T_N t), m_i(T_N t)\right), t \geq 0\right)$$

under Assumption 2 is the same as the distribution of $((x_i(t), m_i(t)), t \geq 0)$ under Assumption 1. So if we write $\tilde{M}^\downarrow(t)$ for the ranked sequence of the masses at time t in the cyclic system under Assumption 1, then

$$\left(\tilde{M}^\downarrow\left(1 - e^{-t}/\sqrt{N}\right), t \geq \log N\right)$$

converges in distribution towards $\Pi^\downarrow$.

The connection between $\tilde{M}^\downarrow$ and $\tilde{a}_b$ will follow from an analysis in the same vein as the one lead in the previous subsection. Consider a 1-periodic system, evolving according to (5) and starting from the following initial condition. The initial configuration $(m_i, x_i)_{i\in\mathbb{Z}}$ is 1-periodic and there are N particles of mass $1/N$ on $[0,1]$ whose locations are given by N i.i.d. uniform variables on $[0,1]$. The right labels at time t then correspond to the labels of the contact points between

$$\mathcal{P}'_N = \left\{\left(\frac{k}{N}, \frac{t}{2} + \int_0^{k/N} (\tilde{X}_N(y) - ty)\, dy\right), k \in \mathbb{Z}\right\}$$

and its convex hull $\mathcal{C}'_N$, where

$$\tilde{X}_N(y) = x_k \quad \text{for} \quad \frac{k-1}{N} < y \leq \frac{k}{N}\,.$$

So, as before, the sequence $\tilde{M}^\downarrow(1 - e^{-s}/\sqrt{N})$ corresponds to $\Gamma^\downarrow_{b_N}(-e^{-s})$ where

$$b_N(y) := \sqrt{N}\left(\tilde{X}_N(y) - y\right)$$

converges in law towards a 1-periodic brownian bridge b. Forthcoming Lemma 2 ensures the convergence of $\Gamma^\downarrow_{b_N}$ to $\Gamma^\downarrow_b$ in the finite dimensional distribution sense, and due to the uniqueness of the limit, the process $\left(\Gamma^\downarrow_b(-e^{-s}), s \in \mathbb{R}\right)$ is distributed as $\Pi^\downarrow$.

Lemma 2 *Assume that b_N belongs to the space of 1-periodic càdlàg path endowed with the uniform metric. If b_N converges in distribution towards a 1-periodic brownian bridge b, then $\Gamma^\downarrow_{b_N}$ converges in the sense of finite dimensional distribution to $\Gamma^\downarrow_b$.*

The proof of Lemma 2 is postponed to the appendix.

4 An infinite system

In view of Bertoin's work [5] on the evolution of self-attracting Poisson clouds in an expanding universe, we are tempted to lead a similar analysis in our case. We will first properly define the evolution of an infinite system and then investigate its statistics when the initial data are "Poissonian".

4.1 Evolution of an infinite system

We will label the clusters of an infinite system on $\mathbb{Z}$, from left to right. The system will be described by the sequence $(m_i, d_i)_{i\in\mathbb{Z}}$, where m_i represents the mass of the cluster number i and d_i the distance between the clusters i and $i+1$. Next lemma formalizes the idea that between collisions the system evolves according to (5). As in [5], this is done via an approximation of the infinite system by a finite or periodic one.

Indeed, to an infinite configuration $((m_i, d_i), i \in \mathbb{Z})$, we will associate a truncated configuration $((m_i, d_i), -N \le i \le N)$ and a $2N+1$-periodic configuration $\big((m_i^{[N]}, d_i^{[N]}), i \in \mathbb{Z}\big)$ defined by $m_i^{[N]} = m_{[i]_N}$ and $d_i^{[N]} = d_{[i]_N}$, where $[i]_N$ stands for the main value of i modulo $2N+1$. The sequences

$$\left(\left(m_i^{(N)}(t), d_i^{(N)}(t)\right), i \in \{l_t, \dots, k_t\}\right) \quad \text{and} \quad \left(\left(m_i^{[N]}(t), d_i^{[N]}(t)\right), i \in \mathbb{Z}\right)$$

will then correspond to the configurations obtained at time t from these initial data by the gravitational dynamic, with the convention that the cluster with label 0 at time t is the cluster that contains the particle whose initial label is 0.

Lemma 3 *Consider an infinite configuration $(m_i, d_i)_{i\in\mathbb{Z}}$ with infinite mass $\sum_i m_i = \infty$ and assume that*

$$t^* := \liminf_{|k|\to\infty} \frac{\sum_{i=0}^{k-1} d_i}{\sum_{i=0}^{k} m_i} > 0 \,.$$

Then for any $t < t^*$ there exists a configuration $(m_i(t), d_i(t))_{i\in\mathbb{Z}}$, such that

$$\begin{aligned} m_i(t) &= m_i^{(N)}(t) = m_i^{[N]}(t) \\ \textit{and} \quad d_i(t) &= d_i^{(N)}(t) = d_i^{[N]}(t) \end{aligned}$$

whenever N is large enough.

Proof: We focus on the truncated case. Write $\mathcal{I}_N(i,t)$ for the set of the initial indices of the particles that have merged at time t to form the cluster i. It suffices to show that $\mathcal{I}_N(i,t)$ remains bounded for any $i \in \mathbb{Z}$.

Consider first $\mathcal{I}_N(0,t) = \{j_N, \dots, k_N\}$, with j_N non-increasing and k_N non-decreasing with N. For $0 \le s \le t$, write $D(s)$ for the distance at time s between the particles that contain the particle with initial label j_N and k_N, and t_N for the first time s where $D(s) = 0$. For any time $s \le t_N$

$$D'(s) \ge -\sum_{i=j_N}^{k_N} m_i \,,$$

so that

$$0 = D(t_N) \ge \sum_{i=j_N}^{k_N-1} d_i - t\sum_{i=j_N}^{k_N} m_i \,,$$

or equivalently

$$\frac{\sum_{i=j_N}^{k_N-1} d_i}{\sum_{i=j_N}^{k_N} m_i} \le t \,. \tag{9}$$

As a consequence, for $t < t^*$ the indices k_N and j_N have to remain bounded. Assume now that $\mathcal{I}_N(0,t), \ldots, \mathcal{I}_N(i-1,t)$ are bounded. Then, j_N and k_N defined by $\mathcal{I}_N(i,t) = \{j_N, \ldots, k_N\}$ also fulfill condition (9), with j_N bounded and k_N non-decreasing. This enforces again k_N to remain bounded and the lemma is obtained by induction. ■

4.2 Evolution of Poisson clouds

In the spirit of [5], we will focus on a special family of initial data, we call Poisson clouds. Consider a finite measure μ on $]0,\infty[$, with finite first moment $\langle \mathrm{id}, \mu\rangle < \infty$ (in the following "id" refers to the identity $x \mapsto x$ on $]0,\infty[$). A Poisson cloud with intensity μ, is a random configuration $(m_i, d_i)_{i\in\mathbb{Z}}$ such that $(m_i)_{i\in\mathbb{Z}}$ and $(d_i)_{i\in\mathbb{Z}}$ are two independent sequences of independent variables, fulfilling[3]

- each distance d_i follows an exponential law of parameter $\langle 1, \mu\rangle$,
- for $i \neq 0$, the masses m_i are distributed according to $\mu/\langle 1, \mu\rangle$,
- the mass m_0 follows the size-biased distribution $\mathrm{id}\mu/\langle \mathrm{id}, \mu\rangle$.

Next theorem shows that the gravitational dynamic preserves the Poissonian structure of a configuration, and links the evolution of its intensity to the Smoluchowski coagulation equation with multiplicative and additive kernel A^* and A^+. The latter are defined by

$$\begin{aligned} \langle f, A^*(\mu)\rangle &= \frac{1}{2}\int\int_{\mathbb{R}^{+2}} [f(x+y) - f(x) - f(y)]\, xy\, \mu(dx)\mu(dy) \\ \text{and} \quad \langle f, A^+(\mu)\rangle &= \frac{1}{2}\int\int_{\mathbb{R}^{+2}} [f(x+y) - f(x) - f(y)]\, (x+y)\, \mu(dx)\mu(dy)\ , \end{aligned}$$

for any $f :]0,\infty[\to \mathbb{R}$ borelian and bounded.

Theorem 2 *Assume that the initial configuration $(m_i, d_i)_{i\in\mathbb{Z}}$ is distributed as a Poisson cloud with intensity μ_0, where μ_0 is a finite measure on $]0,\infty[$, with finite first moment $\langle \mathrm{id}, \mu_0\rangle$. Then the infinite system is well-defined for any time $t < T := \langle \mathrm{id}, \mu_0\rangle^{-1}$ and the configuration $(m_i(t), d_i(t))_{i\in\mathbb{Z}}$ is distributed as a Poisson cloud with intensity μ_t, where $(\mu_t/\mathrm{id}, 0 \le t < T)$ is solution of the multiplicative Smoluchowski equation*

$$\langle f, \frac{\mu_t}{\mathrm{id}}\rangle = \langle f, \frac{\mu_0}{\mathrm{id}}\rangle + \int_0^t \langle f, A^*\left(\frac{\mu_s}{\mathrm{id}}\right)\rangle\, ds \tag{10}$$

for any measurable function f such that f/id is bounded, or equivalently

$$\mu_t = \frac{1}{T-t}\, \eta_{-\log(1-t/T)} \tag{11}$$

where η_t is the solution started from $\eta_0 = T\mu_0$ of the additive Smoluchowski equation

$$\langle f, \eta_t\rangle = \langle f, \eta_0\rangle + \int_0^t \langle f, A^+(\eta_s)\rangle\, ds\ , \tag{12}$$

for any bounded and measurable function f.

[3] note that our definition of Poisson clouds slightly differs of the one in [5], since the law of the particle 0 is biased by the mass instead of the (artificial) size.

Before starting the proof of Theorem 2, we explain why equation (11) should appear. Consider the periodic configuration $\left(m_i^{[N]}(t), d_i^{[N]}(t)\right)_{i\in\mathbb{Z}}$ introduced in the previous subsection and write n_t for its period. Then, in view of Fact 1 (Section 2), the ranked sequence of $m_1^{[N]}(t), \ldots, m_{n_t}^{[N]}(t)$, denoted here by

$$M_N^{\downarrow}(t) = \left(m_1^{[N]\downarrow}(t), \ldots, m_{n_t}^{[N]\downarrow}(t)\right),$$

evolves as $C^{\downarrow}(\sigma_t)$, where $C^{\downarrow}$ is an additive coalescent started from $M_N^{\downarrow}(0)$. On the one hand, according to the law of large numbers, the number n_t of clusters behaves as $n_t \approx N'(1-t/T)$, with $N' = 2N+1$ and from (6)

$$\sigma_t \approx -\frac{cT}{N'}\log\left(1-\frac{t}{T}\right),$$

with $c = \langle 1, \mu_0\rangle$. On the other hand, for an additive coalescent $C^{\downarrow}$ started from $M_N^{\downarrow}(0)$, Norris' theorem (see [14] Theorem 4.1) ensures the weak convergence

$$\frac{1}{N'}\sum_i \delta_{C_i^{\downarrow}(t/N')} \Longrightarrow \nu_t$$

where $(\nu_t, t\geq 0)$ is solution of (12) started from $\nu_0 = \frac{1}{c}\,\mu_0$. Hence, putting pieces together, if the system is assumed to be still a Poisson cloud at time t then its intensity should be given by

$$\begin{aligned}
\mu_t &= \lim_{N\to\infty} c \times \frac{1}{n_t}\sum_{i=1}^{n_t} \delta_{m_i^{[N]\downarrow}(t)} \\
&= \lim_{N\to\infty} \frac{cN'}{n_t} \times \frac{1}{N'}\sum_{i=1}^{n_t} \delta_{m_i^{[N]\downarrow}(t)} \\
&= \frac{cT}{T-t} \times \nu_{-cT\log(1-t/T)}\,.
\end{aligned}$$

Equation (11) is then a consequence of the fact that $\eta_s := cT\nu_{cTs}$ is again a solution of (12), started from $\eta_0 = T\mu_0$.

Of course previous argument is dubious. Instead of trying to make it rigorous, we prefer to give a direct proof of Theorem 2. We split it in three lemmas. Before going further, notice that the law of large numbers combined with Lemma 3 ensures that the dynamic is a.s. well-defined up to time $t^* = T$.

Lemma 4 $\Big($conditional law of $\left(m_i^{[N]}(t), d_i^{[N]}(t)\right)_{i\in\mathbb{Z}}$ given $M_N^{\downarrow}(t)\Big)$
Fix a time $t < T$.

1. *Conditionally on $n_t = n$, the distances $d_1^{[N]}(t), \ldots, d_{n_t}^{[N]}(t)$ are independent of the masses $m_1^{[N]}(t), \ldots, m_{n_t}^{[N]}(t)$ and are distributed as n independent exponential variables with parameter $\langle 1, \mu_0\rangle$.*

2. For $k \leq n-1$ the conditional law of $m_0^{[N]}(t), \ldots, m_k^{[N]}(t)$ given $M_N^{\downarrow}(t) = (m_1^{\downarrow}, \ldots, m_n^{\downarrow})$ is specified by

$$\mathbb{E}\left(f_0\left(m_0^{[N]}(t)\right)\cdots f_k\left(m_k^{[N]}(t)\right)\Big|M_N^{\downarrow}(t) = (m_1^{\downarrow}, \ldots, m_n^{\downarrow})\right) = \frac{\sum m_{i_0}^{\downarrow} f_0(m_{i_0}^{\downarrow})\cdots f_k(m_{i_k}^{\downarrow})}{\sum m_{i_0}^{\downarrow}}$$

for any bounded borelian functions $f_0, \ldots, f_k$, where the summation are taken over the indices $i_0 \neq \cdots \neq i_k$ taking values in $\{1, \ldots, n\}$.

Proof: The first claim follows from a slight adaptation of the argument of the proof of Proposition 1. Due to the definition of Poisson clouds, it should be plain that the second claim holds true at time $t = 0$. Moreover, the way we label the particles after a collision entails that the second claim still holds true after the first collision, and thus at any time t by induction, see Lemma 6 in [5] for a very close argument. ■

The next lemma states the convergence of the empirical measure

$$\rho_t^N := \frac{1}{n_t}\sum_{i=1}^{n_t}\delta_{m_i^{[N]\downarrow}(t)}$$

associated to $M_N^{\downarrow}(t)$ to a deterministic measure.

Lemma 5 (convergence of ρ_t^N)
For any time $t < T$, the empirical measure ρ_t^N converges a.s. in total variation towards a deterministic probability measure ρ_t, with finite first moment $\langle \mathrm{id}, \rho_t\rangle = \lim_{N\to\infty}\langle \mathrm{id}, \rho_t^N\rangle$. Moreover, the $((m_i(t), d_i(t)), i \in \mathbb{Z})$ form a Poisson cloud of intensity $\mu_t := \langle 1, \mu_0\rangle\rho_t$.

Proof: It follows from the previous lemma that the conditional laws of $m_0^{[N]}(t)$ and $m_1^{[N]}(t)$ given $M_N^{\downarrow}(t)$ are

$$\frac{\mathrm{id}}{\langle \mathrm{id}, \rho_t^N\rangle}\,\rho_t^N \quad \text{and} \quad \frac{n_t}{n_t - 1}\left(1 - \frac{\mathrm{id}}{n_t\langle \mathrm{id}, \rho_t^N\rangle}\right)\rho_t^N\ .$$

Using the fact that with probability one $m_0^{[N]}(t)$ and $m_1^{[N]}(t)$ do not depend on N when N is large enough, we obtain the convergence as $N \to \infty$ of

$$\left\langle f, \frac{\mathrm{id}}{\langle \mathrm{id}, \rho_t^N\rangle}\,\rho_t^N\right\rangle \quad \text{and} \quad \frac{n_t}{n_t - 1}\left\langle f, \left(1 - \frac{\mathrm{id}}{n_t\langle \mathrm{id}, \rho_t^N\rangle}\right)\rho_t^N\right\rangle, \tag{13}$$

for any bounded measurable function f. Moreover, according to Fact 1 the period n_t is distributed conditionally on $M_N^{\downarrow}(0)$ as $2N+1-p_t$, where p_t is a Poisson process with parameter $c\,\Sigma_N$ (with $c := \langle 1, \mu_0\rangle$ and $\Sigma_N := \sum_{i=-N}^{N} m_i(0)$) stopped when reaching $2N$. The strong law of large numbers then ensures that $c\Sigma_N/2N+1$ converges a.s. towards $1/T$, so that for any $t < T$, when $N \to \infty$, the sequence n_t tends to infinity a.s. In view of (13), this entails the existence of a probability measure ρ_t such that $\langle f, \rho_t^N\rangle \to \langle f, \rho_t\rangle$ a.s. for any measurable bounded function f. The measure ρ_t depends only on the tail algebra of the sequence $((m_i, d_i), i \in \mathbb{Z})$, so that according to Kolmogorov 0-1 law, ρ_t is deterministic. Finally, for any f measurable such that $(\mathrm{id}f)$ is bounded, when N goes to infinity, $\langle \mathrm{id}f, \rho_t^N\rangle \to \langle \mathrm{id}f, \rho_t\rangle$ and if moreover f is bounded

$$\frac{\langle \mathrm{id}f, \rho_t^N\rangle}{\langle \mathrm{id}, \rho_t^N\rangle} \to \mathbb{E}\left[f(m_0(t))\right]\ .$$

Since $\rho_t \neq \delta_0$, this enforces the existence and finitude of the limit $l = \lim_{N\to\infty}\langle \mathrm{id}, \rho_t^N\rangle$. Taking $f(x) = (x \wedge b)/x$ and letting b goes to infinity yields to $l = \langle \mathrm{id}, \rho_t\rangle$. Finally, making $N \to \infty$ in Lemma 4 ensures that $((m_i(t), d_i(t)), i \in \mathbb{Z})$ is a Poisson cloud of intensity $\mu_t = \langle 1, \mu_0\rangle \rho_t$. ■

Lemma 6 *The measure μ_t has mass $\langle 1, \mu_0\rangle$ and is solution of*

$$\frac{\langle f, \mu_t\rangle}{\langle \mathrm{id}, \mu_t\rangle} = \frac{\langle f, \mu_0\rangle}{\langle \mathrm{id}, \mu_0\rangle} + \int_0^t \frac{\langle f, A^+(\mu_s)\rangle}{\langle \mathrm{id}, \mu_s\rangle}\, ds, \tag{14}$$

for any measurable bounded function f. As a consequence, μ_t is solution of (10) and can be written as (11).

Proof: Kolmogorov's equation for the cyclic system reads

$$\mathbb{E}(\langle f, n_t \rho_t^N\rangle) = \mathbb{E}(\langle f, (2N+1)\rho_0^N\rangle) + \int_0^t \mathbb{E}\left(\frac{1}{n_s - 1}\langle f, A^+(n_s\rho_s^N)\rangle \mathbf{1}_{n_s\geq 2}\right)\frac{ds}{c}\,,$$

with $c = \langle 1, \mu_0\rangle$. Dividing both side by $2N+1$ leads to

$$\mathbb{E}\left(\frac{n_t}{2N+1}\langle f, \rho_t^N\rangle\right) = \mathbb{E}(\langle f, \rho_0^N\rangle) + \int_0^t \mathbb{E}\left(\frac{n_s}{2N+1}\,\frac{n_s}{n_s-1}\,\langle f, A^+(\rho_s^N)\rangle \mathbf{1}_{n_s\geq 2}\right)\frac{ds}{c}\,.$$

For $t < T$, we have the convergence

$$\frac{n_t}{2N+1}\,\langle f, \rho_t^N\rangle = \frac{\langle \mathrm{id}, \rho_0^N\rangle}{\langle \mathrm{id}, \rho_t^N\rangle}\,\langle f, \rho_t^N\rangle \to \frac{\langle \mathrm{id}, \rho_0\rangle}{\langle \mathrm{id}, \rho_t\rangle}\langle f, \rho_t\rangle$$

a.s. and in $L^1(\mathbb{P})$ by dominated convergence. In the same manner

$$\frac{1}{n_s-1}\langle f, A^+(n_s\rho_s^N)\rangle \mathbf{1}_{n_s\geq 2} \to \frac{\langle \mathrm{id}, \rho_0\rangle}{\langle \mathrm{id}, \rho_s\rangle}\langle f, A(\rho_s)\rangle$$

a.s. and in $L^1(\mathbb{P})$. Putting pieces together and using the relation $c = \langle 1, \mu_0\rangle$ leads to equation (14). Since $\langle 1, \mu_s\rangle = \langle 1, \mu_0\rangle$, taking $f = 1$ one obtains

$$\frac{1}{\langle \mathrm{id}, \mu_t\rangle} = \frac{1}{\langle \mathrm{id}, \mu_0\rangle} - t \tag{15}$$

It follows that μ_t is solution of

$$\langle f, \mu_t\rangle = \frac{\langle f, \mu_t\rangle}{\langle \mathrm{id}, \mu_t\rangle}\langle \mathrm{id}, \mu_t\rangle = \int_0^t \big(\langle f, A^+(\mu_s)\rangle - \langle f, \mu_s\rangle\,\langle \mathrm{id}, \mu_s\rangle\big)\; ds\,, \tag{16}$$

or it is equivalent, μ_t is solution of (10). According to Theorem 3.9 in [7], such a solution can be written in the form (11). We give here another argument for this point. Define the time τ_t by

$$\int_0^{\tau_t} \langle \mathrm{id}, \mu_s\rangle\, ds = t\,.$$

It follows from (15) that $\tau_t = T\left(1 - e^{-t}\right)$. A change of variable $\tau_u = s$ in (14) gives

$$\begin{aligned}
\left\langle f, \frac{\mu_{\tau_t}}{\langle \mathrm{id}, \mu_{\tau_t}\rangle}\right\rangle &= \left\langle f, \frac{\mu_0}{\langle \mathrm{id}, \mu_0\rangle}\right\rangle + \int_0^t \frac{\langle f, A^+(\mu_{\tau_u})\rangle}{\langle \mathrm{id}, \mu_{\tau_u}\rangle^2}\, du \\
&= \left\langle f, \frac{\mu_0}{\langle \mathrm{id}, \mu_0\rangle}\right\rangle + \int_0^t \left\langle f, A^+\left(\frac{\mu_{\tau_u}}{\langle \mathrm{id}, \mu_{\tau_u}\rangle}\right)\right\rangle du\,.
\end{aligned}$$

and the use of the equalities $\langle \mathrm{id}, \mu_{\tau_t}\rangle = 1/(T - \tau_t)$ and $\tau_t^{-1} = -\log(1 - t/T)$ leads to (11). ■

5 Appendix: proof of Lemma 2

For the sake of simplicity, we will only shows that for t negative, $\Gamma^{\downarrow}_{b_N}(t)$ converges in law to $\Gamma^{\downarrow}_{b}(t)$ when $N \to \infty$. And since the uniform convergence is equivalent to the pointwise convergence in $\mathcal{S}^{\downarrow}_1$, again, we will only focus on the convergence of the first term $m_{b_N}(t)$ of $\Gamma^{\downarrow}_{b_N}(t)$. We write $\mathbf{D}_{\text{per}}$ for the space of 1-periodic càdlàg path endowed with the uniform metric.

Lemma 7 *Assume that $b_N \in \mathbf{D}_{\text{per}}$ converges uniformly to b. Then*

$$\limsup_{N\to\infty} m_{b_N}(t) \leq m_b(t).$$

Proof: if it is not true, then up to an extraction we can assume that there exists $x_N \in [0,1]$ that converges to x^* such that

$$\tilde{a}_{b_N}(t, x_N) \to a_+, \quad \tilde{a}_{b_N}(t, x_N-) \to a_-$$

and

$$m_{b_N}(t) = \tilde{a}_{b_N}(t, x_N) - \tilde{a}_{b_N}(t, x_N-) \geq m_b(t) + \varepsilon$$

with $\varepsilon > 0$. Now using the uniform convergence $b_N \to b$, we have that a_+ and a_- minimizes

$$z \mapsto \int_0^z b - \frac{t}{2}z^2 - x^* z$$

and therefore

$$\tilde{a}_b(t, x^*-) \leq a_- < a_+ \leq \tilde{a}_b(t, x^*).$$

In particular $\tilde{a}_b(t, x^*) - \tilde{a}_b(t, x^*-) \geq m_b(t) + \varepsilon$, which is impossible. ■

Lemma 8 *Assume that $\tilde{a}_b(t, x-) < \tilde{a}_b(t, x)$, $x \in [0,1]$ and that the only minimizers of*

$$z \mapsto \int_0^z b - \frac{t}{2}z^2 - xz$$

are $\tilde{a}_b(t, x-)$ and $\tilde{a}_b(t, x)$. Consider $b_N \in \mathbf{D}_{\text{per}}$ that converges uniformly to b, and α, β such that $\tilde{a}_b(t, x-) < \alpha < \beta < \tilde{a}_b(t, x)$. Then, when N is large enough there exists no y in $[0,1]$ such that $\tilde{a}_{b_N}(t, y) \in [\alpha, \beta]$. In particular,

$$\liminf_{N\to\infty} m_{b_N}(t) \geq m_b(t).$$

Proof: if it is not true, then up to an extraction we can assume that there exists $x_N \in [0,1]$ that converges to x^* such that $\tilde{a}_{b_N}(t, x_N) \to a^* \in [\alpha, \beta]$. Again using the uniform convergence $b_n \to b$, we obtain that a^* minimizes

$$z \mapsto \int_0^z b - \frac{t}{2}z^2 - x^* z.$$

Since $\tilde{a}_b(t, x-) < a^* < \tilde{a}_b(t, x)$ and $x \mapsto \tilde{a}_b(t, x)$ is non decreasing, then $x = x^*$ and this contradicts that the only minimizers of

$$z \mapsto \int_0^z b - \frac{t}{2}z^2 - xz$$

are $\tilde{a}_b(t, x-)$ and $\tilde{a}_b(t, x)$. ■

To conclude, consider a 1-periodic brownian bridge b and let x_1 be such that $m_b(t) = \tilde{a}_b(t, x_1) - \tilde{a}_b(t, x_1-)$. Then, standard arguments involving Millar's path decomposition [13] for Markov processes ensure that with probability one the only minimizers of

$$z \mapsto \int_0^z b - \frac{t}{2}z^2 - x_1 z$$

are $\tilde{a}_b(t, x_1-)$ and $\tilde{a}_b(t, x_1)$. Applying Lemmas 7 and 8, we obtain that when b_N converges to b in distribution, then $m_{b_N}(t)$ also converges to $m_b(t)$ in distribution.

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References

[1] D.J. Aldous. Deterministic and stochastic models for coalescence (aggregation, coagulation): a review of the mean-field theory for probabilists. *Bernoulli* **5** (1999), 3–48.

[2] D.J. Aldous and J. Pitman. The standard additive coalescent. *Ann. Probab.* **26** (1998), 1703–1726.

[3] J. Bertoin. Clustering statistics for sticky particles with Brownian initial velocity. *J. Math. Pures Appl.* **79** (2000), 173–194.

[4] J. Bertoin. Eternal additive coalescents and certain bridges with exchangeable increments. *Ann. Probab.* **29** (2001), no. 1, 344–360.

[5] J. Bertoin. Self-attracting Poisson clouds in an expanding universe. *Comm. Math. Phys.* **232** (2002), no. 1, 59–81.

[6] J.D. Cole. On a quasi linear parabolic equation occurring in aerodynamics. *Quart. Appl. Math.* **9** (1951), 225–236.

[7] M. Deaconu and E. Tanré. Smoluchowski's coagulation equation: probabilistic interpretation of solutions for constant, additive and multiplicative kernels. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), no. 3, 549–579.

[8] S.N. Evans and J. Pitman. Construction of Markovian coalescents. *Ann. Inst. H. Poincaré, Probabilités Statistiques* **34** (1998), 339–383.

[9] C. Giraud. Clustering in a self-gravitating one-dimensional gas at zero temperature. *J. Statist. Phys.* **105** (2001), 585–604.

[10] E. Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.* **3** (1950), 201–230.

[11] M. Lifshits and Z. Shi. Aggregation rates in one-dimensional stochastic systems with adhesion and gravitation. Preprint 2003.

[12] P.A. Martin and J. Piasecki. Aggregation dynamics in a self-gravitating one-dimensional gas. *J. Statist. Phys.* **84** (1996), 837–857.

[13] P. W. Millar. A path decomposition for Markov processes. *Ann. Probab.* **6** (1978), no. 2, 345–348.

[14] J.R. Norris. Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.* **9** (1999), no. 1, 78–109.

[15] R. Sheth and J. Pitman. Coagulation and branching process models of gravitational clustering. *Mon. Not. R. Astron. Soc.* (1997).

[16] G.R. Shorack and J.A. Wellner. Empirical processes with applications to statistics. Wiley, New York, 1986.

[17] Ya. Sinaï. Statistics of shocks in solution of inviscid Burgers equation. *Comm. Math. Phys.* **148** (1992), 601–621.

[18] M. Vergassola, B. Dubrulle, U. Frisch and A. Noullez. Burgers' equation, devil's staircases and the mass distribution function for large-scale structures. *Astron. Astrophys.* **289** (1994), 325–356.