# Symplectic Topology in the cotangent bundle through Generating functions

*Paris, Spring 2021*

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February 13, 2023 0:17

So do I gather strength and hope anew;
For well I know thy patient love perceives
Not what I did, but what I strove to do,–
And though the full, ripe ears be sadly few,
Thou wilt accept my sheaves.

---

Bringing Our Sheaves with Us (1858)
by Elizabeth Chase Allen

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CHAPTER 1

# Introduction

Symplectic topology can be traced back to Poincaré's last geometric theorem (see [Poi12]) stated in 1912 and proved by Birkhoff in 1913 (see [Bir13]). This states

THEOREM 1.1. *(Poincaré-Birkhoff) Let $\mathbb{A}$ be the annulus $S^1 \times [0,1]$ and $\varphi : \mathbb{A} \longrightarrow \mathbb{A}$ a continuous area-preserving map rotating the two boundary curves in opposite directions. Then $\varphi$ has at least two distinct fixed points.*

We first explain the meaning of "rotating the two boundary curves in opposite directions". Indeed let $\pi : \mathbb{R} \times [0,1] \longrightarrow S^1 \times [0,1]$ be the standard covering, induced by the covering of $S^1$ by $\mathbb{R}$. Then $\pi \circ \varphi : \mathbb{R} \times [0,1] \longrightarrow S^1 \times [0,1]$ has a lift to a map $\widetilde{\varphi} : \mathbb{R} \times [0,1] \longrightarrow \mathbb{R} \times [0,1]$ since $\mathbb{R} \times [0,1]$ is simply connected. Then the rotating conditions says that for $i \in \{0,1\}$, $\widetilde{\varphi}(t,i) = (f_i(t), i)$ where $f_0(t) < t < f_1(t)$. Note that the proof of the theorem shows that there are at least two fixed points for $\widetilde{\varphi}$ having different projections on the annulus. One can use this to prove that $\varphi$ has infinitely many periodic points on the annulus. Note that both assumptions - area preservation or opposite rotation - are necessary. Indeed, if we do not assume area preservation, then the map $(t,u) \mapsto (f_u(t), v(u))$ where $f_0(t) < t < f_1(t)$ and $v(u) > U$ for $u \in ]0,1[$ has no fixed points, while if we drop the opposite rotation condition the map $(t,u) \mapsto (t + \frac{1}{2})$ is area preserving with no fixed points.

What is the right generalization of this theorem to higher dimensions ? The obvious answer : replace area preserving by volume preserving is (un)fortunately wrong. Indeed, this question was open for more than 50 years, until Arnold ([Arn65]), in the russian edition of the complete works of Poincaré, proposed an extension that we now partlially describe. First we have to define what is the extension of "area preserving". Let $H(t,q,p)$ be a smooth function of $(t,q,p) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n)$ and consider the differential equation

$$\dot{q}_j = \frac{\partial H}{\partial q_j}(t, q(t), p(t)\,, \dot{p}_j = -\frac{\partial H}{\partial p_j}(t, q(t), p(t))$$

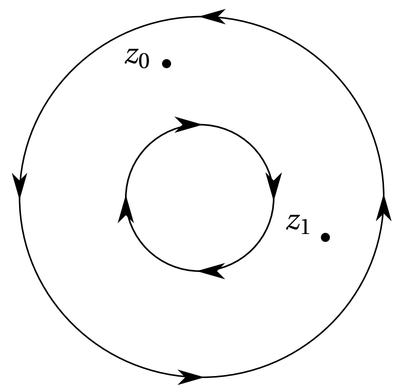


FIGURE 1. A map of the annulus "rotating the two boundary curves in opposite directions" and its fixed points

We call the map $\varphi = \varphi^1$ a Hamiltonian map. A Hamiltonian map preserves the volume, but more than that, it preserves the symplectic form $\sum_{j=1}^{n} dp_j \wedge dq_j$, a fact more or less known to Lagrange (see [Lag11]) and stated in modern form by Poincaré in [Poi90], chapter II: "Théorie des invariants intégraux" (but the term "symplectic" had only appeared in Hermann Weyl's 1939 book [Wey39]). In other words, and as explained by Lagrange, the sum of the projections of the algebraic areas of a surface is preserved by the flow. Note that if $H(t,q,p)$ is unchanged when we replace $q_j$ by $q_{j+1}$ and $p_j$ by $p_{j+1}$, we obtain a map defined on the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, again called a Hamiltonian map. We may now state the conjecture due to Arnold and proved by Conley and Zehnder in 1983 (see [CZ83])

THEOREM 1.2 (Arnold conjecture/Conley-Zehnder's theorem). *Any Hamiltonian map of $T^{2n}$ has at least $2n+1$ fixed points.*

There are generalization of the conjecture in various form as we shall see in the lectures. But this together with Gromov's introduction of holomorphic curves (see [Gro85]) and then Floer's invention of Floer Homology (see [Flo88a; Flo89]) was the starting point of a new branch of mathematics: symplectic topology.

Why are fixed points or periodic points so important ? According to Poincaré: "Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable" ([Poi92], chap. 36)[1].

As we shall see, time proved his statement was even more appropriate than he thought. Indeed, the study of such periodic orbits ushered the way to symplectic

[1]What makes periodic orbits so valuable is that they are the only breach, so to speak, through which we can try to enter a fortress up to now deemed unbreachable. (Translated by the author)

topology, aiming to understand the topological properties of symplectic maps which are not shared by mere volume preserving ones. Whether such properties exist at all was a long standing conjecture- related to the Arnold conjectur-, the so-called "Rigidity/Flexibility" alternative of Gromov. This claims that in higher dimensions, either symplectic (or Hamiltonian) maps cannot be distinguished topologically from the volume preserving ones (i.e. any volume preserving map can be approximated in the $C^0$ topology by symplectic ones), or the only volume preserving maps that can be arbitrarily approximated are the symplectic ones. It turns out that the latter is true as Gromov's non-squeezing theorem shows

THEOREM 1.3 (Gromov's non-squeezing [Gro85]). *Let $D^{2k}(r)$ be the ball of radius r in $\mathbb{R}^{2k}$. Then there is a symplectic embedding of the symplectic ball $B^{2n}(r)$ in $B^2(r') \times \mathbb{R}^{2n-2}$ if and only if $r \leq r'$.*

We leave to the reader to prove that there is always a volume preserving embedding of $B^{2n}(r)$ into $B^2(r') \times \mathbb{R}^{2n-2}$ no matter the values of $r, r'$. As a result, the Gromov alternative is decided as follows

THEOREM 1.4. *If a diffeomorphism is the $C^0$-limit if symplectic diffeomorphisms, then the diffeomorphism is itself symplectic.*

This answers a question but opens a new one:

**Question 1.5.** Describe the homeomorphims that are $C^0$-limits of symplectic diffeomorphisms.

In dimension two these are the are preserving homeomorphism, but not much is known in higher dimensions (see [BHS21; HLS15], etc..), where the study of such maps goes under the name of $C^0$-symplectic topology.

Our goal in these notes will be to reach a proof of the Arnold conjecture for tori. We shall however explore different aspects of symplectic geometry and its many connections going from dynamical systems, to PDE and even number theory.

We shall not use the tools of Gromov and Floer, but instead use the so called "Generating function" approach extending the original approach of Conley and Zehnder and developed by Chaperon, Laudenbach, Sikorav and the author (see [Cha84a][LS85] [Sik87][Vit92]). Its main advantage is that, provided we restrict ourselves to the class of cotangent bundles, this approach is much simpler technically and yields essentially the same results (with very few exceptions) than the more technically involved "holomorphic curves techniques". Moreover proofs using generating functions can often be translated in a more general setting, in proofs using Floer theory.

Another justification t restricting ourselves to cotangent bundles, is that they are privileged objects in symplectic geometry, since they appear as

(1) The phase space of classical mechanics and more generally for variational problems in one dimension
(2) The space where Hamilton-Jacobi equations live

(3) Where the singular supports of sheaves (see [KS90]) or of Fourier integral operators (see [Hör71]) live

Moreover it is then only a technical step (but a steep one, see [AD14]) to use Floer homology to extend the results obtained in cotangent bundles to general manifolds. We shall conclude with an example in which symplectic topology shed light on two-dimensional questions, that is also a great success for symplectic topology[2] by mentioning the recent proof of an old conjecture (see [Fat80]) by D. Cristofaro-Gardiner, V. Humilière and S. Seyfaddini

THEOREM 1.6. *([CHS20] The group of compact supported area preserving homeomorphisms of the* $2$*-disc is not simple.*

[2]Also as fitting to our search for periodic orbits by returning to the two dimensional case

CHAPTER 2

# Symplectic linear algebra

> Money, mechanization, algebra. The three monsters of contemporary civilization.
>
> Simone Weil, Gravity and Grace, 1947

## 1. Basic facts

Even though there is a point in studying infinite dimensional symplectic spaces, and in particular Hilbert symplectic spaces (see [Kuk95; Bus19]) we shall in this book restrict ourselves, unless otherwise stated, to finite dimensional vector spaces over the field $\mathbb{K}$. We shall also assume the field $\mathbb{K}$ is of characteristic different from 2.

DEFINITION 2.1. *Let $V$ be a $\mathbb{K}$-vector space. A symplectic form on $V$ is a bilinear form with values in $\mathbb{K}$, satisfying:*

*(1) It is skew-symmetric:*

$$\forall x, y \in V, \;\; \omega(x, y) = -\omega(y, x)$$

*(2) It is non-degenerate:* $\forall x \in V \setminus \{0\}, \exists y \in V$ *such that* $\omega(x, y) \neq 0$.

REMARKS 2.2.

(1) Since $\omega(x, x) = -\omega(x, x)$ and 2 is invertible in $\mathbb{K}$ we have for all $x$ in $V$, $\omega(x, x) = 0$. If $\omega(x, x) = 0$ we say that $\omega$ is **alternating** . If $\omega$ is alternating, expanding $\omega(x + y, x + y) = 0$ we obtain that $\omega$ is skew-symmetric. When $\mathbb{K}$ has characteristic 2, the notion of symmetric and skew-symmetric coincide (see Exercise 35 for more on this) For example on $\mathbb{Z}/2\mathbb{Z}$ the form $(x, y) \mapsto xy$ is (skew)-symmetric, but not alternating. .

(2) The second condition can be rephrased as requiring that the $\omega$-duality

$$\omega^\sharp : V \longrightarrow V^*$$

given by $x \mapsto \omega(x, \bullet)$ is an isomorphism. This is often considered an "isotropy" condition[1], i.e. there is no preferred direction. Indeed, vectors in the kernel of the $\omega$-duality map, denoted by $\mathrm{Ker}(\omega)$ are special (we shall see a more precise statement in Proposition 2.19, (3)).

[1] Not to be confused with isotropy in the sense of Definition 2.7.

EXAMPLES 2.3.

(1) $V = \mathbb{K}^2$ with the symplectic form $\sigma_1((x, y), (x', y')) = xy' - x'y$.
(2) If $(V_1, \omega_1), (V_2, \omega_2)$ are symplectic spaces, $V = V_1 \oplus V_2$, $\omega = \omega_1 \oplus \omega_2$ defined by $\omega(v_1 + v_2, v_1' + v_2') = \omega_1(v_1, v_1') + \omega_2(v_2, v_2')$ for $v_i, v_i' \in V_i$ is also symplectic. It is called the **symplectic direct sum** of the $(V_i, \omega_i)$. In particular, combining the above example, we get a symplectic structure $\sigma_n = \sigma_1 \oplus .... \oplus \sigma_1$ on $\mathbb{K}^{2n}$.
(3) If $(V, \omega)$ is a symplectic space, then $(V, -\omega)$ is a symplectic space.
(4) If $L$ is a vector space, and $L^*$ its dual space, $V = L \oplus L^*$ endowed with

$$\sigma_L((x, x^*), (y, y^*)) = x^*(y) - y^*(x)$$

is symplectic. Taking $L = \mathbb{K}$ we recover the symplectic form $-\sigma_1$ on $\mathbb{K}^2$ and taking $L = \mathbb{K}^n$ we get $(\mathbb{K}^{2n}, -\sigma_n)$. If $(e_1, ..., e_n)$ is the canonical basis of $\mathbb{K}^n$ then $(f_1, ..., f_n)$ corresponds to the dual basis in $(\mathbb{K}^n)^*$ and we can write, using the tensor notation $\sigma_n = \sum_{j=1}^n e_j^* \wedge f_j^*$, where $(e_1^*, .., e_n^*, f_1^*, .., f_n^*)$ is the dual canonical basis of $\mathbb{K}^{2n}$.
(5) Let $\omega$ be a skew-symmetric form on $W$. Then $\omega$ induces a symplectic form on $W/\mathrm{Ker}(\omega)$, denoted $\overline{\omega}$.

Note that for the above standard spaces we often omit the symplectic form. For example $\mathbb{K}^{2n}$ means $(\mathbb{K}^{2n}, \sigma_n)$, $L \oplus L^*$ means $(L \oplus L^*, \sigma_L)$. We shall use the notation $V_1 \overset{\omega}{\oplus} V_2$ to denote $(V_1, \omega_1) \oplus (V_2, \omega_2)$ and $\overline{V}_1$ to denote $(V_1, -\omega_1)$.,

REMARK 2.4. Some people and books use different sign conventions, not necessarily coherent. Note that with the standard identification of $\mathbb{K}^2$ to $\mathbb{K} \oplus \mathbb{K}^*$ the symplectic form $\sigma_1$ corresponds to $-\sigma_{\mathbb{K}}$. See Section 6 for a detailed discussion.

DEFINITION 2.5. *For a general skew-symmetric form $\omega$ on a vector space, $V$, and $W$ a vector subspace of $V$ we define*

$$W^\omega = \{x \in V \mid \forall y \in W\ \omega(x, y) = 0\}$$

*and we denote by* $\mathrm{Ker}(\omega)$ *the subspace $V^\omega$. The space $W^\omega$ is called the $\omega$-orthogonal*[2] *of $W$.*

For $\omega$ a symplectic form, the second condition implies that $\mathrm{Ker}(\omega)$ reduces to zero. Notice that the skew-symmetric form $\omega$ always induces a symplectic form on $V/\mathrm{Ker}(\omega)$, so we can often reduce questions about general skew-symmetric forms to questions about symplectic forms. When $(V, \omega)$ is symplectic, Grassmann's formula applied to the surjective duality map $\omega_F^\sharp : V \to F^*$ given by $\omega_F^\sharp(v) = \omega(v, \bullet)$, implies that $\dim(F^\omega) = \dim(\mathrm{Ker}(\omega_F^\sharp)) = \mathrm{codim}(F) = \dim(V) - \dim(F)$ proving the first statement of the next Proposition. The proof of the following formulas in the Proposition is classical and left to the reader (recall that $\dim(F) < +\infty$).

[2]Or just the orthogonal if $\omega$ is implicit and there is no scalar product floating around that could fuel confusion.

PROPOSITION 2.6. *Let $F, G$ be subspaces of a symplectic space $(V, \omega)$. Then we have*

*(1)* $\dim(F^\omega) = \operatorname{codim}(F)$
*(2)* $(F^\omega)^\omega = F$
*(3)* $(F+G)^\omega = F^\omega \cap G^\omega$
*(4)* $(F \cap G)^\omega = F^\omega + G^\omega$

In a symplectic vector space we have the following distinguished type of linear subspaces:

DEFINITION 2.7. *A subspace $F$ of $(V, \omega)$ is*

- *isotropic if $F \subset F^\omega$ ($\Longleftrightarrow \omega|_F = 0$ or $\operatorname{Ker}(\omega_{|F}) = F$)*
- *coisotropic if $F^\omega \subset F$*
- *Lagrangian if $F^\omega = F$.*
- *symplectic if $\ker(\omega_{|F}) = F \cap F^\omega = \{0\}$ or equivalently $F \overset{\omega}{\oplus} F^\omega = V$.*

REMARK 2.8. Clearly the $\omega$-orthogonal of a symplectic subspace is symplectic. The $\omega$-orthogonal of an isotropic (resp. coisotropic) subspace is coisotropic (resp. isotropic) and this allows us to reduce many situations to one of this two cases (usually the isotropic case is easier). We also notice that for an isotropic (resp. coisotropic) space $F$ we have $2\dim(F) \leq \dim(V)$ (resp. $2\dim(F) \geq \dim(V)$) and for a Lagrangian $2\dim(F) = \dim(V)$. Thus to check a space is Lagrangian it is enough to verify that it is isotropic and satisfies $2\dim(L) = \dim(V)$.

This last remark implies that Lagrangian can only exist if $\dim(V)$ is even (we shall soon see that this is always the case).

EXAMPLE 2.9. (1) In $L \oplus L^*$, the spaces $L \oplus 0, 0 \oplus L^*$ are Lagrangian.

(2) Let $N$ be a subspace of $L$. Then $N \oplus 0$ is isotropic, $N \oplus L^*$ is coisotropic,
$$\nu^* N = N \oplus N^\perp = \{(x,p) \mid x \in N, p_{|N} = 0\} =$$
is Lagrangian where $N^\perp$ denotes the set of $p \in L^*$ vanishing on $N$, then $(N \oplus L^*)^\omega = 0 \oplus N^\perp$ and $(N \oplus 0)^\omega = V \oplus N^\perp$.

(3) If $M$ is a complement of $N$, then $N \oplus M^\perp$ is symplectic. It is trivially isomorphic to $N \oplus N^*$ since the restriction of the duality map $M^\perp \longrightarrow N^*$ is an isomorphism.

EXERCISES 2.10. (1) Let $S$ be a symplectic subspace of the symplectic space $(V, \omega)$. Prove that if $F$ is isotropic (resp. coisotropic) then $S \cap F$ is isotropic (resp. coisotropic) in $(S, \omega_{|S})$.

(2) Let $S$ be a subspace of the symplectic vector space $(V, \omega)$. If $S + S^\omega = V$ then $S$ is symplectic.

We now define the morphisms between symplectic spaces

DEFINITION 2.11. *A map $\varphi$ between the symplectic vector spaces $(V_1,\omega_1)$ and $(V_2,\omega_2)$ is a* **symplectic map** *if $\varphi^*(\omega_2)=\omega_1$ that is*

$$\forall x,y\in V_1, \omega_2(\varphi(x),\varphi(y))=\omega_1(x,y)$$

*It is a* **symplectomorphism** *if and only if it is bijective- its inverse is then necessarily symplectic. Finally we denote by $Sp(V,\omega)$ the group of symplectic automorphisms of the symplectic space $(V,\omega)$.*

REMARK 2.12. A symplectic map is necessarily injective, since an element $x$ in $\ker(\varphi)$ will also belong to $\ker(\varphi^*(\omega_2))=\ker(\omega_1)$.

EXAMPLE 2.13.
(1) If $\varphi: L_1 \longrightarrow L_2$ is an isomophism, then $\varphi\oplus(\varphi^*)^{-1}$ is a symplectomorphism from $L_1\oplus L_1^*$ to $L_2\oplus L_2^*$.
(2) The map $\varphi: V_1\longrightarrow V_2$ is symplectic if and only if its graph $\Gamma_\varphi\subset \overline{V}_1\overset{\omega}{\oplus} V_2$ defined as $\Gamma_\varphi=\{(x,\varphi(x))\mid x\in V_1\}$ is isotropic. It is a symplectomorphism if and only if $\Gamma_\varphi$ is Lagrangian.

A useful tool in the sequel will be the following **decomposition theorem**

THEOREM 2.14 (Decomposition theorem). *Let $W$ be a subspace of the symplectic space $(V,\omega)$ and let $K=(W\cap W^\omega)=\mathrm{Ker}(\omega_{|W})$. Then for any complement $S$ of $K$ in $W$, $S$ is symplectic and we have a decomposition*

$$W=K\oplus S$$

*Moreover there is an isotropic subspace $K'\subset S^\omega$ uniquely determined by $S$ such that $(K\oplus K',\omega_{K'\oplus K})$ is symplectic, $\omega$-orthogonal to $S$ and symplectomorphic to $(K\oplus K^*,\sigma_K)$ through the map $(x,y)\mapsto(x,\omega(y,\bullet))$. Finally there is a symplectic space $T$, uniquely defined by $S$, such that $V$ can be decomposed as*

$$V=T\overset{\omega}{\oplus}(K'\oplus K)\overset{\omega}{\oplus}S=T\oplus K'\oplus W$$

We shall first prove the

LEMMA 2.15. *Let $C$ be a coisotropic (resp. $I$ an isotropic ) subspace in the symplectic space $(V,\omega)$. Then there exists an isotropic subspace $I$ (resp. a coisotropic subspace $C$) such that $C\oplus I=V$.*

PROOF. Let $I$ be a maximal isotropic subspace such that $C\cap I=\{0\}$ so by duality $C^\omega+I^\omega=V$. Assume $C\oplus I\neq V$ and consider an element $x=u+c$ in $V\setminus(C\oplus I)=(C^\omega+I^\omega)\setminus(C\oplus I)$, with $u\in I^\omega, c\in C^\omega\subset C$. Then $u\in I^\omega\setminus(C\oplus I)$ otherwise, since $c\in C\subset C\oplus I$ we would have $x=u+c\in C\oplus I$, so $I\oplus\mathbb{K}u$ is isotropic and $(I\oplus\mathbb{K}u)\cap C=\{0\}$ since $y\in C$ and $y=v+t\cdot u$ with $v\in I$ implies $tu=y-v\in C\oplus I$, so $t=0, y=0, v=0$. This would contradict the maximality of $I$. $\square$

PROOF OF THE PROPOSITION. Indeed let $S$ be any complement of $K$ in $W$. Then $(S,\omega_{|S})\simeq(W/K,\overline{\omega})$ which is symplectic, so $S$ is symplectic and $S^\omega\cap S=\{0\}$.

Now $S^\omega$ is a complement of $S$ in $V$, is also symplectic and $K^\omega \cap S^\omega$ is coisotropic in $S^\omega$, because the orthogonal of $K^\omega \cap S^\omega$ in $S^\omega$ is $(K+S)\cap S^\omega = W \cap S^\omega = K$ is isotropic. Let $K'$ be given by the above Lemma applied to the coisotropic $K^\omega \cap S^\omega$ in $S^\omega$. Then $K'$ is isotropic and $K' \oplus (K^\omega \cap S^\omega) = S^\omega$. Now clearly $K, K'$ are contained in $S^\omega$ and the map $\varphi$ from $(K \oplus K', \omega_{K\oplus K'})$ to $(K \oplus K^*, \sigma_K)$ given by $\varphi(x, x') = (x, \omega(x', \bullet))$ is

- symplectic, since for all $(x, x'), (y, y') \in K \times K'$ we have
$$\sigma_K((x, \omega(x', \bullet), (y, \omega(y', \bullet))) = \omega(x', y) - \omega(y', x) = \omega(x + x', y + y')$$
using that $\omega(x, y) = \omega(x', y') = 0$ because $K, K'$ are isotropic, .
- onto because the duality map from $S^\omega$ to $K^*$ is onto and vanishes on $(K^\omega \cap S^\omega)$ so is onto on its complement, $K'$.

Therefore $\varphi$ is a symplectomorphism.

□

LEMMA 2.16. *Any symplectic space has even dimension. Let $(V_1, \omega_1), (V_2, \omega_2)$ be two symplectic spaces. Then there is a symplectic map from $(V_1, \omega_1)$ to $(V_2, \omega_2)$ if and only if* $\dim(V_1) \le \dim(V_2)$.

PROOF. We argue by induction. Let $(V_1, \omega_1)$ be a symplectic space. Let $x, y \in V_1$ such that $\omega(x_1, y_1) \neq 0$. Dividing $y$ by $\omega(x, y)$ we can assume $\omega(x_1, y_1) = 1$. Then
$$V_1 = (\mathbb{K}x_1 \oplus \mathbb{K}y_1) \oplus (\mathbb{K}x_1 \oplus \mathbb{K}y_1)^\omega$$
so $\dim(V_1) = 2 + \dim[(\mathbb{K}x_1 \oplus \mathbb{K}y_1)^\omega]$ and $(\mathbb{K}x_1 \oplus \mathbb{K}y_1)^\omega$ is obviously symplectic. By the induction assumption it has even dimension, and then so does $V_1$. Now for the second statement, we argue again by induction on $\dim(V_1)$, the Lemma being obvious for $\dim(V_1) = 0$. Let $x, y \in V_1$ such that $\omega(x_1, y_1) = 1$. Similarly choose $x_2, y_2 \in V_2$ such that $\omega(x_2, y_2) = 1$. Then
$$V_1 = (\mathbb{K}x_1 \oplus \mathbb{K}y_1) \overset{\omega}{\oplus} (\mathbb{K}x_1 \oplus \mathbb{K}y_1)^\omega$$
and
$$V_2 = (\mathbb{K}x_2 \oplus \mathbb{K}y_2) \overset{\omega}{\oplus} (\mathbb{K}x_2 \oplus \mathbb{K}y_2)^\omega$$
The map $\varphi : (\mathbb{K}x_1 \oplus \mathbb{K}y_1) \longrightarrow (\mathbb{K}x_2 \oplus \mathbb{K}y_2)$ sending $x_1$ to $x_2$ and $y_1$ to $y_2$ is symplectic, and by the induction assumption, there is a symplectic map $\psi : (\mathbb{K}x_1 \oplus \mathbb{K}y_1)^\omega \longrightarrow (\mathbb{K}x_2 \oplus \mathbb{K}y_2)^\omega$. Then $\varphi \oplus \psi$ is the required map. A symplectic map is injective, the condition $\dim(V_1) \le \dim(V_2)$ is clearly necessary. □

The Lemma is a special case of the next result, which is fundamental in linear symplectic geometry.

THEOREM 2.17 (Witt's theorem). *Let $(V_1, \omega_1), (V_2, \omega_2)$ be symplectic spaces such that $\dim(V_1) \le \dim(V_2)$, $W_1, W_2$ be subspaces in $V_1, V_2$ respectively and $\varphi : W_1 \longrightarrow W_2$ a linear isomorphism such that $\varphi^*(\omega_{2|W_2}) = \omega_{1|W_1}$. Then $\varphi$ extends to a symplectic map* $\widetilde{\varphi} : (V_1, \omega_1), \longrightarrow (V_2, \omega_2)$.

PROOF. Let us write down the decomposition from Proposition 2.14, $W_1 = K_1 \oplus S_1$. Since $\varphi^*(\omega_{|W_2}) = \omega_{|W_1}$, $\varphi$ must preserve kernels, we have $\varphi(K_1) = K_2 = \ker(\omega_{|W_2}$. Since we may choose for $S_2$ any complement of $K_2$, we may choose $S_2 = \varphi(S_1)$. Then we get from the decomposition theorem that $K_1', K_2'$ which are identified by duality to $K_1^*, K_2^*$ and we may define $\widetilde{\varphi} : K_1' \longrightarrow K_2'$ as the map corresponding to $(\varphi_{K_1}^*)^{-1} : K_1^* \longrightarrow K_2^*$. It is easy to check that $\varphi_{|K_1} \oplus (\varphi_{|K_1})^*)^{-1} : K_1 \oplus K_1^* \longrightarrow K_2 \oplus K_2^*$ yields a symplectic map from $X_1 = (K_1' \oplus K_1) \overset{\omega}{\oplus} S_1$ to $X_2 = (K_2' \oplus K_2) \overset{\omega}{\oplus} S_2$ extending $\varphi$. Since $(X_1, \omega_1)$ and $(X_2, \omega_2)$ are symplectic, we have $(V_1, \omega_1) = (X_1^{\omega_1}, \omega_1) \overset{\omega}{\oplus} (X_1, \omega_1)$, $(V_2, \omega_2) = (X_2^{\omega_2}, \omega_2) \overset{\omega}{\oplus} (X_2, \omega_2)$, we have a symplectomorphism from $(X_1, \omega_1)$ to $(X_2, \omega_2)$, and according to Lemma 2.16 a symplectic map from $(X_1^{\omega_1}, \omega_1)$ to $(X_2^{\omega_2}, \omega_2)$, since $\dim(X_1^{\omega_1}) = \dim(V_1) - \dim(X_1) \leq \dim(V_2) - \dim(X_2)$. Taking the $\omega$-orthogonal direct sum of these maps we obtain a symplectic map from $(V_1, \omega_1)$ to $(V_2, \omega_2)$ extending $\varphi$. □

EXERCISE 2.18. Use exterior calculus to prove that a symplectic map has determinant 1.

Let us state some easy consequences of the above results

PROPOSITION 2.19.

(1) *If* $(V_1, \omega_1)$, $(V_2, \omega_2)$ *are symplectic vector spaces of the same dimension, they are symplectomorphic (and therefore symplectomorphic to* $(\mathbb{K}^{2n}, \sigma_n)$ *or* $(L \oplus L^*, \sigma_L)$ *where* $2n = \dim(V_1) = 2\dim(L)$*).*

(2) *Any isotropic subspace is contained in a Lagrangian subspace and any coisotropic subspace contains a Lagrangian subspace.*

(3) *If* $(V_1, \omega_1)$, $(V_2, \omega_2)$ *are symplectic vector spaces with* $\dim(V_1) \leq \dim(V_2)$
- *if* $x_j \in V_j \setminus \{0\}$ *there is a symplectic map from* $V_1$ *to* $V_2$ *sending* $x_1$ *to* $x_2$
- *If the* $I_j$ *are isotropic subspaces in* $V_j$ *and* $\dim(I_1) \leq \dim(I_2)$*, then there is a symplectic map sending* $I_1$ *in* $I_2$
- *if* $C_i$ *are coisotropic subspaces in* $V_i$*, with* $\dim(V_1) \leq \dim(V_2)$ *and* $\dim(C_1) - \operatorname{codim}(C_1) \leq \dim(C_2) - \operatorname{codim}(C_2)$ *then there is a symplectic map from* $V_1$ *to* $V_2$ *sending* $C_1$ *to* $C_2$*.*
- *if* $\dim(V_1) = \dim(V_2)$ *and* $L_i$ *are Lagrangian subspaces in* $V_i$*, then there is a symplectomorphism from* $V_1$ *to* $V_2$ *sending* $L_1$ *to* $L_2$*.*

PROOF. Statement (1) follows from Lemma 2.16, since for $\dim(V_1) = \dim(V_2)$, a symplectic map is an isomorphism. For (2), we may assume, according to (1) that we are in $(L \oplus L^*, \sigma_L)$. Let $I$ be isotropic and let $J \subset L$ be a subspace with $\dim(J) = \dim(I)$ (this is possible since $\dim(I) \leq \dim(L)$, see Remark 2.8). An isomorphism from $J$ to $I$ pulls-back $\omega_{|I}$ to $\omega_{|J}$, since both vanish, hence by Witt's Theorem this map can be extended to a symplectomorphism from $L \oplus L^*$ to itself. Then the image of $L \oplus 0$ is a Lagrangian containing $I$. For (3) the first statement is just Witt's theorem applied to the isomorphism $\varphi$ from $\mathbb{K}x_1$ to $\mathbb{K}x_2$ sending $x_1$ to $x_2$. The second statement to an embedding of $I_1$ to $I_2$, the pull-back condition being obvious since the $\omega_i$ vanish on $I_i$.

For the third statement set $\dim(V_i) = 2n_i, \dim(C_i) = n_i + c_i$ so that $\mathrm{codim}(C_i) = n_i - c_i$. We assume $n_1 \le n_2$ and $c_1 \le c_2$. Then $C_i$ decomposes as $C_i^{\omega_i} \oplus S_i$, and $\dim(S_i) = 2c_i$ so we have a symplectic map $\varphi$ from $S_1$ to $S_2$. Set $S_2 = \varphi(S_1) \overset{\omega}{\oplus} T_2$ where $T_2$ is symplectic of dimension $2(c_2 - c - 1)$. Let $I_2$ be a Lagrangian subspace in $T_2$. Then the map $C_1^{\omega_1}$ can be sent by an isomorphism to $C_2^{\omega_2} \oplus I_2$ since both are isotropic and $n_1 - c_1 \le n_2 - c_2 + (c_2 - c_1) = n_2 - c_1$. Since $\varphi(S_1)$ is orthognal to $C_2^{\omega_2} \oplus I_2$, $\varphi$ extends to a symplectic map from $C_1^{\omega_1} \oplus S_1$ to $C_2^{\omega_2} \oplus \varphi(S_1) \oplus I_2$, hence, by Witt's theorem to a symplectic map from $V_1$ to $V_2$.

The last statement is obtained by applying Witt's theorem to any isomorphism $L_1 \longrightarrow L_2$. In both cases the pull-back condition is obvious since $\omega_i$ vanishes on all these spaces. Its symplectic extension $\widetilde{\varphi} : V_1 \longrightarrow V_2$ is then the required symplectomorphism.

$\square$

EXERCISE 2.20. Prove that if there is a symplectic map from $V_1$ to $V_2$ sending the coisotropic $C_1$ to the coisotropic $C_2$ we must have $\dim(V_1) \le \dim(V_2)$ and $\dim(C_1) - \mathrm{codim}(C_1) \le \dim(C_2) - \mathrm{codim}(C_2)$.

Let us remind the reader that if $\alpha, \beta$ are linear forms then $\alpha \wedge \beta$ is the skew-symmetric form defined by $\alpha \wedge \beta(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$.

COROLLARY 2.21. *Any vector space endowed with a skew-symmetric form $\omega$ has the decomposition $W = S \oplus K$ with $(S, \omega_{|S})$ symplectic and $K = \mathrm{Ker}(\omega)$. The rank of $\omega$ is by definition the dimension of $S$. Then we can write*

$$\omega = \sum_{j=1}^{r} \alpha_j \wedge \beta_j$$

*where the $\alpha_1, \dots \alpha_r, \beta_1, \dots \beta_r$ are linearly independent one forms.*

PROOF. The proof is identical to that of the first part of Proposition 2.14 (there was no need for an ambient symplectic space): if $S$ is a complement of $K$, then $(W/K, \overline{\omega}) \simeq (S, \omega_{|S})$. This shows that the decomposition holds . According to Proposition 2.19, $(S, \omega)$ is then isomorphic to $(\mathbb{K}^{2r}, \sigma_r)$, and if the $\tilde{\alpha}_j, \tilde{\beta}_j$ are the images of the canonical basis of $\mathbb{K}^{2r}$, $e_j^*, f_j^*$, we get that $\omega_{|S} = \sum_{j=1}^r \tilde{\alpha}_j \wedge \tilde{\beta}_j$. Denoting by $\alpha_j, \beta_j$ the extension of the $\tilde{\alpha}_j\, \tilde{\beta}_j$ vanishing on $K$, we get the above formula. The linear independence of $\alpha_1, \dots \alpha_r, \beta_1, \dots \beta_r$ follows from the linear independence of the $e_1^*, \dots, e_r^*, f_1^*, \dots, f_r^*$. $\square$

COROLLARY 2.22. *A skew-symmetric form, $\omega$, is symplectic if and only if* $\mathrm{rank}(\omega) = \dim(E)$. *If $\mathbb{K}$ has characteristic zero, then a skew-symmetric form on a vector space $E$ has rank $2r$ if and only if $\omega^r \neq 0$ and $\omega^{r+1} = 0$.*

PROOF. The first claim is obvious since then $\omega$ is isomorphic to $\sigma_n$ It is is enough to check this for $\omega = \sum_{j=1}^r \alpha_j \wedge \beta_j$ where the $\alpha_1, \dots, \alpha_r, \beta_1, \dots \beta_r$ are linearly independent. But $\omega^r = (r!)\alpha_1 \wedge \beta_1 \wedge \dots \wedge \alpha_r \beta_r$ and $\omega^{r+1} = 0$.

$\square$

DEFINITION 2.23. *We denote by* $\Lambda(V,\omega)$ *the Lagrangian Grassmannian, that is the set of Lagrange subspaces in* $(V,\omega)$.

Since any symplectic vector space is isomorphic to $(\mathbb{K}^{2n},\sigma_n)$, the group of symplectic automorphisms of $(V,\omega)$, $Sp(V,\omega)$ is isomorphic to $Sp(2n) = Sp(\mathbb{K}^{2n},\sigma_n)$ and the Lagrangian Grassmannian is isomorphic to $\Lambda(n) = \Lambda(\mathbb{K}^{2n},\sigma_n)$.

We now give a better description of the set of Lagrangian subspaces of $(V,\omega)$.

PROPOSITION 2.24. *Let $L$ be a Lagrangian subspace in* $(V,\omega)$. *Recall that two vector subspaces $V,W$ in $E$ are said to be* **transverse** *if* $V+W=E$.

(1) *The space*

$$\Lambda_L(V,\omega) = \left\{T \mid T \text{ is Lagrangian and } T\cap L = \{0\}\right\}$$

*is isomorphic to the space of quadratic forms on $L^*$. When $\mathbb{K}=\mathbb{R}$, $\Lambda_L(V,\omega)$ is contractible, and $\Lambda(V,\omega)$ is a smooth manifold of dimension* $\frac{n(n+1)}{2}$.

(2) *The action of $Sp(V,\omega)$ on the set of pairs of transverse Lagrangians is transitive. Moreover given $L_1,L_2$ and $L_1',L_2'$ two pairs of transverse Lagrangians and an isomorphism $\varphi: L_1 \longrightarrow L_1'$ there is a unique symplectic map extending $\varphi$ and sending $L_2$ to $L_2'$.*

(3) *The set of Lagrangians transverse to both $L_1$ and $L_2$, where $L_1,L_2$ are a fixed pair of transverse Lagrangians in $V$, can be identified to the set of non-degenerate quadratic forms on $L^*$. If $\mathbb{K}=\mathbb{R}$ this space has exactly $n+1$ connected components.*

PROOF. First notice that $V$ is symplectomorphic to $L\oplus L^*$ with the symplectic form $\sigma_L$ and that $L\oplus 0$ is a Lagrangian subspace. According to Proposition 2.19 there is a symplectic map $\psi: V \longrightarrow L\oplus L^*$ such that $\psi(L) = L\oplus 0$, so we can work in $L\oplus L^*$ and identify $L$ with $L\oplus 0$.

Let $T$ be a Lagrangian in $L\oplus L^*$ with $T\cap L=\{0\}$. Then $T$ is the graph of a linear map $A_T: L^* \to L$, more precisely

$$T = \{(A_T y^*, y^*) | y^* \in L^*\}.$$

The subspace $T$ is Lagrangian if and only if

$$\sigma((A_T y_1^*, y_1^*), (A_T y_2^*, y_2^*)) = 0, \text{ for all } y_1^*, y_2^* \in L^*$$

i.e. if and only if

$$\langle y_1^*, A_T y_2^*\rangle = \langle y_2^*, A_T y_1^*\rangle$$

that is if $\langle\cdot, A_T\cdot\rangle$ is a bilinear symmetric form on $L^*$. But bilinear forms are in 1-1 correspondence with quadratic forms[3]. The second part of the statement immediately follows from the fact that the set of quadratic forms on an $n$-dimensional vector space is a vector space of dimension $\frac{n(n+1)}{2}$, and the fact that to any Lagrangian $L_0$ we may associate a transverse Lagrangian $L_0'$, and $L_0$ is contained in the open set of Lagrangians

[3]We again need that $\mathbb{K}$ has characteristic $\neq 2$.

transverse to $L_0'$ (Well we still have to check the change of charts maps are smooth, this is left as an exercise). This proves 1.

To prove (2) let $(L_1, L_2)$ and $(L_1', L_2')$ be two pairs of transverse Lagrangians. By the transitivity of $Sp(V,\omega)$ on $\Lambda(V,\omega)$ we may assume $V = (L \oplus L^*, \sigma_L)$ and $L_1 = L_1' = L$. It is enough to find $\varphi \in Sp(V,\omega)$ such that $\varphi_{|L} = \mathrm{Id}_L, \varphi(L_2) = L^*$. The map $x + y \mapsto (x, \omega(y, \bullet))$ where $x \in L, y \in L^*$ is the required map. Uniqueness follows from the fact that the only symplectic map $(x, x^*) \mapsto (x, Tx^*)$ is obtained for $T = \mathrm{id}$.

Finally for (3), we may assume by (2) that two of three Lagrangians are $L \oplus 0$ and $0 \oplus L^*$ and the third $T$ being transverse to $L$, is the graph of the quadratic form $A_R$ on $L^*$. Then $T \cap L^*$ can be identified to the set of $y^* \in L^*$ such that $A_T y^* = 0$, that is the kernel of $A_T$. Thus $T$ is transverse to $L^*$ if and only if $A_T$ is non-degenerate. But the set of non-degenerate quadratic form has $n+1$ connected components defined by the index (that is the number of negative eigenvalues). □

REMARKS 2.25.
(1) We refer to Corollary 2.46 for a more precise statement of (3).
(2) We shall see in Exercise 13 that the quadratic forms associated to two pairs of triples of Lagrangians are conjugate if and only if the triple of Lagrangians are in the same $Sp(V,\omega)$-orbit.

EXERCISES 2.26.

(1) Prove that if $L_0, L_1$ are transverse Lagrangian subspaces in $(V,\omega)$, then $\omega$ induces a well-defined isomorphism $L_1 \longrightarrow (L_0)^*$ through the map $\omega^\sharp : x \mapsto \omega(x, \bullet)$ and this extends to a symplectic map from $(V,\omega)$ to $(L_0 \oplus L_0^*, \sigma_L)$ sending $L_0$ to itself and $L_1$ to $L_0^*$.
(2) Compute the dimension of the space of Lagrangians containing a given isotropic subspace, $I$. Hint: show that it is the space of Lagrangians in $I^\omega / I$.
(3) Prove that $Sp(2n)$ acts transitively on the set of isotropic subspaces (resp. coisotropic subspaces) of given dimension.

## 2. Symplectic reduction

Let $(V,\omega)$ be a symplectic vector space and $C$ a coisotropic subspace. Then $\omega$ naturally induces a symplectic form on $C/C^\omega$. We denote it by $\omega_C$ or simply $\omega$ if there is no ambiguity.

DEFINITION 2.27. *The space $(C/C^\omega, \omega_C)$ is a symplectic vector space of dimension $2\dim(C) - \dim(V)$. It is called the symplectic reduction of $V$ by $C$. Let $X$ be a vector space such that*

*(1) $X + C = V$ or equivalently $X^\omega \cap C^\omega = \{0\}$*
*(2) $X \cap C^\omega = \{0\}$*

*Then $X_C = X \cap C / C^\omega$ is called the symplectic reduction of $X$ by $C$*

The dimension count is left to the reader. Note that it is often convenient to write $\dim(C) = \frac{1}{2}\dim(V) + c = n + c$ where $\dim(V) = 2n$. Then $C/C^\omega$ has dimension $2c$.

Symplectic reduction is an important operation in symplectic geometry. We shall establish some of its basic properties. The main point of symplectic reduction is that certain maps and submanifolds in $V$ can be "reduced" to $(C/C^\omega, \omega_C)$. We start with

LEMMA 2.28 (Orthogonality commutes with reduction). *Let $X$ and $C$ satisfy properties (1) and (2) of Definition 2.27. Then we have $(X_C)^{\omega_C} = (X^\omega)_C$.*

PROOF. First of all we have an inclusion $(X^\omega)_C \subset (X_C)^{\omega_C}$. Indeed, an element in $(X^\omega)_C$ is just the projection of an element in $X^\omega \cap C$, hence is orthogonal to any element of $X \cap C$. But two orthogonal subspaces in $C$ are still orthogonal after projection on $C/C^\omega$. Now $\dim X_C = \dim(X) - \operatorname{codim}(C)$ and we notice that taking orthogonals, we also have $X^\omega + C = V$ and $X^\omega \cap C^\omega = \{0\}$. Now the above argument proves that $\dim(X^\omega)_C = \dim(X^\omega) - \operatorname{codim}(C) = \operatorname{codim}(X) - \operatorname{codim}(C) = \dim(C) - \dim(X)$. On the other hand $\dim(X_C^{\omega_C}) = \dim(C/C^\omega) - \dim(X_C)(\dim(C) - \operatorname{codim}(C)) - (\dim(X) - \operatorname{codim}(C)) = \dim(C) - \dim(X)$ hence by dimension count we must have $(X^\omega)_C = (X_C)^\omega$. $\square$

EXERCISE 2.29. Prove directly that $(X_C)^{\omega_C} \subset (X^\omega)_C$.

The next lemma shows that for some special spaces, only one of the two properties ( 1), ( 2) from Definition 2.27 needs to be checked.

LEMMA 2.30 (Automatic transversality). *Let $I$ be isotropic (resp. $K$ be coisotropic), $C$ be coisotropic in the symplectic vector space $(V, \omega)$. Then $I + C = V$ (resp. $K \cap C^\omega = \{0\}$) implies $I^\omega \cap C^\omega = \{0\}$ (resp. $K + C = V$). Moroever if $I$ is isotropic (resp. $K$ is coisotropic) and satisfies the transversality conditions ( 1), ( 2) from Definition 2.27 then so does $I^\omega$ (resp. $K^\omega$).*

PROOF. This is obvious by taking orthogonals, since $I + C = V$ implies $I \cap C^\omega \subset I^\omega \cap C^\omega = \{0\}$ in the isotropic case, and $V = K^\omega + C \subset K + C$ in the coisotropic case. For the second statement it follows immediately (in the isotropic case) because we just proved that conditions ( 1), ( 2) are equivalent to $I + C = V$ which obviously implies $I^\omega \cap C^\omega = \{0\}$ which is in turn equivalent to $I^\omega$ satisfying ( 1), ( 2). The coisotropic case follows by the same argument. $\square$

From this we may prove

PROPOSITION 2.31. *Let $I$ be isotropic, $C$ be coisotropic in $(V, \omega)$ and assume $I + C = V$. Then $I_C = (I \cap C)/C^\omega$ is isotropic. Similarly if $K$ is coisotropic and $K \cap C^\omega = \{0\}$ then $K_C$ is coisotropic. As a result if $L$ is Lagrangian, so is $L_C$.*

PROOF. Indeed, using Lemma 2.28 we have

$$(I_C)^{\omega_C} = (I^\omega)_C \supset I_C$$

so $I_C$ is isotropic. The coisotropic case follows from the same argument:

$$(K_C)^{\omega_C} = (K^\omega)_C \subset K_C$$

so $K_C$ is coisotropic. Since a Lagrangian is just a space both isotropic and coisotropic, the last claim clearly follows. □

EXERCISE 2.32. Let $S$ be a symplectic subspace of $(V,\omega)$. Assume $S+C=V$ and $S\cap C^\omega=\{0\}$. Is $S_C$ symplectic ?

There is a converse operation. Given $L\subset C/C^\omega$ we can construct a Lagrangian in $V$ as follows. Remember from Lemma 2.15 that a coisotropic space always has an isotropic complement. For a subspace $X$ of $C/C^\omega$ by a lift of $X$ we mean a space $Y\subset C$ such that the projection of $Y$ on $C/C^\omega$ is an isomorphism onto $X$.

PROPOSITION 2.33. *Let $J$ be an isotropic complement to $C$. Then if $I$ is isotropic in $C/C^\omega$, there is a unique lift $I_J$ of $I$ to $C$ such that $I_J\oplus J$ is isotropic in $V$. The same holds for $K$ coisotropic in $C/C^\omega$. In particular if $L$ is Lagrangian in $C/C^\omega$ then $L_J\oplus J$ is Lagrangian in $V$.*

PROOF. According to the Decomposition Theorem (Proposition 2.14) we can write $C=C^\omega\oplus S$ with $S$, symplectic being any complement of $C^\omega$. Let us choose $S$ such that $S\subset J^\omega$, by taking $S=(J\oplus C^\omega)^\omega=J^\omega\cap C$. Clearly $S\subset C$ and $S\cap C^\omega=J^\omega\cap C^\omega=\{0\}$ because $J\oplus C=V$. So $S$ is a complement of $C^\omega$ and is thus symplectic.

Now since $S\longrightarrow C/C^\omega$ is a symplectic isomorphism, we have a unique lift of $I_J$ to $S_J$. Then $I_J\oplus J$ is isotropic, since $I_J\subset S\subset J^\omega$ and both $J$ and $I_J$ are isotropic. In the Lagrangian case, we only have to check dimensions.

In the coisotropic case, we just apply the above to $K^{\omega_C}$, we get an isotropic space $I_J\oplus J$ and set $K=(I_J\oplus J)^\omega$. Then $J\subset(I_J\oplus I_J)^\omega=I_J^\omega\cap J^\omega$ since $J\subset J^\omega$ and $J\subset S^\omega\subset I_J^\omega$ since $I_J\subset S$. So we can write $K=K_J\oplus J$. □

Note that choosing $J$ determines $S$ as $(J^\omega\cap C)$. We shall equivalently write $I_J$ or $I_S$ for the lift of $I$ to $S$. We may now describe the reduction on Lagrangians.

PROPOSITION 2.34. *Let $C$ be coisotropic and $\Lambda_C(V)$ be the set of Lagrangians transverse to $C$. Then the map $\Lambda_C(V)\longrightarrow\Lambda(C/C^\omega)$ obtained by symplectic reduction is a fibration with fibre the product $\mathscr{L}(L_C,C^\omega)\times\Lambda_{C_K}(K/K^\omega)$ where $\mathscr{L}(L_C,C^\omega)$ is an affine space with underlying vector space the set of linear maps from $L_C$ to $C^\omega$, $K=L+C^\omega$, $C_K=(C\cap K)/K^\omega$. For $\mathbb{K}=\mathbb{R}$ the fibre is contractible.*

PROOF. Let $L,L'$ have the same reduction. Note that $L+C=L'+C=V$ so $L\cap C^\omega=L'\cap C^\omega=\{0\}$. Our assumption means that $(L\cap C)\oplus C^\omega=(L'\cap C)\oplus C^\omega$, so $L'\cap C$ is the graph of a map from $L\cap C$ to $C^\omega$. Conversely any such graph yields an isotropic subspace of $C$, since for $I=L\cap C$ and $A:I\longrightarrow C^\omega$, any two elements $x,y\in I$ we have

$$\omega(x+Ax,y+Ay)=\omega(x,y)+\omega(x,Ly)+\omega(Lx,y)+\omega(Lx,Ly)$$

But $\omega(x,y)$ vanishes by assumption and since $I\subset C\cap C^\omega$ the terms $\omega(x,Ly),\omega(Lx,y)$ and $\omega(Lx,Ly)$ vanish. Then the set of $L\cap C$ with fixed reduction is an affine space and its underlying vector space is the set of linear maps $\mathscr{L}(L_C,C^\omega)$.

Now if we fix $I = L\cap C$ isotropic, then $L$ is defined by a subspace $X$ in $I^\omega = K$ such that $L = X \oplus (L\cap C)$ well defined modulo $L\cap C = I$. So we need only to keep track of the image of $X$ in $I^\omega/I$, that is $X_K$ in $\Lambda(K/K^\omega)$. Moreover $X$ must satisfy the condition $X\cap C = \{0\}$ and since $I\subset C$, this is equivalent to $X_K \cap C_K = \{0\}$. But $C\cap K = C\cap I^\omega = C\cap(L+C^\omega) = (C\cap L)+C^\omega = I + C^\omega = (I^\omega\cap C)^\omega$. As a result $C_K$ is Lagrangian, and so $X_K \in \Lambda_{C_K}(K/K^\omega)$. It is easy to see that given such a pair $(I,Y)$ where $I$ is isotropic such that $I\cap C^\omega = \{0\}$ and $Y\in\Lambda_{C_K}(K/K^\omega)$ wherer $K = I^\omega$, and for any lift $X$ of $Y$ to $I^\omega$ we have $I\oplus X = L$ is in $\Lambda_C(V)$, and $L$ does not depend on the choice of the lift $X$ of $Y$.

The last property follows from the fact that both $\mathscr{L}(L_C, C^\omega)$ and $\Lambda_{C_K}(K/K^\omega)$ are contractible, so their fibre product is contractible. □

Finally notice that symplectic reduction is a transitive operation :

PROPOSITION 2.35. *Let $K\subset C$ be two coisotropic spaces. Let us denote by $\overline{K} = K/C^\omega \subset C/C^\omega$ Then for any subspace $X$ for which $X_C$ and $(X_C)_{\overline{K}}$ are defined, we have that $X_K$ is defined and*

$$X_K = (X_C)_{\overline{K}}$$

PROOF. Note that $\overline{K}$ is not the reduction of $K$ since $K$ does not satisfy the transversality condition. By assumption $X_C$ satisfies the transversality assumption for $\overline{K}_C$ that is $X_C\cap \overline{K}^{\omega_C} = X_C^{\omega_C}\cap\overline{K}^{\omega_C} = \{0\}$. First

$$X_C\cap\overline{K}^{\omega_C} = ((X\cap C)\cap K^\omega)/C^\omega = X_C\cap(K^\omega/C^\omega) = (X\cap K^\omega)/C^\omega$$

because $C\supset K^\omega\supset C^\omega$. Then since we assume $X$ satisfies the transversality assumption for $C$ we have $X\cap C^\omega = \{0\}$. So we may conclude $X\cap K^\omega = \{0\}$. Similarly for $X_C^{\omega_C} = (X^\omega)_C$, we prove from $X_C^{\omega_C}\cap\overline{K}^{\omega_C} = \{0\}$ that $X^\omega\cap K^\omega = \{0\}$ and thus $X_K$ is well defined. Finally

$$(X_C)_{\overline{K}} = \left[(X\cap C)/C^\omega))\cap(K/C^\omega)\right]/K^\omega = ((X\cap C)\cap K)/K^\omega = (X\cap K)/K^\omega = X_K$$

□

## 3. Complex structures

In this section we assume $\mathbb{K} = \mathbb{R}$.

DEFINITION 2.36. *Let $V$ be a complex vector space. A complex valued map $h : V\times V\longrightarrow\mathbb{C}$ is a* **sesquilinear form**[4] *if:*

*(1) $h(z,z') = \overline{h(z',z)}$;*
*(2) $h(\lambda z, z') = \lambda h(z,z')$ for $\lambda\in\mathbb{C}$;*
*(3) $h(z,\lambda z') = \bar\lambda h(z,z')$ for $\lambda\in\mathbb{C}$ (note that (1) and (2 ) imply 3))*

[4]This is latin for "one and a half" linear by which we mean linear in the first variable and antilinear in the second one. Physicists usually take different conventions: they impose linearity on the left and antilinearity on the right.

*We say that $h$ is* **hermitian** *if moreover $h(z,z) > 0$ for all $z \neq 0$. We write*

$$h(z,z') = g(z,z') - i\omega(z,z'),$$

*where $g$ is a (real-valued) scalar product and $\omega$ is (real valued) symplectic*[5].

Note that $\omega$ is skew symmetric since $h(z,z)$ is real according to (1) and non-degenerate since $\omega(iz,z) = g(z,z) > 0$ .

EXAMPLE 2.37. On $\mathbb{C}^n$, define

$$h((z_1,\cdots,z_n),(z'_1,\cdots,z'_n)) = \sum_{j=1}^{n} \varepsilon_j z_j \bar{z}'_j \in \mathbb{C}.$$

where $\varepsilon_j \in \{0,-1,1\}$. This is a sesquilinear form and is hermitian if all the $\varepsilon_j$ equal $+1$. If there are $p$ positive and $q$ negative $\varepsilon_j$, we say that $h$ has signature $(p,q)$. This is invariant by a complex change of variable and it is well known that any sesquilinear form is isomorphic to such a form.

When $h$ is hermitian, (i.e. all the $\varepsilon_j = +1$) the symmetric part is the usual scalar product on $\mathbb{C}^n = \mathbb{R}^{2n}$ given by

$$g((z_1,...,z_n),(z_1,..,z_n)) = \sum_{j=1}^{n} |z_j|^2$$

while $\omega$ is the standard symplectic form

$$\omega((z_1,...,z_n),(z'_1,..,z'_n)) = \sum_{j=1}^{n} (x_j y'_j - x'_j y_j)$$

where $z_j = x_j + iy_j, z'_j = x'_j + iy'_j$ and corresponds to $\sigma_n$.

We shall denote by $J_0$ the multiplication by $i$, so that in standard real coordinates any hermitian vector space can be reduced to the above with $J_0$ given by the diagonal bloc matrix with blocs $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or in the decomposition $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ by the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The following Proposition is immediately checked

PROPOSITION 2.38. *Let $h$ be a hermitian structure on the complex vector space $V$ and $J$ be the endomorphism corresponding to multiplication by $i$. Then setting $h = g - i\omega$ we have*

$$\begin{cases} g(Jz,z') = -\omega(z,z') \\ \omega(Jz,z') = g(z,z') \end{cases}$$

*In particular $J$ is an isometry for $g$ and a symplectic map for $\omega$.*

[5] The minus sign in front of $\omega$ of the above formula allows us to get from the standard hermitian form on $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$, the standard symplectic form on $\mathbb{R}^n \oplus \mathbb{R}^n$. Had we adopted the physicist convention, we would have had a + sign...

REMARK 2.39. The skew-symmetric form $\omega$ is non-degenerate because $\omega(Jz,z) = g(z,z) > 0$ for all $z \neq 0$.

**Conclusion:** Any hermitian space $V$ has a canonical symplectic form.

We will now answer the following question: can a symplectic vector space be made into a hermitian space? In how many different ways?

DEFINITION 2.40. *A complex structure on a real vector space $W$ is an automorphism $J$ of $W$ such that $J^2 = -\mathrm{Id}$. This makes $W$ into a complex vector space with $(a+ib)\cdot v = a\cdot v + b\cdot Jv$. If $\omega$ is a symplectic structure on $W$, then $J$ is* **tame** *if $\omega(J\xi,\xi) > 0$ for all $\xi \neq 0$. It is* **compatible** *if moreover $\omega(J\xi,J\eta) = \omega(\xi,\eta)$*

Note that if $(V,\omega)$ is symplectic and $J$ compatible, then $h(z,z') = \omega(Jz,z') - i\omega(z,z')$ defines a hermitian structure on $V$ with associated symplectic structure $\omega$. Tameness is a weaker version that is moire flexible, and can be useful.

REMARK 2.41. The complex structure $J$ already gives us a convenient way to write that a matrix is symplectic. Indeed, $\omega(Rx,Ry) = -g(JRx,Ry)$ so the condition to be symplectic is

$$R^*JR = J$$

where $R^*$ is the adjoint of $R$, defined by $g(Rx,y) = g(x,R^*y)$ for all $x,y \in E$. This implies in particular that preserving two of the three objects $g,\omega,J$ implies preserving the third[6]. So for example

$$GL(n,\mathbb{C}) \cap Sp(2n,\mathbb{R}) = U(n)$$

because commuting with $J$ and preserving the symplectic form implies preservation of the scalar product hence of the hermitian form

$$O(2n,\mathbb{R}) \cap Sp(2n,\mathbb{R}) = GL(n,\mathbb{C}) \cap O(2n,\mathbb{R}) = U(n)$$

since preserving $g$ and $\omega$ means preserving $J$ and of course $h$, or commuting with $J$ and preserving $g$ implies preserving also $\omega$ hence the hermitian form.

PROPOSITION 2.42. *Let $(V,\omega)$ be a symplectic vector space. Then the set $\mathscr{J}(\omega)$ of complex structures compatible with $\omega$ can be identified to the set $S(n)$ of symplectic symmetric and positive matrices. It is therefore contractible.*

PROOF. (cf. [MS07]) Notice that the fact that $\omega(J\xi,\eta)$ is a scalar product implies that $\omega(J\xi,J\eta) = (\xi,J\eta) = (J\eta,\xi) = \omega(J^2\eta,\xi) = \omega(\xi,\eta)$. In other words $J \in Sp(V,\omega)$. We can always assume $(V,\omega) = (\mathbb{R}^{2n},\sigma_n)$, so the standard identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ gives a standard complex structure $J_0$ and scalar product $(\xi,\eta)_0$ and $\sigma_n(\xi,\eta) = (J_0\xi,\eta)_0$. Notice that $J_0^* = -J_0$. We are looking for $J$ such that

- $J^2 = -\mathrm{Id}$
- $\omega(J\xi,J\eta) = (J_0J\xi,J\eta)_0 = (J_0\xi,\eta)_0$

[6] By preserving $J$ we mean commuting with $J$: this means that the map is complex-linear and not only real-linear

- $\omega(J\xi,\xi) = (J_0 J\xi,\xi)_0 > 0$ for all $\xi \neq 0$

The second condition can be rewritten as $J^* J_0 J = J_0$ i.e. $J$ is symplectic. So we are looking for $J$ symplectic such that the first and third condition hold. But then $J_0 J = P$ is symplectic as the composition of two symplectic maps, symmetric since $P^* = J^* J_0^* = -J^* J_0 = -J_0 J^{-1} = J_0 J$, and positive according to the third condition. Conversely if we set $J = J_0^{-1} P$ with $P$ positive symmetric and symplectic we get $J^2 = J_0^{-1} P J_0^{-1} P = -J_0^{-1}(P^* J_0 P) = -J_0 J_0^{-1} = -\mathrm{Id}$. As a result the map from $J \in \mathscr{J}(\omega)$ to $P \in S(n)$ yields a diffeomorphism. That $S(n)$ is contractible follows from Exercise 2.

□

See Exercise 20 for an alternative proof which is also valid in the tame setting.

EXERCISE 2.43. Let $L$ be a Lagrangian subspace, show that $JL$ is also a Lagrangian and $L \cap JL = \{0\}$. Conversely given two Lagrangians $L_1, L_2$ such that $L_1 \cap L_2 = \{0\}$ there is a complex structure such that $JL_1 = L_2$.

## 4. The symplectic group

We finally study the structure of the symplectic group, for $\mathbb{K} = \mathbb{R}$.

PROPOSITION 2.44. *The group $Sp(2n,\mathbb{R})$ of linear symplectic maps of $(\mathbb{R}^{2n}, \sigma_n)$ is contained in $SL(2n,\mathbb{R})$ and contains the unitary group $U(n)$.*

1. *(Iwasawa decomposition) Each element $R$ of $Sp(2n,\mathbb{R})$ can be written uniquely as $R = QP$ with $Q \in U(n) = O(2n) \cap Sp(2n,\mathbb{R})$ and $P \in S(n)$ symmetric, positive definite and symplectic.*
2. *$Sp(2n,\mathbb{R})$ has $U(n)$ as a deformation retract. It is therefore connected, and has fundamental group isomorphic to $\mathbb{Z}$.*
3. *The subgroup $U(n)$ of $Sp(2n,\mathbb{R})$ acts transitively on the set $\widetilde{\Lambda}(n)$ of oriented Lagrangians. Hence $\widetilde{\Lambda}(n)$ can be identified to $U(n)/SO(n)$ and the unoriented Grassmannian Lagrangian $\Lambda(n)$ to $U(n)/O(n)$.*

PROOF. According to Corolllary 2.22, a symplectic map preserves $\sigma_n^n$ which is (up to a factor $n!$) a volume form, so it belongs to $SL(2n,\mathbb{R})$. Let $\sigma_n(x,y) = (Jx,y)$ where $J$ is the standard complex structure. Since elements of $U(n)$ preserve both the scalar product $(\bullet,\bullet)$ and the complex structure $J$, they preserve the symplectic form hence belong to $Sp(2n,\mathbb{R})$. Let $R \in Sp(2n)$, then $\sigma(Rx,Ry) = \sigma(x,y)$ i.e.

$$(JRx,Ry) = (x,y)$$

Thus as we pointed out in Remark 2.41, $R \in Sp(2n)$ is equivalent to $R^* JR = J$.

Now if $R$ is symplectic, so is $R^*$, since $(R^*)JRJ = J^2 = -\mathrm{Id}$ hence[7]

$$(R^*)^{-1}[(R^*)JRJ]R^* = -\mathrm{Id} \Leftrightarrow JRJR^* = -\mathrm{Id}$$

so that multiplying by $J$ on the left, we get $RJR^* = J$.

Now decompose $R$ as $R = QP$ with $P$ symmetric and $Q$ orthogonal, by setting $P = (R^*R)^{1/2}$ and $Q = RP^{-1}$. Since $R, R^*$ are symplectic so is $P$ (see Exercise 2)and hence $Q$. Thus $Q$ is both symplectic and orthogonal, which means that it preserves the hermitian product (since $Q$ preserves both its real part - the scalar product- and its imaginary part-the symplectic form), and must then be unitary. This proves (1). Then since $P$ is also positive definite, the map $s \mapsto P^s$ is well defined (as $\exp(s\log(P))$ and $\log(P)$ is well defined for a positive symmetric matrix, see again Exercise 2) for $s \in \mathbb{R}$ and the path $T_{1-s} : QP \mapsto QP^s$ yields a retraction form $Sp(2n)$ to $U(n)$ since $P^0 = \mathrm{Id}$. This proves the second claim. For the last one, let $v_1, ..., v_n$ be an orthonormal basis of a Lagrangian $L$. Then $h(v_i, v_j) = \delta_i^j$ since $\omega(v_i, v_j) = 0$. Hence there is a unique unitary map sending the canonical basis $e_1, ..., e_n$ of $\mathbb{C}^n$ identified to $\mathbb{R}^{2n}$ to $(v_1, ... v_n)$. Now two basis describe the same oriented Lagrangian if and only if they differ by an element of $SO(n)$ and the same unoriented one, if and only if they differ by an element of $O(n)$. □

REMARK 2.45. It may be useful to explicit the embeddings of $U(n)$ and $O(n)$ in $Sp(2n, \mathbb{R})$ as matrices. If $M = A + iB$ is in $U(n)$, then we have $A^*A + B^*B = \mathrm{Id}_n$ and $B^*A + A^*B = 0$ and

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

is in $Sp(2n, \mathbb{R})$.

If $R \in O(n)$, then

$$\begin{pmatrix} R & 0 \\ 0 & R^{-1} \end{pmatrix}$$

is in $Sp(2n, \mathbb{R})$.

COROLLARY 2.46. *The set of triples of pairwise transverse Lagrangians has $n+1$ connected components.*

PROOF. We already know that this is the case when we fix $L_1, L_2$. Let now $(L_1, L_2, L_3)$ and $(L_1', L_2', L_3')$ be two triples of pairwise transverse Lagrangians. We can find $R \in Sp(V, \omega)$ sending $L_1$ to $L_1'$ and $L_2$ to $L_2'$ according to Proposition 2.24 (2). Since $Sp(2n, \mathbb{R})$ is connected, using a path $R_t$ from id to $R$, we can deform $L_1$ to $L_1'$ then $L_2$ to $L_2'$ and $L_3$ to $L_3''$ so that they remain pairwise transverse. Then $L_3''$ can be deformed to $L_3'$ if and only if they are in the same connected component of the set of Lagrangians transverse to $L_1'$ and $L_2'$. □

[7] A word of caution: we cannot just take any scalar product on a symplectic space, and claim that the transpose of a symplectic map is symplectic. We must use a scalar product compatible with the symplectic form. Of course in general we mean the standard scalar product, and the standard symplectic form.

Finally the Lie algebra of the symplectic group, that is the algebra of left invariant vector fields on $Sp(2n,\mathbb{R})$ (canonically identified to $T_{\mathrm{Id}}Sp(2n,\mathbb{R})$) endowed with the Lie bracket is given by

$$\mathfrak{sp}(2n,\mathbb{R}) = \{M \in M(2n,2n) \mid M^* J + JM = 0\}$$

the Lie bracket being given by $[A,B] = AB - BA$. Indeed the linearization at $R = \mathrm{Id}$ of the relation $R^* JR = J$ is $M^* J + JM = 0$. In particular

PROPOSITION 2.47. *The solutions $R(t)$ of the equation $\dot{R}(t) = M(t)R(t)$, $R(0) = \mathrm{Id}$ belong to $Sp(2n,\mathbb{R})$ if and only if for all $t$, $M(t)$ satisfies $M(t)^* J + JM(t) = 0$ or equivalently $M(t) = JA(t)$ where $A(t)$ is symmetric.*

PROOF. We just saw that $T_{\mathrm{Id}}Sp(2n,\mathbb{R}) = \{M \in M(2n,2n) \mid M^* J + JM = 0\}$ and applying multiplication to the right by $R \in Sp(2n,\mathbb{R})$ (which induces a diffeomorphism on $Sp(2n,\mathbb{R})$) we get

$$T_R Sp(2n,\mathbb{R}) = \{MR \in M(2n,2n) \mid M^* J + JM = 0\}$$

Now $Sp(2n,\mathbb{R})$ is smooth submanifold in $M(2n,2n)$ and a path $R(t)$ remains in $Sp(2n,\mathbb{R})$ if and only if $R(0) \in Sp(2n,\mathbb{R})$ and for each $t$ the vector $\dot{R}(t)$ belongs to $T_R Sp(2n,\mathbb{R})$. This is exactly the first condition. Now writing $M(t) = JA(t)$ we have

$$JM(t) + M(t)^* J = -A(t) + A(t)^* J^* J = -A(t) + A(t)^*$$

and this vanishes if and only if $A(t)$ is symmetric, so we get the equivalence with the second condition. □

**4.1. Normal form of real Symplectic matrices. The Krein type.** We refer for this section to [Kre50; Eke90; Abb01]. Let $R$ be an element of $Sp(2n,\mathbb{R})$. Our goal is to analyze the eigenvalues of $R$ and also their possible bifurcations. It will be useful to complexify $R$, that is we consider it as an endomorphism of $\mathbb{C}^{2n}$, so we can talk about complex eigenvalues[8]. We denote by $V_\lambda$ the eigenspace corresponding to the eigenvalue $\lambda$ and by $E_\lambda = \mathrm{Ker}(R - \lambda\mathrm{Id})^{2n}$ the characterisitc space so that $V_\lambda \subset E_\lambda$. We call $\dim(V_\lambda)$ the geometric multiplicity of $\lambda$ and $\dim(E_\lambda)$ the algebraic multiplicity. We still denote by $\omega$ the extension of $\omega$ to $\mathbb{C}^n$ as a bilinear (not hermitian !) form. One should be careful because $J$ is now different from multiplication by $i$ : $J$ corresponds to the matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, and has eigenvalues $\pm i$ each with multiplicity $n$) while $i$ corresponds to the diagonal matrix $\begin{pmatrix} i & \dots & 0 & \\ 0 & i & .. & 0 \\ \dots & & & \\ 0 & 0 & .. & i \end{pmatrix}$ (with a eigenvalue $i$ of multiplicity $2n$). Besides the standard hermitian form denoted by $\langle \bullet, \bullet \rangle$, we can consider the sesquilinear form $h(v,w) = \langle -iJv, w \rangle$. It is non-degenerate (since $iJ$ is invertible) and has signature $(n,n)$

[8]We do not use the notation $Sp(2n,\mathbb{C})$ since this has different possible meanings.

since the matrix $iJ$ has eigenvalues $i \cdot i = -1$ and $i \cdot (-i) = +1$ both with multiplicities $n$. Moreover $R$ preserves $h$ since

$$h(Rv, Rw) = \langle -iJRv, Rw \rangle = \langle -iR^*JRv, W \rangle = \langle -iJv, w \rangle$$

PROPOSITION 2.48. *If $\lambda$ is an eigenvalue of $R$, then so are $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ and the corresponding $V_\lambda$ (resp. $E_\lambda$) all have the same dimension. Moreover for $\lambda = \pm 1$ the dimension of the characteristic space has even multiplicity. Finally for $\lambda\bar{\mu} \neq 1$, the spaces $E_\lambda$ and $E_\mu$ are $h$-orthogonal. In particular if $|\lambda| \neq 1$, the space $E_\lambda$ is $h$-isotropic.*

PROOF. We have $R^*JR = J$, so that $R^* = JR^{-1}J^{-1}$ means that $R^*$ and $R^{-1}$ are conjugate, so $\dim V_\lambda = \dim V_{\lambda^{-1}}, \dim E_\lambda = \dim E_{\lambda^{-1}}$. Since $R$ is real, we have $\dim V_\lambda = \dim V_{\bar{\lambda}}, \dim E_\lambda = \dim E_{\bar{\lambda}}$ and this proves the first claim. Note that because the determinant of $R$ is $+1$, $-1$ must have even multiplicity. Since the sum of the algebraic multiplicities is even, then $+1$ must also have even multiplicity. Now we have if $Ru = \lambda u, Rv = \mu v$ that $h(u, v) = h(Ru, Rv) = \lambda\bar{\mu} h(u, v)$. So if $\lambda\bar{\mu} \neq 1$ we have $h(u, v) = 0$ so $V_\lambda, V_\mu$ are $h$-orthogonal. Now set $E_\lambda^{(k)} = \ker(R - \lambda \mathrm{id})^k$ and let us argue by induction on $k + l$ to prove that $E_\lambda^{(k)}$ and $E_\mu^{(l)}$ are $h$-orthogonal. We just proved this for $k + l = 2$. Assume this is proved whenever $k + l = m$ and let us prove it for $m + 1$. Let $x \in E_\lambda^{(k)}, y \in E_\mu^{(l)}$ with $k + l = m + 1$. Then $(R - \lambda \mathrm{Id})x \in E_\lambda^{(k-1)}, (R - \mu \mathrm{Id})y \in E_\mu^{(l-1)}$ so $Rx = u + \lambda x, Ry = v + \mu y$ where $u \in E_\lambda^{(k-1)}, v \in E_\mu^{(k-1)}$. By assumption $h((R - \lambda \mathrm{Id})x, y) = 0$ that is

$$h(x, y) = h(Rx, Ry) = h(\lambda x + u, \mu y + v) = \lambda\overline{\mu} h(x, y) + h(u, \mu y + v) + h(\lambda x + u, v)$$

but by induction hypothesis $h(u, \mu y + v) = 0$ since $u \in E_\lambda^{(k-1)}, \mu y + v \in E_\mu^{(l)}$ and $(k - 1) + l = m$, and the same argument proves $h(\lambda x + u, v) = 0$. So we get

$$h(x, y) = h(Rx, Ry) = h(\lambda x + u, \mu y + v) = \lambda\overline{\mu} h(x, y)$$

and if $\lambda\overline{\mu} \neq 1$ we must have $h(x, y) = 0$.

For the last statement apply our result to $\lambda = \mu$. □

LEMMA 2.49. *We have*

*(1) the form $h$ is non-degenerate on $E_\lambda$ for $\lambda$ on the unit circle*
*(2) $h$ is non-degenerate on $E_\mu \oplus E_{1/\bar{\mu}}$, but $E_\mu$ is isotropic for $\mu$ outside the unit circle.*

PROOF. Since any two spaces $E_\alpha, E_\beta$ are $h$ orthogonal provided $\alpha\bar{\beta} \neq 1$, we have that $E$ is an $h$-orthogonal direct sum of $E_1, E_{-1}, E_\lambda$ for $\lambda \in S^1 \setminus \{\pm 1\}$ or $E_\mu \oplus E_{1/\bar{\mu}}$ for $\mu \in \mathbb{C} \setminus S^1$. Since $h$ is non-degenerate, it must be non-degenerate on each term of the orthogonal direct sum. □

As a result[9], setting

$$E_{S^1} = \overset{h}{\bigoplus}_{|\lambda|=1,\lambda\neq\pm1} E_\lambda$$
$$E_D = \overset{h}{\bigoplus}_{|\mu|<1} (E_\mu \oplus E_{1/\bar{\mu}})$$

we can write

$$E = E_1 \overset{h}{\oplus} E_{-1} \overset{h}{\oplus} E_{S^1} \overset{h}{\oplus} E_D$$

Note that on $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$ we have $\Im(h(x,y)) = -\omega(x,y)$, so that $h$-orthogonal real spaces are $\omega$-orthogonal. Since

$$E_1, E_{-1}, E_\lambda \oplus E_{\overline{\lambda}}, (E_\mu \oplus E_{\overline{\mu}}), (E_{1/\bar{\mu}} \oplus E_{1/\mu})$$

are real spaces for $\lambda \in S^1 \setminus \{\pm 1\}, |\mu| < 1$, we have the symplectic decomposition[10]

$$E_{S^1} = \overset{\omega}{\bigoplus}_{|\lambda|=1,\Im\lambda>0} E_\lambda \oplus E_{\overline{\lambda}}$$
$$E_D = \overset{\omega}{\bigoplus}_{|\mu|<1,\Im\mu>0} (E_\mu \oplus E_{\overline{\mu}}) \oplus (E_{1/\bar{\mu}} \oplus E_{1/\mu})$$

and

$$E = E_1 \overset{\omega}{\oplus} E_{-1} \overset{\omega}{\oplus} E_{S^1} \overset{\omega}{\oplus} E_D$$

Setting $E_D^\pm = \bigoplus_{|\mu|^{\pm1}<1} (E_\mu \oplus E_{\bar{\mu}})$ we see from Proposition 2.48 that $E_D^\pm$ are Lagrangians in $E_D$, hence isotropic in $E$. We summarize our findings in

PROPOSITION 2.50. *We have the decomposition*

$$E = E_1 \overset{\omega}{\oplus} E_{-1} \overset{\omega}{\oplus} E_{S^1} \overset{\omega}{\oplus} E_D$$

*and* $E_D = E_D^+ \oplus E_D^-$ *where each factor is Lagrangian in the symplectic space* $E_D$.

We now want to understand the possible bifurcations of the eigenvalues along a path of symplectic maps. This will be in particular useful in our study of the Maslov and Conley-Zehnder indices (see Chapter **??**). The crucial tool is the following notion :

DEFINITION 2.51. *Let* $|\lambda| = 1$ *and* $E_\lambda$ *be the characteristic space associated to* $\lambda$. *The signature* $(p,q)$ *of the restriction of* $h$ *to* $E_\lambda$ *is called the* **Krein type** *(or* **Krein signature***) of* $\lambda$.

There is no point in defining a Krein type for $\mu$ outside the unit circle, since $h$ vanishes on $E_\mu$ and has signature $(d,d)$ on $E_\mu \oplus E_{1/\bar{\mu}}$, where $d = \dim(E_\mu)$.

Thus a symplectic matrix has eigenvalues at $\pm 1$, then conjugate pairs on the unit circle and finally quadruples away from the unit circle. In a continuous family of symplectic matrices, the eigenvalues will move continuously, respecting the above constrains. Bifurcations can occur for example if two eigenvalues on the unit circle collide

[9]where $\overset{h}{\oplus}$ denotes $h$-orthogonal direct sum

[10]where $\Im(\lambda)$ is the imaginary part of $\lambda$.

(by symmetry this will happen at the same time for its conjugate pair), and give rise to two pairs of eigenvalues such that $\mu, \mu^{-1}, \bar{\mu}, \bar{\mu}^{-1}$ are all distinct. Whether this can happen for all collisions on the circle is a question that was addressed by M. Krein. The answer is negative, as the notion of Krein signature implies.

It is easy to see that if $\lambda$ has Krein type $(p,q)$, then $\bar{\lambda}$ has Krein type $(q,p)$. Also for $\lambda = \pm 1$, since $E_{\pm 1}$ is the complexification of a real space, it has Krein type $p,p$. We saw that if $|\lambda| \neq 1$, the hermitian form $h$ is of signature $(p,p)$ on $E_\lambda$. We now prove that two eigenvalues on the unit circle that collide and move to the complement of the unit circle must have signature $(p_1,q_1),(p_2,q_2)$ such that $p_1+p_2 = q_1+q_2$. More generally we have

PROPOSITION 2.52. *Let $R$ be an element in $Sp(2n,\mathbb{R})$ having an eigenvalue $\lambda$ of module $1$ with signature $(p,q)$. Then if $S$ is close enough to $R$, we have*

1. *the eigenvalues of $S$ close to $\lambda$ are $\lambda_1, \dots \lambda_r$ of module $1$ and Krein type $(p_j,q_j)$, pairs of eigenvalues $\mu_1, 1/\bar{\mu}_1, \dots, \mu_s, 1/\bar{\mu}_s$ of algebraic multiplicity $d_1, .., d_s$ where $|\mu_j| \neq 1$ and they must satisfy the relation*
$$\sum_{j=1}^{r} p_j = p - d, \sum_{j=1}^{r} q_j = q - d$$
*where $d = \sum_{j=1}^{s} d_j$.*

COROLLARY 2.53. *In particular, with the assumptions of the Pproposition, if $\lambda$ has algebraic multiplicity $2$*

1. *If $\lambda$ has Krein type $(2,0)$ (resp. $(0,2)$) and $S$ has distinct simple eigenvalues, then $S$ must have eigenvalues $\lambda_1, \lambda_2$ of module $1$, and both of Krein type $(1,0)$ (resp. $(0,1)$)*
2. *If $\lambda$ has Krein type $(1,1)$ and $S$ has distinct simple eigenvalues, then $S$ must have either eigenvalues $\lambda_1, \lambda_2$ of module $1$, and of Krein type $(1,0)$ and $(0,1)$, or a pair of eigenvalues $\mu, 1/\bar{\mu}$ with $|\mu| \neq 1$.*

In other words when two simple eigenvalues on the unit circle collide, they can only leave the unit circle if they have complementary Krein type.

PROOF. Indeed, let $\gamma$ be a loop on $\mathbb{C}$ bounding a small neighborhood of $\lambda$ and containg no other eigenvalue of $R$. We shall assume $S$ is close enough to $R$ so that it has no eigenvalue on $\gamma$. Then let $\chi$ be equal to 1 inside $\gamma$ and zero outside. Then $\pi(S) = \frac{1}{2i\pi}\int_\gamma \chi(z)(z-S)^{-1}dz$ is the projector on the sum of the characteristic spaces corresponding to eigenvalues of $S$ close to $\lambda$. The restriction of $h$ to the image of $\pi(R)$ has signature $(p,q)$. The same holds for the image of $\pi(S)$ for $S$ close enough to $R$. But the image of $\pi(S)$ decomposes as the $h$-orthogonal sum of characteristic spaces on which the signature of $h$ is $(p_j,q_j)$ for those on the unit circle, or $(d_j,d_j)$ for the sums $E_{\mu_j} \oplus E_{1/\bar{\mu}_j}$ and therefore $p = d + \sum_{j=1}^{r} p_j, \sum_{j=1}^{r} q_r + d = q$. The particular case is obtained by noticing that if $q$ or $p$ vanishes, then $d = 0$. $\square$

More generally if $p = 0$ or $q = 0$ we shay that $\lambda$ is Krein definite. In this case for $S$ close to $R$, no eigenvalue can leave the unit circle. In general, counting with algebraic multiplicity, starting from an eigenvalue on the unit circle of Krein type $(p, q)$, only $\min\{p, q\}$ eigenvalues can leave the unit circle.

## 5. Symplectic vector bundles

This is a first step towards the non-linear situation. A vector bundle should be thought as a parametrized family of vector spaces, possibly with some extra structure (complex, orthogonal, volume, symplectic, etc...).

DEFINITION 2.54. *Let $G$ be a subgroup of $GL(n,\mathbb{R})$. A $G$-vector bundle over a manifold $M$ is defined by a space $E$ a projection $\pi : E \longrightarrow M$ such that there is a covering $(U_j)_{j\in I}$ of $M$ and charts $\varphi_j : U_j \times \mathbb{R}^{n_j} \longrightarrow \pi^{-1}(U_j)$ such that on $U_i \cap U_j$ the map*

$$\varphi_j^{-1}\varphi_i : (U_i \cap U_j) \times \mathbb{R}^{n_i} \longrightarrow (U_i \cap U_j) \times \mathbb{R}^{n_j}$$

*is of the form $(x, v) \mapsto (x, g_{i,j}(x)v)$ where $g_{i,j}$ is a smooth map from $U_i \cap U_j$ to $G$. Two $G$-vector bundles are equivalent if there is a map $f : E_1 \longrightarrow E_2$ such that $\pi_2 \circ f = \pi_1$ and $f : \pi_1^{-1}(x) \longrightarrow \pi_2^{-1}(x)$ is symplectic.*

Note that if $M$ is connected, the dimension $n$ is fixed, and we may thus replace $\mathbb{R}^{n_j}$ it by $\mathbb{R}^n$. In a fancy terminology the $g_{i,j} \in C^\infty(U_i \cap U_j, G)$ form a cocycle[11] with values in $G$), and two cocycles define equivalent vector bundles if and only if they coincide in $H^1(M, G)$. Note that often one thinks of a vector bundle as just the triple $(E, M, \pi)$ but then the group $G$ is not explicit. In fact any $G$ bundle can be considered as a $GL(n,\mathbb{R})$ vector bundle. For example a $GL(n,\mathbb{C})$ vector bundle $E$ can be considered as a $GL(2n,\mathbb{R})$-vector bundle. One has to be careful,though, since such a vector bundle can be equivalent to a trivial bundle as $GL(n,\mathbb{R})$ bundle, but not as a $GL(n,\mathbb{C})$-vector bundle (see Remark 2.62).

It is often convenient to stress the structure group of the vector bundle. For this one introduces the notion of **principal bundle**

DEFINITION 2.55. *Let $G$ be a Lie group. A smooth $G$ principal bundle is given by a smooth manifold $P$, and a proper right $G$-action on $P$. Then we denote by $B = P/G$ the* **base** *of the bundle and $\pi$ is the* **projection** *$\pi : P \longrightarrow P/G$. A morphism between two principal bundles with total spaces $P, P'$ is an equivariant map from $P$ to $P'$. We then have a commutative diagram*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"] & P' \arrow[d, "\pi'"] \\ P/G \arrow[r, "{[f]}"] & P'/G \end{tikzcd}$$

[11] Because $\varphi_j^{-1}\varphi_i \circ \varphi_i^{-1}\varphi_k \circ \varphi_k^{-1}\varphi_i = \mathrm{Id}$ on $U_i \cap U_j \cap U_k$ we have $g_{i,j} \circ g_{j,k} \circ g_{k,i} = \mathrm{Id}$ : this is what begin a cocycle means.

Principal bundles are ubiquitous in differential geometry, and often appears as **frame bundles**

EXAMPLE 2.56. Let $E$ be a a vector bundle on $B$. We denote by $E_b$ the fiber over $b$. Let $F(E)_b$ be the set of bases of $E_b$. Then for $n = \dim(E_b)$ $GL(n,\mathbb{R})$ acts on $F(E)_b$ as follows: a basis $(u_1,...,u_n)$ of $E_b$ is nothing else than a linear isomorphism $i_u : \mathbb{R}^n \longrightarrow E_b$, with $i_u(t_1,...,t_n) = t_1u_1+...+t_nu_n$. Then for $M \in GL(n,\mathbb{R})$ we associate the basis defined by $i_{v\cdot M} = i_U \circ M$. This obviously defines a proper right action of $GL(n,\mathbb{R})$ on $F(E_b)$, so that $F(E)$ becomes a principal $GL(n,\mathbb{R})$-bundle. Note that if $E$ is a complex bundle, then taking $F(E)_b$ to be the set of complex bases of $E_b$, makes $F(E)$ a principal $GL(n,\mathbb{C})$-bundle, if $E_b$ is symplectic, and we consider $F(E_b)$ the set of symplectic bases, we get a principal $Sp(2n,\mathbb{R})$-bundle.

DEFINITION 2.57. *Consider a $G$ action on a smooth manifold $F$. Let $E = P \times F/G$ where the right $G$-action is given by $(p, f)\cdot g = (p\cdot g, \rho(g^{-1)} f)$. Then $E$ is the fiber bundle associated to $P$ and the $G$ action on $F$.*

For example if $F$ is a vector space and $\rho : G \longrightarrow GL(F)$ is a representation we obtain the vector bundle associated to the principal bundle (so we do not need that $G$ be a subgroup of $GL(n,\mathbb{R})$).

EXERCISE 2.58. Prove that for $P = F(E)$ the frame bundle, and the canonical representation of $GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ given by the identity map, the associated bundle is $E$. Same question in the complex, unitary and symplectic case.

An important notion for fiber bundles is that of the reduction of the structural group. If $H$ is a subgroup of $H$ a $G$-vector bundle can have its structural group reduced to $H$ if the cocycles can be assumed to take values in $H$. In other words

DEFINITION 2.59. *Let us consider a principal $G$-bundle with total space $P$. Let $\rho : H \longrightarrow G$ be a morphism. We say that the structural group of $P$ can be reduced to $H$ if there exists an $H$-bundle $Q$ such that $P \simeq Q\times_\rho G$ the isomorphism being a $G$-equivariant diffeomorphism. Here $Q \times_\rho G = (Q \times G)/H$ where the $H$ action is given by $(q, g)\cdot h = (q\cdot h, \rho(h^{-1})g)$ and the $G$-action on $Q \times_\rho G$ is given by $[(q, g)]\cdot g' = [(q, g\cdot g')]$.*

Note that $Q \times_\rho G$ is the associated bundle to $Q$ for the $H$-action on $G$ given by $h : g \mapsto \rho(h)g$. Most of the time $\rho$ is an inclusion, this is why we talk of “reduction” : this means that the cocycle defining $P$ can be chosen to take its values in $H$ instead of $G$. But the case where $\rho$ is a covering is interesting in its own right. For example $Spin(n)$ is the double cover of $SO(n)$, and a lifting of an $SO(n)$ bundle to $Spin(n)$ is called a spin structure on the vector bundle (which may or may not exist, depending on the vanishing of the second Stiefel-Whitney class). In the symplectic case, we will be interested in the double cover of $SP(2n,\mathbb{R})$, the metaplectic group $MP(2n,\mathbb{R})$. Note that a principal bundle is trivial if and only if its structural group can be reduced to the identity. This is equivalent to the existence of a section of $\pi : P \longrightarrow M$, since we can then find an isomorphism $M \times G \longrightarrow P$ by setting $f(m, g) = g \cdot s(m)$.

PROPOSITION 2.60. *If $H$ is subgroup of $G$ and $H$ is a deformation retract of $G$ then any $G$-vector bundle is equivalent to a unique (up to equivalence) $H$-vector bundle. In particular any vector bundle can be reduced to a $O(n)$ vector bundle ($SO(n)$ if it is orientable). As a result any $Sp(2n)$ vector bundle can be reduced to a $U(n)$ bundle.*

PROOF. Assume $H$ is a subgroup of $G$ and we have an equivariant map $s: P \longrightarrow G/H$ such that $s(p\cdot g) = g^{-1}s(p)$. Then the restriction to $H$ of the $G$-action on $s^{-1}(e) = Q$ makes $Q$ into a principal $H$-bundle. It is easy to see that $P = Q\times_\varphi G$, so that we can reduce the structural group of $P$ to $H$. The existence of $s$ is equivalent to the existence of a section of the fibre bundle of fibre $G/H$ associated to $P$ where the action of $G$ on $G/H$ is the obvious one. Indeed if $\sigma(b) = [(p,\gamma)$ with $p\in P, \gamma\in G/H$, then for each $p\in\pi^{-1}(b)$ there is a unique $\gamma\in G/H$ such that $[(p,\gamma)]\in\sigma(b)$ and we set $s(p) = \gamma$. Since $[(p\cdot g, g^{-1}\gamma] = [(p,g)$ we have $s(p\cdot g) = g^{-1}\gamma$. Now if $G/H$ is contractible, this fibre bundle has a section, so the structure group can be reduced to $H$. □

One should be a little careful because *a priori* there can be several non-isomorphic $H$ bundle such that $Q\times_\varphi G = P$ as a $G$-bundle.

COROLLARY 2.61. *Any symplectic vector bundle has a well defined complex structure up to homotopy, hence well defined Chern classes, which are the Chern classes of any of the corresponding $U(n)$ bundle.*

PROOF. Since the inclusion $U(n)\longrightarrow Sp(2n,\mathbb{R})$ is a homotopy equivalence, this follows from Proposition Prop-2.31 (2). □

For the definition of Chern classes we refer to [Mil74] or [Hus94].

REMARK 2.62. A complex (or symplectic ) bundle can be non-trivial, while the underlying real bundle is trivial. For example the tangent space to $S^2 = \mathbb{C}P^1$ is a complex vector bundle. As a real vector bundle, it is the tangent space to $S^2$, and adding to it its normal bundle in $\mathbb{R}^3$, which is trivial, we get a bundle $TS^2\oplus\varepsilon^1_{\mathbb{R}}\simeq\varepsilon^3_{\mathbb{R}}$ hence $TS^2\oplus\varepsilon^2_{\mathbb{R}}\simeq\varepsilon^4_{\mathbb{R}}$, where $\varepsilon_{\mathbb{R}}$ (resp. $\varepsilon_{\mathbb{C}}$) denotes the trivial one dimension real (resp. complex) bundle. On the other hand $T\mathbb{C}P^1\oplus\varepsilon_{\mathbb{C}}$ has first Chern class equal to 2, hence is not trivial[12].

Note that since $\pi_1(Sp(2n)) = \pi_1(U(n)) = \mathbb{Z}$, there is a double cover of $Sp(2n)$ called the metaplectic group . The question of lifting a symplectic bundle to the metaplectic group is of importance in representation theory. The metaplectic group has no finite dimensional faithful representation, but some infinite dimensional ones. As a group, it is a $\mathbb{Z}/2\mathbb{Z}$ extension of $Sp(2n,\mathbb{R})$. This construction also exists for other local fields. We refer to Chapter **??** for more on this.

[12]Alternative proof: assume $T\mathbb{C}P^1\oplus\varepsilon_{\mathbb{C}}\simeq\varepsilon^2_{\mathbb{C}}$. Given the canonical section on the trivial factor $\varepsilon^1_{\mathbb{C}}$, taking its image in $\varepsilon^2_{\mathbb{C}}$ yields a section of $\varepsilon^2_{\mathbb{C}}$. But these are all homotopic, since $\pi_2(S^3) = 0$. As a result $T\mathbb{C}P^1\simeq T\mathbb{C}P^1\oplus\varepsilon_{\mathbb{C}}/\varepsilon_{\mathbb{C}}\simeq\varepsilon^2_{\mathbb{C}}/\varepsilon_{\mathbb{C}} = \varepsilon_{\mathbb{C}}$. But this is impossible since the Euler characteristic of $T\mathbb{C}P^1 = TS^2$ (as a real bundle) is $2\neq 0$.

## 6. Normalization issues

As we already pointed out, the definition of the "canonical" from $\sigma$ on $\mathbb{R}^{2n}$ and $T^*\mathbb{R}^n$ can have different signs depending on the author. Recall that we choose the following normalizations

(1) On $\mathbb{R}^2$ with coordinates $(x, y)$ the standard form should induce the standard orientation, the trigonometric circle should have positive area so that we must set $\sigma_1 = dx \wedge dy$. More generally the standard form on $\mathbb{R}^{2n}$ is $\sigma_n = \sum_{j=1}^n dx_j \wedge dy^j)$.

(2) On $T^*\mathbb{R}$ the symplectic area below $p = f'(x)$ between 0 and $x = x_0$ should be $f(x_0) - f(0)$, so the form $\sigma_{\mathbb{R}}$ is $dp \wedge dq$. Similarly we define the standard form $\sigma_V$ on $T^*V$ that we denote, when $V$ is a linear space as $V \oplus V^*$ and $\sigma_V((x, x^*)(y, y^*)) = \langle y^*, x\rangle - \langle x^*, y\rangle$.

With these definitions, we get that the "standard" isomorphism from $\mathbb{R}^2$ to $T^*\mathbb{R}$ given by $(x, y) \longrightarrow (q, p)$ $q = x, y = p$ is anti-symplectic.

Thus one should be careful, with our conventions $\sigma_n = -\sigma_{\mathbb{R}^n}$

## 7. Exercises and Problems

*Unless otherwise specified, vector spaces are on a field of characteristic $\neq 2$.*

### 7.1. Skew-symmetric forms.

(1) Let $B$ be a bilinear form yielding a symmetric relation of orthogonality, that is $B(x, y) = 0 \Leftrightarrow B(y, x) = 0$ then $B$ is either symmetric or antisymmetric.

HINT. *We assume $B$ to be non-degenerate (otherwise just divide by the kernel of $B$, that is well-defined, since $B(x, y) = 0$ for all $y$ if and only if $B(y, x) = 0$ for all $y$). Then $B(x,\bullet)$ and $B(\bullet, x)$ are non zero and have the same kernel so there is a $c_x \neq 0$ such that $B(x, y) = c_x B(y, x)$ for all $y$. If $B(x, x) \neq 0$ we have, setting $y = x$, that $c_x = 1$. Now if $B(x, x) = 0$ assume for some $z$ we have $B(z, z) \neq 0$. By the same argument $B(z, x) = B(x, z)$ and $B(z, y) = B(y, z)$. Now the first equality implies $c_x = 1$ if $B(z, x) \neq 0$. On the other hand if $B(z, x) = 0$ we have $B(z + x, z + x) = B(z, z) \neq 0$ hence $B(z + x, y) = B(y, z + x)$ but since we already know that $B(z, y) = B(y, z)$, we get $B(x, y) = B(y, x)$. Finally if for all $z$ we have $B(z, z) = 0$ then $B$ is skew-symmetric.*

(2) (The space of positive symplectic matrices is contractible) Let $A$ be a positive definite symmetric linear map on a real vector space $V$. Let us assume $\|A\| < 1$ (for the norm $\|A\| = \sup\{\|Ax\| \mid \|x\| \leq 1\}$) and set

$$\log(I + A) = \sum_{j=1}^{+\infty} (-1)^{j+1} \frac{A^j}{j}$$

and for all $A$,

$$\exp(A) = \sum_{j=0}^{+\infty} \frac{A^j}{j!}$$

(a) Prove that that if $A$ is positive then so is $\log(I + A)$ and that when defined, the following identity holds :

$$\exp(\log(A)) = \log\exp(A) = A$$

HINT. *Use the fact that the identity holds in $\mathbb{R}_+$.*

(b) Prove that if $A, B$ are symmetric and $A \leq B$ means $B - A$ is non-negative, then $0 \leq A \leq B$ implies $\|A\| \leq \|B\|$.

HINT. *Use the fact that for a symmetric non-negative matrix*

$$\|A\| = \sup\{(Ax, x) \mid \|x\| \leq 1\}$$

(c) Prove that if $M$ is positive symmetric then we can define $\log(M)$ such that $\exp(\log(M)) = M$ and such that $\log(M)$ is positive.

HINT. *Use that for $\varepsilon > 0$ small enough there is a positive $\eta$ such that* $\eta\mathrm{Id} < \varepsilon M < I$ *and then* $\varepsilon M = \mathrm{Id} + (\varepsilon M - \mathrm{Id})$ *with* $(1-\eta)\mathrm{Id} < (\mathrm{Id} - \varepsilon M) < \mathrm{Id}$ *so that* $\|\varepsilon M - \mathrm{Id}\| < 1$. *As a result we can define* $\log(\varepsilon M)$ *hence* $\log(M) = \log(\varepsilon M) - \log(\varepsilon)\mathrm{Id}$.

(d) Show that if moreover $M$ is symplectic, that is $JMJ = M^*$ (here $M^* = M$), then $A = \log(M)$ satisfies $A^* = A$ and $JA + AJ = 0$. Prove that conversely if $A^* = A$ and $JA + AJ = 0$ then $\exp(A)$ is symplectic, symmetric and positive.

HINT. *Prove that if two symmetric matrices $A, B$ commute we have* $\exp(A+B) = \exp(A)\exp(B)$ *and if they are positive,* $\log(AB) = \log(A) + \log(B)$ *using the fact that it is true in* $\mathbb{R}_+^*$. *Use also that* $\log(T^{-1}AT) = T^{-1}\log(A)T$ *for all $A, T$ such that $A$ is positive symmetric and $T$ invertible.*

(e) Prove that the set $S(n)$ of real matrices which are both positive symmetric and symplectic is contractible

HINT. *If* $M = \exp(A)$ *where* $A = \log(M)$, *the map* $s \mapsto \exp(sA) = M^s$ *defines a retraction from $S(n)$ to* $\mathrm{Id}$.

(3) Use the previous Exercise to show that any positive definite matrix $M$ has a unique positive square root, i.e. a matrix $P$ such that $P$ is positive and $P^2 = M$ and that $P$ and $M$ commute. Prove that if $M$ is moreover symplectic (i.e. $M \in S(n)$)) then $P$ is also symplectic.

(4) Use Exercise 2 to show that $U(n)$ is the maximal compact group of $Sp(2n, \mathbb{R})$

HINT. *Prove that if $U(n) \subsetneq G \subset Sp(2n, \mathbb{R})$ then $G$ contains a non-trivial element of $S(n)$. Prove that the set of $P^n$ for $n \in \mathbb{Z}$ is non-compact.*

(5) Let $\omega$ be a symplectic form on a real vector space, $\tau$ be a skew-symmetric form on $E$ and let $C$ be a constant such that

$$\forall x, y \in E \ |\tau(x, y)| \leq C|\omega(x, y)|$$

(a) Prove that $\tau = c\omega$ for some constant $c$

(b) Does the conclusion still holds if we only assume $\omega$ (but not $\tau$) to be skew-symmetric ?

(6) ([MH73]) Let $M$ be a module over a commutative ring $R$. We assume 2 is invertible in $R$ and $M$ is projective, that is there exists a module $N$ such that $M \oplus N$ is a free module, i.e. is isomorphic to $R^n$ for some $n$. s A bilinear skew-symmetric form on $M$, $B$ is symplectic if any linear form $\alpha : M \longrightarrow R$ can be written as $\alpha : y \mapsto B(x, y)$ for some $x \in M$. We shall assume the ring is such that any projective module is actually free (this is particular the case for Principal Ideal Domains).

(a) Prove that $M$ has a symplectic basis $(e_1, ..., e_n, f_1, ..., f_n)$, that is $e_j$ such that $B(e_i, f_j) = \delta_i^j, B(e_i, e_j) = B(f_i, f_j) = 0$. That is $(M, B)$ is isomorphic to $P \oplus P^*, \sigma)$ where $\sigma$ is the canonical symplectic form on $P \oplus P^*$.

(b) Prove that the same holds for Dedekind domains, that is rings such that any projective module is of the form $R^n \oplus \mathfrak{a}$ where $\mathfrak{a}$ is an ideal in $R$ (this is not the usual definition for a Dedekind domain, but one of its properties due to Steinitz). From now on we assume we are in one of the two cases above.

(c) Let $L$ be a maximal projective module such that $B$ vanishes on $L$. Prove that there is a basis as above such that $(e_1, ...., e_n)$ is a basis of $L$.

### 7.2. Symplectic linear algebra.

(7) Let $\omega$ be a skew-symmetric form of rank $2r$ on a vector space $E$. Prove that if $T$ has codimension $q$, then the rank of $\omega_{|T}$ is at least $2r-2q$. Prove that if $\omega$ vanishes on $T$ then $T$ has codimension at least $q$.

(8) Prove that a codimension 1 subspace is always coisotropic. Prove that a codimension 2 subspace is either symplectic or coisotropic.

(9) Prove that a space is coisotropic if and only if it contains a Lagrangian subspace.

(10) Prove that a space is coisotropic if and only if it is not contained in any proper symplectic subspace.

(11) Let $(V,\omega) = (L \oplus L^*, \sigma_L)$. We shall write linear maps as block matrices $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A : L \longrightarrow L, B : L^* \longrightarrow L$, etc. We write $A^*$ for the adjoint map $A^* : L^* \longrightarrow L^*$ defined by $\langle y, Ax\rangle = \langle A^* y, x\rangle$ for all $x \in L, y \in L^*$.

(a) Prove that $R$ is symplectic if and only if $A^*C, B^*D$ are symmetric and $A^*D - C^*B = \mathrm{Id}$.

(b) Consider the group of symplectic matrices preserving $L \oplus 0$, so if $R$ is of the above form we have $C = 0$. What are the relations satisfied by $A, B, D$ ?

(c) Prove that this group has the homotopy type of $GL(n,\mathbb{R})$ (i.e. of $O(n,\mathbb{R})$).

(12) Let $\omega$ be a symplectic form on a vector space $V$ of dimension $2n \geq 4$. Prove that the map

$$\Lambda^1(V) \longrightarrow \Lambda^3(V)$$
$$\alpha \mapsto \alpha \wedge \omega$$

is injective.

(13) Let $(L_1, L_2, L_3)$ and $(L_1', L_2', L_3')$ be two triples of Lagrangian subspaces in $(V,\omega)$ such that for all $i \neq j$, $L_i \cap L_j = L_i' \cap L_j' = \{0\}$. Let $Q$ (resp. $Q'$) be the quadratic forms on $L_1^* \simeq L_3$ (resp. $(L_1')^* \simeq L_3'$) associated to the first (resp. second) triple. We want to prove that there is an element $\varphi \in Sp(V,\omega)$ such that $\varphi(L_j) = L_j'$ for $j \in \{1,2,3\}$ if and only if the quadratic forms $Q, Q'$ are conjugate, that is there is an invertible linear map $P : L_1' \longrightarrow L_1$ such that $Q(P^*x^*) = Q'(x^*)$ or in terms of the associated symmetric linear maps $A : L_1^* \longrightarrow L_1$ (resp. $A' : (L_1')^* \longrightarrow L_1'$) that $PAP^* = A'$.

(a) Prove that we may assume $(V,\omega) = (L\oplus L^*, \sigma_L)$ and $L_1 = L_1' = L, L_2 = L_2' = L^*$

(b) Prove that if $M = \begin{pmatrix} P & R \\ S & T \end{pmatrix}$ is symplectic and preserves $L$ and $L^*$, we have $R = S = 0$ and $T = (P^*)^{-1}$ and that conversely such a map is symplectic (see Exercise 11) and preserves $L$ and $L^*$.

(c) Prove that such an $M$ sends the graph of $A$ to the graph of $PAP^*$. Conclude.

(14) $^\star$ We want to prove that if an injective linear map $\varphi : (E,\omega) \longrightarrow (F,\rho)$ between symplectic vector spaces sends Lagrangian subspaces to Lagrangian subspaces, then it is conformally symplectic (i.e. $\varphi^*\rho = c\omega$ for some $c \neq 0$).

(a) Prove that $\omega(x,y) = 0$ implies $\rho(\varphi x, \varphi y) = 0$

(b) We assume $\rho(\varphi x,\varphi y) = c\omega(x,y)$ with $\rho(\varphi x,\varphi y) \neq 0$ and $z$ linear independent from $x,y$. Then $v = \omega(x,y)z - \omega(x,z)y$ satisfies $\omega(x,v) = 0$ hence $\omega(x,y)\rho(\varphi x,\varphi z) - \omega(x,z)\rho(\varphi x,\varphi y) = 0$, so that there is a constant $c_x$ such that $c_x\omega(x,z) = \rho(\varphi x,\varphi z)$ for all $z$

(c) Prove that for all $u \in E$ we have $\rho(\varphi x,\varphi u) = c_x\omega(x,u)$

(d) Prove that if $\omega(x,y) \neq 0$ we have $c_x = c_y$

(e) Prove that for any $x, y \in E$ there exists $z$ such that $\omega(x,z), \omega(y,z)$ are both non zero, unless $E$ is the union of two proper hyperplanes.

(f) Prove that $E$ cannot be the union of two proper hyperplanes. Conclude.

(15) Let $U(n)$ be the group of unitary matrices on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and $S(n)$ be the group of symmetric matrices in the symplectic Lie algebra $\mathfrak{sp}(2n)$, that is $S(n) = \{A \in M(2n,2n) \mid {}^tA = A, AJ + J^tA = 0\}$. Then the map

$$\begin{aligned} U(n)\times S(n) &\longrightarrow Sp(2n,\mathbb{R}) \\ (U,A) &\mapsto U\exp(A) \end{aligned}$$

is a diffeomorphism.

(16) Prove that the action of $Sp(2n,\mathbb{K})$ on the Grassmannian of $q$-dimensional subspaces of $(V,\omega)$ has for orbits the set of subspaces $T$ where $\omega_{|T}$ has rank $2r$ for $2r \leq q$. What is the dimension (in whatever sense you like) of the orbits.

(17) (Iwasawa and $D\cdot N$ decomposition) Prove that a real symplectic matrix can be written as $M = KAN$ where $K \in U(n) = O(2n)\cap Sp(2n)$, $A$ is diagonal with diagonal terms $(a_1,\dots,a_n,a_1^{-1},\dots,a_n^{-1})$ and $N$ symplectic and upper triangular, with 1 on the diagonal.

HINT. *In view of Proposition 2.44, (1) it is enough to deal with the case of a symmetric, positive symplectic matrix.*

(18) (a) Prove that any invertible matrix $M$ can be written as $D\cdot U$ where $D\cdot U = U\cdot D$, and $D$ is semi-simple (we say that a matrix is semi-simple if it can

be diagonalized in the algebraic closure of the field[13]) and $U$ is unipotent (i.e. $(U-\mathrm{Id})^{k+1}=0$ for some $k\geq 0$). This decomposition is unique (and is called the Dunford or Jordan-Chevalley decomposition[14]). (see any classical algebra book, for example [Lan02])

(b) (see [CD77], lemma 1.1) Prove that when $M$ is symplectic, then $D$ and $U$ are symplectic.

HINT. *Set $D'=J^{-1}(D^*)^{-1}J, U'=J^{-1}(U^*)^{-1}J$ and prove that $D',U'$ are respectively semi-simple and unipotent and $D'U'=U'D'=M$.*

(c) Prove that if a symplectic matrix is semi-simple, then we can write it, up to a symplectic base change, as a diagonal of blocks of the following types:

(i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) $R_\theta=\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

(iv) $M_{r,\theta}=\begin{pmatrix} r\cos(\theta) & -r\sin(\theta) & 0 & 0 \\ r\sin(\theta) & r\cos(\theta) & 0 & 0 \\ 0 & 0 & r^{-1}\cos(\theta) & -r^{-1}\sin(\theta) \\ 0 & 0 & r^{-1}\sin(\theta) & r^{-1}\cos(\theta) \end{pmatrix}$

### 7.3. Complex aspects of linear symplectic geometry.

(19) (Isotropic subspaces of hermitian forms) This is a standard theory for quadratic forms, that we need in the hermitian setting in Section 4.1.

(a) Prove (or lookup in [Art57]) the analogue of Witt's theorem for a hermitian form: given $W_1\subset(V_1,h_1)$ and $W_2\subset(V_2,h_2)$ and an isometry $u:W_1\longrightarrow W_2$, then $u$ extends to an isometry $U:(V_1,h_1)\longrightarrow(V_2,h_2)$.

(b) Prove that all maximal isotropic subspaces (i.e. spaces where $h$ vanishes) in $(V,h)$ have the same dimension, equal to $\min(p,q)$ where $(p,q)$ is the signature of $h$.

(20) (Contractibility of tame complex structures) Let $(V,\omega)$ be a real symplectic vector space. Let $L_0$ be a fixed Lagrangian subspace.

(a) Prove that $J$ is determined by $L_J=JL_0\in\Lambda_{L_0}(V,\omega)$ and by the isomorphism $J:L_0\longrightarrow JL_0$ satisising $\omega(Jx,x)>0$ for $x\neq 0$.

[13] This is not the official definition, which would be that any invariant subspace has an invariant complement. The two properties are however classically equivalent.

[14] In fact this is usually written as $M=D+N$ with $N$ nilpotent (see [Lan02], p. 559.), but we can rewrite it as $D(I+D^{-1}N)$ and $U=I+D^{-1}N$ is unipotent. Chevalley in [Che51] shows that one can recover $D$ and $N$ by a Newton-type algorithm, without actually computing the eigenvalues of $M$

(b) Conversely prove that given $L \in \Lambda_{L_0}(V,\omega)$ and an isomorphism $F : L_0 \longrightarrow L$ such that $\omega(Fx,x) > 0$ for $x \neq 0$, we can find $J$ tame such that $L = JL$ and $F = J_{|L_0}$. Prove that such a $J$ is unique.

(c) Writing $\omega(x,y) = \big(J_0 x, y\big)$ prove that $\omega(Fx,x) > 0$ for $x \neq 0$ is equivalent to $F^* J_0$ is positive (a matrix -non-necessarily symmetric- is positive if $(Mx,x) > 0$ for all $x \neq 0$).

(d) Using the fact that $\Lambda_{L_0}(V,\omega)$ is contractible, and $J \mapsto JL_0$ is a fibration of the space $\widetilde{\mathscr{J}}(\omega)$ of almost complex structures tame with respect to $\omega$ with contractible fiber, prove that $\widetilde{\mathscr{J}}(\omega)$ is contractible.

REMARK 2.63. The above proof can also be adapted to the case of compatible almost complex structures.

(21) Let $L, L'$ be Lagrangian subspaces in $(V,\omega)$. Prove that there is a symplectic basis $(e_1,..,e_n,f_1,..,f_n)$ of $(V,\omega)$ such that $L$ has basis $(e_1,...,e_k,e_{k+1},...,e_n)$ and $L'$ has basis $(f_1,...,f_k,e_{k+1},..,e_n)$.

(22) (Pfaffian) Let $(\bullet,\bullet)$ be a symmetric non-degenerate product on the $\mathbb{K}$-vector space $V$. Let $A$ be a skew-symmetric matrix (i.e. ${}^tA = -A$). We shall write $\omega_A(x,y) = \big(Ax,y\big)$.

(a) Prove that $\omega_A$ is skew-symmetric, and it is symplectic if and only if $A$ is invertible

(b) Prove that if $V$ is odd-dimensional, then $\det(A) = 0$. From now on we assume $V$ has dimension $2n$ and $A$ is invertible.

(c) Let $J$ be written in an orthonormal basis as a diagonal of blocs of the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and indicate by $\sigma$ the corresponding symplectic form. Prove that if $\omega_A(x,y) = \sigma(Px,Py)$ - we shall write $\omega_A = P^*\sigma$- then $\det(A) = \det(P)^2$.

(d) Prove that if $\omega_A = P^*\sigma$ then $\omega_A^n = P^*\sigma^n = (n!)\det(P) e_1^* \wedge ... \wedge e_{2n}^*$ where $(e_1,...,e_{2n})$ is an orthonormal basis and $(e_1^*,...,e_{2n}^*)$ the dual basis.

(e) Conclude that there is a polynomial in the coefficients of $A$, denoted $Pf(A)$, such that $\det(A) = Pf(A)^2$.

(f) Prove that $Pf(A)$ is a polynomial with integral coefficients in the coefficients of $A$

HINT. *Choose $t_{i,j}$ to be independent variables for $i < j$ and set $A = (a_{i,j})$ with $a_{i,j} = t_{i,j}$ for $i < j$, $a_{i,i} = 0$ and $a_{i,j} = -t_{j,i}$ for $j < i$. Set $\mathbb{K} = \mathbb{Q}(t_{i,j})$ be the field of fractions of the $t_{i,j}$ (its elements are quotients of rational polynomials in the $t_{i,j}$). Use the previous result to prove that* $\det(A) = R(t_{i,j})^2$ *with $R(t_{i,j}) \in \mathbb{Q}(t_{i,j})$. Use the fact that $\mathbb{Z}[t_{i,j}]$ is a Unique Factorization Domain (see e.g. [Lan02] p. 111 and 183) to prove that since* $\det(A) \in \mathbb{Z}[t_{i,j}]$ *the same holds for $R$.*

(23) Let $E$ be a complex vector bundle and $\overline{E}$ the complex vector bundle having the same underlying real bundle, but with complex structure $-J$. Using your favourite defintion of the first Chern class, prove
   (a) we have $c_1(\overline{E}) = -c_1(E)$
   (b) if $E = V \otimes \mathbb{C}$ then $2c_1(E) = 0$

### 7.4. The structure of the symplectic group.

(24) (The structure of $Sp(2n,\mathbb{K})$) A shear map is a map of the type $S(x) = x + m(x)\cdot v$ where $m$ is a linear form and $v$ a fixed vector.
   (a) Prove that a shear map is symplectic if and only if $m(x) = c\cdot\omega(v,x)$. We den ote such a map by $S_{c,v}$.
   (b) Prove that for a shear map $S_{c,v}$ (defined by $S_{c,v}(x) = x + c\omega(v,x)v$) the vector $v$ is given by $\mathrm{Ker}(S_{c,v} - \mathrm{Id}) = (\mathbb{K}v)^{\omega}$ or $\mathrm{Im}(S_{c,v} - \mathrm{Id}) = \mathbb{K}v$.
   (c) Prove that $S_{c,v}$ and $S_{d,w}$ commute if and only if $\omega(v,w) = 0$. We shall call $T$ the subgroup of $Sp(2n,V)$ generated by shear maps.
   (d) Let $x, y$ be such that $\omega(x,y) \neq 0$. Prove that there is a shear map sending $x$ to $y$.
   (e) Prove that for all $x, y$ in $V$, there is a composition of two shear maps sending $x$ to $y$, so that $T$ acts transitively on $V$.
   (f) Prove that $T$ acts transitively on pairs of vectors such that $\omega(x,y) = 1$.
   (g) Let $M \in Sp(2n,\mathbb{K})$. Prove that there exists $z$ such that $\omega(z,x) = 1$ and an element $S$ in $T$ such that $Sx = Mx$ and $Sz = Mz$.
   (h) Prove by induction on the dimension that any $M \in Sp(2n,\mathbb{K})$ is a product of shear maps (and in fact at most $2n$ of them).
   (i) Prove by induction that unless $n = 1, \mathbb{K} = \mathbb{F}_3$ (we excluded[15] the case $\mathbb{K}$ has characteristic 2) the subgroup of commutators in $Sp(2n,\mathbb{K})$ contains all shear maps, hence is equal to $Sp(2n,\mathbb{K})$. Deduce Hint:
   (j) Show that $Sp(2n,\mathbb{K})$ is uniformly perfect, that is any element is the product of a uniformly bounded number of commutators.

(25) (Simplicity of the projective symplectic group[16], see [Die48]) We want to use the previous exercise to prove that $PSp(2n,\mathbb{K}) = Sp(2n,\mathbb{K})/\{\pm\mathrm{Id}\}$ is a simple group. Let $N$ be a normal subgroup of $Sp(2n,\mathbb{K})$ not contained in $\{\pm\mathrm{Id}\}$. .
   (a) Prove the relation $[f,g][g,hfh^{-1}] = f[g,[f^{-1},h]]f^{-1}$
   (b) Let $v \neq w$ and consider the shear maps $S_{c,v}, S_{d,w}$. We want to prove that $[S_{c,v}, S_{d,w}] = S_{c,v}\circ S_{d,w}\circ S_{c,v}^{-1},\circ S_{d,w}^{-1} \in N$. Let $h \in N$ be such that $\omega(v,h(w)) = 0$. Using the above relation, and the fact that if $h \in N$ and $f \in Sp(2n\mathbb{K})$ we have $[f,h] \in N$ prove that $[S_{c,v}, S_{d,w}] \in N$.
   (c) Prove that for given $h \neq \pm\mathrm{Id}$ and any $v, w \in \mathbb{R}^{2n}$ non colinear, there is a map $g$ in $T$ such that $\omega(g(v), h(g(w))) = 0$.

[15] In fact if $\mathbb{K}$ has characteristic 2, only $Sp(2,\mathbb{F}_2)$ and $Sp(4,\mathbb{F}_2)$ are not perfect. See [OMe78], p.

[16] Our proof is inspired from [CHS20] itself inspired by the classical works of Fathi, Epstein, Higman.

HINT. *Let $y$ be such that $h(y)$ is non-colinear to $y$. Let $x \in h(y)^{\omega}$ be non-colinear to $x$ such that $\omega(x,y) = \omega(v,w)$. Since according to Witt's theorem, $Sp(2n,\mathbb{K})$ acts transitively on pairs of vectors with the same symplectic product, we can find $g \in Sp(2n,\mathbb{K})$ such that $gv = x, gw = y$.*

(d) Prove that unless $n = 1, \mathbb{K} = \mathbb{F}_3$, $Sp(2n,\mathbb{K})$ is a simple group.

(26) (The structure of $Sp(2n,\mathbb{K})$, II) Let $M = \begin{pmatrix} \mathrm{Id}_n & A \\ 0 & \mathrm{Id}_n \end{pmatrix}$ be a $2n \times 2n$ matrix where we decompose $\mathbb{K}^{2n} = H \oplus V$, where $H, V$ are Lagrangian subspaces, and $M : H \longrightarrow V$ is a linear map. We shall identify $V$ to $H^*$ via the map $x \mapsto \sigma(x, \bullet)$.
   (a) What is the condition on $A$ so that $M$ is in $Sp(2n,\mathbb{K})$ ?
   (b) Describe the set of symplectic maps which preserve $H$

(27) (Conformal symplectic maps) Let $CSp(2n,\mathbb{K})$ be the group of conformally symplectic maps, that is maps such that $\sigma(Cx, Cy) = k(C)\sigma(x,y)$ for all $x, y$, with $k(C) \neq 0$.
   (a) Prove that $CSp(2n,\mathbb{K}) \simeq \mathbb{K}^*\mathrm{Id} \times Sp(2n,\mathbb{K})$.
   (b) Show that the Lie algebra $csp(2n,\mathbb{R})$ of $CSp(2n,\mathbb{R})$ is given by the set of matrices satisfying $\sigma(Ax,y) + \sigma(x,Ay) = k(A)\sigma(x,y)$ for some $k(A) \in \mathbb{R}$.
   (c) We assume from now on that $\mathbb{K} = \mathbb{R}$. Prove that the characteristic spaces of $C \in CSp(2n,\mathbb{R})$ satisfy $E_\lambda = E_{k(C)/\lambda} = E_{k(C)/\bar{\lambda}}$ and those of $A \in csp(2n,\mathbb{R})$ satisfy $E_\lambda = E_{k(A)-\lambda} = E_{k(A)-\bar{\lambda}}$

(28) Prove that if the eigenvalue $\lambda$ of $M \in Sp(2n,\mathbb{K})$ has Krein type $(p,q)$ then $\overline{\lambda}$ has Krein type $(q,p)$.

### 7.5. Hotchpotch.

(29) (Symplectic reduction of ellipsoids) Let $Q$ be a positive definite quadratic form on $(\mathbb{R}^{2n},\sigma)$. We want to prove that $Q$ can be reduced to the form $\sum_{j=1}^n \frac{1}{r_j^2}(x_j^2 + y_j^2)$ where $(x_1,\ldots,x_n,y_1,..y_n)$ are symplectic coordinates (i.e. $\sigma = \sum_{j=1}^n dx_j \wedge dy_j$) and $r_j \in \mathbb{R}_+^*$.
   (a) Prove that this is equivalent to showing that given the standard scalar product $(\bullet,\bullet)$ and a symplectic form, $\omega$, we can find real numbers $(a_1,\ldots,a_n)$ and an orthonormal basis $(e_1,\ldots,e_n,f_1,\ldots,f_n)$ such that $\omega = \sum_{j=1}^n a_j e_j^* \wedge f_j^*$.
   (b) We denote by $(\bullet,\bullet)$ the scalar product defined by $Q$ and define $A$ by $\omega(x,y) = \big(Ax,y\big)$. Prove that $A$ is a skew-symmetric matrix (i.e.${}^tA = -A$) and that if $A$ preserves some vector space $S$ then it preserves $S^\perp$ (this is a standard argument for normal operators i.e. operators commuting with their adjoint)
   (c) Prove that for $A$ as above, there is always a 2-dimensional invariant subspace for $A$ and we can find an orthonormal basis $(e,f)$ of $S$ such that $\omega_{|S} = ae^* \wedge f^*$

   HINT. *Look for a complex eigenvector of $A$.*

(d) Conclude by induction.

(e) Show that any ellipsoid in $E$ can be reduced through a symplectic map to the standard

$$E(r_1,...,r_n) = \{(x_1, y_1,..,x_n, y_n) \mid \sum_{j=1}^{n} \frac{1}{r_j^2}(x_j^2 + y_j^2) \leq 1$$

(f) Prove that the $(a_1,...,a_n)$ are unique, up to permutation. Hint: Assuming $a_1 \leq a_2 \leq \ldots \leq a_n$, Show that $a_{n-k}...a_n$ is equal to the the maximal volume (for $\omega^k$) of a $2k$-dimensional unit cube (for $k = n$ this is Hadamard's inequality, then argue by induction).

(g) Prove (e.g. by induction) that the symplectic maps preserving $E(r_1,...,r_n)$ for $r_1 < ... < r_n$ are reduced to products of rotations in the planes $(e_j, f_j)$

Remark: the situation is much more complicated for non-definite quadratic forms (see [Wil36; Hör95] and [Arn97], p.381 )

(30) Show that Exercise 29 can be translated to the following:

(a) Let $A$ be a positive definite matrix. Prove that there exists $S \in Sp(2n,\mathbb{R})$, $D$ a diagonal complex matrix (i.e. commuting with $J$) such that $A = S^*DS$.

(b) Show that this implies that for $T \in SL(2n,\mathbb{R})$, there exists $S \in Sp(2n,\mathbb{R})$ and $O \in O(2n,\mathbb{R})$ and a diagonal complex matrix $D$ such that $T = ODS$

HINT. *(see [AO06], also for some applications) Let $T^*T = S^*DS$. Prove that if $B$ is the unit ball $DS(T^{-1}(B)) = B$, hence there exists $O \in O(2n,\mathbb{R})$ such that $O^{-1} = DST^{-1}$.*

(31) (Linear non squeezing theorem) This Exercise makes use of Exercise 29. Let $T \in GL(2n,\mathbb{R})$ be an isomorphism such that whenever the ellipsoid $E$ in $\mathbb{R}^{2n}$ has reduced form $\sum_{j=1}^{n} \frac{1}{r_j^2}(x_j^2 + y_j^2) \leq 1$ where $r_1 \leq r_2 \leq ... \leq r_n$ then $TE$ is an ellipsoid with reduced form $\sum_{j=1}^{n} \frac{1}{s_j^2}(x_j^2 + y_j^2)$ where $s_1 \leq s_2 \leq ... \leq s_n$, we must have $r_1 = s_1$. We want to prove that then $T$ is either symplectic or antisymplectic. We shall assume we are in the non-trivial case $n > 1$.

(a) Prove that our assumption implies that a cylinder $C(r) = D^2(r) \times \mathbb{R}^{2n-2}$ has for image by $T$ a cylinder, symplectomorphic to $C(r)$.

(b) Prove that for $S \in Sp(2n,\mathbb{R})$, the maximal affine subspaces contained in $S(C(r))$ are the $\{z\} \times V$ where $V$ is symplectic of dimension $2n-2$ and $z \in S(D^2(r))$. Deduce that $T$ sends a codimension 2 symplectic subspace to a codimension 2 symplectic subspace.

(c) Prove by using the adjoint $T^*$ of $T$ that $T^*$ sends a symplectic plane to a symplectic plane.

(d) Prove that if $T^*$ is neither symplectic nor antisymplectic, we can find a pair of vectors $u, v$ such that $\omega(u, v) = 1$ and $\omega(Tu, Tv) = 0$.
*Hint: Let $e_1, e_2, f_1, f_2$ be part of a canonical basis, we may assume $a_1 =$*

$\omega(Te_1, Tf_1) \neq \omega(Te_2, Tf_2) = a_2$. *Then* $\omega(aTe_1+bTe_2, cTf_1+dTf_2) = aca_1 - bda_2$.

(e) Conclude that $T^*$ and hence $T$ is symplectic or anti symplectic.

(32) (Convexity theorem for the moment map of a torus action) Let $T$ be a finite dimensional torus (i.e. $T = (S^1)^d$) acting on the finite dimensional symplectic space $(V,\omega)$, that is a group morphism $T \longrightarrow Sp(V,\omega)$. This is also called a symplectic representation of $T$.

(a) Prove that there is a complex structure compatible with $\omega$ and preserved by $T$. We denote this structure by $i$.

(b) Using the fact that irreducible representation of a torus are 1-dimensional (on the complex numbers), prove that there is a symplectic decomposition $V = V_0 \oplus V_1 \oplus ... \oplus V_k$ where $V_0$ is the set of fixed points (i.e. $x \in V_0$ if and only if $gx = x$ for all $g \in T$) and $V_j$ are two-dimensional, the action of $T$ on $V_j$ being given by $(\theta_1,...,\theta_d)$ $(\theta, v) \mapsto e^{i\langle \chi_j, \theta\rangle} v$ where $\theta = (\theta_1,...,\theta_d) \in T$, $v \in V_j$ and $\chi_j \in \mathbb{Z}^d \setminus \{0\}$ for $j \neq 0$ (we set $\chi_0 = 0$). The $\chi_j$ are called the weights of the representation.

(c) Prove that $V_0$ is symplectic and that if the action is effective (i.e. the set $\{g \in T \mid \forall v \in V, gx = x\}$ is finite) then $\dim(V) \geq 2d$.

(d) Let $Q$ be a non-degenerate quadratic form on $V$ invariant by $T$. Prove that we can write $Q(v_0,...,v_k) = q_0(v_0) + \sum_{j=1}^d a_j |v_j|^2$ where $q_0$ is non-degenerate and the $a_j$ are non-zero real numbers and $|v_j|^2 = \omega(iv, v)$ corresponds to the standard norm on $V_j$ .

(33) Prove that if $K$ is coisotropic and transverse to a coisotropic $C$, then $K_C = (K \cap C)/C^\omega$ is coisotropic.

(34) Using the fact that for a Lagrangian $L$ in $(V,\omega)$ the space $\Lambda_L(V)$ is contractible, prove, using the exact homotopy sequence of a fibration (see [Spa66], p. 377) that for $C$ coisotropic $\pi_k(\Lambda_C(V)) \simeq \pi_k(\Lambda(C/C^\omega))$.

(35) (Arf invariant) Let $\mathbb{K}$ be a commutative field of characteristic 2. Note that symmetric or skew-symmetric forms are then equivalent objects, and we call such an object a form. However what follows classically belongs to the theory of quadratic forms. Notice also that the only place in Section 1 where we used the fact that the characteristic of $\mathbb{K}$ is different form 2 is to claim that $\omega(x,x) = 0$. A quadratic form on a $\mathbb{K}$-vector space is a map $q : \mathbb{K} \longrightarrow \mathbb{K}$ such that there exists a bilinear form $a$ such that $q(x) = a(x,x)$. We now associate to such a quadratic form [17] the bilinear form $b(u,v) = q(u+v) - q(u) - q(v)$. We notice that $b(u,u) = 0$ (so $b$ is alternating).

(a) Prove that there is a decomposition for the bilinear from $b$ of $V$ as a direct sum of $\ker b$ and the $(V_j)_{1\leq j\leq k}$ where $V_j$ is 2-dimensional and there is a basis $(u_j, v_j)$ such that $b(u_j, v_j) = 1$.

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[17] the usual bilinear form associated to a quadratic form is given by $\frac{1}{2}(q(u+v) - q(u) - q(v))$ but of course the factor $\frac{1}{2}$ makes it useless in characteristic 2.

(b) Prove that the decomposition above is also an orthogonal decomposition of $q$ as a sum of $q_0$ on $\ker(b)$ and of $q_j$ on $V_j$. In other words we can write $x = x_0 + x_1 + ... + x_k$, $y = y_0 + y_1 + ... + y_k$ with $x_0, y_0$ in $\mathrm{Ker}(b)$, $x_j, y_j \in V_j$ and

$$q(x, y) = q(x_0, y_0) + ... + q(x_k, y_k)$$

(c) Prove that on $\ker(b)$, we can find a basis such that $q_0(x_1e_1 + ... + x_se_s) = c_1x_1^2 + .... + c_sx_s^2$

(d) Prove that we may, up to a linear change of variables, assume $q_j(su_j + tv_j) = a_js^2 + st + b_jt^2$

(e) Prove that if $U$ is the additive subgroup of $\mathbb{K}$ of elements of the form $x+x^2$, then $a_jb_j$ mod $U$ does not depend on the choice of the basis and that if $\mathbb{K} = \mathbb{F}_2$, $U = \{0\}$ then this number is equal to the prevailing value taken by $q_j$ on $V_j \setminus \{0\}$ (note that $\mathrm{Card}(V_j \setminus \{0\}) = 3$)

(f) Prove that if $\mathbb{F}_{2^r}$ is the unique field with $2^r$ elements, we have an exact sequence of additive groups $0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_{2^r} \longrightarrow U \longrightarrow 0$ given by the maps $z \mapsto z \cdot 1$ and $x \mapsto x^2 + x$, so when $\mathbb{K}$ is a finite field, we have $F_{2^r}/U \simeq \mathbb{Z}/2\mathbb{Z}$.

(g) Prove that $\mathrm{Arf}(q) = \sum_{j=1}^k a_jb_j$ and is a well defined invariant of the quadratic form when $\mathbb{K} = \mathbb{F}_{2^r}$

(h) Prove that in general $\mathrm{Arf}(q)$ is well defined as an element of $\mathbb{K}$ modulo $U$.

### 7.6. Heisenberg group and Fourier transform.

(36) (Heisenberg group and representation, see [How80]) Let $(V,\omega)$ be a symplectic space and $H(V)$ be the Heisenberg group defined as the space $V \oplus \mathbb{K} \cdot E$ ($E$ is a formal element) endowed with the group law

$$(v, t) \star (v', t') = (v + v', \frac{1}{2}\omega(v, v') + t + t')$$

(a) Prove that $H(V)$ endowed with the above law is indeed a group

(b) Prove that $\mathbb{K} \cdot E$ is the centre of $H(V)$

(c) Let $\chi : \mathbb{K} \longrightarrow \mathbb{C}^*$ be an additive unitary character of $\mathbb{K}$ (i.e. such that $\chi(t + t') = \chi(t)\chi(t')$). Prove that $\chi$ has a unique unitary extension to $H(V)$ where $\mathbb{K}$ is mapped in $H(V)$ as its center(i.e. by $t \mapsto \chi(t)E$ ).

(d) Prove that if $X$ is a vector subspace of $V$, then $X \oplus \mathbb{K}$ is an abelian subgroup of $H(V)$ if and only if $X$ is isotropic. Prove that maximal abelian subgroups of $H(V)$ are in one to one correspondance with the Lagrangian subspaces of $(V,\omega)$.

(e) Let $(e_1, ..., e_n, f_1, ...f_n)$ be a canonical basis of $(V,\omega)$ and identify $V$ to $V \oplus 0 \subset H(V)$. Prove the Heisenberg commutation relations $e_i \star f_j - f_j \star e_i = \delta_i^j E$

(f) Prove that the conformal symplectic group $CSp(V,\omega)$ is contained in $Aut(H(V))$.

(g) Prove that the inner automorphisms are given by $(v, t) \star (w, s) \star (v, t)^{-1} = (w, s + \omega(v, w))$

(h) Prove that $Aut(H(V))$ is generated by the conformal symplectic group, $CSp(V,\omega)$ and the inner automorphisms.
(i) Let $\mathscr{H} = L^2(\mathbb{R}^n)$ and let $\mathscr{A}$ be the algebra of unbounded operators on $\mathscr{H}$. Set $Q(e_j)f = x_j \cdot f, Q(f_j)f = -i\frac{\partial}{\partial x_j}f$. Then $Q$ is a representation of $H(V)$ into the algebra $\mathscr{A}$.

(37) (see [Wei64]) Let $G$ be an abelian locally compact group noted additively. We denote by $\widehat{G}$ the Pontryagin dual of $G$ that is the set of groups morphisms from $G$ to $T = \mathbb{S}^1 = \{t \in \mathbb{C} \mid |t| = 1\}$. The pairing between $\widehat{G}$ and $G$ is denoted by $\langle x^*, x\rangle$. Then the map $G \longrightarrow \widehat{\widehat{G}}$ given by $x \mapsto (x^* \mapsto \langle x^*, x\rangle)$ is an isomorphism (Pontryagin duality).

For $G, H$ locally compact abelian groups, a **bicharacter** is a map $\chi : G \times H \longrightarrow T$ such that the maps $x \mapsto \chi(x, y)$ and $y \mapsto \chi(x, y)$ are characters (of $G$ and $H$ respectively). We have natural bicharacters on $G \times \widehat{G}$ given by the formulas

$$\lambda((x, x^*), (y, y^*)) = \langle y^*, x\rangle$$

$$B((x, x^*), (y, y^*)) = \langle y^*, x\rangle \cdot \langle x^*, y\rangle^{-1}$$

A **quadratic character** $\xi : G \longrightarrow T$ is a continuous map such that

$$\xi(x + y)\xi(x)^{-1}\xi(y)^{-1}$$

is a bicharacter.

We define $H(G)$, the Heisenberg group of $G$, to be $G \times \widehat{G} \times T$ with the law

$$(x, x^*, s) \star (y, y^*, t) = (x + y, x^* + y^*, s \cdot t \cdot \langle y^*, x\rangle)$$

(a) Prove that the above law makes $H(G)$ into a group
(b) Prove that the set of maps from $G \times \widehat{G}$ to itself preserving $B$ forms a group. We call it the symplectic group of $G \times \widehat{G}$.
(c) Prove that if $G = V$ is a real vector space, $\widehat{G}$ can be identified to $V^*$ with $v^*$ corresponding to $v \mapsto e^{2i\pi\langle v^*, v\rangle}$ and then $H(G) = H(V)/\mathbb{Z}$ where $H(V)$ has been defined in Exercise 36 and $\mathbb{Z}$ acts on the center $\mathbb{R}$ of $H(V)$.
(d) Prove that if $G = V$ is a real vector space, bicharacters are of the form $(x, y) \mapsto e^{2i\pi b(x,y)}$ where $b$ is a bilinear form and a quadratic character is of the form $\xi(x) = e^{2i\pi Q(x)}$ where $Q$ is a quadratic form on $V$.
(e) Prove that if $G = \mathbb{R}^n \times T^q$ (where $T^q$ is the $q$-fold product of $T = S^1$), a quadratic character is of the form $\xi(x, y) = e^{2i\pi Q(x)}\chi(x, y)$ where $Q$ is quadratic on $\mathbb{R}^n$ and $\chi$ is a character of $\mathbb{R}^n \times T^q$ (that is of the form $(x, y) \mapsto e^{2i\pi(\langle x^*, x\rangle + \langle y^*, y\rangle)}$ where $x^* \in (\mathbb{R}^n)^*$, and $y^* \in \mathbb{Z}^q$).
(f) Prove that the center of $H(G)$ is $\{(0, 0, t) \mid t \in T\}$ that we shall identify to $T$.
(g) Prove that an automorphism of $H(G)$ induces either $t \mapsto t$ or $t \mapsto t^{-1}$ on $T$
(h) We denote by $A_0(G)$ the group of automorphisms of $H(G)$ inducing the identity on $T$. Prove that such an automorphism descends to a symplectic map of $G \times \widehat{G}$

(i) Given a symplectomorphism $u$ of $G \times \widehat{G}$ prove that the automorphisms of $H(G)$ lifting $u$ are of the form $(z, t) \mapsto (u(z), f(z) \cdot t)$ where $f$ satisfies

$$f(z_1 + z_2) f(z_1)^{-1} f(z_2)^{-1} = \lambda(u(z_1), u(z_2)) \lambda(z_1, z_2)^{-1}$$

(j) Prove that $f$ is a quadratic character of $G$.

(k) A classical result of A. Weil [Wei40] (see also [Car40] for a proof without the axiom of choice) claims that $G$ has a Haar measure, that is a a regular, locally finite Borel measure invariant by translation, and that such a measure is unique up to a positive scalar factor. We denote it by $dx$. It then makes sense to define $L^2(G, \mathbb{C})$ where the Haar measure is understood. Prove that there is a representation of $H(G)$ in $L^2(G, \mathbb{C})$ defined by the formula:

$$(U(x, x^*, t)\varphi)(z) = t\varphi(z + x)\langle x^*, x\rangle$$

(l) Let $\mathscr{F} : L^2(G) \longrightarrow L^2(\widehat{G})$ be the Fourier transform:

$$(\mathscr{F}\varphi)(x^*) = \int_G \varphi(x) \cdot \langle x^*, x\rangle dx$$

(m) Prove that for suitable choices of Haar measures on $G$ and $\widehat{G}$ (i.e. adjusting the scalar factors), the Fourier transform is an isometry.

HINT. *Prove this for a a dense subsets of functions, like compact supported ones.*

(n) Let $\rho$ be a morphism[18] from $G$ to $\widehat{G}$. Prove that $f(x) = \langle \rho(x), x\rangle$ is a quadratic character.

(o) Assume $\rho$ is an isomorphism from $G$ to $\widehat{G}$ and $f_\rho$ be the corresponding quadratic character on $G$. We also denote by $g_\rho$ the bicharacter $g_\rho(x^*) = \langle x^*, \rho^{-1}x^*\rangle^{-1}$ on $\widehat{G}$. Prove that there exists a unique constant $|\rho|$ such that the following change of variable formula holds

$$|\rho| \int_G \varphi(\rho x) dx = \int_{\widehat{G}} \varphi(x^*) dx^*$$

(p) We have for $\varphi \in \mathscr{S}(G)$ the Schwartz space, the formula

$$\mathscr{F}(\varphi * f_\rho)(x^*) = \mu(f)|\rho|^{-1/2} \mathscr{F}(\varphi) g_\rho(x^*)$$

where $\mu(f)$ has modulus 1 and $\varphi * \psi(y) = \int_G \varphi(x)\psi(y - x)dx$. This is formally rewritten as

$$\mathscr{F}(f_\rho)(x^*) = \mu(f)|\rho|^{-1/2} f_\rho(\rho^{-1}x^*)^{-1}$$

(q) Deduce the formula

$$\int_G \left( \int_G \varphi(x - y) f(y) dy \right) dx = \mu(f)|\rho|^{-1/2} \int_G \varphi(x) dx$$

[18] Caveat: there does not always exist a non-trivial one ! For example for $G = S^1$.

Note that $\mu(f)$ can be obtained by applying this formula to any $\varphi$ provided it has nonzero integral.

(r) Apply the above to the case $G = \mathbb{R}$ as a real vector space, and $f(x) = e^{i\pi a x^2}$ and $\varphi(x) = e^{-\pi x^2}$.

HINT. *Note that the Fourier transform of* $e^{-\frac{a\pi x^2}{2}}$ *is* $e^{-\frac{\pi x^2}{2a}}$. *One can write (first changing variable to* $z = y - x$*)*

*Check !!*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\pi(x-y)^2} e^{i\pi y^2}\, dx dy = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\pi z^2} e^{i\pi(z+x)^2}\, dx dz =$$
$$\int_{\mathbb{R}} e^{i\pi x^2} \left(\int_{\mathbb{R}} e^{2i\pi z x} e^{-\pi z^2}\, dz\right) dx$$

## 8. Comments

All of the linear algebra is classical. The term “symplectic” was coined by Herman Weyl in 1939 (see [Wey39]) to replace the term “complex” that was previously used and was a source of confusion[19]. A more detailed study of symplectic linear algebra can be found in [Art57], including the structure of the symplectic group, outlined in Exercise 24 and due to Dieudonné ([Die48]) and more advanced material is in [OMe78]. The reduction theory of symplectic maps in the semi-simple case (i.e. no Jordan block) is classical, and the general case is rather delicate (see [Wil36; LM74; Gut14]. The results from Krein's theory are due to Krein ([Kre50] ) and are very useful in stability questions in mechanics but also in computations of the Conley index for iterations (already in [Bot56; Edw64] in the case of geodesics, and [Eke84; Vit89] in the general case. Modern treatments and applications can be found in the books by Ekeland, Long or Abbondandolo (see [Eke90; Lon02; Abb01]). We shall see in Chapter **??** how this theory is related with iteration formulas for the Conley-Zehnder index (see [CZ84]). The Pfaffian from Exercise 22 goes back to Cayley in [Cay49]. The emphasis on symplectic reduction, is due to Weinstein in [Wei71; Wei77]. Witt's theorem is from [Wit37] in 1937 and is a basic tool in the theory of quadratic form that we shall encounter again in Chapter **??**, Exercise **??** and Exercise **??**. We refer to [Ker00] for a short biography of Ernst Witt. Exercise 35 is about the work of the Turkish mathematician Cahit Arf ([Arf41]), represented since 2009 on the 10 liras turkish banknote. Both Witt and Arf were in Göttingen around 1936-38. Arf first went to study in France, with a scholarship to attend Ecole normale supérieure. After returning to Turkey, he went to Göttingen for his PhD under the direction of Hasse. At the same time, since Emmy Noether had been expelled from the university by the Nazis, her student, Witt became Hasse's assistant(see [LR10]). So all this aspect of the theory of quadratic forms appeared around 1937-39 even though Arf's paper was finally published in 1941 (and 1943 for the second part).

Exercise 32 is the beginning of the proof of the convexity theorem of Atiyah and Guillemin-Sternberg (see [Ati82; GS82]) has its origin in Kostant's convexity theorem concerning the Iwasawa decomposition (see [Kos73]). Exercise 36 is from the first pages of [LV80].

[19]*The name “complex group” formerly advocated by me in allusion to line complexes, . . . has become more and more embarrassing through collision with the word “complex” in the connotation of complex number. I therefore propose to replace it by the Greek adjective “symplectic.” ([Wey39] p.165)*

CHAPTER 3

# Symplectic differential geometry

If God really exists and if he really has created the world, then, as we all know, he created it in accordance with the Euclidean geometry, and he created the human mind with the conception of only the three dimensions of space. And yet there have been and there still are mathematicians and philosophers, some of them indeed men of extraordinary genius, who doubt whether the whole universe, or, to put it more wildly, all existence was created only according to Euclidean geometry and they even dare to dream that two parallel lines which, according to Euclid can never meet on earth, may meet somewhere in infinity. I, my dear chap, have come to the conclusion that if I can't understand even that, then how can I be expected to understand about God?

Brothers Karamazov, F. Dostoievsky (1880)

## 1. Basic results of differential geometry

**1.1. Basic facts about differential forms.** Remember that the vector space of $p$-forms on a smooth manifold $M$ is the set of sections of the vector bundle $\Lambda^p(TM)$, such that $(\Lambda^p(TM))_x = \Lambda^p(T_xM)$, the space of exterior $p$-forms on $T_xM$. We denote by $\Omega^p(M)$ this vector space. There are two main operations on $\Omega^*(M) = \bigoplus_{p=0}^n \Omega^p(M)$:

(1) the wedge product $\wedge$, sending $\Omega^p(M) \otimes \Omega^q(M)$ to $\Omega^{p+q}(M)$
(2) The exterior differential $d$ sending $\Omega^p(M)$to $\Omega^{p+1}(M)$

They satisfy the basic properties

(1) $\alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta)}\beta \wedge \alpha$
(2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta$

In local coordinates $x_1, ..., x_n$ we have the 1-forms $dx_j$, differentials of the coordinates functions $x_j$, and a general $p$-form can locally be written as

$$\sum_{1 \leq i_1 < .. < i_p \leq n} a_{i_1..i_p}(x)dx_{i_1} \wedge ... \wedge dx_{i_p}$$

where the $a_{i_1..i_p}(x)$ are smooth functions. From this and a partition of unity argument, it easily follows that the (anti-commutative, or graded commutative) algebra $\Omega^*(M)$ is generated by its element of the form $f \in \Omega^0(M) = C^\infty(M,\mathbb{R})$ and $df \in d\Omega^0(M) \subset \Omega^1(M)$. There are also

(1) The interior product operation, associating to a vector field $X$ the operator $i_X : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ given by $(i_X\alpha)(x)(v_1,....,v_{p-1}) = \alpha(x)(X(x),v_1,....,v_{p-1})$ It satisfies $i_X(\alpha\wedge\beta) = i_X\alpha\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge i_X\beta$.
(2) The pull-back associating to a smooth map $f : M \longrightarrow N$ the operator $f^* : \Omega^*(N) \longrightarrow \Omega^*(M)$ defined by $f^*\alpha)(x)(v_1,..,v_p) = \alpha(f(x))(df(x)v_1,...,df(x)v_p)$

If $\varphi^t$ is the flow of the time dependent vector field $X_t(x)$ we have the Cartan formula[1]

$$\frac{d}{dt}(\varphi^t)^*_{|t=t_0}\alpha = (\varphi^{t_0})^*(di_X + i_Xd)\alpha$$

We denote by $L_X$ the operator $di_X + i_Xd$. The operator $L_X$ also applies to vector fields by the formula

$$L_XY = \frac{d}{dt}(\varphi^t)_*Y_{|t=0}$$

where $\varphi_*(Y) = d\varphi(x)^{-1}Y(\varphi(x))$. Traditionally we set $[X,Y] = L_XY$ and is called the Lie bracket. It satisfies the Jacobi identity (see Exercise 18) :

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Finally we have for all smooth vector fields $X,Y$ the formula

$$L_Xi_Y - i_YL_X = i_{[X,Y]}$$

as well as Palais's formula

$$d\omega(X_1,..,X_{p+1}) =$$
$$\sum_{j=1}^{p+1}(-1)^{j+1}L_{X_j}\omega(X_1,...,\hat{X}_j,..,X_{p+1}) + \sum_{j<k}(-1)^{j+k}\omega(L_{X_j}X_k,X_1,..,\hat{X}_j,...,\hat{X}_k,...X_{p+1})$$

Here is a useful extension of the Cartan formula

PROPOSITION 3.1. *Let $X,Y$ be smooth manifolds and $f : X\times]-\varepsilon,\varepsilon[\longrightarrow Y$ a smooth map and $\alpha$ a differential form on $Y$. Then denoting by $f_t$ the restriction of $f$ to $\{t\}\times X$*

$$\frac{d}{dt}f_t^*(\alpha)_{|t=0} = f_0^*(i_{\dot{f}_0}d\alpha + d(i_{\dot{f}_0}\alpha))$$

*where* $\dot{f}_0(x) = \frac{\partial f_t}{\partial t}(x)_{|t=0} \in T_xY$, *and* $f_0^*(i_{\dot{f}_0}\alpha) = \alpha(f_t(x))(\dot{f}_0(x),df_0(x)\xi_1,...,df_0(x)\xi_p)$ *and* $f_0^(d\alpha) = df_0^*(\alpha)$

PROOF. This is obtained by applying the formula for $X = \frac{\partial}{\partial t}$ to $f^*\alpha$ and using that $i_{\frac{\partial}{\partial t}}f^*\alpha = i_{\dot{f}_t}\alpha$.

□

[1]Named after Élie Cartan (1869-1951). All other references to Cartan in this book refer to his son, Henri Cartan (1904-2008).

**1.2. Basic facts of differential geometry.** Let $M$ be a manifold. We endow it with a Riemannian metric denoted by $g$, or for $v, w \in T_xM$ we write $g(x)(v,w) = \langle v, w\rangle_x$. This defines geodesics (see Exercise 42 for a symplectic viewpoint) and an exponential map defined for $v \in T_xM$ by the property that $t \mapsto exp(tv)$ is the unique geodesic starting from $x$ with speed $v$. We assume $(M,g)$ is complete, so by Hopf-Rinow's theorem, this is defined for all $t$.

1.2.1. *Tubular neighbourhood theorem.*

THEOREM 3.2. *Let $N$ be a closed submanifold of $M$. Let $D\nu N = \{(x,v) \in TM \mid x \in N, v \in (T_xN)^{\perp}, |v| \leq 1\}$. Given a positive function $\varepsilon$ on $N$ we set $U_\varepsilon(N) = \{\exp(tv) \mid (x,v) \in D\nu N, |t| \leq \varepsilon(x)\}$. Then for $\varepsilon$ small enough, $U_\varepsilon(N)$ is a neighbourhood of $N$ in $M$, and any neighbourhood of $N$ in $M$ contains such a neighbourhood.*

1.2.2. *Basic facts on vector fields.* Let $X(t,z)$ be a time dependent vector field on the manifold $M$. We consider the equation $\dot{x}(t) = X(t, x(t))$ with initial condition $x(t_0) = u_0$. We endow the set of vector fields with the $C^p$ topology $(p \geq 1)$.

THEOREM 3.3 (Cauchy-Lipschitz). *For each $(t_0, u_0)$ there is a neighbourhood such that the map $(t_0, u_0, X) \mapsto x(t)$ from $]t_0-\delta, t_0+\delta[\times B(u_0,\delta)\times B(X_0,\delta) \longrightarrow C^{p+1}(]t_0-\delta, t_0+\delta[, M)$ is continuous. In particular the maximal existence time of a solution depends continuously (as a map from $M$ to $]0,+\infty]$) from the initial condition $u_0$.*

1.2.3. *Frobenius's theorem.*

DEFINITION 3.4. *A distribution $D$ on a manifold $M$ is given by the following data: for each point $x \in M$ we are given a subspace $D(x)$ in $T_xM$ such that in a neighbourhood of $x$ there are $k$ linearly independent vector fields $X_1, ...., X_k$ such that $D(x) = \langle X_1(x), ..., X_k(x)\rangle$. In particular $\dim(D(x))$ is locally constant.*

*A distribution is integrable if through each point $x$ there is a submanifold $S_x$ of $M$ such that for each $x$ in $S$, $T_xS = D(x)$. The submanifold $S$ (or rather the maximal ones for inclusion) are called integral submanifolds.*

DEFINITION 3.5. *A foliation of $M$ is a decomposition of $M$ as a union of submanifolds, such that locally there is a diffeomorphism $\varphi_U : U \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$ and each submanifold $S$ is such that*

$$\varphi(S \cap U) = \bigcup_{x \in S_U} (\{x\} \times \mathbb{R}^{n-k} \cap \varphi(U))$$

*and the right hand side is the decomposition of $\varphi(S \cap U)$ into connected components.*

THEOREM 3.6 (Frobenius's theorem). *A distribution $D$ is integrable if and only if whenever $X, Y$ are vector fields tangent to $D$, we have that $[X,Y]$ is tangent to $D$. In this case the integral submanifolds constitute a foliation of $M$.*

1.2.4. *Basic facts about transversality.* We shall only need the smooth case of Sard's lemma. Let $f : M^m \longrightarrow N^n$ be a smooth map between manifolds. All "genericity" statements rely on Baire's theorem that claims that in a complete metric space, a countable intersection of open dense sets is itself dense.

DEFINITION 3.7. *We say that $x$ is a critical point of $f$ if* $\mathrm{rank}(df(x)) < \min\{m,n\}$. *We say that $y \in N$ is a critical value of $f$ if $f^{-1}(y)$ contains a critical point. If $y$ is not a critical value, we say it is a regular value.*

DEFINITION 3.8. *Let $f : M^m \longrightarrow N^n$ be a smooth map between manifold and $Y^l$ a submanifold of $N$. We say that $f$ is transverse to $Y$ (abbreviated as $f \pitchfork Y$ if at each point $x$ such that $f(x) = y \in Y$ we have*

$$df(x)T_xM + T_{f(x)}Y = T_{f(x)}N$$

Note that even though a manifold has no canonical measure, stating that a set has measure zero is a well-defined concept. A set $A \subset N$ has zero measure if for any chart, the image of $A$ has measure zero. This notion is of course invariant by diffeomorphism.

THEOREM 3.9 (Sard's lemma). *The set of critical values of $f \in C^\infty(M,N)$ has measure zero. It is a countable union of closed sets of empty interior.*

For the proof we refer to [Mil97]. We now set

DEFINITION 3.10. *We define the equivalence relation on $C^\infty(M,N)$ as follows: for $f, g \in C^\infty(M,N)$ we define $f \underset{x,r}{\simeq} g$ means that $f(z) - g(z) = o(d(x,z)^r)$. In other words, in any local chart, $f$ and $g$ have the same Taylor expansion up to order $r$. We denote by $J^k_{(x,y)}(M,N)$ the set of $k$ jets of maps sending $x$ to $y$. Then $J^k(M,N)$ the set of all $k$-jets of maps form $M$ to $N$ is a bundle over $X \times Y$ with fiber $J^k_{(x,y)}(M,N)$.*

One should be careful, given a chart, the Taylor expansion identifies $J^k_{(x,y)}(M,N)$ with a vector space, but this identification is not natural : changing chart yields a different vector space structure (unless $k = 1$), so $J^k(M,N)$ is not a vector bundle. The main result we shall use is

THEOREM 3.11 (Thom transversality theorem, see [GG73], thm 4.9). *Let $W$ be a smooth submanifold in $J^k(M,N)$. Then the set*

$$T_W = \left\{ f \in C^\infty(M,N) \mid j^k f \pitchfork W \right\}$$

*is a countable intersection of open dense sets in $C^\infty(M,N)$ hence is dense. If $W$ is compact then $T_W$ is open.*

PROPOSITION 3.12 (Morse lemma). *A function is said to be a Morse function if for all critical points $d^2 f(x)$ is non-degenerate. Then Morse functions are generic in $C^\infty(M,\mathbb{R})$, i.e. their complement is a countable union of closed sets of empty interior.*

PROOF. Apply Thom's theorem to $J^1(M,\mathbb{R})$ and $W$ the set of jets of functions having a critical point at $x$, or the set of jets of the constant functions. One checks that $j^1 f \pitchfork W$ if and only if $f$ is Morse. $\square$

## 2. Definition and examples

DEFINITION 3.13. *A two-form $\omega$ on a smooth manifold $M$ is symplectic if and only if*

*(1) $\forall x \in M$, $\omega(x)$ is symplectic on $T_xM$;*

*(2) $d\omega = 0$ ($\omega$ is closed).*

EXAMPLES 3.14.

(1) $(\mathbb{R}^{2n}, \sigma_n)$ is symplectic manifold.

(2) (The cotangent bundle or the phase space of classical mechanics) If $N$ is a manifold, then

$$T^*N = \{(q,p) | p \text{ linear form on } T_qM\}$$

is a symplectic manifold. Let $q_1, \cdots, q_n$ be local coordinates on $N$ and let $p^1, \cdots, p^n$ be the dual coordinates. Then the symplectic form is defined by

$$\omega = \sum_{i=1}^{n} dp^i \wedge dq_i.$$

One can check that $\omega$ does not depend on the choice of coordinates and is a symplectic form. Indeed, define a one-form, called the Liouville form

$$\lambda = p\,dq = \sum_{i=1}^{n} p^i dq_i.$$

It is well defined since if $\pi : T^*N \longrightarrow N$ is the projection, $\lambda$ can be alternatively defined by

$$\lambda(q,p)(\xi) = p \cdot d\pi(\xi)$$

This makes sense since $p \in T_q^*N$ and $d\pi(q,p)\xi) \in T_qN$. Then $d\lambda = \omega$.

(3) Projective algebraic manifolds (or the space of Algebraic geometry)

The complex projective space, $\mathbb{C}P^n$ is defined as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\mathbb{C}^*$ the action being by multiplication. We denote a point in $\mathbb{C}P^n$ by $[z_0, z_1, ..., z_n]$, with $(z_0, z_1, ..., z_n) \neq (0, 0, ..., 0)$, so for any non-zero $\lambda$ in $\mathbb{C}$, we have $[\lambda \cdot z_0, ...., \lambda \cdot z_n] = [z_0, ..., z_n]$. It has a canonical symplectic structure $\sigma_{FS}$ called the Fubini-Study form, defined by setting $|z|^2 = \sum_{j=0}^{n} |z_j|^2$ and

$$\sigma(z) = \frac{i}{2|z|^2} \sum_{j=0}^{n} dz_j \wedge d\bar{z}_j$$

This actually defines a symplectic form on $\mathbb{C}^{n+1} \setminus \{0\}$ which is invariant by the action of $\mathbb{C}^*$, so defines a 2-form on $\mathbb{C}P^n$. This can be seen in a slightly different way, by considering $\mathbb{C}P^n$ as the quotient of $S^{2n+1}$ by the action of $S^1$ given by

$$\theta \star (z_0, ..., z_n) = (e^{i\theta} z_0, ...., e^{i\theta} z_n)$$

where $\sum_{j=0}^{n} |z_j|^2 = 1$. Then $\sigma = \frac{i}{2} \sum_{j=0}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=0}^{n} dx_j \wedge dy_j$ is $S^1$ invariant, and clearly closed, so induces a closed form on the quotient $\mathbb{C}P^n$ (this is an example of Marsden-Weinstein reduction, see Chapter 4, Section 2). The vector field generating the $S^1$ action is given by $X_\theta(z_0,...,z_n)=(iz_0,...,iz_n)$. The tangent space to $S^{2n+1}$ is given in complex notations[2] by

$$T_{(z_0,...,z_n)}S^{2n+1}=\left\{(\zeta_0,...,\zeta_n)\mid \Re\left(\sum_{j=0}^n \overline{z}_j\zeta_j\right)=0\right\}$$

The orthogonal $H_{(z_0,...,z_n)}$ to $X_\theta$ in this tangent space is given by

$$H_{(z_0,...,z_n)}=\left\{(\zeta_0,...,\zeta_n)\mid \Re\left(\sum_{j=0}^n \overline{z}_j\zeta_j\right)=0, \Re\left(\sum_{j=0}^n i\overline{z}_j\zeta_j\right)=0\right\}=$$
$$\left\{(\zeta_0,...,\zeta_n)\mid \sum_{j=0}^n \overline{z}_j\zeta_j=0\right\}$$

This is a complex space, and these sub-bundles of the tangent bundle of $S^{2n+1}$ are invariant by the $S^1$-action. We can thus identify $T_{[z_0,...,z_n]}\mathbb{C}P^n$ to $H_{(z_0,...,z_n)}$. Since multiplication by $i$ and the $S^1$ action commute, $i$ defines a complex structure on $\mathbb{C}P^n$ that we shall denote $J_0$. An alternative proof that $\mathbb{C}P^n$ is a complex manifold is by noticing that $\mathbb{C}^{n+1}\setminus\{0\}$ has a complex structure invariant by the action of $\mathbb{C}^*$. Since for $\zeta\neq 0$ we have $\sigma_{FS}(\zeta,J_0\zeta)=\sum_{j=0}^n|\zeta_j|^2>0$, this yields a complex structure compatible with $\sigma_{FS}$. In particular $\sigma_{FS}$ is non-degenerate, hence symplectic.

Note that with this normalisation, the area of $\mathbb{C}P^1$, that is $\int_{\mathbb{C}P^1}\sigma=\pi$.

Now let $V$ be a complex submanifold in $\mathbb{C}P^n$, that is such that its tangent space $T_zV$ is a complex subspace of $T_z\mathbb{C}P^n$. Then the restriction of $\sigma_{FB}$ is of course closed, and since for $\zeta\neq$ in $T_zV$ we have again $\sigma_{FS}(\zeta,J_0\zeta)>0$ it is symplectic. In particular when $V$ is compact, Chow's theorem (see [Cho49]) claims that $V$ is algebraic, i.e. defined by homogeneous polynomial equations. These are called projective manifolds and are a special case of Kähler manifolds, defined as complex manifolds having a complex structure and a hermitian metric such that its imaginary part is symplectic.

REMARKS 3.15.

(1) For the above manifolds the symplectic form is often implicit, and we shall write $\mathbb{R}^{2n}$ instead of $(\mathbb{R}^{2n},\sigma_n)$, $T^*N$ instead of $(T^*N,d\lambda_N)$. Again if $M$ is a symplectic manifold with symplectic form $\omega_M$ we write $\overline{M}$ for the symplectic manifold $(M,-\omega_M)$.
(2) Standard conventions are often incompatible. On $\mathbb{R}^2$ we want a symplectic form such that the standard circle with standard orientation has positive area : we must choose $dx\wedge dy$. On $T^*S^1$, we want that the circle $p=1$ oriented as its projection, has positive area: so we must choose $dp\wedge dx$ ! So again the

[2] for a complex number $z$, $\Re(z)$ and $\Im(z)$ are its real and imaginary part

symplectic form on $T^*\mathbb{R}^n$ and $\mathbb{R}^{2n}$ have opposite signs. should have positive area : $\omega = \frac{i}{2} dz \wedge d\overline{z}$.

DEFINITION 3.16. *A submanifold $V$ in symplectic manifold $(M,\omega)$ is isotropic, Lagrangian or coisotropic if for each $x \in V$, $T_xV$ is respectively isotropic, Lagrangian or coisotropic in $(T_xM,\omega(x))$.*

EXAMPLES 3.17.

(1) A Lagrangian linear subspace in $\mathbb{R}^{2n}$ is Lagrangian.
(2) Let $\alpha$ be a one-form on $N$. Then its graph
$$G_\alpha = \{(x,\alpha(x)) \mid x \in N\}$$
is a submanifold of $T^*N$. The restriction of $\Lambda_N$ to $G_\alpha$ is is $\alpha$ (this is why $\lambda_N$ is sometimes called the tautological one-form), so $(\sigma_N)_{|G_\alpha} = d(\lambda_{N|G_\alpha}) = d\alpha$. So $G_\alpha$ is Lagrangian if and only if $\alpha$ is closed. Note that conversely, any Lagrangian submanifold of $T^*N$ that is a graph over $N$ is of the form $G_\alpha$ with $\alpha$ closed.
(3) Consider $\mathbb{R}P^n \subset \mathbb{C}P^n$ defined by
$$\mathbb{R}P^n = \{[z_0, z_1, ..., z_n] \in \mathbb{C}P^n \mid \forall j, z_j \in \mathbb{R}\}$$
then $\mathbb{R}P^n$ is a Lagrangian submanifold of $(\mathbb{C}P^n, \sigma_{FS})$. This is in fact the case for any fixed point set of an antisymplectic involution : let $s : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ given by $s([z_0,, z_1, ..., z_n] = [\bar{z}_0, \bar{z}_1, ..., \bar{z}_n]$. Then $s$ satisfies $s^*\sigma_{FS} = -\sigma_{FS}$ and $\mathbb{R}P^n$ is the fixed point set of $s$. As a result if $i : \mathbb{R}P^n \longrightarrow \mathbb{C}P^n$ is the inclusion, since $s \circ i = i$, we have $i^*(\sigma_{FS}) = i^*s^*(\sigma_{FS}) = -i^*\sigma_{FS}$ so $i^*(\sigma_{FS}) = 0$. The same argument show that if $V(\mathbb{C})$ is the set of complex points of a real projective smooth manifold, that is the set of complex zeros in $\mathbb{C}P^n$ of a set of real homogeneous polynomials, then $V(\mathbb{R})$ the set of real zeros contained in $\mathbb{R}P^n$ is a Lagrangian submanifold.
(4) Let $M$ be a submanifold in $N$. Then $T^*_NM = \{(q,p) \in T^*N \mid q \in M\}$ is coisotropic. The set
$$\nu^*M = \{(q,p) \in T^*N \mid q \in M, p_{|T_qM} = 0\}$$
is a Lagrangian submanifold.
(5) Let $\pi : P \longrightarrow N$ be a submersion. Then if $K_\pi$ is the set of cotangent vectors vanishing on the fibers of $\pi$, that is
$$K_\pi = \{(y, p_y) \in T^*P \mid p_y = 0 \text{ on } \pi^{-1}(\pi(y)\}$$
then $K_\pi$ is coisotropic.

Finally we we shall often encounter symplectic maniflds such that $\omega = d\lambda$. This leads to

DEFINITION 3.18. *(Exact symplectic and Lagrangian manifolds) An exact symplectic manifold is a pair $(M,\lambda)$ such that $d\lambda$ is a symplectic form. We often write $(M,d\lambda)$ even* *though $\lambda$ is well determined. Given an exact symplectic manifold an exact Lagrangian submanifold is a pair $(L, f_L)$ such that L is Lagrangian and $\lambda_{|L} = df_L$.*

When we talk about $L$ being exact we do not need to specify $f_L$, but $\lambda$ must be explicit : for example on $(T^1 \times \mathbb{R}, d(p \wedge dx))$ (i.e. $\lambda = pdx$) the curve $p = 0$ is an exact Lagrangian, while on on $(T^1 \times \mathbb{R}, d((p+1) \wedge dx))$ (i.e. $\lambda = (p+1)dx$) it is not, since $dx$ is not exact on $S^1$.

Finally note that to carry a symplectic structure, there are two necessary requirements. We first set

DEFINITION 3.19. *An almost complex structure on a manifold M is a smooth section $J$ of $End(TM)$ such that for each $z \in M$ we have $J(z)^2 = -\mathrm{Id}$. If $(M,\omega)$ is a symplectic manifold, the almost complex structure is said to be* **tame** *if for each $z$ $J(z)$ is tame for $\omega(z)$ i.e. $\omega(J\xi,\xi) > 0$ and* **compatible** *if moreover $\omega(J\xi, J\eta) = \omega(\xi,\eta)$.*

PROPOSITION 3.20. *For the smooth manifold $M$ to carry a symplectic form, we must have*

*(1) if M is compact, there must be a cohomology class $u \in H^2(M,\mathbb{R})$ such that $u^n \neq 0$ in $H^{2n}(M,\mathbb{R})$*
*(2) The tangent bundle to $M$ must carry an almost complex structure*

*More precisely if $(M,\omega)$ is a symplectic manifold, it carries a tame (or compatible) almost complex structure and the set of such structures is contractible.*

PROOF. The first condition is easy since $\omega$ being closed represents a cohomology class, and since $\omega$ is symplectic, $\omega^n$ is a volume form, hence in the compact case is not exact. The second condition follows from the fact that acccording to Proposition 2.42, the set of compatible almost complex structures is contractible. Therefore the set $\mathscr{J}(M,\omega)$ of compatible almost complex structures can be identified to the set of sections of a bundle with fiber over $z \in M$ given by $\mathscr{J}(\omega(z))$ of compatible almost complex structures on $T_zM, \omega(z)$. We therefore are looking for the set of sections of a bundle with contractible fibers, and this is a contractible set (in particular is non-empty). □

### 3. Moser's lemma and the local triviality of symplectic differential geometry

We are going to prove that, contrary to the Riemannian case, symplectic manifolds "have no local geometry". In fact the closeness assumption on $\omega$ should be compared to a flatness condition on a Riemannian metric. A very important tool, with applications reaching much beyond symplectic geometry is

LEMMA 3.21 (Moser, [Mos65]). *Let $N$ be a closed submanifold in $M$. Let $\omega_t$ be a family of symplectic forms defined on $M$ such that $\omega_t|_{TN}$ is constant. Then there is a diffeomorphism $\varphi$ defined near N such that $\varphi^*\omega_1 = \omega_0$ and $\varphi|_N = id|_N$. If M is closed, the diffeomorphism is globally defined.*

PROOF. We will construct a time dependent vector field $X(t,x) = X_t(x)$ whose flow $\varphi^t$ satisfies $\varphi^0 = id$ and $(\varphi^t)^*\omega_t = \omega_0$. Differentiating this equality, we see that this is equivalent to

$$(\frac{d}{dt}(\varphi^t)^*)\omega_t + (\varphi^t)^*(\frac{d}{dt}\omega_t) = 0.$$

Then

$$(\varphi^t)^* L_{X_t}\omega_t + (\varphi^t)^*(\frac{d}{dt}\omega_t) = 0.$$

Since $\varphi^t$ is diffeomorphism, this is equivalent to

$$L_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0.$$

Using Cartan's formula $L_X = d\circ i_X + i_X \circ d$, we get

$$d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t = 0.$$

Since $\omega_t$ is non-degenerate, the map $T_xM \to (T_xM)^*$ which maps $X$ to $\omega(X,\cdot)$ is an isomorphism. Therefore, for any one-form $\beta$, the equation $i_X\omega = \beta$ has a unique solution $X_\beta$. It suffices to solve for $\beta_t$,

$$d\beta_t = -\frac{d}{dt}\omega_t.$$

with the requirement that $\beta_t = 0$ on $T_NM$ for all $t$, because we want $\varphi_{|N} = \mathrm{Id}_{|N}$, that is $X_t = 0$ on $N$. On the other hand, the assumption that $\omega_t = \omega_0$ on $TN$ implies $(\frac{d}{dt}\omega_t) \equiv 0$ on $TN$. Denote the right hand side of the above equation by $\alpha_t$, then $\alpha_t$ is defined in a neighbourhood $U$ of $N$. The solution $\beta_t$ is given by Poincaré's Lemma on the tubular neighbourhood of $N$. Here by a tubular neighborhood we mean a neighborhood of $N$ in $M$ diffeomorphic to the unit disc bundle $D\nu_MN$ of $\nu_MN$ the normal bundle of $N$ in $M$ (i.e. $\nu_MN = \{(x,v)\in T_NM \mid v \perp TN\}$).

LEMMA 3.22. *(Parametrized Poincaré's Lemma) Let $p \geq 1$ and $k \mapsto \alpha_k$ be smooth family of $p$-form on $U$, a tubular neighbourhood of $N$, parametrized by $k \in K$ a compact set. Assume that $\alpha_k$ is closed and vanishes on $TN$, then there exists a smooth family of $(p-1)$-form $\beta_k$ defined on a neighbourhood $V$ of $N$ such that*

*(1) $\alpha_k = d\beta_k$*
*(2) $\beta_k$ vanishes on $T_NM$*

PROOF. We omit the parameter since our construction will obviously depend smoothly on $k$. It is of course sufficient to prove this on the disc normal bundle of $M$, $D\nu_MN$. We denote by $(x,\xi)$ an element in $D\nu_MN$ where $x \in M, \xi \in D\nu_xM$. Consider the radial vector field $X(x,\xi) = (0,-\xi)$ with flow $r_s(x,\xi) = (x, e^{-s}\xi)$. Note that $\lim_{s\to+\infty} r_s(x,\xi) = (x,0)$. Now

$$\frac{d}{ds} r_s^*\alpha = r_s^*(L_X\alpha) = r_s^*(d i_X\alpha) = d(r_s^*(i_X\alpha))$$

so that

$$r_u^*\alpha - \alpha = d\int_0^u r_s^*(i_X\alpha)ds$$

To prove this we claim that

(1) $\lim_{s\to+\infty} r_s^*\alpha = 0$
(2) the integral $\int_0^{+\infty} r_s^*(i_X\alpha)ds$ converges

First $r_s^*\alpha(z)(v_1,...,v_p) = \alpha(r_s(z))(dr_s(z)v_1,...,dr_s(z)v_p)$ but $\lim_{s\to+\infty} r_s(x,\xi) = (x,0)$ and $\lim_{s\to+\infty} dr_s(x,\xi)(w,\eta) = (w,0)$, so if $\alpha$ vanishes on $TN$ we indeed have $\lim_{s\to+\infty} r_s^*\alpha = 0$. For the second statement, we have

$$r_s^*(i_X\alpha)(z)(v_1,...,v_{p-1}) = \alpha(r_s(z))(X(r_s(z)), dr_s(z)v_1,...,dr_s(z)v_{p-1}) = \alpha(x,e^{-s}\xi)((0,-e^{-s}\xi),(w_1,e^{-s}\eta_1),...,(w_{p-1},e^{-s}\eta_{p-1}))$$

where $v_j = (w_j,\eta_j)$ so the last term is an $0(e^{-s})$ and the integral converges. As a result the equality $r_s^*\alpha - \alpha = d\int_0^s r_s^*(i_X\alpha)$ holds. Now on $N$ we have $i_X\alpha = 0$ since $X$ vanishes on $N$, hence $\beta = -\int_0^{+\infty} r_s^*(i_X\alpha)ds$ satisfies $\beta = 0$ on $T_NU$ and $d\beta = \alpha$. The same argument proves the parametrized version. □

We may now conclude the proof of Moser's lemma. Since $\frac{\partial}{\partial t}\omega_t$ vanishes on $TN$ we may apply Poincaré's lemma and get $\beta_t$ such that $\frac{\partial}{\partial t}\omega_t + d\beta_t = 0$. Since $\omega_t$ is non-degenerate, we may find $X_t$ such that $i_{X_t}\omega_t = \beta_t$ and $X_t$ is unique. Since $\beta_t$ vanishes on $T_NU$ we have $X_t = 0$ on $N$. This implies, possibly reducing $U$, that the flow $\varphi^t$ of $X$ is defined for $x, t$ as long as $\varphi^t(x)$ belongs to $U$. But since $X$ is continuous and vanishes on $N$ there is a neighbourhood $U$ of $N$ such that $|X_t(x)| \leq Cd(x,N)$ on $U$. Then $d(\varphi^t(x),N) \leq e^{Ct}d(x,N)$ so if $U$ contains the set of points defined by $d(x,N) \leq \varepsilon$ the flow $\varphi^t$ is defined for $0 \leq t \leq 1$ on the set defined by $d(x,N) \leq e^{-C}\varepsilon(x)$. This concludes our proof. □

EXERCISES 3.23.

(1) (Homotopy formula) Prove that if $\varphi^t$ is the flow of a time dependent vector field $X_t(x)$, then for all $\alpha \in \Omega^p(M)$ we have

$$(\varphi^1)^*(\alpha) - (\varphi^0)^*(\alpha) = dK_X\alpha + K_Xd\alpha$$

where $K_X\alpha = \int_0^1(\varphi^t)^* i_{X_t}\alpha dt$.

(2) Apply the above for the flow of $\frac{\partial}{\partial t}$ on $N\times\mathbb{R}$ and prove that if $i_0, i_1$ are the canonical injection of $N$ into $N\times\{0\}$ and $N\times\{1\}$ respectively, then

$$(i_1)^*\alpha - (i_0)^*(\alpha) = Kd\alpha + dK\alpha$$

(3) Let $F : N\times[0,1] \longrightarrow M$ be a smooth map, let $i_0, i_1$ be the canonical injection of $N$ into $N\times\{0\}$ and $N\times\{1\}$ respectively, and $f_0 = F\circ i_0, f_1 = F\circ i_1$

$$f_1^*\alpha - f_0^*\alpha = dK_F\alpha + K_Fd\alpha$$

where now $K_F\alpha = K(F^*\alpha)$.

(4) Prove using the above lemma that if $N$ is a submanifold of $M$, the relative de Rham cohomology, $H^*(M,N)$, can be defined in three equivalent ways:
   (a) As the set of closed forms vanishing on $TN$ modulo the differential forms vanishing on $TN$
   (b) As the set of closed forms vanishing on $T_N M$ modulo the differential forms vanishing on $T_N M$
   (c) As the set of closed form vanishing in a neighborhood of $N$ modulo the differential of forms vanishing near $N$.

(5) Prove that Moser's lemma holds also if $\omega_t$ is a family of volume forms.

Moser's lemma has many applications in symplectic geometry. Let's start with

PROPOSITION 3.24 (Darboux). *Let $(M,\omega)$ be a symplectic manifold. Then for each $z \in M$, there is a local diffeomorphism $\varphi$ from a neighborhood of $0$ in $(\mathbb{R}^{2n},\sigma_n)$ to to a neighborhood of $z_0$ in $M$ such that $\varphi^*\omega = \sigma$.*

PROOF. According to Proposition 2.19 (1), there exists a linear map $L : \mathbb{R}^{2n} \to T_z M$ such that $L^*\omega(z) = \sigma$. Hence, using a local diffeomorphism $\varphi_0 : U \to W$ such that $d\varphi_0(0) = L$, where $U$ and $W$ are neighborhoods of $0 \in \mathbb{R}^{2n}$ and $z_0 \in M$ respectively, we are reduced to the case where $\varphi_0^*\omega$ is a symplectic form defined in $U$ and $\omega(z_0) = (\varphi_0^*)\sigma$.

Define $\sigma_t = (1-t)\varphi_0^*\omega + t\sigma$ in $U$. We readily see that $\sigma_t$ satisfies the assumptions of Moser's Lemma for $N = \{0\}$, therefore, there exists $\psi$ such that $\psi^*\sigma_1 = \sigma_0$ and $\psi(0) = 0$, i.e.

$$\psi^*\sigma = \varphi_0^*\omega.$$

Then $\varphi = \varphi_0 \circ \psi^{-1}$ is the required diffeomorphism. □

EXERCISES 3.25.

(1) Prove that if $(M,\omega)$ is a closed symplectic manifold and $\omega_t$ is a smooth family of symplectic forms such that $[\omega_t] \in H^2(M,\mathbb{R})$ is constant, then there is a symplectomorphism from $(M,\omega_0)$ to $(M,\omega_1)$. There are examples of continous families of symplectic forms $\omega_t$ (not in the same cohomology class) such that $\omega_0$ and $\omega_1$ are non-symplectomorphic (see [McD87]).

(2) Let $\omega_1$, $\omega_2$ be symplectic forms on a connected closed surface. Then there exists a diffeomorphism $\varphi$ such that $\varphi^*\omega_1 = \omega_2$ if and only if $\int\omega_1 = \int\omega_2$.

Note that in contrast to the Riemannian case, where the neighbourhood of points can be distinguished, for example in dimension a flat, positively or negative curved surface do not have isometric neighborhoods, no such distinction is possible in symplectic geometry. In fact the condition of being closed on the symplectic form is the analogue of a flatness condition on a metric.

PROPOSITION 3.26 (Weinstein's neighbourhood theorem). *(Weinstein, see [Wei73a]) Let $L$ be a closed Lagrangian immersed submanifold in $(M,\omega)$. Then the immersion $i_L$ extends to an immersion of a neighborhood of $0_L \subset T^*L$ where $0_L = \{(q,0) \mid q \in L\}$ is the zero section.*

PROOF. The idea of the proof is the same as that of Darboux Lemma, but the linear argument requires to work on a vector bundle. Let $T_L M = \{(x, v) \in TM \mid x \in L\}$. First, for any $x \in L$, find a sub-bundle of $T_L M$, $V$, such that the space $V(x)$ in $T_x M$ satisfies

(1) $V(x) \subset T_x M$ is Lagrangian subspace;
(2) $V(x) \cap T_x L = \{0\}$;
(3) the map $x \to V(x)$ is smooth.

For example if we define a compatible complex structure $J$ on $T_L M$, we see that $J(x)T_x L$ satisfies (1), (2), and (3), so existence of compatible almost complex structures (see Proposition 2.42) proves the existence of the sub-bundle $V$. An alternative approach is to consider the bundle $\Lambda_H$ defined by $\Lambda_H(x) = \Lambda_{H(x)}(T_x M)$ where $\Lambda_{H(x)}(T_x M)$ is the set of Lagrangians subspaces transverse to $H(x) = T_x L$. The space $\Lambda_H(x)$ is contractible according to Proposition 2.24 (1). So the bundle has a smooth section, and this defines $V(x)$. Abusing notations a little, we write $L$ for the zero section in $T^*L$. Denote by $T_L(T^*L)$ the restriction of the tangent bundle $T^*L$ to $L$. Denote by $T_L M$ the restriction of the bundle $TM$ to $L$. Both are symplectic vector bundles over $L$. For $x \in L$, their fibres are

$$T_x(T^*L) = T_x L \oplus T_x(T_x^*L)$$

and

$$T_x M = T_x L \oplus V(x).$$

We then construct a bundle map $A_0 : T_L(T^*L) \to T_L M$ which restricts to identity on the factor $T_x L$ and sends $T_x(T_x^*L)$ to $V(x)$. Moreover, we require

$$\omega(A_0 u, A_0 v) = \sigma(u, v),$$

where $u \in T_x(T_x^*L) = T_x^*L$ and $v \in T_x L$. We set $A_0 = \mathrm{id} : T_x L \longrightarrow T_x L$ and then this defines $A_0$ uniquely because $\omega$ identifies $V(x)$ to the dual of $T_x L$ (see Exercise 2.26 (1)). Then we can find $\varphi_0$ from a neighborhood of $L$ in $T^*L$ to a neighborhood of $L$ in $M$ such that $d\varphi_0|_{T_L(T^*L)} = A_0$. This is done as follows, using the exponential map (for the reader who has not seen the exponential map in Riemannian geometry, we refer to Exercise 42)

$$\begin{tikzcd} T_L(T^*L) \arrow[r, "A_0"] \arrow[d, "exp"] & T_L M \arrow[d, "exp"] \\ T^*L \arrow[r, "\varphi"] & M \end{tikzcd}$$

By the construction of $A_0$

$$\varphi_0^*\omega = \sigma \text{ on } T_L(T^*L).$$

Define

$$\omega_t = (1-t)\varphi_0^*\omega + t\sigma, \quad t \in [0,1].$$

$\omega_t$ is a family of symplectic forms in a neighborhood of $0_L$ : they are obviously closed, and since they are non-degenerate on $L$ they are still non-degenerate in a neighbourhood of $L$. Moreover, $\omega_t \equiv \omega_0$ on $T_L(T^*L)$. By Moser's Lemma, there exists $\Psi$ defined near $0_L$ such that $\Psi^*\omega_1 = \omega_0$, i.e. $\Psi^*\sigma = \varphi_0^*\omega$. Then $\varphi_0 \circ \Psi^{-1}$ is the diffeomorphism we need. □

Note that the above theorem (in fact the first lines of its proof) imply

COROLLARY 3.27. *Let L be a Lagrangian immersed submanifold in a symplectic manifold M. Then its normal bundle $\nu L$ is isomorphic to its tangent bundle.*

PROOF. The almost complex structure $J$ yields the isomorphism between $T_xL$ and $\nu_x L = T_xM/T_xL$. □

For example if $L$ has a Lagrangian immersion in $\mathbb{R}^{2n}$, we must have $TL \oplus TL = \varepsilon_{\mathbb{R}}^{2n}$. It has been shown in [Aud88] that for $n = 2$ this implies that for $L$ to be embedded $L$ must either either be a torus or a non-orentable surface of Euler characteristics divisible by 4. Givental in [Giv86] gave examples of such embeddings except for the Klein bottle. Then non existence of a Lagrangian embedding of the Kelin bottle was proved much later by Shevchishin in [She09b] (see also [Nem09]).

We thus proved that even in the neighbourhood of a Lagrangian there is no local geometry. We can also describe all Lagrangians submanifolds near a given one: since in $T^*L$ Lagrangians submanifolds close to the zero sections are graphs of closed one-forms, Lagrangians near $L$ will be obtained as images of closed one-forms by the symplectomorphism of the above Proposition.

COROLLARY 3.28. *Let L be a Lagrangian submanifold in $(M,\omega)$. Consider $\Phi$ a symplectomorphism given by Weinstein's neighbourhood theorem from a neighbourhood of $0_L$ in $T^*L$ to $(M,\omega)$ sending $O_L$ to L. Then any Lagrangian $C^1$-close to L is of the form $\Phi(G_\alpha)$ where $\alpha$ is a closed 1-form on L.*

The most general such "no local geometry" theorem of this kind is

THEOREM 3.29. *(Darboux-Weinstein-Givental theorem a.k.a. non-linear Witt theorem. See [Wei71], theorem 4.1 for a weaker version and [AG90] page 26).*

*Let $S_1, S_2$ be two closed submanifolds in $(M_1,\omega_1),(M_2,\omega_2)$. Assume there is a diffeomorphism $\varphi : S_1 \longrightarrow S_2$ which lifts to bundle map*

$$\Phi : T_{S_1}M_1 \longrightarrow T_{S_2}M_2$$

*coinciding with $d\varphi$ on the sub-bundle $TS_1$, and preserving the symplectic structures, i.e. $\Phi^*(\omega_2) = \omega_1$. Then there is a symplectic diffeomorphism between a neighborhood $U_1$ of $S_1$ and a neighborhood $U_2$ of $S_2$.*

PROOF. Using the exponential map for some Riemannian metric (see Exercise 42), we prove that there is a map $\Psi : U_1 \longrightarrow U_2$ sending $S_1$ to $S_2$ and such that $D\Psi : T_{S_1}M_1 \longrightarrow T_{S_2}M_2$ coincides with $\Phi$. Then we consider the family of symplectic forms

$\tau_t = (1-t)\omega_1 + t\Psi^*\omega_2$. This is a family of symplectic forms in a neighborhood of $S_1$ since $\tau_t$ coincides with $\omega_1$ on $T_{S_1}M_1$. Now according to Moser's Lemma, we can find a diffeomorphism $\Xi_t$ in a neighborhood of $S_1$ such that $\Xi_t^*(\tau_t) = \tau_0 = \omega_1$. So for $t = 1$ we get $\Xi_1^*\Psi^*(\omega_2) = \omega_1$ and $\Phi \circ \Xi_1$ is the symplectomorphism we are looking for. □

EXERCISE 3.30. Let $I_1$, $I_2$ be two diffeomorphic isotropic submanifold in $(M_1, \omega_1)$, $(M_2, \omega_2)$. Let $E_1 = (TI_1)^{\omega_1}/(TI_1)$ and $E_2 = (TI_2)^{\omega_2}/(TI_2)$. Then $E_1, E_2$ are symplectic vector bundles over $I_1$ and $I_2$. Show that $I_1$ and $I_2$ have symplectomorphic neighborhoods if and only if $E_1 \cong E_2$ as symplectic vector bundles.

## 4. The groups $\mathrm{DHam}(M,\omega)$ and $\mathrm{Diff}(M,\omega)$

According to Klein's Erlangen's program, geometry is the study of the symmetry group of some structure. Let $(M,\omega)$ be a symplectic manifold. The group playing the first role here is

DEFINITION 3.31. *The group of symplectic diffeomorphisms of the symplectic manifold* $(M,\omega)$*, denoted by* $\mathrm{Diff}(M,\omega)$ *is the set of diffeomorphisms satisfying* $\varphi^*\omega = \omega$*. We denote by* $\mathrm{Diff}^0(M,\omega)$ *the connected component of* $\mathrm{Id}$ *and* $\mathrm{Diff}_c(M,\omega)$ *the set of compact supported ones.*

This is a huge group since it contains $\mathrm{DHam}(M,\omega)$, which we will now define.

Let $H(t,x)$ be any smooth function on $[0,1]\times M$ and $X_H$ the unique vector field such that[3]

$$\omega(X_H(t,x),\xi) = -d_x H(t,x)\xi, \quad \forall \xi \in T_x M.$$

Here $d_x$ means exterior derivative with respect to $x$ only. From now on it will be understood that $d$ applies only to the spatial coordinates.

DEFINITION 3.32. *Let* $H \in C^\infty([0,1]\times M, \mathbb{R})$ *and consider the vector field* $X_H(t,x)$*. It is called the Hamiltonian vector field associated to* $H$*. If its flow from* $s$ *to* $t$*,* $\varphi_H^{[s,t]}$ *is well defined for* $s = 0, t = 1$ *then* $\varphi_H^1 = \varphi_H^{[0,1]}$ *is called the Hamiltonian map associated to* $H$*.*

PROPOSITION 3.33. *The flow of* $X_H$ *is in* $\mathrm{Diff}^0(M,\omega)$*.*

PROOF. To see this,

$$\begin{aligned}
\frac{d}{dt}(\varphi^t)^*\omega &= (\varphi^t)^*(L_{X_H}\omega) \\
&= (\varphi^t)^*(d\circ i_{X_H}\omega + i_{X_H}\circ d\omega) \\
&= (\varphi^t)^*(d(dH)) = 0.
\end{aligned}$$

□

REMARKS 3.34.

[3]See the Comments section for this choice of sign.

(1) It is worth mentioning that we can replace $dH$ by a closed one-form, $\alpha$ and define $X_\alpha$ by the identity $\omega(X_\alpha,\xi) = -\alpha(x)(\xi)$. The same argument shows that the flow of $X_\alpha$ is symplectic. For reasons that will appear in the following chapters, such flows are not as fundamental as the one associated to $H$, and anyway they become Hamiltonian once we pass to the universal cover of $M$, since $\alpha$ becomes exact.

(2) In local coordinates in $T^*N$ we obtain the classical equations of Hamiltonian mechanics. Indeed $X_H(q,p) = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$ and the flow equations become

$$\dot{q}(t) = \frac{\partial H}{\partial p}(t,q(t),p(t)),\ \dot{p}(t) = -\frac{\partial H}{\partial q}(t,q(t),p(t))$$

For example $H(q,p) = \frac{1}{2m}|p|^2 + V(q)$. The equation then become $\dot{q} = \frac{1}{m}p,\ \dot{p} = -\nabla V(q)$, that is $m\ddot{q} + V(q) = 0$, that is the usual Newton equation. More generally if we choose local coordinates $q_1,\dots,q_n$ and their dual $p_1,\dots,p_n$ in the cotangent space, so that $\lambda = \sum_{j=1}^n p_j, q_j$, the flow is given by the ordinary differential equation

$$\begin{cases} \dot{q}_j = \frac{\partial H}{\partial p_j}(t,q,p) \\ \dot{p}_j = -\frac{\partial H}{\partial q_j}(t,q,p) \end{cases}$$

In particular for a particle in a force field derived from the potential $V$, the Hamiltonian is $H(q,p) = \frac{1}{2m}|p|^2 - V(q)$ the flow is given by $\dot{q} = p, \dot{p} = \nabla V(q)$ that is equivalent to $\ddot{q} + \nabla V(q) = 0$ so for a force field deriving from a potential, we recover Newton's equation. For the $N$-body problem, that is $N$ particles attracted to each other by the gravitational force, the potential is $\sum_{1\leq i<j\leq N} \frac{m_i m_j}{|q_i - q_j|}$ we obtain the gravitational potential of $N$ masses of mass $m_i$.

Note that in the previous example, $H$ is the total energy of the particle, and as we know from any physics course, it is conserved. This is a general phenomenon:

PROPOSITION 3.35. *Let $X_H$ be the Hamiltonian vector field associated to a time independent Hamiltonian $H$ (also called autonomous Hamiltonian). Then the levels of $H$ are preserved by the flow $\varphi_H^t$.*

PROOF. This immediately follows from the computation

$$\frac{d}{dt}H(\varphi_H^t(x)) = dH(\varphi_H^t(x))\frac{d}{dt}\varphi_H^t(x) =$$
$$dH(\varphi_H^t(x))X_H^t(\varphi^t(x)) = -\omega(X_H(\varphi_H^t(x)), X_H(\varphi_H^t(x))) = 0$$

□

REMARKS 3.36.

(1) This does not hold for time dependent Hamiltonians.

(2) On a given energy surface the trajectories do not depend on $H$ but only on the energy hypersurface. Let now $H : M \longrightarrow \mathbb{R}$ be a smooth function and $c$ be a regular value. Then for $z \in \Sigma = H^{-1}(0)$, the restriction of $\omega(z)$ to the hyperplane $T_z\Sigma$ has kernel $\mathbb{R}X_H(z)$. So the trajectories of $X_H$ define a foliation on $\Sigma$ independent from the choice of $H$ : it only depends on $\Sigma$. Changing the Hamiltonian changes the speed at which the trajectories are described, but not the trajectories themselves. An important question in symplectic topology is the existence of closed characteristics. We refer to Section 7 for more on this.

As the following Lemma will imply, the set of Hamiltonian maps make a group

LEMMA 3.37. *(Composition formulas)*

*(1) Let $\psi$ be a symplectic map and $H \in C^\infty([0,1] \times M, \mathbb{R})$. Set $K(t,x) = H(t,\psi(x))$. Then*

$$X_K(t,x) = d\psi(x)^{-1} X_H(t,\psi(x))$$

*(2) The flow $(\varphi_H^t)^{-1}$ is the flow associated to the Hamiltonian $\overline{H}(t,x)) = -H(t,\varphi_H^t(x))$*

*(3) If $\varphi_H^t, \varphi_K^t$ are associated to $H, K$ then $\varphi_H^t\varphi_K^t$ is associated to*

$$L(t,x) = H(t,x) + K(t,(\varphi_H^t)^{-1}(x))$$

PROOF. For (1) we write

$$dK(x) = dH(t,\varphi(x))d\varphi(x)v = \omega(\varphi(x))(X_H(t,\varphi(x)), d\varphi(x)v)$$

and since $d\varphi(x)$ is symplectic, this is equal to $\omega(x))(d\varphi(x)^{-1}X_H(t,\varphi(x)), v)$, which means that $X_K(t,x) = d\varphi(x)^{-1}X_H(t,\varphi(x))$.

For (3) we write

$$\frac{d}{dt}\left(\varphi_H^t\varphi_K^t(x)\right) = \left(\frac{d}{dt}\varphi_H\right)(\varphi_K^t(x)) + d\varphi_H(\varphi_K^t(x))\left(\frac{d}{dt}(\varphi_K^t(x)\right) =$$
$$X_H(t,\varphi_H^t\varphi_K^t(x)) + d\varphi_H(\varphi_K^t(x))X_K(t,\varphi_K^t(x)) = X_L(t,\varphi_H^t\varphi_K^t(x))$$

Setting $z = \varphi_H^t\varphi_K^t(x)$ this becomes

$$X_H(t,z) + d\varphi_H((\varphi_H^t)^{-1}(z))X_K(t,(\varphi_H^t)^{-1}(z))$$

and the second term, according to (1) is the Hamiltonian vector field associated to $K(t,(\varphi_H^t)^{-1}(x))$.

Now we may prove (1) since by (2) $\varphi_H^t\varphi_{\overline{H}}^t(x)$ is associated to the Hamiltonian

$$H(t,x) + \overline{H}(t,(\varphi_H^t)^{-1}(x)) = H(t,x) - H(t,(\varphi_H^t)^{-1}\varphi_H^t(x)) = 0$$

$\square$

PROPOSITION AND DEFINITION 3.38. *The set of all Hamiltonian maps is a normal subgroup of* $\mathrm{Diff}(M,\omega)$*, contained in the connected component of the identity. It is denoted by* $\mathrm{DHam}(M,\omega)$*. We denote by* $\mathrm{Ham}(M,\omega)$ *the set of Hamiltonian isotopies starting from* Id.

PROOF. This follows immediately from the above Lemma which implies that the inverse and composition of Hamiltonian maps is a Hamiltonian map. That the subgroup is normal follows from (1), since the flow of $d\psi(x)^{-1}X_H(t,\psi(x))$ is $\psi^{-1}\varphi_H^t\psi$. □

REMARK 3.39. Denote by $\mathrm{Diff}_0(M,\omega)$ the component of $\mathrm{Diff}(M,\omega)$ containing the identity. It's obvious that $\mathrm{DHam}(M,\omega) \subset \mathrm{Diff}(M,\omega)$. Since two different smooth functions yield different flows, we see that $\mathrm{DHam}(M,\omega)$ hence $\mathrm{Diff}_0(M,\omega)$ are pretty big groups (in particular they are infinite dimensional groups).

**Question 3.40.** How big is the quotient $\mathrm{Diff}_0(M,\omega)/\mathrm{DHam}(M,\omega)$?

In the general case we set

DEFINITION 3.41. *Let $(\varphi^t)_{t\in[0,1]}$ be a path in $\mathrm{Diff}_0(M,\omega)$ generated by the vector field $X_t$. We set*

$$\widetilde{\mathrm{Flux}}(\varphi^t)_{t\in[0,1]} = \int_0^1 i_{X_t}\omega\, dt \in H^1(M,\mathbb{R})$$

PROPOSITION 3.42. *The class $\widetilde{\mathrm{Flux}}(\varphi^t)_{t\in[0,1]}$ only depends on the homotopy class with fixed endpoints of $(\varphi^t)_{t\in[0,1]}$. Thus $\widetilde{\mathrm{Flux}}$ defines a morphism from the universal cover $\widetilde{\mathrm{Diff}_0(M,\omega)}$ of $\mathrm{Diff}_0(M,\omega)$ to $H^1(M,\mathbb{R})$. If $\Gamma(M,\omega)$ is the image of the set of closed loops in $\mathrm{Diff}_0(M,\omega)$ by $\mathrm{Flux}$, we have a morphism*

$$\mathrm{Flux}: \mathrm{Diff}_0(M,\omega) \longrightarrow H^1(M,\mathbb{R})/\Gamma(M,\omega)$$

*The same holds for the identity component of the set of compact supported symplectomorphisms, $\mathrm{Diff}_{c,0}(M,\omega)$ and we get a morphism*

$$\mathrm{Flux}_c : \mathrm{Diff}_{c,0}(M,\omega) \longrightarrow H_c^1(M,\mathbb{R})/\Gamma_c(M,\omega)$$

LEMMA 3.43. *Let $\gamma$ be a loop representing some class $[\gamma]$ in $\pi_1(M)$. Set $u(s,t) = \varphi^t(\gamma(s))$. Then*

$$\langle \mathrm{Flux}(\varphi^t), [\gamma]\rangle = \int_{S^1\times[0,1]} u^*\omega$$

*In other words, $\langle \mathrm{Flux}(\varphi^t), [\gamma]\rangle$ is the area fo the cylinder $u(S^1\times[0,1])$*

PROOF. Indeed,

$$\frac{d}{dt}\int_{S^1\times[0,t]} u^*\omega = \int_{S^1\times\{t\}} i_{\frac{\partial}{\partial t}} u^*\omega = \int_{S^1} \omega(X_t(\gamma(s)), \dot\gamma(s))\, ds$$

Integrating both sides fro $t=0$ to $t=1$, we get our Lemma. □

PROOF OF THE PROPOSITON. Clearly if we deform $(\varphi^t)_{t\in[0,1]}$ with fixed endpoints the cylinder is deformed but the boundary does not move. Stokes formula implies that the area of the cylinder does not change. Thus $\widetilde{\mathrm{Flux}}$ is defined on the universal cover $\widetilde{\mathrm{Diff}_0}(M,\omega)$ of $\mathrm{Diff}_0(M,\omega)$. it is a morphism, since the composition on $\widetilde{\mathrm{Diff}_0}(M,\omega)$ can be defined by concatenation. By definition, if $\Gamma(M,\omega)$ is the set of images of loops, $\widetilde{\mathrm{Flux}}$ descends to a map $\mathrm{Flux}: \mathrm{Diff}_0(M,\omega) \longrightarrow H^1(M,\mathbb{R})/\Gamma(M,\omega)$ □

To prove that $\widetilde{\mathrm{Flux}}(\varphi^t)_{t\in[0,1]}$ only depends on the homotopy class with fixed endpoints of $(\varphi^t)_{t\in[0,1]}$ it is enough to prove that if a loop of symplectic maps extends to a family parametrized by a disc, then the flux is zero.

We may now characterize the Hamiltonian maps among the symplectic diffeomorphisms :

PROPOSITION 3.44. *For $M$ a compact (resp. general) symplectic manifold, the subgroup* $\mathrm{DHam}(M,\omega)$ *(resp.* $\mathrm{DHam}_c(M,\omega)$ *coincides with the kernel of* Flux *(resp.* $\mathrm{Flux}_c$*). In particular if $H^1(M,\mathbb{R})=0$ (resp. $H^1_c(M,\mathbb{R})=0$), then* $\mathrm{DHam}(M,\omega)=\mathrm{Diff}(M,\omega)$ *(resp.* $\mathrm{DHam}_c(M,\omega)=\mathrm{Diff}_c(M,\omega)$

PROOF. See Exercise 31. □

REMARK 3.45. Note that the Flux is onto. Indeed if $\alpha$ represents a given class in $H^1(M,\mathbb{R})$, and $X$ is defined by $i_X\omega=\alpha$ then the time one flow of $X$ has Flux given by $[\alpha]$.

EXERCISE 3.46. Prove that if $\varphi^t$ is a path in $\mathrm{DHam}(M,\omega)$, it is defined as a Hamiltonian flow: $\varphi^t$ satisfies $\frac{d}{dt}\varphi^t = X_H(\varphi^t)$ for some function $H$ on $[0,1]\times M$.

HINT. *Let $X_s$ be the vector field defining $\varphi^s$, that is $\varphi^s$ is the time-one-flow of $X_s(t,x)$. Prove that $\alpha_t=i_{X_t}\alpha$ is closed. Use the fact that the Flux between $t$ and $s$ is exact for any pair $0\le t<s\le 1$ to conclude that $\alpha_t$ must be exact.*

We now assume $\omega=d\lambda$. Then there is an alternative definition of Flux as follows

PROPOSITION 3.47. *If $\omega$ is exact then* $\mathrm{Flux}(\varphi^t)_{t\in[0,1]})=[(\varphi^1)^*\lambda-\lambda]\in H^1(M,\mathbb{R})$. *In particular* $\Gamma(M,\omega)=\{0\}$

PROOF. Note that $\varphi^*\lambda-\lambda$ is closed for all $\varphi\in\mathrm{Diff}(M,\omega)$, since

$$d(\varphi^*\lambda-\lambda)=\varphi^*\omega-\omega=0.$$

If $\varphi^t$ is the flow of $X_t$ we have $i_{X_T}\omega=i_{X_t}d\lambda=L_{X_t}\lambda-d(i_{X_t}\lambda)$. Note also that for a closed form, the homotopy formula (Exercise 3.23 (3)) implies $(\varphi^t)^*\alpha=\alpha$ in cohomology.

So in cohomology we have

$$\int_0^1(i_{X_t}d\lambda)dt=\int_0^1 L_{X_t}\lambda dt=\int_0^1(\varphi^t)^*(L_{X_t}\lambda)dt\int_0^1\frac{d}{dt}[(\varphi^t)^*\lambda]dt=\varphi^*\lambda-\lambda$$

Since the Flux only depends on the endpoint of the path, it is zero for a loop, i.e. $\Gamma(M,\omega)=\{0\}$.

□

EXAMPLES 3.48.

(1) On $T^*T^1$ the translation $\varphi:(x,p)\longrightarrow(x,p+p_0)$ is symplectic, but $\mathrm{Flux}(\varphi)=p_0$, so $\varphi$ is not a Hamiltonian map.

(2) Similarly if $M = T^2$ and $\sigma = dx \wedge dy$, the map $(x, y) \longrightarrow (x, y + y_0)$ is not in $\mathrm{DHam}(T^2, \sigma)$ for $y_0 \not\equiv 0 \mod 1$.

Indeed, since the projection $\pi : T^*T^1 \longrightarrow T^2$ is a symplectic covering, any Hamiltonian isotopy on $T^2$ ending in $\varphi$ would lift to a Hamiltonian isotopy on $T^*T^1$ (if $H(t,z)$ is the Hamiltonian on $T^2$, $H(t,\pi(z))$ is the Hamiltonian on $T^*T^1$) ending to some lift of $\varphi$. But the lifts of $\varphi$ are given by $(x, y) \longrightarrow (x + m, y + y_0 + n)$ for $(m,n) \in \mathbb{Z}^2$, with Flux given by $y_0 + n \neq 0$.

Since $H^1(M,\mathbb{R})$ is abelian, this implies that in particular that $\mathrm{DHam}(M,\omega)$ contains the commutator subgroup of $\mathrm{Diff}_0(M,\omega)$. A difficult theorem by Banyaga [Ban78] states that this inclusion is in fact an equality and that the group $\mathrm{DHam}(M,\omega)$ is simple.

REMARK 3.49. The following notation is useful. First of all note that the Hamiltonian $H : [0,1] \times M \longrightarrow \mathbb{R}$ defines a path in $\mathrm{DHam}(M,\omega)$ starting from Id. The set of such paths will be denoted by $\mathscr{P}\mathrm{DHam}(M,\omega)$. An element in $\mathscr{P}\mathrm{DHam}(M,\omega)$ defines a Hamiltonian $H : [0,1] \times M \longrightarrow \mathbb{R}$, unique up to a constant shift, and an element in the universal cover $\widetilde{\mathrm{DHam}}(M,\omega)$ of $\mathrm{DHam}(M,\omega)$. So we have exact sequences

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Ham}(M,\omega) \longrightarrow \mathscr{P}\mathrm{DHam}(M,\omega) \longrightarrow 0$$

and

$$0 \longrightarrow \Omega\mathrm{DHam}(M,\omega) \longrightarrow \mathscr{P}\mathrm{DHam}(M,\omega) \longrightarrow \mathrm{DHam}(M,\omega) \longrightarrow 0$$

Note that in the compact supported version, $H$ is well-defined (we cannot add a constant as $H$ would not be compact supported anymore) so we get $\mathrm{Ham}_c(M,\omega) = \widetilde{\mathrm{DHam}}_c(M,\omega)$

A subtle fact is whether the subgroup $\mathrm{DHam}(M,\omega)$ is $C^1$-closed in $\mathrm{Diff}(M,\omega)$. This is the so-called $C^1$ -Flux conjecture. It was proved to hold if and only if $\Gamma(M,\omega)$ is a discrete subgroup of $H^1(M,\mathbb{R})$ and was solved by K. Ono ([Ono06]). The question of the $C^0$-flux conjecture, that is whether a $C^0$-limit of Hamiltonian maps is Hamiltonian, called the $C^0$-Flux conjecture is still open in most cases (see [LMP98] and [Buh14]).

Finally note that since $\omega^n$ is a volume form, an element preserving $\omega$ is volume preserving. We denote by $\mathrm{Diff}(M,\omega^n)$ the group of volume preserving diffeomorphisms. One of the main questions at the origin of symplectic topology is to understand the difference between $\mathrm{Diff}(M,\omega)$ and $\mathrm{Diff}(M,\omega^n)$.

## 5. Lagrangian and Hamiltonian dynamics

The Hamiltonian formulation of mechanics has its origin in the resolution of variational problems describing the time evolution of mechanical systems. We shall se that in turn Hamiltonian systems can be described as solutions of a variational problem, so that for one-parameter (i.e. describing the time evolution of finite dimensional quantities) variational and Hamiltonian systems are equivalent. Let us be more explicit. For $L$ a smooth function on $[0,1] \times TM$. We consider the quantity

$$\mathscr{L}(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t))\,dt$$

where $\gamma \in C^1([0,1],M)$. Existence of minimizers for $\mathscr{L}$ (under suitable assumptions on $L$) is the fundamental problem of the calculus of variations. The first examples are Newton's minimal resistance problem (find a solid of revolution having the least resistance as it moves in a fluid) and the more famous brachistochrone (find a curve connecting two points, such that a ball sliding without friction on the curve goes from $A$ to $B$ in the shortest time), around the end of the 17th century. We shall only deal with finding a critical point of $\mathscr{L}$, without investigating the conditions for being a minimum. Note that formally, $\gamma$ is a critical point if it satisfies the Euler-Lagrange equation. Indeed, to be a critical point, we need that for any smooth family $\gamma_\varepsilon(t)$ we have $\frac{d}{d\varepsilon}\mathscr{L}(\gamma_\varepsilon)=0$. This can be rewritten as

$$\frac{d}{d\varepsilon}\mathscr{L}(\gamma_\varepsilon)_{|\varepsilon=0} = \int_0^1 \left[\frac{\partial L}{\partial x}(t,\gamma(t),\dot\gamma(t))\frac{\partial \gamma}{\partial \varepsilon}_{|\varepsilon=0} + \frac{\partial L}{\partial v}(t,\gamma(t),\dot\gamma(t))\frac{\partial \dot\gamma}{\partial \varepsilon}_{|\varepsilon=0}\right]$$

Integration by part of the second term yields

$$\frac{d}{d\varepsilon}\mathscr{L}(\gamma_\varepsilon)_{|\varepsilon=0} = \int_0^1 \left[\frac{\partial L}{\partial x}(t,\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\frac{\partial L}{\partial v}(t,\gamma(t),\dot\gamma(t))\right]\frac{\partial \gamma}{\partial \varepsilon}_{|\varepsilon=0} + \\ \frac{\partial L}{\partial v}(1,\gamma(1),\dot\gamma(1))\frac{\partial \gamma}{\partial \varepsilon}(1)_{|\varepsilon=0} - \frac{\partial L}{\partial v}(0,\gamma(0),\dot\gamma(0))\frac{\partial \gamma}{\partial \varepsilon}(0)_{|\varepsilon=0}$$

Choosing a family with $\frac{\partial \gamma}{\partial \varepsilon}(t)_{|\varepsilon=0}$ arbitrary and vanishing for $t=0,1$ (check that this is always possible), we see that $[\frac{\partial L}{\partial x}(t,\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\frac{\partial L}{\partial v}(t,\gamma(t),\dot\gamma(t))$ is a continuous function which vanishes when integrated against any continuous function vanishing at the end points. This implies that

$$\frac{\partial L}{\partial x}(t,\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\frac{\partial L}{\partial v}(t,\gamma(t),\dot\gamma(t)) = 0 \tag{EL}$$

We shall assume $L$ is a Tonelli Lagrangian , that is

DEFINITION 3.50. *$L$ is a Tonelli Lagrangian if*

*(1) $L \in C^2(TM,\mathbb{R})$*
*(2) $L$ is strictly convex on $T_qM$ for all $q \in M$, that is $\frac{\partial^2 L}{\partial v^2}(q,v)$ is positive definite*
*(3) $L$ is superlinear in the fibers, that is $\lim_{|v|\to+\infty}\frac{L(t,q,v)}{|v|}=+\infty$*
*(4) The flow of the Euler-Lagrange equation* (EL) *is complete (i.e. solutions exist for all time)*

REMARK 3.51. Note that the last assumption is superfluous when $L$ does not depend on $t$. It is necessary in general (see [BM85]).

The equation (EL) is a second order ordinary differential operator. It can be reduced to a first order equation in many ways, however there is a preferred one. Set for $p \in T_x^*M$, $H_L(t,x,p) = \sup_{v\in T_XM}\left[\langle p,v\rangle - L(t,x,v)\right]$. The maximum is achieved at a single point, due to the strict convexity of $L$ in $v$, and this point satisfies $p=\frac{\partial L}{\partial v}(t,x,v)$. On the other hand $\frac{\partial H}{\partial p}(t,x,p) = v$, the point where the maximum is achieved. This proves that

$$p = \frac{\partial L}{\partial v}(t,x,v) \Leftrightarrow \frac{\partial H}{\partial p}(t,x,p) = v$$

that is the maps $p \mapsto \frac{\partial H}{\partial p}(t,x,p)$ and $v \mapsto \frac{\partial L}{\partial v}(t,x,v)$ are inverse to each other. Note that the variables $(t,x)$ play no active role here. Given a smooth strictly convex function $\ell : \mathbb{R}^n \longrightarrow \mathbb{R}$, we can define its Legendre dual, $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the maps $x \mapsto d\ell(x)$ and $p \mapsto dh(p)$ are inverse to each other. Said differently, the graph $\{(x, d\ell(x)) \mid x \in \mathbb{R}^n\}$ coincides with the graph $\{(dh(p), p) \mid p \in \mathbb{R}^n\}$.

We say that $h$ is the Legendre transform of $\ell$. Now equation (EL) becomes $\dot{p}(t) = \frac{\partial L}{\partial x}(t,x,v) = -\frac{\partial H}{\partial x}(t,x,p)$. Recall that $\dot{x} = v = \frac{\partial H}{\partial p}(t,x,p)$ so we get the first order equation

$$\begin{cases} \dot{p}(t) = -\frac{\partial H}{\partial q}(t,x,p) \\ \dot{q}(t) = \frac{\partial H}{\partial p}(t,x,p) \end{cases}$$

In other words the Euler-Lagrange equation is equivalent to the Hamiltonian equation for $H_L$.

EXAMPLES 3.52. (1) The motion of a particle in a potential force field is given by minimizing the integral of $T - V$ where $T$ is the kinetic energy, $V$ the potential energy. So $L(t,q,\dot{x}) = \frac{1}{2}|\dot{x}|^2 - V(q)$, and $H(q,p) = \frac{1}{2}|p|^2 + V(q)$. The equations of motion are $\dot{x} = p, \dot{p} = -\nabla V(x)$, that is equivalent to $\ddot{x} + V(x) = 0$.

(2) Let us consider a Riemannian metric $g(x)(v,v)$ that is a quadratic form in $v \in T_xM$, depending smoothly on $x \in M$. The metric yields a vector bundle isomorphism between $TM$ and $T^*M$. Thus any function on $TM$ yields a function on $T^*M$. In particular the metric itself yields a function $H_g(x,p)$ on $T^*M$. The flow of $H_g$ corresponds to the geodesic flow, that is the flow such that $T_t(x,v) = (y,w)$ if and only if the unique geodesic starting from $x$ with speed $v$ reaches $y$ after time $t$ and has speed $w$ in $y$.

## 6. The Poisson brackets viewpoint

Invariant subsets are important objects in the study of a dynamical system. For an autonomous Hamiltonian vector field, $X_H$ a regular hypersurface $\{f = 0\}$ is invariant if and only if $df(x)(X_H(x)) = 0$ on $\Sigma = f^{-1}(0)$ (we assume 0 is a regular value of $f$). This can be rewritten as $\omega(X_H, X_f) = 0$. We then set

DEFINITION 3.53. *The Poisson bracket of $F, G \in C^\infty(M,\mathbb{R})$ is defined as $\{F,G\} = \omega(X_F, X_G) = -dF(X_G) = dG(X_H)$. By a slight abuse of language for $F,G \in C^\infty(\mathbb{R}\times M,\mathbb{R})$ and denoting $F_t, G_t$ the restrictions of $F,G$ to $\{t\}\times M$ we set $\{F,G\}(t,x) = \{F_t, G_t\}(x)$.*

One obvious property is that $\{G,H\} = -\{H,G\}$. In fact the Poisson bracket induces a Lie algebra structure on $C^\infty(M,\mathbb{R})$.

PROPOSITION 3.54. *We have*

*(1)* $\{F,G\} = -\{G,F\}$
*(2)* $X_{\{F,G\}} = [X_F, X_G]$
*(3)* $\{F,G\}\omega^n = n dF \wedge dG \wedge \omega^{n-1}$
*(4)* *If one of the two functions F, G is compact supported, we have* $\int_M \{F,G\}\omega^n = 0$
*(5)* *In local symplectic coordinates* $q_j, p_j$ *we have*

$$\{F,G\}(q,p) = \sum_{j=1}^{n} \left( \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial G}{\partial q_j}\frac{\partial G}{\partial q_j} \right)(q,p)$$

*(6)* *(Leibniz identity)* $\{FG,H\} = F\{G,H\} + G\{F,H\}$
*(7)* *(Jacobi identity)* $\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$

PROOF. We already proved the first statement. For the second, we use the formula $L_X i_Y - i_Y L_X = i_{[X,Y]}$. Thus $i_{[X_F,X_G]} = L_{X_F} i_{X_G} - i_{X_G} L_{X_F}$. Apply this to $\omega$ and we get

$$i_{[X_F,X_G]}\omega = L_{X_F} i_{X_G}\omega - i_{X_G} L_{X_F}\omega = -L_{X_F} dG - i_{X_G} d i_{X_F}\omega - i_{X_G} i_{X_F} d\omega =$$
$$-d i_{X_F} dG + i_{X_F} ddG - i_{X_G} ddF = -d(\omega(X_F, X_G)) = d\{F,G\}$$

Statement ( 3) follows from

$$-i_{X_G}(dF \wedge \omega^n) = \{F,G\}\omega^n - n dF \wedge dG\omega^n$$

but since $dF \wedge \omega^n$ vanishes so does the right hand side. Then (4) follows by integration by parts, since $dF \wedge dG \wedge \omega^{n-1} = d(FdG \wedge \omega^{n-1})$. Formulas (5) and (6) are straightforward verification. The Jacobi identity (7) follows from (3)

$$0 = i_{X_F}\left(\{G,H\}\omega^n - n dG \wedge dH \wedge \omega^{n-1}\right) = -n\{G,H\}dF \wedge \omega^{n-1} - n dG(X_F) dH \wedge \omega^{n-1} +$$
$$n dH(X_F) dG \wedge \omega^{n-1} + n(n-1) dG \wedge dH \wedge dF \wedge \omega^{n-2} =$$
$$-n\{G,H\}dF \wedge \omega^{n-1} - n\{F,G\}dH \wedge \omega^{n-1} +$$
$$n\{F,H\}dG \wedge \omega^{n-1} + n(n-1) dG \wedge dH \wedge dF \wedge \omega^{n-2}$$

Taking the differential of the above expression, we get

$$0 = -d\{G,H\} \wedge dF \wedge \omega^{n-1} - d\{F,G\} \wedge dH \wedge \omega^{n-1} + d\{F,H\} \wedge dG \wedge \omega^{n-1} =$$
$$dF \wedge d\{G,H\}\omega^{n-1} + dH \wedge d\{F,G\}\omega^{n-1} + dG \wedge d\{H,F\}\omega^{n-1}$$

and using again (3) we get

$$\{F,\{G,H\}\} + \{H,\{F,G\}\} + \{G,\{H,F\}\} = 0$$

Jacobi's identity. ☐

REMARKS 3.55.

(1) One should be careful with the different sign conventions in the litterature.
(2) It follows form the above Proposition that the Poisson bracket induces a Lie algebra structure on $C^\infty(M,\mathbb{R})$. This naturally leads to one of the formulations of **quantization**, that is going from classical mechanics, as described by

Hamiltonian dynamical systems, to quantum mechanics, as described by operator theory. In an ideal (but unrealistic) world, this should be a Lie algebra morphism from the Lie algebra $(C^{\infty}(M,\mathbb{R}),\{\bullet,\bullet\})$ to the Lie algebra of self-adjoint operators of a Hilbert space $H$, that is $(Op(H),[\bullet,\bullet])$ where $[A,B]=\frac{1}{i\hbar}(AB-BA)$. Unfortunately such a morphism does not exist as soon as we want to satisfy some physically sound properties, but one can try to approximate this ideal situation, and approximate representations do exist.

The Lie algebra $(C^{\infty}(M,\mathbb{R},\{\bullet,\bullet\})$ is an example of a Poisson algebra

DEFINITION 3.56. *A Poisson algebra is an algebra $A$ toghether with a bilinear map $(X,Y)\mapsto[X,Y]$ satisfying the Jacobi and Leibniz identity.*

A nice feature of Poisson brackets is that it allows one to write Hamiltonian flows as transport equations. Indeed, if $\varphi_H^t$ is the flow of $H$, and $F$ is any function, then

PROPOSITION 3.57. *For $H$ a Hamiltonian on $(M,\omega)$ and $F$ a smooth function, the function $F_t(z)=F(\varphi_H^t(z))$ satisfies the equation*

$$\frac{d}{dt}F_t=\{H,F_t\}$$

PROOF. We have, setting $H_t(x)=H(t,x)$

$$\frac{d}{dt}F(\varphi_H^t(x))_{|t=t_0}=dF(\varphi_H^{t_0}(x))d\varphi_H^{t_0}(x)X_H(t,x)=dF_{t_0}(x)X_H(t_0,x)=\{H,F_t\}$$

□

In particular we see that $F$ is invariant by $\varphi_H^t$ if and only if $\{F,H\}=0$. We then say that $F$ is an **integral of motion** for the Hamiltonian $H$. We claim that if $F,G$ are integrals of motion for $H$, then so is $\{F,G\}$. Indeed, $\{\{F,G\},H\}=-\{\{G,H\},F\}-\{\{H,F\},G\}$, so if $\{G,H\}=\{F,H\}=0$ we have $\{\{F,G\},H\}=0$. In other words we proved the first statement of the following

PROPOSITION 3.58.

(1) *The set of functions invariant by the flow is a Poisson sub-algebra of $C^{\infty}(M,\mathbb{R})$*
(2) *The submanifold $C$ is coisotropic if and only if the set of functions vanishing on $C$ is a Poisson sub-algebra.*
(3) *A diffeomorphism $\varphi:(M,\omega_M)\longrightarrow(N,\omega_N)$ is symplectic if and only if for all functions $F,G$ we have $\{F\circ\varphi,G\circ\varphi\}=\{F,G\}\circ\varphi$*

PROOF. The first statement follows from the well-known that $\varphi_F^t,\varphi_G^s$ commute if and only if $[X_F,X_G]=0$, the formula $[X_F,X_G]=X_{\{F,G\}}$ and the fact that $X_H=0$ if and only if $H$ is constant. For the second statement assume first that $C$ is coisotropic. Then $H=0$ on $C$ implies that $T_zC\subset\mathrm{Ker}(dH(z))$ for all $z\in C$, hence $X_H(z)\subset(T_zC)^{\omega}\subset T_ZC$. So if $F,G$ vanish on $C$, we have $\{F,G\}=\omega(X_F,X_G)=0$ on $C$. Conversely, assume the set of functions vanishing on $C$ is a Poisson sub-algebra. Assume $T_zC$ is not coisotropic, so there exists $X \in T_zC^\omega \setminus T_zC$. Let us consider $H$ vanishing on $C$ such that $(\mathbb{R}X)^\omega = \ker dH(z)$. Since $T_zC \subset (\mathbb{R}X)^\omega$, this is possible. Now $X_H(z) = X$, and $F$ be a function vanishing on $C$ such that $dF(z)X \neq 0$(this is possible since $X \notin T_zC$). Then $\{F,H\} \neq 0$ which contradicts the assumption.

For a symplectic map, we have $\varphi_* X_{H\circ\varphi} = X_H$, so

$$\{F\circ\varphi, G\circ\varphi\} = \omega(\varphi^*(X_F), \varphi^*(X_G)) = (\varphi^*\omega)(\varphi^*(X_F), \varphi^*(X_G)) = \{F,G\}\circ\varphi$$

Now $\varphi$ is by definition a symplectomorphism from $(M, \varphi^*\omega_N)$ to $(N, \omega_N)$. Setting $\rho = \varphi^*\omega_M$ and $\{F,G\}_\rho$ for the Poisson bracket corresponding to $\rho$, we have $\{F\circ\varphi, G\circ\varphi\}_\rho = \{F,G\}_{\omega_M}\circ\varphi$. If we assume $\{F,G\}_{\omega_N}\circ\varphi = \{F\circ\varphi, G\circ\varphi\}_{\omega_M}$ we have $\{F\circ\varphi, G\circ\varphi\}_{\omega_M} = \{F\circ\varphi, G\circ\varphi\}_\rho$ for all functions $F,G$. Since $\varphi$ is a diffeomorphism, this is equivalent to stating that $\{F,G\}_\rho = \{F,G\}_{\omega_M}$ for all functions $F,G$. Then $dF(X_G^\rho) = dF(X_G^{\omega_M})$ where $X_G^\rho$ is the Hamiltonian for $\rho$. Since we can choose at each point $dF$ to be any linear form, this implies $X_G^\rho = X_G^{\omega_M}$ for all functions $G$, and this implies that the duality maps $T_zM \longrightarrow T_z^*M$ induced by $\rho$ and $\omega_M$ coincides, so that $\rho = \omega_M$, i.e. $\varphi^*(\omega_N) = \omega_M$. $\square$

EXERCISE 3.59. Prove that if a map $\varphi$ preserves the Poisson brackets then it is an immersion (i.e. $d\varphi(x)$ is injective for all $x$)

**6.1. Hamiltonian group action and moment maps.** Let $G$ be a Lie group action on $M$ by symplectic maps, that is for all $g$, the map $m \mapsto g\cdot m$ is symplectic. A special case is when this map is in fact Hamiltonian, so we have a map $g \mapsto H_g$ so that $g\cdot m = \varphi^1_{H_g}(m)$. Let now $X \in \mathfrak{g} = T_eG$ and $g_t = \exp(tX)$. Then we set $H_X = \lim_{t\to 0}\frac{1}{t}H_{\exp(tX)}$. This is not well defined, because $H_g$ is only defined up to a constant. In the compact case, we can get rid of the indeterminacy by assuming $\int_M H_X(m)dm = 0$ (if $M$ is not connected assume this for each connected component). In any case, if there is such a map, we can assume it is linear

DEFINITION 3.60. *(see [Sou97; Kos66]) Let $X \mapsto H_X$ be a linear map such that $\varphi^t_{H_X}(m) = \exp(tX)\cdot m$. We say that the action is Hamiltonian if it is a Lie algebra homomorphism, that is*

$$H_{[X,Y]} = \{H_X, H_Y\}$$

*The moment map is then defined as the map $\mu \in C^\infty(M, \mathfrak{g})$ given by $\langle \mu(x), X\rangle = H_X(x)$*

EXAMPLES 3.61. (1) Let $G$ be the action of the group of euclidean motions that is generated by rotations $SO(3)$ and translations. So $G$ is the cross product of $SO(3)$ and $\mathbb{R}^3$. Its Lie algebra is the product $\mathbb{R}^3\times\mathbb{R}^3$ but the structure is given by the exterior product of vectors on the first factor (i.e. $[u,v] = u\wedge v$) and trivial on the second factor. More precisely

$$[(u,x),(v,y)] = (u\wedge v, u\wedge y - v\wedge x)$$

The moment map is then given by the angular momentum and the linear momentum.

One should be careful: for the action to be Hamiltonian, it is not sufficient that each element acts by a Hamiltonian flow. If this is the case, we have a linear map $\mathfrak{g} \longrightarrow C^\infty(M,\mathbb{R})$, but we demand that this map be a Lie algebra morphism. Note that it may happen that while this is not a Lie algebra morphism, it becomes one by changing each $H_X$ by a constant : if there is a linear map $c : \mathfrak{g} \longrightarrow \mathbb{R}$ such that replacing $H_X$ by $H_X + c(X)$ we get a Lie algebra morphism, in other words $H_{[X,Y]} - \{H_X, H_Y\} = c([X,Y])$. In terms of Lie algebra cohomology, maps like $(X,Y) \mapsto c([X,Y])$ are $\delta c$ where $c$ is a 1-cycle. On the other hand the map $R : (X,Y) \mapsto H_{[X,Y]} - \{H_X, H_Y\}$ satisfies

$$R(X,[Y,Z]) + R(Y,[Z,X]) + R(Z,[X,Y])$$

thanks to Jacobi's identity, and thus [4] defines an element in $H^2(\mathfrak{g},\mathbb{R})$. As a result $R$ is a 2-cocycle and is well defined by the action modulo an element in $\delta C^1(\mathfrak{g})$, in other words $[R] \in H^2(\mathfrak{g})$. This is the obstruction to the existence of the moment map. We shall see more about this in Exercise 25 and Chapter 4, Section 2).

## 7. Contact geometry and Homogeneous symplectic geometry

Contact geometry is the study of hyperplane distributions $\xi$ on the manifold $M$ which are "maximally non-integrable". A hyperplane distribution, by definition attributes to each point a hyperplane of its tangent space, and this hyperplane varies smoothly with the point. In other words the hyperplane is locally defined by a 1-form $\alpha$ such that $\xi = \mathrm{Ker}(\alpha)$

DEFINITION 3.62. *A contact structure on the manifold $M$ is a hyperplane distribution $\xi$ defined locally as the kernel of a 1-form $\alpha$ such that $\alpha \wedge (d\alpha)^n$ is a volume form.*

Note that a contact manifold has dimension $\dim(M) = 2n+1$. The one-form $\alpha$ is not always defined globally, this is however the case if (and only if) the contact structure is co-orientable (i.e. there is a globally defined vector field transverse to $\xi$). If $\xi$ is defined by $\alpha$ the other one-forms defining $\xi$ are the $f\alpha$ with $f$ a non-vanishing function, so on each hyperplane $\xi$ there is a conformal symplectic structure[5] that is a symplectic structure well defined up to a positive constant factor. Indeed, if we replace $\alpha$ by $f\alpha$ we replace $d\alpha$ by $f d\alpha$. Note also that $d\alpha$ has rank $n$, and $\ker(d\alpha)$ is one-dimensional (this easily follows from Corollary 2.21).

DEFINITION 3.63. *Let $\alpha$ be a contact form. The unique vector field $R_\alpha$ such that $\alpha(R_\alpha) = 1$, $i_{R_\alpha} d\alpha = 0$ is called the Reeb vector field of $\alpha$.*

PROPOSITION 3.64. *A hyperplane distribution is a contact structure if and only if $X$ is a vector field defined in the neighbourhood of a point and tangent to $\xi$, we can find another vector field tangent to $\xi$ such that $[X,Y]$ is not tangent to $\xi$ at the point.*

---

[4] By definition, a Lie algebra cochain is an alternating map $c : \mathfrak{g}^p \longrightarrow \mathbb{R}$ and $\delta c(X_1,....,X_{p+1}) = \sum_{1\leq i<j\leq p+1}(-1)^j(-1)^{i+j} c([X_i,X_j],X_1,.,X_{i-1},X_{i+1},....,X_{j-1},X_{j+1},..,X_{p+1})$. The Lie algebra cohomology is, as usual, the space $\mathrm{Ker}(\delta)/\mathrm{Im}(\delta)$.

[5] We mean symplectic as a vector space.

PROOF. Indeed, we have

$$d\alpha(X,Y) = X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha([X,Y])$$

so if $X,Y$ are tangent to $\xi$ we get $d\alpha(X,Y) = -\alpha([X,Y]) \neq 0$, hence $d\alpha$ is symplectic. □

Note that the Reeb vector field is transverse to the contact hyperplanes. Moreover $L_{R_\alpha}\alpha = d(i_{R_\alpha}\alpha + i_{R_\alpha}d\alpha) = 0$. In other words the flow of $R_\alpha$ preserves the contact form. This is much stronger than preserving the contact structure which is given by the condition $L_X\alpha = f\alpha$. Note also that the Reeb vector field depends on the 1-form $\alpha$ and not only on $\xi$.

EXERCISES 3.65.
(1) Changing $\alpha$ to $f\alpha$ changes $R_\alpha$ to $\frac{1}{f}R_\alpha + \frac{1}{f^2}Y_f$ where $d\alpha(Y_f,Z) = df(Z)$ for all $Z$ in $\xi$
(2) In dimension 3 prove that it is enough that there exist locally a pair of vector fields $X,Y$ tangent to $\xi$ such that $[X,Y]$ is not tangent to $\xi$.

Contact geometry is very much related to symplectic geometry and many books are devoted to its study. Modern theory goes back to Bennequin's proof of the existence of exotic structures in $\mathbb{R}^3$ ([Ben83]) with many spectacular results by Eliashberg ([Eli91; Eli93; Eli90]), Giroux ([Gir91; Gir94]), etc. Note however that contact geometry has many features quite different from symplectic geometry.

EXAMPLES 3.66.
(1) Let $N$ be a smooth manifold and $J^1(N,\mathbb{R})$ be the set of triples $(x,p,z)$ where $p \in T^*_xN$, $z \in \mathbb{R}$. This can be identified with the set of 1-jets of functions from $N$ to $\mathbb{R}$ at $x$ where $p = df(x), z = f(x)$. By definition the 1-jet of a function at $x$ is its equivalence classe for the relation

$$f \simeq g \Leftrightarrow (f-g)\text{vanishes to second order}$$

Then $J^1(N,\mathbb{R}) = T^*N \times \mathbb{R}$ and $\alpha = dz - \lambda_N$ defines a 1-form whose kernel is a contact structure. Indeed in local coordinates, $(dz - pdq)\wedge(dp\wedge dq)^n = dz\wedge dq_1\wedge dp_1\wedge..\wedge dq_n\wedge dp_n$.
(2) Let us consider $PT^*N$ the projectivized cotangent bundle, or the bundle of contact elements of $N$ to be the quotient of $T^*N\setminus 0_N$ by the dilations $(x,p)\mapsto(x,t\cdot p)$. The vector field $X$ corresponding to this action is conformal, in other words it satisfies $L_X\lambda = \lambda$, so its flow satisfies $d_t^*\lambda = e^t\lambda$. As a result its kernel $\xi_N$ is invariant by the dilation, hence defines a hyperplane distribution on $PT^*N$ which is a contact structure.
(3) Let $ST^*N$ be the unit sphere bundle in $T^*N$, that is we fix a metric $g$ on $N$ and set for $p\in T^*_xN$, $|p|_g$ to be the norm of the linear form $p$ for the metric $g$. Then

$$ST^*N = \{(x,p) \mid |p|_g = 1\}$$

with the one form $\lambda_{N|ST^*N}$. Note that a priori the contact structure depends on the choice of the metric, but they will all be contactomorphic. One can see $ST^*N$ as the double cover of $PT^*N$. We could also have defined $ST^*N$

as the bundle of cooriented contact elements on $N$, i.e. the set of cooriented hyperplanes in $T_qN$.

(4) Let $(M,\omega)$ be a symplectic manifold with $[\omega] \in H^2(M,\mathbb{Z})$ and let $P$ be the $U(1) = S^1$ principal bundle having $c_1(P) = [\omega]$. This bundle has a connection corresponding to the 1-form $\alpha$ such that $d\alpha = \pi^*(\omega)$ ($\pi : P \longrightarrow M$ is the projection) and $\alpha(X_\theta) = 1$ where $X_\theta)$ is the generator of the $S^1$ action. This is called the **geometric prequantization** of $(M,\omega)$.

(5) Let $\Sigma$ be a smooth hypersurface in $\mathbb{C}^n$. We may assume it is defined by a smooth function $\varphi$ having 0 as a regular value:

$\Sigma = \{z \in \mathbb{C}^n \mid \varphi(z) = 0\}$ and its interior defined by $\varphi < 0$. Consider $\xi(z) = T_z\Sigma \cap \overline{T_z\Sigma}$ that is the only complex hyperplane contained in $T_z\Sigma$. One can show that $(\Sigma, \xi_\Sigma)$ is a contact manifold if and only if it is CR Levi convex, that is the Levi form $h = i\partial\overline{\partial}\varphi$ is positive definite on all complex subspaces. Here $\partial, \overline{\partial}$ are respectively the complex and anti-complex part of the exterior differential $d$ defined on complex valued differential forms see Exercice 48 for details and extensions in the almost complex case. Note that a function satisfying $i\partial\overline{\partial}\varphi > 0$ everywhere is called strictly plurisubharmonic. It is easy to show (Exercise 49) that a convex function is plurisubharmonic.

An important general class of contact structures is obtained as follows

DEFINITION 3.67. *Let $\Sigma$ be a smooth hypersurface in a symplectic manifold $(M,\omega)$. We say that $\Sigma$ is of* **contact type** *if there is a vector field $X$ defined in a neighbourhood of $\Sigma$ such that*

*(1) $L_X\omega = \omega$ (its flow satisfies $(\varphi^t)^*(\omega) = e^t\omega$ so $X$ is conformal)*
*(2) $X$ is transverse to $\Sigma$.*

This is equivalent to $\alpha = i_X\alpha$ satisfying $d\alpha = \omega$ and $\alpha$ restricts to a contact form on $\Sigma$. Indeed, $L_X\omega = d\alpha = \omega$ and $\alpha \wedge (d\alpha)^{n-1} = (i_X\omega)\wedge\omega^{n-1} = \frac{1}{n} i_X(\omega^n)$. But $\omega^n$ is a volume form if and only if $i_X\omega^n$ is a volume form on $\Sigma$.

EXERCISE 3.68. In the above list of examples, which ones are contact type hypersurfaces of a symplectic manifold ?

A remarkable result about contact structures is the stability theorem of Gray

THEOREM 3.69 (Gray's stability theorem). *The set of contact structures is open in the set of hyperplane distributions. Moreover two contact structures in the same connected component of the set of contact structures are diffeomorphic.*

PROOF. Indeed the condition $\alpha \wedge (d\alpha)^n \neq 0$ is obviously open. Now let $\alpha_t$ be a smooth family of contact forms. We look for $\varphi^t$ such that $(\varphi^t)^*(\alpha_t) = \exp(f_t)\alpha_0$. By taking the $t$ derivative, we get $(\varphi^t)^* L_{X_t}\alpha = \frac{d}{dt} f_t \exp(f_t)\alpha_0$. Using Cartan's formula, we get

$$(\varphi^t)^*[L_{X_t} d\alpha_t + \frac{d}{dt}\alpha_t] = \frac{d}{dt} f_t \exp(f_t)\alpha_0 = \frac{df_t}{dt}(\varphi^t)^*(\alpha_t)$$

that is equivalent to

$$d(i_{X_t}\alpha_t) + i_{X_t}d\alpha_t + \beta_t = h_t\alpha_t$$

This is equivalent to requiring that the left hand side restricted to the contact hyperlane $\xi_t = \mathrm{Ker}(\alpha_t)$ vanishes, that is

$$d(i_{X_t}\alpha_t) + i_{X_t}d\alpha_t + \beta_t = 0 \text{ on } \xi_t$$

Set $X_t \in \xi_t$ so that $i_{X_t}\alpha_t$ vanishes. Since the restriction of $d\alpha_t$ to $\xi_t$ is symplectic we can find $X_t$ solving $i_{X_t}d\alpha_t + \beta_t = 0$ on $\xi_t$, and we found our vector field hence our isotopy. □

The analogue of Lagrangian is given by Legendrians

DEFINITION 3.70. *A submanifold in a contact manifold* $(M^{2n+1},\xi)$ *is Legendrian if it is tangent to the contact structure and has dimension* $n$.

Note that there cannot be submanifolds $V$ tangent to the contact structure of dimension greater than $n$, since $d\alpha$ will also vanish on $V$ and has rank $2n$. We now show that contact structures are equivalent to homogeneous symplectic structures. Indeed,

DEFINITION 3.71. *A homogeneous symplectic manifold is a symplectic manifold* $(M,\omega)$ *endowed with a smooth proper and free action of* $(\mathbb{R},+)$*, such that denoting by* $X$ *the vector field associated to the action, we have* $L_X\omega = \omega$.

DEFINITION 3.72. *Let* $(M,\xi)$ *be a contact manifold with a coorientable contact structure* $\xi = \mathrm{Ker}(\alpha)$*. Then* $(M\times\mathbb{R}, d(e^t\alpha))$ *is a homogenous symplectic manifold called the symplectization of* $(M,\alpha)$.

Clearly the symplectization of a contact manifold is a homogeneous symplectic manifold with the homogenous map $\lambda\cdot(z,t) = (z,\lambda+t)$.

EXAMPLE 3.73. Let $M$ be a smooth manifold. We denote by $\mathring{T}^*M$ the manifold $T^*M\setminus 0_M$ endowed with the obvious action $\lambda\cdot(q,p) = (q, e^\lambda\cdot p)$. This is the symplectization of $ST^*M$.

PROPOSITION 3.74. *(Darboux for contact forms) Let* $\alpha$ *be a* $1$*-form on a neighbourhood of* $0$ *in* $\mathbb{R}^{2n-1}$ *such that* $\alpha\wedge(d\alpha)^{n-1}$ *does not vanish. Then we can find local coordinates such that* $\alpha = dz - \sum_{j=1}^{n-1} y_j dx_j$.

PROOF. We shall reduce this to the Darboux-Weinstein-Givental theorem (Theorem 3.29). First of all notice that near the origin in $\mathbb{R}^{2n+1}\times\mathbb{R}$ the form $\omega = d(e^t\alpha)$ is symplectic. Here $t$ is the coordinate corresponding to the last factor. Applying Darboux to $\omega$ shows that there is a local diffeomorphism $\varphi : (\mathbb{R}^{2n+1}\times\mathbb{R},0) \longrightarrow (\mathbb{R}^{2n},0)$ such that $\varphi^*(\sigma) = \omega$. We now consider the hypersurface $\Sigma$ given by $\varphi(\{t=0\})$ and apply the Darboux-Weinstein-Givental theorem to $(\Sigma,\sigma_{|\Sigma})$ to prove that there is a diffeomorphism $\psi : \Sigma \longrightarrow \mathbb{R}^{2n-1}$ such that $\psi^*(\sum_{j=1}^{n-1} dx_j\wedge dy_j) = d\alpha$. Since $\ker(\sigma_{|\Sigma})$ is one-dimensional and locally spanned by a non-vanishing vector field, the rectification theorem shows that we can first send the vector field $X$ generating $\ker(\sigma)$ to $\frac{\partial}{\partial t}$. Using that $\left(\alpha - \sum_{j=1}^{n-1} x_j dy_j\right)$ is closed hence locally exact, it is of the form $df$ for some function $f$, and $\alpha = df - \sum_{j=1}^{n-1} x_j dy_j$. It is then easy to see that $df$ is linearly independent from $dx_j, dy_j$ and we conclude our proof. □

The following is also consequence of the Darboux-Weinstein-Givental theorem :

PROPOSITION 3.75. *Let $\Sigma$ be a contact type hypersurface in a symplectic manifold $(M,\omega)$. Then there is a symplectomorphism from a neighbourhood of $\Sigma$ in $(M,\omega)$ to a neighbourhood of $\Sigma$ in its symplectization extending $\mathrm{Id}_\Sigma$. In other words a neighbourhood of $\Sigma$ in $M$ is symplectomorphic to $(\Sigma\times]-\varepsilon,\varepsilon[, d(e^t\alpha))$.*

PROOF. We have a map $T_\Sigma M \longrightarrow T_{\Sigma\times\{0\}}(\Sigma\times]-\varepsilon,\varepsilon[)$ that is the identity on $T\Sigma$ and sends $X$ such that $i_X\omega = \alpha$ to $\frac{\partial}{\partial t}$. One checks that this is a symplectic map. Indeed, this is obvious for pairs of vectors in $T\Sigma$, and we only have to check that

$$\omega(X,u) = \sigma(\frac{\partial}{\partial t}, u)$$

for $u \in T\Sigma$. But by assumption $i_X\omega = \alpha = i_{\frac{\partial}{\partial t}}\sigma$. □

As we mentioned in Section 4, Remark 3.36(2), a hypersurface in a symplectic manifold has a characteristic line distribution, and an important question is whether this distribution has closed trajectories. A famous conjecture due to Weinstein states

**Conjecture** (Weinstein). *If $\Sigma$ is a contact type hypersurface in $(M,\omega)$ then it has a closed characteristic.*

The conjecture has been proved in a number of cases (see [Vit87a; HV88; HV92; Tau07] and also [Vit99] for the case of cotangent bundles of simply connected manifolds), and counterexamples have been found if the contact-type condition is omitted (see [Gin95; Gin97; Her99] ). We shall prove it for $\mathbb{R}^{2n}$ in Chapter 7, Proposition 7.33.

There is also a contactization operation, in fact two of them. For details on the first Chern class we refer to [Mil74].

DEFINITION 3.76. *Let $(W, d\lambda)$ be an exact symplectic manifold. Then $W\times\mathbb{R}$ endowed with the one-form $\alpha = dz - \lambda$ is a contact manifold called the contactization of $(W, d\lambda)$. If $(W,\omega)$ is a symplectic manifold with $[\omega] \in H^2(W,\mathbb{Z})$ then there is a unique circle bundle over $W$ with Chern class $[\omega]$. This circle bundle has a connection, $\alpha$ that is $S^1$ invariant and defines a contact structure on $P$. Its Reeb vector field is the vector field generating the circle action. The contact manifold $(P,\alpha)$ is called a prequantization of $(W,\omega)$.*

The Chern-Weil theory in this elementary case tells us that $d\alpha = \pi^*(\omega)$. Two connections differ by a one form on the base, here the form will be closed, i.e. any two such connections differ by an element of $H^1(W,S^1) = \mathrm{Hom}(\pi_1(W), S^1)$ which vanishes if $W$ is simply connected. We shall see later why the name “prequantization”.

PROPOSITION 3.77 (Homogeneous symplectic geometry is contact geometry). *Let $(M,\omega)$ be a homogeneous symplectic manifold and $X$ be the generator of the action of $(\mathbb{R},+)$. Then $(M,\omega)$ is symplectomorphic (by a homogeneous map) to the symplectization of $(M/\mathbb{R}, i_X\omega)$*

PROOF. Consider the form $\alpha(\xi)=\omega(X,\xi)$ which is well defined on the quotient $\Sigma = M/\mathbb{R}$. This is a contact form on $\Sigma$, since

$$i_X\omega \wedge (d(i_X\omega))^{n-1} = i_X\omega\wedge(L_X\omega)^n = i_X\omega\wedge\omega^{n-1} = \frac{1}{n}i_X(\omega^n)$$

and since tangent vectors to $\Sigma$ are identified to tangent vectors to $M$ transverse to $\Sigma$, this does not vanish. Let $t$ be a coordinate on $M$ such that $dt(X)=1$, and $\widetilde{\omega} = d(e^t\pi^*(\alpha))$, then $(M,\omega)$ is equal to $(\Sigma\times\mathbb{R}, d(e^t\alpha))$. Indeed, let us consider two vectors, first of all in the case where one is $X$ and the other is in $dt(Y)=0$. Then $\widetilde{\omega}(X,Y) = e^t(dt\wedge\alpha + td\alpha)(X,Y) = e^t dt(X)\alpha(Y) = e^t(i_X\omega)(Y) = \omega(X,Y)$. Now assume $Y,Z$ are both in $\ker(dt)$. Then $\widetilde{\omega}(Y,Z) = dt\wedge t\alpha(Y,Z) + td\alpha(Y,Z)$ but $d\alpha = di_X\omega = \omega$ so that $\widetilde{\omega}(Y,Z) = \omega(Y,Z)$. ☐

EXERCISE 3.78. Prove that $\mathring{T}^*(N\times\mathbb{R})$ is symplectomorphic to $T^*N\times\mathbb{R}\times\mathbb{R}_+^*$, the symplectization of $J^1(N)$.

HINT. *Prove that the contact manifold $J^1(N,\mathbb{R})$ is contactomorphic to an open set of $ST^*(N\times\mathbb{R})$.*

EXERCISES 3.79.
(1) Prove that the above lift is functorial, that is the lift of $\Phi\circ\Psi$ is $\widetilde{\Phi}\circ\widetilde{\Psi}$.
(2) Let $\varphi : T^*M \to T^*M$ be an exact symplectic map, that is a map such that $\varphi^*(\lambda)-\lambda$ is exact. Prove that there is a lift of $\varphi$ to a contact map $\widetilde{\varphi} : J^1M \to J^1M$. Prove that if $(N,\alpha)$ is a contact manifold and $\psi$ a diffeomorphism of $N$ such that $\psi^*(\alpha)=\alpha$ (note that this is stronger than requiring that $\psi$ is a contact diffeomorphism, that is $\psi^*(\alpha) = f\cdot\alpha$ for some nonzero function $f$) then $\psi$ lifts in turn to a homogeneous symplectic map $(N\times\mathbb{R}_+^*, d(t\alpha))$ to itself.
(3) Prove that the symplectization of $J^1(M)$ is $T^*(M)\times\mathbb{R}\times\mathbb{R}_+^*$ and explicit the symplectomorphism obtained from the above $\widetilde{\varphi}$ by symplectization. Thus to any symplectomorphism $\varphi : T^*M\to T^*M$ we may associate a homogeneous symplectomorphism
$$\Phi : T^*(M\times\mathbb{R}_+^*) = T^*M\times\mathbb{R}_+\times\mathbb{R}^* \to T^*(M)\times\mathbb{R}\times\mathbb{R}_+^*$$
Prove that the lift is functorial. That is the lift of $\varphi\circ\psi$ is $\Phi\circ\Psi$.

There is an analog of Hamiltonian vector fields, these are contact Hamiltonian vector fields.

DEFINITION 3.80. *Let $\alpha$ be a contact form on $M$. Let $H\in C^\infty(M,\mathbb{R})$. The* **contact Hamiltonian** *associated to $H$ is the vector field $Z_H$ defined by*

$$i_{Z_H}\alpha = -H,\ \ i_{Z_H}d\alpha = dH - dH(R_\alpha)\alpha$$

Indeed if a flow preserves a contact structure, we must have $L_Z\alpha = f\alpha$ for some non-vanishing $f$. Rewriting it as $d(i_Z\alpha)+i_Z d\alpha = f\alpha$ we see that setting $H=-i_Z\alpha$ we must have $i_Z d\alpha = dH$ on $\xi$. Moroever $L_{Z_H}\alpha = f\alpha$ with $f = dH(R_\alpha)$.

This corresponds to the Hamiltonian $K(x,s)=e^s H(x)$ on $(\Sigma\times\mathbb{R}, d(e^s\alpha))$. Then

$$X_K = Z_H + dH(R_\alpha)\frac{\partial}{\partial s}$$

Conversely a homogeneous Hamiltonian on $(\Sigma\times\mathbb{R}, d(e^s\alpha))$ yields a contact Hamiltonian on $\Sigma$. Note that $i_{Z_H}d\alpha = dH$ on $\{\alpha=0\}=\xi$ and we have

PROPOSITION 3.81. *The contact Hamiltonian has the following properties*

*(1)* $\alpha(Z_H)=H$
*(2)* $L_{Z_H}\alpha = -dH(R_\alpha)\alpha$, *so the flow preserves the contact structure, but in general not the contact form (unless* $dH(R_\alpha)=0$*).*
*(3)* $L_{Z_H}H = -H^2$, *so* $H$ *is not preserved by the flow (even in the autonomous case), but the level* $H^{-1}(0)$ *is preserved.*

Note that the first property allows us to recover easily $H$ from $Z_H$. In local coordinates $(q,p,z)$ we have

$$Z_H(q,p,z) = \\ \sum_{j=1}^n \frac{\partial H}{\partial p_j}(q,p,z)\frac{\partial}{\partial q_j} - \sum_{j=1}^n\left(\frac{\partial H}{\partial q_j}(q,p,z) - p_j\frac{\partial H}{\partial z}(q,p,z)\right)\frac{\partial}{\partial p_j} - \\ \left(\sum_{j=1}^n p_j\frac{\partial H}{\partial p_j}(q,p,z) - H(q,p,z)\right)\frac{\partial}{\partial z}$$

The analog of the Poisson bracket is called the Lagrange bracket

DEFINITION 3.82. *Let* $F,G$ *be smooth functions on* $(M,\xi)$ *where* $\xi$ *is defined as the kernel of the one form* $\alpha$. *We set*

$$[F,G] = \alpha([Z_F,Z_G]) = dG(Z_F) - dF(R_\alpha)G$$

*We call* $[F,G]$ *the Lagrange brackets of* $F$ *and* $G$

One should be careful : this does not define a Poisson structure (see Exercice 20 for the definition of Poisson structure !

EXERCISE 3.83. Find the property of Poisson brackets that is not satisfied by the Lagrange bracket.

Note that if $K(x,s)=e^sF(x), L(x,s)=e^sG(x)$, then $X_K = Z_F + dF(R_\alpha)\frac{\partial}{\partial s}, X_L = Z_G + dG(R_\alpha)\frac{\partial}{\partial s}$, so

$$X_{\{K,L\}} = [X_K,X_L] = [Z_F,Z_G] + [dF(R_\alpha)\frac{\partial}{\partial s}, Z_G] - [dG(R_\alpha)\frac{\partial}{\partial s}, Z_F]$$

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We also have as an easy consequence of Proposition 3.81 (1)

PROPOSITION 3.84. *We have*

$$Z_{[F,G]} = [Z_F, Z_G]$$

As a result of Proposition 3.77 we have

PROPOSITION 3.85. *Let $(M, d\lambda)$ be an exact symplectic manifold and $L$ an exact Lagrangian submanifold. Then there is a unique -up to translation- Legendrian $\Lambda$ in the contactization of $M$ projecting on $L$. Let $(\Sigma, \xi)$ be a contact manifold and $\Lambda$ a Legendrian submanifold. Then $\Lambda \times \mathbb{R}$ is a Lagrangian in the symplectization of $(\Sigma, \xi)$ and any homogeneous Lagrangian is so obtained.*

PROOF. This is left to the reader. $\square$

COROLLARY 3.86. *An exact Lagrangian submanifold $L$ in $(M, \omega = d\lambda)$ has a unique lift (up to a constant translation) $\widehat{L}$ to the (homogeneous) symplectization of its contactization, $(\widehat{M}, \Omega) = (M \times \mathbb{R}_*^+ \times \mathbb{R}, dt \wedge d\tau - dt \wedge \lambda)$.*

PROOF. Indeed, let $f(z)$ be a primitive of $\lambda$ on $L$. Set $\widehat{L} = \{(z, t, \tau) \mid z \in L, \tau = f(z)\}$. Then, $d(td\tau - t\lambda)$ restricted to $\widehat{L}$ equals zero. $\square$

We finally characterize conical Lagrangians and Homogeneous Hamiltonians

PROPOSITION 3.87. *Let $L$ be an exact Lagrangian. Then $L$ is a conical (or homogeneous) Lagrangian in $T^*X$ if and only if $\lambda_{|L} = 0$.*

PROOF. Let $X$ be the homogeneous vector field, that is the vector fleld such that $i_X\omega = \lambda$. Then since for every vector $Y \in TL$ we have $\lambda(Y) = \omega(X, Y) = 0$ since both $X$ and $Y$ are tangent to $L$, we have $\lambda_L = 0$. Conversely if $\lambda_{|L} = 0$ we have for any $Y \in TL$ that $\omega(X, Y) = 0$. Then $X \in TL^\omega$, hence $X \in TL$ and this implies that $L$ is homogeneous. $\square$

PROPOSITION 3.88. *Let $(M, \omega)$ be a homogenous symplectic manifold and $X$ be the vector field dual to $\lambda$. Then $H$ is a homogenous Hamiltonian of degree one. if and only if $L_{X_H}\lambda = 0$. In other words $H$ is one-homogeneous if and only if it preserves $\lambda$.*

PROOF. Stating that $H$ is one-homogenous is equivalent to $i_Z dH = L_Z H = H$. But this means $i_Z i_{X_H}\omega = H$ so that $i_{X_H} i_Z \omega = -H$ and $L_{X_H}\lambda = d(i_{X_H}\lambda) + i_{X_H} d\lambda = 0$. But this implies $L_{X_H}\lambda = -dH + dH = 0$. Let us consider a Lagrangian $L$ in the symplectization of $(\Sigma, \alpha)$. If $L$ is homogeneous, then it is of the form $\Lambda \times \mathbb{R}$. But if on the contrary $L$ $\square$

## 8. Normalization

There are different possible normalizations. We can take $\omega = d\lambda$ or $\omega = -d\lambda$. Then we can define $X_H$ by $i_{X_H}\omega = dH$ or $i_{X_H}\omega = -dH$. Finally we can define $\{F, G\}$ as $dF(X_G)$ or as $-dF(X_G)$. We would like in any case that

(1) For the standard structure on $T^*N$ we have $X_H(q,p) = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$
(2) for the Poisson brackets $X_{\{F,G\}} = [X_F, X_G]$

Our choice is $d\lambda = \omega$, $i_{X_H}\omega = -dH$ and $\{F,G\} = -dF(X_G)$. This is the same choice as in [GS77; HZ94] but opposite choice from [MS07].

## 9. Appendix: Some properties of Symplectic dynamics

**9.1. Basic properties of Hamiltonian dynamics.** Since a symplectic map is volume preserving, it satisfies two important properties. On a manifold a volume form is a smooth form $\Omega$ of top degree without 0 and its volume is $\int_M \Omega$.

PROPOSITION 3.89 (Poincaré recurrence). *Let $\varphi$ be a volume preserving flow of a domain with finite volume. Let $U$ be a set of positive measure. Then for almost all points $x$ in $U$ the orbit $\{\varphi^n(x), n \geq 0\}$ returns infinitely many times in $U$.*

PROOF. Consider the sets $V_j \subset U$ of points that do not return in $U$ before time $j$. This implies that $V_j, \varphi^1(V_j), ...\varphi^j(V_j)$ are disjoint, otherwise we would have $\varphi^p(V_j) \cap \varphi^q(V_j) \neq \emptyset$ for $0 < p < q$ and this implies $V_j \cap \varphi^{q-p}(V_j) \neq \emptyset$ a contradiction. Thus $\mu(V_j) \leq \mu(M)/j$ and hence $\mu(\bigcap_{j\geq 1} V_j) = 0$. In other words $U_1 = U \setminus \bigcap_j V_j$, the set of points in $U$ that eventually (in the future!) return in $U$ has full measure in $U$. Then we can do the same starting from $U_1$, noticing that if a point $x$ in $U_1$ is returning to $U_1$, then $T^{j_1}(x) \in U_1$ for some $j$, but by assumption $T^j(x)$ returns to $U$, so there is a positive $j_2$ such that $T^{j_2}(T^{j_1}(x) \in U$ i.e. $T^{j_1}(x), T^{j_1+j_2}(x) \in U$ and the orbit of $x$ meets $U$ twice. We denote by $U_2$ this set. By induction we see that points in $U_k$ return at least $k$ times in $U$ and $U_k$ is defined as the set of points in $U_{k-1}$ returning to $U_{k-1}$. We clearly have a decreasing sequence $U_k \subset U_{k-1} \subset ... \subset U_1 \subset U$ and since all $U_k$ have full measure in $U$, we get that $U_\infty = \bigcap_k U_k$ has full measure as announced. $\square$

REMARK 3.90. Poincaré's paradox states that if you put a lump of sugar in a glass of water and wait long enough, then the lump will form it self again. The paradox lies in the fact that "long enough" is longer than the age of the universe.

A second important property of volume preserving maps is an averaging property. We here assume $(M,\mu)$ is a probability space, so for example $(M,\Omega)$ is a manifold with a volume form of finite integral, we set $\mu(U) = \frac{\int_U \Omega}{\int_M \Omega}$.

**9.2. Ergodic properties.**

THEOREM 3.91. *(Kingman's subadditive theorem) Let $(f_n)_{n\geq 1}$ be a sequence in $L^1(M,\mu)$ and $T : M \longrightarrow M$ a measure preserving map such that*

$$f_{n+m}(x) \leq f_m(x) + f_n(T^m(x))$$

*then the sequence $(\frac{1}{n} f_n)_{n\geq 1}$ converges almost everywhere to a $T$-invariant[6] function $\overline{f}$.*

[6] i.e. such that $\overline{f} \circ T = \overline{f}$

This has the following consequence

THEOREM 3.92 (Birkhoff's ergodic theorem). *Let $\varphi^t$ be an autonomous flow preserving a probability measure $\mu$ on $M$. Then for any function $f$ in $L^1(d\mu)$, we set $S_n f(x) = \frac{1}{n}\sum_{j=0}^{n} f(\varphi^n(x))$. Then $S_n f(x)$ converges for $\mu$-almost every point $x$ to $\overline{f}$ which is $\varphi^1$ invariant and such that*

$$\int_M \overline{f}(x)\,d\mu(x) = \int_M f(x)\,d\mu(x)$$

*In particular if the flow is ergodic (i.e. the only sets which are invariant have measure $0$ or full measure) the function $\overline{f}$ is constant almost everywhere.*

We shall not prove these theorem here, several proofs can be found in classical papers or textbooks (see[Ste89], [HK02] p. 66 theorem 3.5.2, [Pet83])

**9.3. Chaos and hyperbolic dynamics.** Finally let $\varphi$ be a symplectic diffeomorphism. Then $d\varphi(x) : (T_x M, \omega(x)) \longrightarrow (T_{\varphi(x)} M, \omega(\varphi(x)))$ is a symplectic map. In particular if we have a fixed point $x$ of $\varphi$, we shall say that it is hyperbolic if the eigenvalues of $d\varphi(x)$ are not on the unit circle. In this situation, there are stable and unstable manifolds which are invariant by the map and tangent to $\mathbb{R}^{2n} \cap \bigoplus_{|\mu|>1} E_\mu$ and $\mathbb{R}^{2n} \cap \bigoplus_{|\mu|<1} E_\mu$, and these spaces are isotropic hence Lagrangian.

PROPOSITION 3.93. *If $x$ is a hyperbolic fixed point of a symplectic map, then $W^u(x)$, $W^s(x)$ are both immersed Lagrangian submanifolds.*

PROOF. It is again classical (see [HK02], p.130)that $W^s(x)$ (resp. $W^u(x)$) is immersed and its tangent space is the sum of the eigenspaces of $d\varphi(x)$ with eigenvalues $< 1$ (resp. $> 1$). Moreover for $z \in W^s(x), \xi \in T_z W^s(z)$, we have $d\varphi^n(z)\xi \longrightarrow 0$ as $n$ goes to $\infty$ and since $d\varphi(z)$ is symplectic,

$$\omega(z)(\xi,\eta) = \omega(\varphi^n(z))(d\varphi^n(z)\xi, d\varphi^n(z)\eta)) \longrightarrow 0$$

so $W^s(x)$ is isotropic. Changing $\varphi$ to $\varphi^{-1}$ exchanges the roles of $W^s(x)$ and $W^u(x)$. Since at $x$ we have $T_x M = T_x W^u(x) \oplus T_x W^s(x)$ they must be Lagrangians. □

Note that the intersections of these Lagrangians are extremely important and useful to detect the chaotic aspect of dynamics. In case they intersect transversely, this gives rise to the "Smale horseshoe" and chaotic dynamics (see [Sma67]). There is also a variational approach, that does not require the transversality (see [CES90; Sér93]).

**9.4. KAM theorem.** Let us consider an integrable system that for now means a Hamiltonian on $T^n \times \mathbb{R}^n$ given by $H(\theta, I) = h(I)$. The flow is then given by $\dot{\theta} = \nabla h(I), \dot{I} = 0$. The solutions are then given by straight lines winding on the torus : $\theta(t) = \theta(0) + t\nabla h(I(0)), I(t) = I(0)$, in particular the tori $I = I_0$ are thus invariant. Poincaré thought that once this is perturbed, the structure of the invariant tori disappears. This is not so, as was discovered by Kolmogorov and then Arnold and Moser (see [Kol54; Arn63;

Mos62]) the first two in the analytic setting, the third one in the smooth case. More precisely as a simple statement of the so called KAM theorem we have

THEOREM 3.94 (see [Bos86]). *Let $H_0(\theta, I) = h_0(I)$ a Hamiltonian such that for $I = I_0$ we have $D^2h(I_0)$ is non-degnerate. Assume that $\alpha = (\alpha_1, ..., \alpha_n) = \nabla h(I_0)$ satisfies the following Diophantine property*

$$\forall k \in \mathbb{Z}^n \setminus \{0\}, \quad \left|\sum_{i=1}^N k_i\alpha_i\right| \geq c\left(\sum_{i=1}^N |k_i|\right)^{-\tau}$$

*Then there is a neighbourhood of $C^\infty(T^n \times \mathbb{R}^n)$ such that for any $H$ in this neighbourhood, there is a smooth Lagrangian $n$-dimensional torus $\mathcal{T}_H$ invariant by the flow of $X_H$ and such that the restriction of $X_H$ to $\mathcal{T}_H$ is smoothly conjugate to $R_\alpha$ defined by $R_\alpha(\theta) = \theta + t\alpha$. Moroever the map $H \mapsto \mathcal{T}_H$ is continuous.*

Note that one may reduce to finite differentiability (in [Mos62], to $C^{333}$) but not below $C^3$ (see [Rüs83; Her83]). Note that in dimension 4 the flow remains on an energy surface and the invariant tori are codimension 1. They separate the space in open sets which are therefore invariants. The flow is then not ergodic.

**9.5. Periods for small autonomous vector fields.** As we shall see, fixed points or periodic orbits are an extremely important feature of Hamiltonian dynamics. If $H(x)$ is a Hamiltonian on a symplectic manifold $(M, \omega)$ its time 1 periodic orbits are on one hand the fixed points of the flow, that is the critical points of $H$, and on the other hand the non-constant periodic orbits. When $H$ is $C^2$ small, only the former exist.

PROPOSITION 3.95. *Let $M$ be a manifold and $U$ a bounded domain in $M$. there exists a constant $C$ such that for any an autonomous vector field $X$ supported in $U$ with $\|X\|_{C^1} \leq C$, then $X$ has no non-constant periodic orbit of period $T \leq 1$.*

PROOF. Indeed, it is enough to prove this in a neighborhood of 0 in $\mathbb{R}^n$ since for $T$ small enough $x(t)$ remains in a chart domain. Consider $x(t)$ a non-constant periodic orbit. Set $v(t) = \frac{\dot{x}(t)}{|x(t)|}$ which is well defined since $\dot{x}(t) = X(x(t))$ does not vanish. Then $v$ has values on the unit sphere and is not contained in any open hemisphere, since this would mean that there is some $h \in S^{n-1}$ such that $\langle h, v(t)\rangle > 0$ for all $t$, hence $\langle \dot{x}(t), h\rangle > 0$, but then $\langle x(T) - x(0), h\rangle = \int_0^T \langle \dot{x}(t), h\rangle dt > 0$ contradicting the assumption $x(T) = x(0)$. Now according to the Cauchy-Crofton formula (see Exercise 37 or 38, or [Cro68; San04]) we have that such a curve has length at least $2\pi$. Now

$$\dot{v}(t) = \frac{\ddot{x}(t)|\dot{x}(t)|^2 - \langle \ddot{x}(t), \dot{x}(t)\rangle \dot{x}(t)}{|\dot{x}(t)|^3}$$

so

$$|\dot{v}(t)| \leq 2\frac{|\ddot{x}(t)|}{|\dot{x}(t)|} \leq 2|DX(x(t))|$$

since $\ddot{x}(t) = DX(x(t))\dot{x}(t)$. Therefore the length of the curve $v$ is at most $2CT$ and we must have $2CT \geq 2\pi$. So if $C < \pi$ non-constant orbits of period $T \leq 1$ cannot exist. □

COROLLARY 3.96. *Let $H$ be a Hamiltonian on a compact symplectic manifold $(M,\omega)$ such that the flow of $X_H$ is defined for all $t$ in $[0,T]$. then there exists a constant C such that if $\|D^2H(x)\| \leq C_T$ all periodic orbits of period less than $T$ for $X_H$ are constant.*

PROOF. Just apply the Proposition to $X_H$. $\square$

## 10. Exercises and Problems

### 10.1. Symplectic manifolds.

(1) Prove that $S^{2n}$ has no symplectic structure for $n>1$.
(2) Prove that in dimension $>2$ if $\omega$ is symplectic, then $f\omega$ is symplectic, for $f \in C^\infty(M,\mathbb{R})$ if and only if $f$ is constant. What happens in dimension 2 ?
(3) Prove that $\mathbb{R}^{2n}$ has no compact symplectic submanifold. Prove that $\mathbb{C}^n$ has no compact complex submanifold.
(4) (Archimedes's theorem) Consider the unit sphere $S^2$ represented as $\{(x,y,z) \mid x^2+y^2+z^2=1\}$ with the standard Fubini-Study metric
   (a) Write down the Fubini-Study form in the coordinates $(x,y,z)$
   (b) Consider the cylinder

$$C=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1, -1\leq z\leq 1\}$$

   with the symplectic form $\sigma=\frac{1}{2}(xdy-ydx)\wedge dz$. Consider the "horizontal projection" of $S^2$ to $C$ given by $f:(x,y,z)\mapsto(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2},z)$. Prove that $f$ is symplectic (a slight abuse of language since $f$ is not defined at the north and south poles: justify why the rest of the exercise makes sense).
   (c) Compute the area of the sphere [7]
(5) (Alternative proof of Darboux) We consider a germ $\omega$ of symplectic form near 0 in $\mathbb{R}^{2n}$.
   (a) Let us consider a function $f$ without critical points near 0. Prove that there are local coordinates $(x_1,....,x_{2n})$ such that $X_f=\frac{\partial}{\partial x_n}$ (use the rectification theorem in any book on dynamical systems e.g. [HS74], [Arn97]).
   (b) Prove that $\{f,x_n\}=1$
   (c) Prove that if we consider $\Sigma=\{f=x_n=0\}$ it is a germ of submanifold of codimension 2, and that $\omega_{|\Sigma}$ is symplectic
   (d) Prove that we can write $\Sigma\times\mathbb{R}^2$ and $\omega=\omega_{|\Sigma}\oplus df\wedge dx_n$
   (e) Prove Darboux's theorem by induction
   (f) Prove the Caratheodory-Jacobi-Lie theorem : given $r$ functions $f_1,...,f_r$ in a neighborhood of the point $x_0$ in the symplectic manifold $(M,\omega)$ such that the $df_j(x)$ are linearly independent and $\{f_i,f_j\}=0$, we can find functions $f_{r+1},..,f_n,g_1,...,g_n$ defined near $x_0$ such that $f_1,...,f_n,g_1,..g_n$ are symplectic local coordinates near $x$. In other words the $df_i(x),dg_j(x)$ are linearly independent and $\{f_i,f_j\}=\{g_i,g_j\}=0,\{f_i,g_j\}=\delta_i^j$.

[7]This was discovered, of course without the symplectic terminology, by Archimedes. According to Plutarch and Cicero, Archimedes requested that the sphere and cylinder be carved on his tombstone (however according to Plutarch, the reference was to the ratio of the volumes enclosed by the sphere and cylinder, being in the ratio of two-thirds, not on their areas). See Plutarch, Marcellus 17.7 and Cicero, Tusculanae Disputationes V, XXIII, 64,65.

(6) (Local structure of coisotropic submanifolds) Prove that if $C$ is a coisotropic submanifold, for each point we can find local coordinates $(q^1,...,q^n,p_1,..,p_n)$ such that $C=\{(q^1,...,q^n,p_1,..,p_n)\mid p_{k+1}=..=p_n=0\}$.

(7) Use the Darboux-Weinstein-Givental theorem to prove that all closed curves have symplectomorphic neighborhoods.

HINT. *Show that all symplectic vector bundles on the circle are trivial.*

(8) Prove that if $\omega_1,\omega_2$ are symplectic forms which are compatible with the same almost complex structure, then if they are cohomologous, they are symplectomorphic.

(9) Let $n\geq 2$ and $\varphi:(M_1^{2n},\omega_1)\longrightarrow(M_2^{2n},\omega_2)$ be a map such that $\varphi^*(\omega_2)=f\cdot\omega_1$ where $f\in C^\infty(M_1,\mathbb{R})$. Prove that $f$ is constant, so that $\varphi$ is conformal.

(10) Let $n\geq 2$ and $\varphi:(M_1^{2n},\omega_1)\longrightarrow(M_2^{2n},\omega_2)$ be a map such that the image of any Lagrangian submanifold is a Lagrangian submanifold. Prove that $\varphi$ is conformal (i.e. $\varphi^*\omega_2=c\omega_1$ for some nonzero constant $c$)

HINT. *Use Exercice 14 from Chapter 14 in Chapter 2 and the fact that for each Lagrangian subspace $V\subset T_xM_1$ we can find a smooth Lagrangian $L$ such that $T_xL=V$.*

(11) (see [GLS09]) Let $E\xrightarrow{p}B$ be a symplectic fiber bundle with fiber $F$ and base $B$, that is there are charts over open sets of a covering $U_i$ such that $\varphi_i:p^{-1}(U_i)\longrightarrow U_i\times F$ and the coordinate changes $\varphi_i\circ\varphi_j^{-1}:(U_i\cap U_j)\times F\longrightarrow(U_i\cap U_j)\times F$ are of the type $(x,f)\mapsto(x,g_{i,j}(x)(f))$ where $g_{i,j}(x)\in Diff_\omega(F)$.
   (a) Prove that the bundle $b\mapsto H^2(F_b)$ is naturally endowed with a flat connection (use the lattice $H^2(F_b,\mathbb{Z})\subset H^2(F_b,\mathbb{Z})$ or look-up "Gauss-Manin connection").
   (b) Assume $F$ is compact and let $(\omega_b)_{b\in B}$ be a family of symplectic forms varying smoothly on $B$. Prove that it defines a symplectic fibration if and only if $[\omega_b]$ defines a flat section for the Gauss-Manin connection.
   (c) Prove that there exists a 2-form $\sigma$ on $E$ such that the restriction of $\sigma$ to a fiber is $\omega_b$ (we do not claim $\sigma$ is closed !).
   (d) Prove that $(T_fF)^\omega$ defines a connection on $E$ and this connection is symplectic (i.e. the holonomy map is symplectic).

(12) (Moser with boundary-volume case) We consider a family $\Omega_t$ of volume forms on the compact manifold $M$ with smooth boundary $\partial M$. We assume $\int_M\Omega_t$ is constant and want to prove that there is a smooth flow $\varphi^t$ such that $(\varphi^t)^*\Omega_t=\Omega_0$.
   (a) Prove that this can be reduced to finding a vector field $X_t$ tangent to $\partial M$ such that $d(i_{X_t}\Omega_t)+\frac{d}{dt}\Omega_t=0$ where with $X_t$ tangent to $\partial M$.
   (b) Prove that this is possible provided there is a form $\Xi_t$ vanishing on $\partial M$ such that $\frac{d}{dt}\Omega_t=d\Xi_t$.

(c) Prove that $\frac{d}{dt}\Omega_t$ is exact, and the existence of $\Xi_t$ follows from the Poincaré Lemma (Lemma 3.22).

(13) (Moser with boundary-symplectic case) We want to prove the analogue of the above Theorem for the case of a symplectic form, i.e. we consider a family $\omega_t$ of symplectic forms on the compact manifold $M$ with smooth boundary $\partial M$. We would like to prove, under some cohomological assumption, that there is a smooth flow $\varphi^t$ such that $(\varphi^t)^*\omega_t = \omega_0$.

(a) Prove that for such a theorem to hold, we must assume the characteristic foliation of $\omega_t$ on $\partial M$ (see Remark 3.36 (2)) does not depend on $t$.

(b) From now on we assume there is a symplectic map $\psi^t : \partial M \longrightarrow \partial M$ such that $(\psi^t)^*(\omega_t)_{|\partial M} = \omega_{0|\partial M}$.

(c) Prove that $\psi^t$ extends to an isotopy (still denoted $\psi^t$) such that $(\psi^t)^*(\omega_t) = \omega_0$ in a neighborhood of $\partial M$.

(d) Prove that it is sufficient to prove the existence of $\varphi^t$ when $\omega_t = \omega_0$ near $\partial M$ to conclude the existence of $\varphi^t$ when $[\omega_t - \omega_0] = 0$ in $H^2(M, \partial M)$.

(14) Consider the action of $Sp(2n,\mathbb{R})$ on $\Lambda(n)$ given by $R \cdot L \mapsto R(L)$. Let us consider a Lagrangian complement $H$ to $L$, so that a Lagrangian near $L$ is identified to a graph of a quadratic form on $L$. We denote by $\Phi_{L,H}$ this map.

(a) Prove that if $R(t)$ is a smooth path in $Sp(2n)$ starting from the identity, such that $\frac{d}{dt}R(t) + JA(t)R(t) = 0$ then $\frac{d}{dt}\Phi_{L,H}(R(t)L)_{|t=0}$ is the restriction of the quadratic form $A(0)$ to $L$. In other words it is also equal to $\omega\left(\frac{d}{dt}R(t)_{|t=0}x, x\right)$

(b) Prove that if $L, T$ are two Lagrangian and $I = L \cap T$ and $R(t)$ is as above, then $\frac{d}{dt}\Phi_{L,H}(R(t)L)_{|t=0} = \frac{d}{dt}\Phi_{T,H}(R(t)L)_{|t=0}$ on $I$.

### 10.2. Poisson structures.

(15) Let $\varphi, \psi$ be two real-valued smooth functions defined on $\mathbb{R}^2$. We set $J(\varphi,\psi)$ to be the determinant of the matrix

$$\begin{pmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\psi}{\partial x} \\ \frac{\partial\varphi}{\partial y} & \frac{\partial\psi}{\partial y} \end{pmatrix}$$

Prove that if $F, G$ are functions defined on a symplectic manifold then $\{\varphi(F,G), \psi(F,G)\} = J(\varphi,\psi)\{F,G\}$.

HINT. *One can compute directly. Alternatively use that* $(\varphi,\psi)$ *sends* $dx \wedge dy$ *to* $J(\varphi,\psi)\,dx \wedge dy$.

(16) Let $(M,\omega)$ be a symplectic manifold and $P$ be defined by $F_1(z) = \ldots = F_{2r}(z) = 0$. Prove that $P$ is symplectic if and only if

$$\det(\{F_j, F_k\}) \neq 0$$

(17) Let $(P,\omega_{|P})$ be a symplectic submanifold of $(M,\omega)$. Let $H$ be a smooth function on $M$ and $X_H$ be the Hamiltonian vector field. Let $K = H_{|P}$ be the restriction of $H$ to $P$.
  (a) Prove that $X_K(z)$ is the projection of $X_H(z)$ on $T_zM$ in the direction of $T_zP^\omega$
  (b) Prove that there are smooth functions $\lambda_1,\dots,\lambda_{2r}$ such that setting $\widetilde{H}(z) = H(z) - \sum_{j=1}^{2r}\lambda_j(z)F_j(z)$ we have $X_K = X_{\widetilde{H}}$ on $P$.
  (c) Assume now that $\{F_j,F_k\} = \delta_{j,k-r}$ for $1 \le j \le k \le 2r$ and $\{F_j,H\} = 0$ for $1 \le j \le r$. Prove that we can take

$$\widetilde{H}(z) = H(z) - \sum_{j=1}^{r}\lambda_j F_j(z)$$

(18) (Jacobi identity for vector fields) Let $X$ be a vector field. We define $D_X : C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$ to be the map $D_X f = i_X df$. We call derivation a map $D : C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$ , that is $D(fg) = gDf + fDg$.
  (a) Prove that for a derivation $D(1) = 0$
  (b) Prove that $D_X$ is a derivation.
  (c) Prove that any derivation is of the form $D_X$ for some vector field $X$. Hint: prove that this is a local result. Then use a chart and Hadamard's lemma (a direct consequence of Taylor's theorem with integral form remainder) stating that any smooth function can be written locally as

$$f(x) = f(x_0) + \sum_{j=1}^{n}(x^j - x_0^j)f_j(x)$$

where the $f_j$ are smooth functions and $f_j(x_0) = \frac{\partial f}{\partial x_j}(x_0)$.
  (d) Prove that the commutator of two derivations is a derivation
  (e) Define $[X,Y]$ by $D_{[X,Y]} = D_XD_Y - D_YD_X$. Prove that this coincides with the other definitions of the Lie bracket of vector fields you have seen (or will find in other textbooks).
  (f) Prove that the Jacobi identity

$$[F,[G,H]] + [G,[H,F]] + [H,[F,G]] = 0$$

holds.

(19) (A silly proof of the Jacobi formula for vector fields) Let $X,Y,Z$ be three vector fields on $N$.
  (a) Prove that $H_X(q,p) = \langle p, X(q)\rangle$ defines a smooth function on $T^*N$.
  (b) Prove that the Hamiltonian vector field of $H_X$ coincides with $X$ on $0_N$.
  (c) Prove that $\{H_X,H_Y\} = H_{[X,Y]}$ on $0_N$
  (d) Use the Jacobi identity for the Poisson bracket to prove the Jacobi identity for vector fields

(20) (Poisson manifolds) Let us consider a bilinear map $\{\bullet,\bullet\} : C^\infty(M)\otimes C^\infty(M) \longrightarrow C^\infty(M)$. We shall say that $\{\bullet,\bullet\}$ is a Poisson structure on $M$ if
- (antisymmetry) $\{F,G\} = -\{G,F\}$
- (Jacobi identity) $\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0$
- (Leibniz identity) $\{FG,H\} = F\{G,H\}+G\{F,H\}$

(a) Prove that there is smooth section $\pi$ of $\bigwedge^2 TM$ such that $\{F,G\} = (dF\wedge dG)(\pi)$

(b) Prove that a symplectic manifold is automatically a Poisson manifold. What is $\pi$ ?

(c) Let $F\in C^\infty(M,\mathbb{R})$, then $G\mapsto \{F,G\}$ is a derivation (according to the Leibniz identity), hence corresponds to a vector field $X_F$ (see Exercise 18,(18c)) .

(d) Prove that $F\longrightarrow X_F$ is a Lie algebra morphism between $(C^\infty(M),\{,\})$ and $(\mathfrak{X}^\infty(M),[,])$ the Lie algebra of vector fields (with the usual Lie bracket $[\bullet,\bullet]$).

(e) Prove that $F\mapsto X_F$ defines a linear bundle map $B : T^*M\longrightarrow TM$. We define the rank of the Poisson structure at $m\in M$ to be the rank of $B_m$.

(f) Prove that if the rank is $\dim(M)$ at every point, then $M$ is symplectic.

(g) Prove the following theorem :

THEOREM. *Let $P$ be a Poisson manifold, and let $p\in P$. Then there exist a neighborhood $U\subset P$ of $p$ and a diffeomorphic Poisson mapping $\varphi = \varphi_S\times\varphi_N : U\longrightarrow S\times N$ where $S$ is a symplectic manifold and $N$ is a Poisson manifold with rank zero at $\varphi_N(p)$.*

Let us provide the

HINT. *Argue by induction on the rank of $\{,\}_P$ at $p\in P$. The case where the rank is zero is easy. The induction step is obtained by first finding $f,g$ such that $\{f,g\}(p)\neq 0$ then showing that using the straightening lemma for vector fields, we can assume $f(p)=g(p)=0$ and $\{f,g\}=1$. Then show that $X_f, X_g$ commute and the map $(f,g)$ from $(P,\{,\}_P)$ to $\mathbb{R}^2$ with the standard symplectic structure is a Poisson map. Moreover the flows of $X_f, X_g$ define a map $\psi : N\times\mathbb{R}^2\longrightarrow P$ where $N = f^{-1}(0)\cap g^{-1}(0)$ : take $(x,s,t)\mapsto \varphi_f^s\varphi_g^t(x)$. Prove that $\psi$ is a Poisson map.*

(h) Prove that the mappings $\varphi$ in the above theorem defines a foliation of $P$ by symplectic leaves.

HINT. *Let $x,y\in P$. If there is a Hamiltonian vector field whose flow sends $x$ to $y$ we write $x\simeq y$ and this generates an equivalence relation still denoted $\simeq$. This is also the equivalence relation generated by $x\simeq y$ if and only if there is path $\gamma$ connecting $x$ to $y$ such that $\dot\gamma(t)\in F_{\gamma(t)}$ where $F_x$ is the vector space generated by the set of $X_f(x)$ for $f\in C^\infty(P,\mathbb{R})$. According to Sussman's theorem (see [Sus73; Ste74]), for such a relation, the equivalence classes are embedded submanifolds, and it is easy to check that these are symplectic.*

(21) Let us consider $T^*N$ and $\mathscr{F}$ be a foliation with Lagrangian leaves defined near $0_N$ and transverse to $0_N$. Then there exists a unique one-form $\alpha$, such that $d\alpha = \omega$, $\alpha$ vanishes on vector fields tangent to $\mathscr{F}$ and $\alpha$ vanishes on $0_N$.

HINT. *Prove first that we may assume by a linear change of variable that $\mathscr{F}_x$ -the leaf through $(x,0) \in L$- is tangent to the cotangent fiber through $x$, $T_x(T_x^* L)$. Then prove that for a smooth path $\gamma$, $\int_\gamma \alpha$ can be defined as follows : connect $\gamma(0)$ (resp. $\gamma(1)$) to $0_L$ by a path contained in $\mathscr{F}_{\gamma(0)}$ (resp. $\mathscr{F}_{\gamma(1)}$) and then connect the two points in $\mathscr{F}_{\gamma(0)} \cap 0_N$ and $\mathscr{F}_{\gamma(1)} \cap 0_N$ respectively. We denote by $\rho$ the closed path thus obtained and set*

$$\int_\gamma \alpha = \int_\rho \lambda$$

*Prove that this definition is indeed independent from the choice of $\rho$, that it actually defines a smooth one-form and using Stoke's formula, that $d\alpha = \omega$.*

HINT (Alternative hint). *Prove that the foliation $\mathscr{F}$ is symplectomorphic in a neighborhood of L to the vertical foliation of $T^*L$. Indeed over a chart of L, for $x \in L$, we see that $\mathscr{F}_x$ is the graph of the differential of a function $f_x(p) = f(x,p)$ such that $f_x(0) = 0, df_x(0) = 0$. In other words near $p = 0$, we have $\mathscr{F}_x = \{(x + \frac{\partial f}{\partial p}(x,p), p) \mid p \in \mathbb{R}^n\}$. Locally the symplectic map*

$$\phi : (x,p) \mapsto (x - \frac{\partial f}{\partial p}(x,p), p + \frac{\partial f}{\partial x}(x,p))$$

*gives a symplectomorphism sending $\mathscr{F}_x$ to the cotangent fiber. Moreover if in some neighbourhood of the zero section in $T^*U$ the foliation $\mathscr{F}_x$ already coincides with the cotangent fiber, then the symplectomorphism is the identity in this region. We then cover L by charts, and apply this deformation on each chart : it moves the foliation to the cotangent fiber. Finally prove that $\phi^*(\lambda) = \alpha$.*

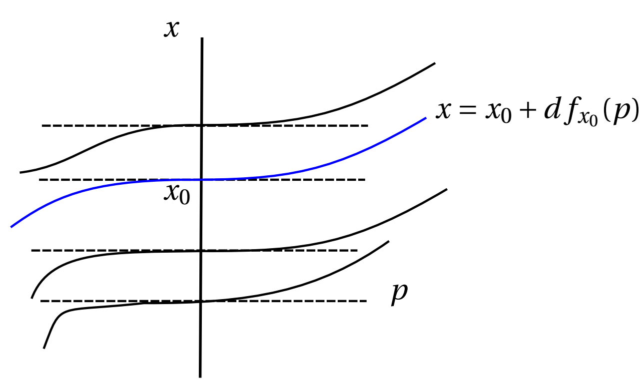


FIGURE 1. The foliation $\mathscr{F}$

(22) We want to prove the following ([GS77] page 230, proposition 3.2.)

PROPOSITION. *Let $\alpha$ be a one form on $T^*N$ such that $\alpha = 0$ on $0_N$, $d\alpha = \omega$. Then there is a unique Lagrangian foliation $\mathscr{F}$ such that $\alpha$ is the one-form associated to $\mathscr{F}$ by Exercice 21*

(a) Prove that if $\xi$ is the vector field such that $i_\xi \omega = \alpha$, we have that $\xi$ is conformal, that is $L_\xi \omega = \omega$. Setting $D\xi(x,0) = A_x$ we have $\omega(A_x u, v) + \omega(u, A_x v) = \omega(u,v)$ for all $u, v \in T_{(x,0)}(T^*N)$.

HINT. *Write the equation relating $\frac{d}{dt} D\varphi^t(z)$, $D\varphi^t(z)$ and $D\xi(z)$*

(b) Prove that the eigenspace for the eigenvalue 0 for $A_x$ contains $T_{(x,0)}0_N$, and the eigenvalues of $A_x = D\xi(x,0)$ are 0 and 1, both with multiplicity $n$.

HINT. *Write the relation found in the first question in terms of $J, A_x, A_x^*$ and use that $A_x$ and $A_x^*$ have the same eigenvalues and their eigenspaces have the same dimension.*

(c) Prove that there is through each point $(x,0)$ an invariant submanifold $\mathscr{F}_x$, invariant by the flow, tangent to the eigenspace of $A_x$ corresponding to the eigenvalue 1 (use the invariant manifold theorem, [Har02], chap. IX).
(d) Prove that $\mathscr{F}_x$ is Lagrangian, and the $\mathscr{F}_x$ constitute a foliation, that we denote by $\mathscr{F}$
(e) Use Exercice (21)) to prove that $\alpha$ is the one-form associated to $\mathscr{F}$.

(23) Prove that Exercise 22 has the following consequences.

COROLLARY. *With the assumptions of the Proposition in Exercise 22, there is a symplectic diffeomorphism $\varphi$ defined near $0_N$ such that $\varphi_{|0_N} = \mathrm{id}_{0_N}$ and $\varphi^*(\lambda) = \alpha$, where $\lambda$ is the canonical Liouville form.*

HINT. *Construct a symplectic diffeomorphism sending $V_x$ to $\mathscr{F}_x$ where $V_x = T_x^*L$ is the cotangent fibre over $x$ by using the "alternative hint" in Exercise 21.*

COROLLARY. *Let $L$ be a Lagrangian submanifold in $(M,\omega)$ and assume $\omega = d\alpha$. Then denoting $i_L : L \longrightarrow M$ the embedding, we have an embedding $\varphi$ of a neighborhood $U$ of the zero section of $T^*L$ such that $\varphi_{|0_L} = i_L$ ($L$ and $0_L$ are canonically identified) and $\varphi^*\alpha = \beta + \lambda$, where $\lambda$ is the Liouville form in $T^*L$, $\beta$ is closed and $\beta_{|L} = i_L^*\alpha_{|L}$.*

HINT. *Identify $U$ with a subset of $M$ (with the induced symplectic form) and consider $\widetilde{\alpha} = \alpha - \pi^* i_L^*(\alpha) - \lambda$. Then $\widetilde{\alpha}_{|L} = 0$, so we may write $\widetilde{\alpha} = dg$ with $g = 0$ on $L$ (use Lemma 3.22). Then $\beta = dg + \pi^* i_L^*(\alpha)$ is such that $\alpha - \beta$ satisfies the assumptions of the previous Corollary.*

(24) Let $(\mathfrak{g}, [,])$ be a Lie algebra and $\mathfrak{g}^*$ be its dual. Given a function $u \in C^\infty(\mathfrak{g}^*)$ we can identify $du(\alpha)$ to an element of $\mathfrak{g}$ (since a linear form on $\mathfrak{g}^*$ is identified to an element of $\mathfrak{g}$). We then set

$$\{u, v\}(\alpha) = -\alpha([du(\alpha), dv(\alpha)])$$

(a) Prove that this defines a Poisson structure on $\mathfrak{g}^*$.
(b) We can identify $\mathfrak{g}$ to a subspace of $C^\infty(\mathfrak{g}^*,\mathbb{R})$ through $X \mapsto u_X$ where $u_X(\alpha) = \alpha(X)$. Prove that $\{u_X, u_Y\} = u_{[X,Y]}$.

(25) (Moment map) Let $G$ be a group acting (smoothly) by Hamiltonian maps on a closed manifold $M$. Let $v \in \mathfrak{g}$. Then, since $\exp(tv)$ is a one-parameter subgroup of $G$, its image in $\mathrm{DHam}(M,\omega)$ is a one parameter subgroup, so is given by an autonomous Hamiltonian $H_v$.
(a) Prove that $X_{\{H_v,H_w\}} = [X_v, X_w]$ so that $\{H_v, H_w\} = H_{[v,w]} + c(v,w)$ where $c(v,w)$ is a constant
(b) Prove using the Jacobi identity that $c$ is a cocycle, that is
$$c([v,w],x]) + c([w,x],v) + c([x,v],w) = 0$$
so defines an element in $H^2(\mathfrak{g},\mathbb{R})$.
(c) Prove that if the above element vanishes in $H^2(\mathfrak{g},\mathbb{R})$ (i.e. if there is a map $a : \mathfrak{g} \longrightarrow \mathbb{R}$ such that $c(u,v) = a[u,v]))$ then we can find a function $a : \mathfrak{g} \longrightarrow \mathbb{R}$ such that $K_v = H_v + a(v)$ defines a Lie algebra morphism from $\mathfrak{g}$ to $C^\infty(M)$ i.e.
$$\{K_v, K_w\} = K_{[v,w]}$$
(d) Prove that if $M$ is compact, then we can normalize $H_v$ by $\int_M H_v(x)dx$ and we always have such a Lie algebra morphism.

We call moment map the map $\mu : M \longrightarrow \mathfrak{g}^*$ defined by $\mu(m)v = H_v(m)$

(26) (Noether's theorem) Let $M$ be a symplectic manifold with a Hamiltonian $G$-action having a moment map $\mu$. Let $H$ be a $G$-invariant Hamiltonian. Prove that its flow $\varphi_H^t$ satisfies $\{H, H_v\} = 0$ where $H_v(m) = \mu(m)v$ is defined in Exercise 25. Deduce that the functions $H_v$ are preserved by the flow of $X_H$. This seems obvious, but the translation in Lagrangian terms is not so obvious. Prove the following
(a) If a Lagrangian is time-independent, then energy is conserved
(b) If the Lagrangian depends only on velocity, then the momenta $mv_i$ are conserved
(c) If the Lagrangian is invariant by rotations around the origin in $\mathbb{R}^3$ (or more generally $\mathbb{R}^n$)) then the angular momenta are conserved (i.e. in dimension 3, $x \wedge v$ where $x$ is position $v$ velocity and $\wedge$ the cross product.

HINT. *The Lie algebra of* $SO(3)$ *can be identified to* $\mathbb{R}^3$ *with the Lie bracket corresponding to the cross product. Use the fact that* $H_u(x,p) = \langle x \wedge p, u\rangle = \det(x,p,u)$.

(d) Prove that for a Hamiltonian depending only on the distance to the origin, the areal velocity is constant (this is the second Kepler law).

(27) (Coriolis force see [Mar+07]) Let $G$ be a connected Lie group acting on $N$. We assume the action is free and proper.

(a) Explain how the action of $G$ on $N$ extends to an action (also free and proper) of $G$ on $T^*N$.
(b) Prove that the action of $G$ on $T^*N$ is Hamiltonian and has a moment map $\mu : T^*N \longrightarrow \mathfrak{g}^*$.
(c) Prove that the symplectic manifolds $\mu^{-1}(0)/G$ and $T^*(N/G)$ are symplectomorphic.

HINT. *Prove that* 0 *is a regular value of* $\mu$ *and the action of* $G$ *on* $\mu^{-1}(0)$ *is proper. Define the map* $\Phi : \mu^{-1}(0) \longrightarrow T^*(N/G)$ *as follows. Notice that* $\alpha_q \in \mu^{-1}(0) \cap T_q^*N$ *is equivalent to* $\langle \alpha_q, \xi \rangle = 0$ *for all* $\xi \in \mathfrak{g}$ *and we may associate to it an element of* $T^*_{[q]}(N/G)$. *Prove that* $\Phi$ *is* $G$ *invariant, and induces a diffeomorphism* $\varphi : \mu^{-1}(0)/G \longrightarrow T^*(N/G)$*, and that it preserves the respective Liouville forms.*

Of course one wonders what happens for $\xi \neq 0$. We refer to [Mar+07], (p. 60 ff.) for a detailed explanation. The goal is to compare $\mu^{-1}(\xi)/G_\xi$ where $G_\xi$ is the isotropy group of $\xi$ for the coadjoint action and $T^*(N/G_\xi)$. These are not equal, but with reasonable assumptions we get en embedding from $(\mu^{-1}(\xi)/G_\xi, \omega_\xi)$ into $(T^*(N/G_\xi), \sigma_\xi - B_\xi)$ where $\sigma_\xi$ is the canonical symplectic form on $T^*(N/G_\xi)$, and $B_\xi$ an extra term. This extra term is for example responsible of the Coriolis force in mechanics.

(28) (Poisson manifolds 2) We consider again a Poisson manifold with Poisson bracket $\{\bullet, \bullet\}$. A Poisson manifold is symplectic if its Poisson structure is given by a symplectic structure as in Exercise 20. A map $\varphi : (M, \{,\}_M) \longrightarrow (N, \{,\}_N)$ is a Poisson mapping if and only of $\{f \circ \varphi, g \circ \varphi\}_M = \{f, g\}_N \circ \varphi$ Prove that if $M, N$ are symplectic, then a Poisson map is symplectic.

(29) We want to prove that the connected components of $\mathrm{Diff}(M, \omega)$ are path connected by smooth paths. In other words if $\varphi_0, \varphi_1$ are in the same connected component of $\mathrm{Diff}(M, \omega)$, there is a smooth path of symplectic maps from $\varphi_0$ to $\varphi_1$.
(a) Prove that the graph of a symplectic map is a Lagrangian in $M \times \overline{M}$
(b) Prove that a Lagrangian $C^1$ close to the diagonal $\Delta_M$ in $M \times \overline{M}$ is the graph of a symplectic map
(c) Using Weinstein's Lagrangian neighbourhood theorem, prove that a Lagrangian $C^1$ close to the diagonal $\Delta$ can be identified to a closed 1-form on $\Delta_M$
(d) Prove that any two 1-forms are joined by smooth path of closed 1-forms
(e) Conclude that any symplectic map close to the identity can be joined to the identity by a smooth path of symplectic maps.

### 10.3. Classification of singularities.

(30) (Morse and Moser) We want first to prove the Morse lemma (see [Mil63])that states the following:

LEMMA. *Assume $f$ is a smooth function defined in a neighbourhood of 0 such that $df(0) = 0$ and $d^2f(0) = Q$ is a non-degenerate quadratic form. We claim that there is a local diffeomorphism near 0 such that $f \circ \varphi(x) = Q(x)$. The reduction of quadratic forms implies that with a further linear change of variables, we may assume $Q(x) = x_1^2 + .. + x_p^2 - x_{p+1}^2 - ... - x_n^2$.*

(a) Consider $f_t(x) = (1-t)f(x) + tQ(x)$. We want to find $\varphi^t$, flow of the vector field $X_t(x)$ such that $f_t(\varphi^t(x)) = f_0(x)$. Prove that this is equivalent to solving
$$(Q-f)(\varphi^t(x)) + df_t(\varphi^t(x))X_t(\varphi^t(x)) = 0$$
with $X_t(0) = 0$.

(b) Prove that the above equation can be rewritten as $(Q-f)(y)+df_t(y)X_t(y) = 0$ for all $y$ in $M$.

(c) Prove that $df_t(0) = 0$ and $d^2f_t(0) = Q$ and there exists a smooth family of matrices $A_t(x) \in M_n(\mathbb{R})$ such that $df_t(x)\cdot v = A_t(x)\cdot x\cdot v$ where $A_t(x) = \int_0^1 d^2 f_t(sx)ds$ (use the formula $\frac{d}{ds}df(sx) = d^2f(sx)\cdot x$) and $A_t(0)$ is invertible.

(d) Prove similarly that $(Q-f)(x) = \langle B(x)x, x\rangle$ where $B(x)$ is in $M_n(\mathbb{R})$.

(e) Show that our equation is now reduced to
$$\langle A_t(x)\cdot X_t(x), x\rangle + \langle B(x)x, x\rangle + 0$$
and we can set $X_t(x) = -A_t(x)^{-1}B(x)x$.

(f) Prove that the theorem suitably modified still holds in a Banach space, and also if $f$ is only $C^{k+2}$ (and then $\varphi$ is only $C^k$). This proof is due to R. Palais ([Pal69])

(g) (Tougeron [Tou68], Arnold [Arn68] section 8) Let $\mathfrak{m}$ be the ideal of the ring of $C^\infty$ germs near 0 of functions vanishing at 0, and $\mathfrak{J}(f)$ the ideal generated by the $\frac{\partial f}{\partial x_j}(x)$. Prove that if $g - f \in \mathfrak{M}\cdot\mathfrak{J}(f)$, then we can find $\varphi$ such that $f\circ\varphi = g$.

(h) Using the analytic version of Hilbert's Nullstellensatz (due to W. Rückert, see [Rüc32] or the more recent proofs in [Dem] (4.22) or [Huy04], 1.1.29 ) stating that if $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ is holomorphic and has 0 as an isolated critical point then $\mathfrak{J}(f) \supset \mathfrak{M}^r$ for some $r$, prove that if a holomorphic germ $f$ has 0 as isolated critical point, then $f$ is conjugate to a polynomial.

HINT. *apply the Nullstellensatz to the Taylor expansion of $f$ at order $r+1$ and then the previous question.*

**10.4. Various geometric results.**

(31) (Flux homomorphism : proof that Ker(Flux) = DHam$(M,\omega)$) Let $(\varphi^t)_{t\in[0,1]}$ be a symplectic isotopy of a connected symplectic manifold $M$. We want to prove that if Flux$((\varphi^t)_{t\in[0,1]} = 0$ then there is a Hamiltonian isotopy with the same

endpoints (in particular $\varphi^1 \in \mathrm{DHam}(M,\omega)$. We set $X_t$ to be the vector field generating $\varphi^t$ and $\alpha_t = i_{X_t}\omega$.

(a) Prove that we can reduce ourselves to the case where $\int_0^1 \alpha_t dt = 0$ in $H^1(M,\mathbb{R})$ (rather than in $H^1(M,\mathbb{R})/\Gamma(M,\omega)$.

(b) Prove that if $\int_0^1 \alpha_t dt$ is exact, then we can concatenate $\varphi^t$ with a Hamiltonian isotopy so that $\int_0^1 \alpha_t dt = 0$, so we are reduced to dealing with this case.

(c) Prove that it is enough to show that $\varphi^t$ can be deformed with fixed end points to a path such that $\mathrm{Flux}(\varphi^t) = 0$ for each $t \in [0,1]$.

HINT. *The property* $\mathrm{Flux}(\varphi^t) = 0$ *for all* $t$ *implies that for each* $t$ *we have* $i_{X_t}\omega$ *is exact, so* $X_t$ *is a Hamiltonian vector field.*

(d) Set $\beta_t = \int_0^t \alpha_s ds$ and $Y_t$ be defined by $i_{Y_t}\omega = \beta_t$. Let $s \mapsto \psi_t^s$ be the flow of $Y_t$ (this is an autonomous vector field ! ). Prove that $\psi_t^1 \circ \varphi^t$ has vanishing Flux for all $t$.

(e) Conclude.

(32) Let us consider $T^{2n}$ with the standard symplectic form obtained from the covering $\mathbb{R}^{2n}/\mathbb{Z}^{2n} \longrightarrow T^{2n}$. We consider a fundamental domain for this cover $U \subset \mathbb{R}^{2n}$ : this is a domain such that it is sent bijectively to $T^{2n}$ by the covering map. For example $\{(x_1,...,x_n,y_1,..y_n) \mid 0 \le x_i < 1, 0 \le y_i < 1\}$. By volume of a domain, we mean $\int_U \sigma_n^n$ and by center of mass, we mean the point with coordinates $\overline{x}_j = \int_U x_j\sigma_n^n, \overline{y}_j = \int_U y_j\sigma_n^n, y_j$ for $1 \le j \le n$. Prove that a symplectic diffeomorphism $\varphi$ of $T^{2n}$, symplectically isotopic to the identity, is Hamiltonian if and only if $U$ and $\varphi(U)$ have the same center of mass modulo $\mathbb{Z}^{2n}$.

(33) (Multi-Transitivity of the action of $\mathrm{DHam}(M,\omega)$) Let $(M,\omega$ be a connected symplectic manifold.

(a) Show that the action of $\mathrm{DHam}(M,\omega)$ is $k$-transitive for all $k$, that is given $x_1,...,x_k$ and $y_1,..y_k$ in $M$ we can find $\varphi \in \mathrm{DHam}(M,\omega)$ such that $\varphi(x_j) = y_j$ for all $j \in \{1,...,k\}$

HINT. *Prove that there is such a Hamiltonian map supported in* $U$ *whenever* $x_1, y_1 \in U$ *and* $U$ *is connected. Then argue by induction.*

(b) Let $x_1,...,x_k$ and $y_1,..y_k$ and $S_j \subset T_{x_j}M, T_j \subset T_{y_j}M$ be linear subspaces such that if $S_j$ is of the same type as $T_j$ then there is a Hamiltonian map $\varphi$ such that $\varphi(x_j) = y_j$ and $d\varphi(x_j)S_j = T_j$.

HINT. *Use the first question to reduce to the case* $x_j = y_j$ *and then to the case* $n = 1$. *Finally use the fact that the linear action of* $Sp(2n)$ *is transitive on linear subspaces of given type and show that for any* $R \in SP(2n)$ *there is a Hamiltonian map with support in a neighbourhood of* $0$ *such that* $d\varphi(0) = R$

(34) (Lagrangian isotopies are induced by Hamiltonian flows) Let $i_t : L \longrightarrow (M,\omega)$ be a continuous family of Lagrangian embeddings (this is called a Lagrangian isotopy). Then $X_t = \frac{d}{d\tau} i_t(x)_{\tau=t}$ is a vector field along $i_t$. In other words if we consider $i : [0,1]\times L \longrightarrow M$ and $E$ to be the bundle $i^*(TM)$ over $[0,1]\times L$, we have that $X$ is a section of $E$ given by $X(t,x) = \frac{d}{d\tau} i_t(x)_{\tau=t}$.
  (a) Prove that there is a natural connection on $E$ such that the horizontal space is $A(t,x) = d i_t(x) T_x L$.
  (b) Prove that $i^*\omega$ vanishes on $A(t,x)$ and is symplectic on the vertical
  (c) Prove that using Weinstein's theorem, that we can extend $i_t$ to a symplectic embedding $j_t : D_\varepsilon T^*L \longrightarrow (M,\omega)$ where $D_\varepsilon T^*L$ is a neighbourhood of the zero section in $T^*L$.
  (d) Prove that provided the family $[\omega] \in H^2(M, i_t(L)) \simeq H^2(M,L)$ is constant, there is a function $H(t,z)$ defined on $[0,1]\times D_\varepsilon T^*L$ such that $X(t,x) = d j_t(x) X_H(t,x)$ for all $t \in [0,1], x \in L$. Find a counterexample if this condition does not hold.
  (e) Conclude that if the family $[\omega] \in H^2(M, i_t(L)) \simeq H^2(M,L)$ is constant, then $i_t(L) = \varphi_H^t(L)$.
  (f) Prove that the same holds if $i_t$ is a continuous family of Lagrangian immersion (this is called a Lagrangian regular homotopy).

(35) Let us consider two subgroups of the group of Hamiltonian maps of $T^*N$ :
  (a) The set of fiber preserving symplectic maps, that is of the form $(q,p) \mapsto (q, p + df(q))$ for some smooth function $f$
  (b) the set of Hamiltonian symplectomorphisms preserving the zero-section $0_N$.

  Prove that the union of these two subgroups generates $\mathrm{DHam}(T^*N,\omega)$

(36) Let $L : V \times L \longrightarrow \mathscr{L}(T^*N)$ be a smooth family of exact Lagrangians and assume there is a smooth function $f$ on $V \times L$ such that $f(v,z)$ satisfies $d_z f = \lambda_{|L_v}$. Prove that there is an exact Lagrangian $\Lambda \in T^*(V\times N)$ such that the reduction of $\Lambda$ by $v = v_0$ is $L_{v_0}$.

(37) (Crofton's formula, see [Cro68; San04]) Let $\gamma : [0,1] \longrightarrow S^{n-1}$ be a smooth curve on the sphere. We want to prove that if we set for a vector $\xi \in S^{n-1}$ $\#(\gamma \cap E_\xi)$ to be the number (in $\mathbb{N}\cup\{+\infty\}$) of intersection points of $\gamma$ with the codimension one sphere $E_\xi = \{x \in S^{n-1} \mid \langle \xi, x\rangle = 0\}$. We want to prove that

$$\mathrm{length}(\gamma) = \pi \int_{S^{n-1}} \#(\gamma\cap E_\xi)\, d\xi$$

where $d\xi$ is the $O(n)$ invariant measure on the sphere with integral one (e.g. for $n=3$ this is $\frac{1}{4\pi}$ the usual measure)
  (a) Show that if $\gamma$ is a great circle on the sphere (i.e. the intersection of a plane through the origin and the sphere) the formula holds
  (b) Show that the formula holds if $\gamma$ is an arc of a great circle

(c) Let $x_i = \gamma(i/N)$, and replace $\gamma$ by the curve $\gamma_N$ obtained by connecting the $x_i$ by the shortest arc of a great circle connecting $x_i$ and $x_{i+1}$

(d) Prove that length$(\gamma_n)$ converges to the length of $\gamma$ and that

$$\pi \int_{S^{n-1}} \#(\gamma_N \cap E_\xi) d\xi$$

converges to

$$\pi \int_{S^{n-1}} \#(\gamma \cap E_\xi) d\xi$$

(e) Conclude

(38) Let $\gamma : S^1 \longrightarrow S^{n-1}$ be a closed smooth curve on the sphere. We want to prove that if $\gamma$ intersects all the $E_n$ (see Exercise 37) then it has length at least $2\pi$.

(a) Prove that this is a consequence of Crofton's formula.
We shall now propose an alternative proof.

(b) Let $x, y$ two points on the curve dividing the curve in two paths $\gamma_1, \gamma_2$ of equal length. Let $n$ be a vector on a great circle through $x, y$ (this is unique unless $x, y$ are antipodal, in which case we can conclude immediately).

(c) Prove that $\gamma_1$ or $\gamma_2$ intersects $E_\xi$. After a possible change of notation we assume it is $\gamma_1$

(d) Let $s_\xi$ be the symmetry with fixed point set $\mathbb{R}\xi$ (and equal to $-\mathrm{Id}$ on $\xi^\perp$). Prove that $\gamma_1 \cup s_N(\gamma_1) = \widetilde{\gamma}$ is a closed curve with the same length as $\gamma$

(e) Prove that $\widetilde{\gamma}$ connects two antipodal points in $E_n$

(f) Deduce that $length(\widetilde{\gamma}) \geq 2\pi$ (we admit that the path with shortest length between antipodal points is a great circle) hence $length(\gamma) \geq 2\pi$.

(39) Let $S$ be a surface in a Riemannian manifold $(M, g)$. We assume $S$ is minimal, that is for any embedded curve $C$ on $S$ the area of any surface bounding $C$ and close to $S$ is greater or equal than the area of $S$. We want to prove that if $x_0 \in S$ and $B(x_0, r)$ is the Riemannian ball of radius $r$ centered at $x_0$, then $\lim_{r\to 0} \frac{\mathrm{area}(S\cap B(x,r))}{\pi r^2} \geq 1$. We shall assume first that the metric in $B(x_0, r)$ is the euclidean metric.

(a) Prove that $\frac{d}{dr}\mathrm{area}(S \cap B(x, r)) = \mathrm{length}(S \cap S(x_0, r))$ for almost all $r$. We set $a(r) = \mathrm{area}(S \cap B(x, r))$, so that $a'(r) = \mathrm{length}(S \cap S(x_0, r))$.

(b) Prove that $S \cap S(x_0, r)$ cannot be strictly contained in a hemisphere. For this we may assume the curve is contained in $x_n \geq \varepsilon > 0$. Consider $S \cap \{x_n = \varepsilon\}$ and assume it is a union of smooth curves (which is the case after small perturbation of $\varepsilon$). These smooth curves bound a union of discs in $\{x_n = \varepsilon\}$ that we denote by $\Delta$.

(c) Prove that $\mathrm{area}(\Delta) < \mathrm{area}(S \cap \{x_n \leq \varepsilon\})$. Now we replace $S$ by $S' = S \cap \{x_n \geq \varepsilon\} \cup \Delta$.

(d) Prove that $S'$ bounds $C$ and $\mathrm{area}(S') < \mathrm{area}(S)$, contradicting the assumption that $C$ is contained in a hemisphere.

(e) Use Crofton formula to show that $a'(r) \geq 2\pi r$ and conclude that $a(r) \geq \pi r^2$.
(f) Prove that $\lim_{r\to 0} \frac{\text{area}(S\cap B(x,r))}{\pi r^2} \geq 1$ holds for any metric by approximating $g$ by the euclidean metric $g(x_0)$.

### 10.5. Lagrangian and Hamiltonians.

(40) (Beltrami's equation) Consider the Lagrangian $L(q,\dot{q})$ independent from $t$. Prove that the Euler-Lagrange equation implies that $L(q,\dot{q}) - \dot{q}\frac{\partial L}{\partial \xi}(q,\dot{q})$ is constant on a trajectory.

HINT. *Write that the Hamiltonian is constant.*

(41) Let $H : M^{2n} \longrightarrow \mathbb{R}$ be an autonomous Hamiltonian on a symplectic manifold $(M^{2n},\omega)$, and $X_H$ the corresponding vector field. Prove that if $\Sigma$ is a regular level of $H$, then there exists a volume form $\mu$ on $\Sigma$ invariant by $X_H$.

HINT. *Prove that there exists $\Omega$, a $2n-1$-volume form on $\Sigma$ and a vector field $Y$ such that $i_Y\Omega = \omega^{(n-1)}$. Prove that $Y$ and $X_H$ must be colinear. Then find $\mu$ such that $i_{X_H}\mu = \omega^n$ and then $L_{X_H}\mu = 0$*

(42) (The exponential map and tubular neighborhood) Let $g(x)(\xi,\xi)$ be a Riemannian metric where $x \in M, \xi \in T_xM$. We set $H(x,p) = g(x)(p,p)$ for the metric induced on the dual space, by identifying $T_x^*M$ to $T_xM$ using $g(x)$ that is the map

$$g(x)^\# : v \in T_xM \longmapsto p = g(x)(v,\bullet) \in T_x^*M$$

We consider the Hamiltonian flow $\varphi_H^t$ on $T^*M$ and we denote by $\Phi_H^t$ the corresponding flow on $TM$. Now we consider the map $E : TM \longrightarrow M$ given by $E(x,v) = \pi\Phi_H^1(x,v)$, $\pi : TM \longrightarrow M$ being the projection. We claim that $\frac{d}{dt}E(x,tv)_{|t=0} = v$.
(a) Show for $(x(t),p(t)) = \varphi_H^t(x(t),p(t))$ that $\frac{d}{dt}x(t) = g(x)^\# p(t)$ where $g(x)^\# : T_x^*M \longrightarrow T_xM$ is the duality map i.e. the inverse to $p \mapsto g(x)p$.
(b) Prove that $\dot{x}(t) = v(t)$ where $v(0) = v$. Conclude that $DE(x,0)(0,v) = v$.
(c) We want to prove that for $M$ compact, $N$ a smooth compact submanifold, $\nu_N M$ the normal bundle, $E : D_\varepsilon \nu_N M \longrightarrow M$ is a diffeomorphism onto its image that is a neighbourhood of $N$. Prove that for $\varepsilon$ small enough, $E$ is injective. One can consider a sequence $(x_n,v_n),(y_n,w_n)$ such that $E(x_n,v_n) = E(y_n,w_n)$ and $v_n,w_n$ go to zero. Then prove that $x_n,y_n$ converges to $z \in M$, and prove that the existence of $(x_n,v_n,(y_n,w_n)$ would contradict the inverse function theorem for $E$ applied at $z$.

Remark: usually the exponential map is defined after Levi-Civita connection has been introduced. This is by no means necessary as we just saw. The advantage of using a Hamiltonian formulation is that we do not have to deal with the Euler-Lagrange equation, which is a second order equation, and this is a slightly complicated mathematical concept, if one wishes to define it intrinsically. Of course connections and/or local coordinates do the job.

(43) Let $L$ be a (non necessarily compact) Lagrangian submanifold in $T^*N$. Let $f : T^*N \longrightarrow \mathbb{R}$ be a smooth function on $N$. Prove that if $(x,p) \in L$ is such that $f$ achieves its maximum on $L$ at the point $(x_0,p_0)$ , we must have $X_f(x_0,p_0) \in T_{(x_0,p_0)}L$.

(44) (Control theory and Pontryagin maximum principle) The following models many phenomenon, but for example how to minimize the cost of sending a rocket to the Moon, how to park a car with minimal effort, etc. Consider $L: N\times\Omega \longrightarrow \mathbb{R}$ where $N$ is manifold and $U$ a domain and $c : N \longrightarrow \mathbb{R}$ be smooth functions. We define on the set of continuous paths the cost function

$$E(x,u) = \int_0^t L(x(t),u(t))dt + c(x(T))$$

Our goal is to minimize $E(x,u)$ over all "controls" $u$ assumed to belong to some domain in a Banach space $B$ and all paths from $x(0) = x_0$

$$\dot{x}(t) = f(t,x(t),u(t))$$

We assume that we have existence for all times and uniqueness for solutions of the above equation (for example $N$ is compact and $f$ is locally Lipschitz in $x$, uniformly for $u \in \Omega$. We look for a function (hereafter named "control") $u$ such that $E(x,u)$ is minimal, $x_0$ being fixed. Note that we do not require $u$ to be continuous in $t$ !

(a) Prove that this can be rewritten as the following problem : consider the equation

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)) \\ \dot{c}(t) = L(x(t),u(t)) \\ x(0) = x_0, c(0) = 0 \end{cases}$$

For each control $u$, this defines a flow on the space of $(x,c)$ and we want to maximize the function $F(x,c) = c$.

(b) Prove that setting

$$H_u(t,x,c,p,\gamma) = \gamma \cdot L(x,u(t)) + \langle p, f(x,u(t))\rangle$$

where $(x,c,p,\gamma) \in T^*(N\times\mathbb{R})$, we have that $L_u = \varphi^T_{H_u}(V_{(x_0,0)})$ is the set of $(x(T),c(T),p(T),\gamma(T))$ such that $x(t),c(t)$ satisfy the above equation and

$$\begin{cases} \dot{p}(t) = \gamma\dfrac{\partial L}{\partial x}(x(t),u(t)) + \langle p, f(t,x(t),u(t))\rangle \\ \dot{\gamma}(t) = 0 \end{cases}$$

(c) Prove that the family of $L_u$ can be bundled toghether in a Lagrangian $\Lambda$ in $T^*(N\times\mathbb{R}\times B)$ and we want to maximize the function $G(x,c,u) = c$ on $\Lambda$.

(d) Write the condition $X_G \in T_{(x,p,c,\gamma,u,v)}\Lambda$ and prove Pontryagin's weak maximum principle : we must have the above equations to be satisfied and also for all $t \in [0,T]$

$$\gamma\frac{\partial L}{\partial u}(x(t),u(t))+\langle p,\frac{\partial f}{\partial u}(x(t),u(t))\rangle=0$$

(45) (Billiard maps) Consider a convex region $\Omega$ with smooth boundary $\Sigma$ in $\mathbb{R}^n$. We think of a ball traveling in straight line, but obeying the Descartes reflexion on the boundary : the speed of the ball and its reflection remain in the same plane containing the normal vector, and the incoming and outcoming speed make the same angle with the normal. This is obviously not a smooth map, since we have some discontinuity at a reflection. But we can look at the map $T^+_\Sigma\mathbb{R}^n$ of pairs $(x,v)$ with $x \in \Sigma$ and $v$ a vector such that $\langle v,\nu(x)\rangle < 0$. We set $\varphi(x,v)=(y,w)$ where $y$ is the point where the line $x+t\cdot v$ meets $\Sigma$ and $w$ the speed of the reflected ball at $y$. Since we can write $v=\sqrt{1-|\tilde{v}|^2}\nu(x)+\tilde{v}$ where $\tilde{v}\in T_x\Sigma$ has norm less than 1, we then see that the billiard map is a map $DT\Sigma \longrightarrow DT_\Sigma$ and we can carry it to a map $DT^*\Sigma \longrightarrow DT^*\Sigma$. Prove that the billiard map is a $C^0$ limit of Hamiltonian maps. Assume $\Omega=\{q\in N \mid f(q)\le 0\}$ and 0 is a regular value of $f$, prove that if $H_\varepsilon(q,p)=\frac{1}{\varepsilon}f(q)+\varepsilon|p|^2$, the limit of the characteristic flow on $H_\varepsilon=1$ as $\varepsilon$ goes to 0 is the billiard map. (see Chapter 8, Section 2 for more)

### 10.6. Contact geometry and Homogeneous symplectic manifolds.

(46) Let $(M,\xi)$ be a 3-manifold with a plane distribution. A curve $\gamma$ is an integral curve if $\dot{\gamma}(t)\in\xi(\gamma(t))$ for all $t$.
(a) Prove that $\xi$ is contact if and only if any two points in a connected open set $U$ can be joined by an integral curve inside $U$.
(b) Give an example to show that the above becomes false if "inside $U$" is removed.
(c) Prove the analog statement in higher dimension i.e. for a hyperplane distribution in $M^{2n+1}$.

(47) Prove that in $T^*N$ a Hamiltonian vector field preserves the Liouville form, i.e. $L_{X_H}\lambda=\lambda$ if and only if $H$ is positively homogeneous of degree one, that is $H(x,\tau p)=\tau H(x,p)$ for all $\tau\in\mathbb{R}$ and $(x,p)\in T^*N$.

(48) Let $(E,J)$ be a complex vector space. Let us consider $\Lambda^p(E)$ the complex vector space of complex valued $p$-forms on $E$.
(a) Prove that $\Lambda^1(E)=\Lambda^{(1,0)}(E)\oplus\Lambda^{(0,1)}(E)$ where $\Lambda^{(1,0)}(E)$ is the set of $\mathbb{C}$-linear 1-forms and $\Lambda^{(0,1)}(E)$ the set of antilinear one-forms.
(b) We shall denote by $\Lambda^{(k,p-k)}(E)$ the space of $p$-forms generated by the exterior products of $k$ forms in $\Lambda^{(1,0)}(E)$ and $k-p$ forms in $\Lambda^{(0,1)}(E)$. Prove

that

$$\Lambda^{(p)}(E) = \bigoplus_{k=0}^{p} \Lambda^{(k,p-k)}(E)$$

(c) We now do the same for an almost complex manifold $(M,J)$. Prove that

$$\Omega^{p}(M) = \bigoplus_{k=0}^{p} \Omega^{(k,p-k)}(M)$$

(d) If $(M,J)$ is complex manifold, i.e. there are holomorphic charts, i.e. holomorphic local coordinates $z_1,...,z_n$. We denote by $\bar{z}_j$ the composition of $z_j$ and the complex conjugation. Prove that $\alpha \in \Omega^{(k,p-k)}$ if and only if we can write locally

$$\alpha = \sum_{i_1,...i_k,j_1...,j_{p-k}} \alpha_{i_1,...i_k,j_1...,j_{p-k}}(z_1,...,z_n)dz_{i_1}\wedge...\wedge dz_{i_k}\wedge d\bar{z}_{j_1}\wedge...\wedge d\bar{z}_{j_{p-k}}$$

(e) Assuming once more that $(M,J)$ is complex, prove that $d\Omega^{k,p-k}(M) \subset \Omega^{k+1,p-k}(M)\oplus\Omega^{k,p+1-k}$. We denote the first component of $d\alpha$ by $\partial\alpha$ and the second by $\overline{\partial}\alpha$

(f) Prove that for $\varphi$ a smooth real function, $i\partial\overline{\partial}\varphi$ is a real valued $(1,1)$-form.

(g) Set $J^*\alpha(z)(\xi_1,...,\xi_p) = \alpha(z)(J\xi_1,...,J\xi_p)$.

(h) Prove that for a smooth function $\varphi$ we have $dJ^*d\varphi = i\partial\overline{\partial}\varphi$

(i) Let us now again assume $(M,J)$ is only almost complex. Prove that if $dJ^*d\varphi$ is positive on complex lines, then it is a symplectic form, and $J$ tames it.

(49) Prove that if $\varphi$ is a convex function on $\mathbb{R}^{2n} = \mathbb{C}^n$, then it is plurisubharmonic.

HINT. *Prove that the restriction of the Levi form to a complex line is the restriction of* $\Delta\varphi$, *where* $(\Delta f)(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)$.

(50) (Pfaff normal form) Use the above Exercise to show that if a 1-form $\alpha$ is such that $d\alpha$ has constant rank, then

(a) if $(d\alpha)^r \neq 0$ and $\alpha\wedge(d\alpha)^r \equiv 0$ there are local coordinates $x_1,...,x_r,y_1,..,y_r,z,t_1,...,t_{n-2r-1}$ such that $\alpha = \sum_{j=1}^r y_j dx_j$

(b) if $\alpha\wedge(d\alpha)^r \neq 0$ and $(d\alpha)^{r+1} \equiv 0$ then there are local coordinates such that $\alpha = dz - \sum_{j=1}^r y_r dx_r$.

(c) $\alpha = 0$ defines a distribution with maximal integral submanifolds (i.e submanifolds $V$ such that $T_zV \subset \mathrm{Ker}(\alpha(z))$ of dimension $r$.

(51) Prove that if $\Lambda$ is a Legendrian submanifold in a contact manifold $(M,\xi)$ it has a neighbourhood contactomorphic to a neighbourhood of the zero section in $J^1(\Lambda,\mathbb{R})$

HINT. *Reduce to Weinstein's neighbourhood theorem (Proposition 3.26)*

(52) Let $L$ be a Lagrangian submanifold in $(M,\omega)$ and $W$ a domain with restricted contact-type boundary $\partial W$. Let $\lambda$ be a one-form on $W$ defining the contact form on $\partial W$, that is $d\lambda = \omega$ and $\lambda_{|\partial W} = \alpha$.
  (a) Prove that that if $\partial W$ is transverse to $L$, we may deform $L$ in a neighbourhood of $\partial W$ so that there exists a Legendrian submaniifold $\Lambda$ in $(\partial W, \alpha)$ such that $L \cap (\partial W \times ]-\varepsilon,\varepsilon[) = \Lambda \times ]-\varepsilon,\varepsilon[$. Here $\omega = d(e^t\alpha)$ in $\partial W \times ]-\varepsilon,\varepsilon[$.
  (b) Let $X$ be the conformal vector field induced by $\lambda$, that is $i_X\omega = \lambda$. Prove that $X$ is $-\frac{\partial}{\partial t}$ on $\partial W \times ]-\varepsilon,\varepsilon[$. Prove that there is a conformal vector field which
    (i) Enters $\partial W$, i.e. $X = f(x,t)\frac{\partial}{\partial t} + Y(x,t)$ where $f \le 0$ everywhere and $f < 0$ away from $\Lambda$ and $Y$ is tangent to $\partial W \times \{t\}$.
    (ii) $f = 0$ on $\Lambda \times \{t\}$ for $t \in ]-\varepsilon,\varepsilon[$

  HINT. *Prove that on $\partial W \times ]-\varepsilon,\varepsilon[$ there is a function $H$ such that $dH + e^t\lambda$ vanishes on $\Lambda \times ]-\varepsilon,\varepsilon[$. (by "$\alpha$ vanishes on $V \subset X$" we mean $\alpha(x)$ vanishes as an element in $T^*X$ for $x \in V$)*

(53) (Contact Hamiltonians) Let $(M,\alpha)$ be a manifold of dimension $2n+1$ endowed with a contact form $\alpha$. Let $H \in C^\infty(M,\mathbb{R})$ and $Z_H$ the Hamiltonian vector field from Definition 3.80 by the equations

$$i_{X_H}\alpha = -H,\ \ i_{Z_H}d\alpha = dH - dH(R_\alpha)\alpha$$

  (a) For $M = J^1N$ and $\alpha = dz - pdq$ prove that $Z_H$ can be written in coordinates as

$$Z_H = \left(\frac{\partial H}{\partial p}(q,p,z), -\frac{\partial H}{\partial q}(q,p,z) - p\frac{\partial H}{\partial z}(q,p,z), p\frac{\partial H}{\partial p}(q,p,z) - H(q,p,z)\right)$$

  (b) Prove in the general case that if $H$ does not depend on $z$, then we recover the Hamiltonian equations of motion, with $z(s)$ is the action, i.e.

$$z(s) = \int_0^s p(t)\dot{q}(t) - H(t,q(t),p(t))dt$$

  (c) Let $\Omega$ be the volume form $\Omega = \alpha\wedge(d\alpha)^n$. Prove that $L_{Z_H}\Omega = -(n+1)dH(R_\alpha)\Omega$
  (d) Assume $dH(R_\alpha)$ does not vanish. Show that $\tilde{\Omega} = H^{-n+1}\Omega$ is invariant by $Z_H$.
  (e) Show that if $H$ is time independent, $L_{Z_H}H = -HdH(R_\alpha)$
  (f) Let $H(q,p,z) = \frac{p^2}{2} + V(q) + cz$ where $(q,p,z) \in J^1\mathbb{R}^n$. Write down the equation of the flow as a second order equation and show that $c$ corresponds to a damping term.
  (g) Let $L$ be a Legendrian submanifold of $(M,\alpha)$ that is $\dim(L) = n$ is maximal for the inclusion among the submanifolds on which $\alpha$ vanishes, then if $H = 0$ on $L$ then the flow of $Z_H$ preserves $L$.

(54) Let $S$ be a smooth hypersurface in $M$, and $\pi : T^*M \to M$ be the projection.

(a) Prove that if $\nu^* S = \{(x,p) \mid x \in S, p_{|T_x S} = 0\}$ is the conormal to $S$, then $S = \pi(\nu^* S)$.
(b) Prove that for any homogeneous Lagrangian, $L \neq 0_N$, in $T^*M$, $\pi_{|L}$ is a map of rank at most $n-1$ (find a trivial kernel).
(c) Prove that if $L'$ is homogeneous Lagrangian and $C^1$ close to $L$ (i.e. $L' \cap DT^*M$ is $C^1$ close to $L \cap DT^*M$), then $L'$ is the conormal of some hypersurface $S'$. Hint: prove that $\pi(L')$ is a (non-empty) smooth hypersurface.

(55) Prove the following

PROPOSITION. *Let $\Sigma$ be a germ of hypersurface near $z$ in a homogeneous symplectic manifold. Then after a homogeneous symplectic diffeomorphism we may assume $\Sigma$ is either locally given by $\{q_1 = 0\}$ or by $\{p_1 = 0\}$.*

(56) Prove that any vector field preserving the contact structure $\xi$ and transverse to it is the Reeb vector field of some contact form defining $\xi$.

(57) Let $X$ be vector field on a contact manifold $(M,\xi)$defined by the one-form $\alpha$.
(a) Prove that the flow of $X$ preserves $\xi$ if a and only if $L_X\alpha = f\alpha$ for some smooth function $f$ on $M$
(b) Prove that if we set $X = Y + hR_\alpha$ where $R_\alpha$ is the Reeb vector field

(58) (Euler transformation (see [Fer]) Let us consider $J^1(S^{n-1},\mathbb{R})$ and $ST^*(\mathbb{R}^n)$ with their canonical contact structures. We use the canonical duality in $\mathbb{R}^n$ to identify $v \in T_u^* S^{n-1}$ to a vector (again denoted $v$) in $\mathbb{R}^n$ by $\langle v,\xi\rangle = v(\xi)$ for all $\xi \in T_x S^{n-1}$.
To a point $(u,v,z)$ in $J^1(S^{n-1},\mathbb{R})$ we associate the point $x = z\cdot u + v \in \mathbb{R}^n$
(a) Prove that the above map, called the **Euler transform** is a contact map
(b) Prove that the image of the above map is dense in $ST^*(\mathbb{R}^n)$.
(c) Let $C$ be the smooth boundary of a convex domain containing the origin in $\mathbb{R}^n$. Let $\nu^*C$ be the submanifold of $ST^*(\mathbb{R}^n)$ of pairs $(x,p)$ with $x \in C$, $p = 0$ on $T_xC$. Show that $\nu^*C$ is Legendrian.
(d) Show that the image of $\nu^*C$ by the Euler transform is the 1-jet of a function $f \in C^\infty(S^{n-1},\mathbb{R})$ called the support function of $C$.

(59) (Euler transformation 2) Let us consider $ST^*(\mathbb{R}^{n+1})$ with the standard contact form $\lambda$ and $J^1(S^n)$ also with its standard form $dz - pdq$. Consider $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$ and the map

$$ST^*(\mathbb{R}^{n+1}) \longrightarrow J^1(S^n)$$
$$E : (x,y) \mapsto (y, x - \langle x,y\rangle y, \langle x,y\rangle) = (q,p,z)$$

(a) Prove that $E$ is a contact transformation.
(b) For $\Sigma$ a hypersurface in $\mathbb{R}^{n+1}$, consider $\nu^+\Sigma$ the unit conormal to $\Sigma$ i.e. $\nu^+\Sigma = \{(x,y) \mid x \in \Sigma, y \perp T_x\Sigma$ and$y$ points outwards$\}$. Prove that $\nu^+\Sigma$ is a Legendrian submanifold.
(c) Determine its image by $E$ for $\Sigma$ convex.

(60) (Legendre transform) Let $f$ be a convex function on a vector space $V$. Let $\Gamma(f)$ be the graph of $df$ in $V \times V^*$.
  (a) Prove that the projection of $\Gamma(f)$ on $V^*$ is a diffeomorphism if $f$ is strictly convex or strictly concave (i.e. $d^2 f(x) \geq \varepsilon \mathrm{Id}$ or $d^2 f(x) \leq -\varepsilon \mathrm{Id}$)
  (b) Prove that if $f$ is strictly convex and we consider $\Gamma(f)$ as the graph of a map $G : V^* \longrightarrow V$ then $G$ is a the graph of the differential of a function $f^*$ defined on $V^*$
  (c) Prove that $f^*(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{R}^n\}$

### 10.7. Integrable systems.

(61) (Arnold-Liouville) Let $(M,\omega)$ be a compact symplectic manifold of dimension $2n$, $H$ a smooth function. We assume there are $n-1$ functions, $F_2, \ldots, F_n$ such that $\{H, F_j\} = \{F_i, F_j\} = 0$ and the vectors $X_H, X_{F_2}, \ldots, X_{F_n}$ are linearly independent on $H(z) = c_1, F_2(z) = c_2, \ldots, F_n(z) = c_n$ for some $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$.
  (a) Prove that the set $\{z \in M \mid H(z) = c_1, F_2(z) = c_2, \ldots, F_n(z) = c_n\}$ is an $n$-dimensional submanifold, $X^n$. We set $F_1 = H$
  (b) Prove that $X^n$ is preserved by the flows of $X_H$ and the $X_{F_j}$ $2 \leq j \leq n$ and that these flows commute
  (c) Prove that for $z_0 \in X^n$, the map

$$(t_1, \ldots t_n) \mapsto \varphi_1^{t_1} \circ \ldots \circ \varphi_n^{t_n}(z_0)$$

  yields a covering from $\mathbb{R}^n$ to $X^n$.
  (d) Conclude that if $X^n$ is compact, it is an $n$-torus.
  (e) Prove that the trajectory of $X_H$ is either a closed curve or an irrational line on the torus (i.e. of the form $t \mapsto (\alpha_1 t, \ldots, \alpha_n t)$ with the $\alpha_j$ not all rational multiples of the same real number.
  (f) Prove that a trajectory is dense in $X^n$ if and only if the $\alpha_j$ are rationally independent (i.e. $\sum_{j=1}^n k_j \alpha_j = 0$ for $k_j \in \mathbb{Z}$ implies $k_1 = \ldots = k_n = 0$).
  (g) Prove that near $X^n$ there are coordinates $\theta_1, \ldots, \theta_n$ with values in $S^1$ and real valued coordinates $I_1, \leq \ldots I_n$ such that $\omega = \sum_{j=1}^n dI_j \wedge d\theta_j$ and $F_j = f_j(I_1, \ldots, I_n)$ for some smooth function $f_j \in C^\infty(\mathbb{R}^n, \mathbb{R})$

(62) (see [MM97])
  (a)

(63) (see [Mos83; Jac66]) Let $E$ be the ellipsoid in $\mathbb{R}^n$ defined by $\langle A^{-1}q, q\rangle = 1$ where $A$ is a symmetric matrix with distinct real eigenvalues $0 < \alpha_1 < \ldots. < \alpha_n$. We want to study the system

$$\ddot{q} = -\nu(q) A^{-1} q$$

where $\nu(q)$ is determined so that $\langle A^{-1}q(t), q(t)\rangle$ remains constant i.e.

$$\nu(q) = |A^{-1}q|^{-2} \langle A^{-1}\dot{q}, \dot{q}\rangle$$

(a) Prove that the equations has a Hamiltonian formulation as the restriction of the Hamiltonian $H(q,p)=\frac{1}{2}|p|^2$ to the tangent bundle of the ellipsoid, that is
$$\langle A^{-1}q,q\rangle=1,\langle A^{-1}q,p\rangle=0$$
(b) Prove that the flow can be written as
$$\begin{cases}\dot{q}(t)=p(t)\\ \dot{p}(t)=-|A^{-1}q|^{-2}\langle A^{-1}p(t),p(t)\rangle A^{-1}q(t)\end{cases}\tag{3.1}$$
(c) Set $Q_z(q,p)=\langle p,(z\mathrm{Id}-A)^{-1}q\rangle$ and $Q_z(q)=\langle q,(z\mathrm{Id}-A)^{-1}q\rangle$ so that $Q_z(q)=1$ defines a family of confocal quadrics and $Q_0(q)=1, Q_0(q,p)=0$ defines the tangent space to $E$. We set
$$\Phi_z(q,p)=(1+Q_z(q))Q_z(p)-Q_z(q,p)^2$$
(d) Using the fact that $Q_z$ has a simple pole in $z$ at $z=\alpha_k$ and the residue formula (see [Car95], show that
$$\Phi_z(q,p)=\sum_{j=1}^n\frac{F_j(q,p)}{z-\alpha_k}$$
where
$$F_k(q,p)=p_k^2+\sum_{j\neq k}\frac{(q_jp_k-p_jq_k)^2}{\alpha_k-\alpha_j}$$
(e) Prove that the $F_k$ Poisson-commute by proving that for $z_1\neq z_2$ the functions $\Phi_{z_1}$ and $\Phi_{z_2}$ commute.
(f) Fix $c_k=F_k(q,p)$ and show that for generic $c_k$, these, together with $\langle p,q\rangle=0$ define an $(n-1)$-torus $\mathscr{T}(c_1,...,c_n)$ (use Exercice 61)
(g) Assume the function $\sum_{k=1}^n\frac{c_k}{z-\alpha_k}$ has $n-1$ real zeros, $\beta_1<....<\beta_{n-1}$
(h) Consider for $(q_1,\dots,q_n)$ on the unit sphere the roots $(\mu_1,\dots,\mu_{n-1})$ of
$$Q_z(q)=\sum_{j=1}^n\frac{q_j^2}{z-\alpha_j}$$
Show that the map $(q_1,\dots,q_n)\mapsto(\mu_1,\dots,\mu_{n-1})$ defines elliptic coordinates on the sphere $\sum_{j=1}^n q_j^2=1$
(i) Setting
$$m(z)=\prod_{j=1}^{n-1}(z-\mu_j), a(z)=\prod_{j=1}^{n}(z-\alpha_j), b(z)=\prod_{j=1}^{n-1}(z-\beta_j)$$
prove that $Q_z(q)=\frac{m(z)}{a(z)}$ and $q_j^2=\frac{m(\alpha_j}{a'(\alpha_j)}$
(j) Prove that $\Phi_z(q,\dot{q})=-Q_z(q,\dot{q})^2$ for $z=\mu_j$ so that for $z=\mu_j$
$$Q_z(q,\dot{q})=\sqrt{-\frac{b(z)}{a(z)}}$$

(k) Prove that in these coordinates, the differential equation becomes

$$\sum_{k=1}^{n-1} \frac{\mu_k^{n-j-1}\dot{\mu}_k}{2\sqrt{-R(\mu_k)}} = \delta_{j,1}$$

for $j = 1,2,...,n-1$ where $R(z) = a(z)b(z)$

(l) Let $w^2 = -4R(z)$ be the hyperelliptic curve of genus $n-1$ projecting on the plane $w$ with branch points at $\alpha_j, \beta_k$. Its Jacobi mapping is given by sending the $g = n-1$ points (a divisor in the classical terminology) $(\mu_1, w_1), \dots (\mu_{n-1}, w_{n-1})$ to

$$\sum_{k=1}^{n-1} \int_{(0,0)}^{(\mu_k, w_k)} \frac{z_k^{n-j-1}\dot{z}_k}{2\sqrt{-R(\mu_k)}} dz = s_j$$

Here the integral is in the complex domain and may depend on the choice of the path from $(0,0)$ to $(\mu_k, w_k)$. This is only well defined if $s = (s_1, ..., s_{n-1})$ is considered as a point in $\mathbb{C}^{n-1}/\Gamma$ where $\Gamma$ is the set of periods of the above differentials

(m) Prove that in the new variables the equation becomes $\dots s_j = \delta_{j,1}$ or $s_j(t) = \delta_{j,1} t + s_j(0)$

(n) Deduce that the linear structure on the Arnold-Liouville torus is the same as the linear structure on the (real part of the ) Jacobian of the algebraic curve.

(64) (Torus actions on symplectic manifolds-Atiyah's convexity) Let us consider a Hamiltonian group action of a compact Lie group $G$ on a symplectic manifold $(M,\omega)$ To any $X \in \mathfrak{g}$ we associate the vector field $X_M$ given by $X_M(x) = \frac{d}{dt}exp(tX)x_{|t=0}$. By assumption there is a map $X \mapsto H_X$ such that $X_{H_X} = X_M$ and $X \mapsto H_X$ is a Lie algebra morphism from $\mathfrak{g}$ with the usual Lie bracket to $C^\infty(M,\mathbb{R})$ with the Poisson bracket. When $G$ is a torus, we set $H_0 = H_{X_0}$ where $X_0$ is chosen so that $\exp tX_0$ is dense in $T$.

(a) Prove that the fixed points of $X_0 = X_{H_0}$ are the same as the fixed points of the action and correspond to the critical points of $H_0$

(b) Prove that if $F$ is a connected component of the set of fixed points of $T$, it is a non-degenerate critical manifold of $H_0$ (i.e. for $x \in F$, $d^2H_0(x)$ is non degenerate on the normal space to $T_xF$)

(c) Prove that for $x \in F$, we have a decomposition of $T_xM$ as $V_0(x) \oplus V_1(x) \oplus .... \oplus V_k(x)$ where the $V_j(x)$ are symplectic, $D^2H_0(x) = a_j|v_j|^2$ on $V_j$ and $T_xF = V_0(x)$ and $a_j \neq 0$ for $j \neq 0$

(d) Prove that the $V_j(x)$ form a symplectic bundle over $F$ and that $F$ is a symplectic manifold.

(e) Prove (using Morse-Bott theory) that $F$ is a non-degenerate critical manifold in the sense of Morse-Bott theory and has even index

(f) Using Morse-Bott theory(see [Mil63]), prove that $H^{-1}(c)$ is always connected
(g) Prove that if $\mu$ is the moment map, $\mu : M \longrightarrow \mathfrak{t}^*$ and $X \in \mathfrak{t}$ has dense exponential (i.e. the image of $t \mapsto \exp(tX)$ is dense in $G$), then the function $H_X(m) = \langle \mu(m), X\rangle$ has connected levels
(h) Prove that the above holds for any $X \in \mathfrak{t}$
(i) Deduce that the image of $\mu$ is convex (Atiyah's convexity theorem, see [Ati82][GS82])

(65) (Moment maps, cohomology of Lie groups and mass)

(66) (Local integrability of the sheaf of symplectic relations) On $C^{\infty}(X,Y)$ we define the equivalence relation $f \overset{r}{\simeq} g$ if in local coordinates we have

$$f(x) - g(x) = o(|x|^r)$$

We define $J^r(X,Y)$ to be the set of equivalence classes for $\overset{r}{\simeq}$. Let $\mathscr{R} \subset J^r(X,Y)$ be a differential relation i.e. $\mathscr{R}$ is a subset of the set of $r$-jets of maps from $X$ to $Y$.

We say that $\mathscr{R}$ is locally integrable if for any element $s_0$ in $J^r(X,Y)_{(x_0,v_0)}$ belonging to $\mathscr{R}$, there is a germ $f \in C^{\infty}(X,Y)$ defined near $x_0$, such that $j^r f(x_0) = s_0$ and $j^r f \in \mathscr{R}$ in a neighbourhood of $x_0$. We assume $(W,\omega_W)$ is a symplectic manifold.

(a) Prove that if if $(V,\omega_V)$ is symplectic and $\mathscr{R}$ is the symplectic relation in $J^1(V,W)$ defined by

$$\mathscr{R} = \{(v,w,L) \mid v \in V, w \in W, L \in \mathscr{L}(T_vV, T_wW),\ L^*(\omega_W(w)) = \omega_V(v)\}$$

Then $\mathscr{R}$ is locally integrable

(b) Prove that if $\dim(V) \le \frac{1}{2}\dim(W)$ and $\mathscr{R}$ is the isotropic immersion relation in $J^1(V,W)$, that is $\mathscr{R} = \{(v,w,L) \mid \mathrm{Ker}(L) = 0 \text{ and } L^*(\omega_W)(w)) = 0\}$ then $\mathscr{R}$ is locally integrable

(67) We say that a differential relation $\mathscr{R}$ is flexible if for any pair of compact polyhedra $(A,B)$ with $B \subset A \subset V$ and any family $F_0 : Op(A) \longrightarrow W$ such that $j^r F_0$ is in $\mathscr{R}$ and $(F_t)_{t\in[0,1]}$ defined in $Op(B)$, such that we can extend $F_t$ to $\widetilde{F}_t$ defined on $Op(A) \times [0,1]$ and coinciding with $F_t$ on $Op(A) \times \{0\} \cup Op(B) \times [0,1]$. The relation is microflexible if with the same assumption there exists $\varepsilon > 0$ such that we can extend $F_t$ to $\widetilde{F}_t$ defined on $Op(A) \times [0,\varepsilon]$ and coinciding with $F_t$ on $Op(A) \times \{0\} \cup Op(B) \times [0,\varepsilon]$. Use Exercise 34 to prove that Lagrangian immersions define a microflexible relation. We refer to [Gro86; EM02] to deduce from this that Lagrangian immersions satisfy the h-principle.

### 10.8. Connections with Physics.

(68) (Caratheodory version of the Second Law of Thermodynamics, see [Pau00])[8] We are given a smooth manifold $M$, called the set of "states" of the system. We are also given two one forms, denoted $\delta Q$ and $\delta W$ (the $\delta$ indicates that they are not exact forms). If the system goes from state $A$ to state $B$ through a path $\gamma$, it yields an amount of heat given by $\int_\gamma \delta Q$ and an amount of mechanical work given by $\int_\gamma \delta W$. The fact that $\delta Q, \delta W$ are not differentials is reflected in the fact that these integrals depend on the path $\gamma$ and not only on its end points.

An adiabatic evolution of a state $x \in M$ is just a smooth path in $M$ tangent to $\delta Q = 0$. The first law of thermodynamics (existence of total energy) is stated as the existence of a function $U$ on $M$ with differential $dU = \delta W + \delta Q$.

(a) Prove that if $\delta W$ and $\delta Q$ were differentials (i.e. exact forms) refrigerators, air conditioning, etc would not exist.

(b) Caratheodory's formulation of the Second Law of thermodynamics reads: *Near a given state of a system there are states that cannot be obtained by an adiabatic process*[9] We assume $d\delta Q$ has constant rank and $\dim(M) = 5$. Using Exercice 50 stating that locally $\delta Q = \sum_{j=1}^r y_j dx_j$ or $\delta Q = dz + \sum_{j=1}^r y_j dx_j$ prove that the Second Law, as stated by Caratehodory, implies that $r = 1$ and $\delta Q = y dx$.

(c) Setting $y = \frac{1}{T}$, show that there exists functions $T, S$ such that $T dS = \delta Q$. The function $S$ is called the entropy, the function $T$ is the temperature.

(d) For a perfect gas, $\delta W = P dV$ and $\delta Q = T dS$, so we are in 5 dimensional space with coordinates $U, P, V, T, S$ describing the state of the system[10], but they are not independent for a given system : such a system is a submanifold on which $dU - P dV - T dS = 0$ so it is a 2 dimensional surface according to Exercise 50. Prove that this surface is Lagrangian in the $P, V, T, S$ space (or Legendrian in $U, P, V, T, S$) space).

(e) For a pure substance, the projection on $P, T$ is usually one-to-one, at least until we reach a phase transition : at this point $P, T$ remain constant but $S, V$ vary : for example when water boils $P, T$ remains constant (e.g. if $P = 1 bar, T = 100^0 C$), but the volume increases (and so does the entropy). In other words there are some vertical pieces on the surface. We assume $n = 2$, then denoting $\pi$ to be the projection of $T^*\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, one can prove that $(L, \pi)$ is generically locally diffeomorphic to one of the following maps

(i) $(x_1, x_2) \longrightarrow (x_1, x_2)$ (regular map)

(ii) $(x_1, x_2) \longrightarrow (x_1, x_2^2)$ (fold)

[8] In this Exercise we shall talk about adiabatic processes. The physically inclined reader who knows what this means (and already heard about entropy in the physicist's sense) will be more prone to enjoying this Exercise, however no such knowledge is required.

[9] To be rigorous, we should add "remaining in the neighbourhhod of the original state".

[10] $U$ is the total energy, $P$ the pressure, $V$ the volume, $T$ the temperature, $S$ the entropy.

(iii) $(x_1, x_2) \longrightarrow (x_1, x_2^3 - x_1 x_2)$ (cusp)

(69) (Deformation quantization) Let $(M,\omega)$ be a symplectic manifold and $\mathscr{A}$ the ring $C^\infty(M)[[h]]$ of formal power series with coefficients in $C^\infty(M)$. Here $C^\infty(M)$ is identified to $C^\infty(M) \otimes h^0$, and its product is denoted by $(f,g) \mapsto f \cdot g$. We are looking for an associative product $\star_h$ on $\mathscr{A}$ such that

$$f \star_h g = f \cdot g + \sum_{j=1}^{+\infty} B_j(f,g) h^j$$

and such that $f \star_h g - g \star_h f = \{f,g\}h + o(h)$. This is called a deformation quantization of $C^\infty(M)$ for the Poisson structure $\{,\}$. Sometimes people assume $B_1(f,g) = \frac{1}{2}\{f,g\}$ that we shall do here.

Since we do note require the convergence of the series, one talks about formal quantization. Note that associativity puts a lot of constrains on the $B_j$. The star product is said to be strongly closed if $M$ is closed and $\int_M f \star_h g \omega^n = \int_M g \star_h f \omega^n$. We denote the Weyl algebra as the algebra generated by a symplectic vector space $(V,\omega)$, starting from a symplectic basis $(e_1,..,e_n, f_1,\dots f_n)$ by adding $1, \nu$ to the basis elements and setting $[\nu, e_i] = [\nu, f_i] = 0$ $[e_i, e_j] = [f_i, f_j] = 0$ $[e_i, f_j] = -\nu\delta_{i,j}$ We set for multi-indices $\alpha, \beta$ $e^\alpha \star f^\beta$ as $e_1^{\alpha_1} \star \dots \star e_n^{\alpha_n} \star f_1^{\alpha_1} \star \dots \star f_n^{\alpha_n}$. We shall grade $W$ by setting $\deg(\nu) = 2, \deg(e_j) = \deg(f_j) = 1$ and $\deg(\nu^{\alpha_0} e^\alpha \star f^\beta) = 2\alpha_0 + \sum_{j=1}^n (\alpha_j + \beta_j)$. We write $W(k)$ for the submodule made of elements of degree $k$, and we have $W(k) \star W(l) \subset W(k+l)$ and $[W(k), W(l)] \subset \nu W(k+l-2)$

(a) Prove that $W$ is isomorphic to the algebra of polynomial differential operators on $\mathbb{R}^n$, where $e_i$ is sent to multiplication by $X_i$, $f_j$ to $\frac{\partial}{\partial X_j}$

We define the non-associative product $a \circ b = \frac{1}{2}(a \star b + b \star a)$

## 11. Comments

The Hamiltonian formulation of mechanics was hinted by Lagrange in [Lag77b; Lag77c; Lag77a], and the second edition of [Lag11] in 1811 (Part II, section V). He considers first the set of trajectories for the Kepler problem as a space (it is 5-dimensional, and adding time, becomes 6-dimensional), with the idea that pertubations of the two body problem will be determined by an evolution of the ellipse defining the Keplerian trajectories. Lagrange also introduces the "Lagrange parenthesis" a dual to the Poisson brackets: the set of Keplerian motions is a symplectic vector space, and in modern terms, Lagrange notices that the coefficients of the symplectic form are given by the "Lagrange parenthesis" that he defines. He also defines the "Hamiltonian" for a mechanical system (i.e. given by Kinetic energy + Potential) and proves that for an autonomous system this is a conserved quantity. He uses the letter $H$, but since Hamilton was then only 6 years old, it is conjectured that the letter $H$ was to honour Christian Huyghens ([Sou86]). The dual point of view, that of POisson brackets was invented by Lagrange's student, Poisson in [Poi09].

The development of general Hamiltonian dynamics may well be due to Cauchy[11] in 1831 (see [Cau31; Cau37]) where Hamiltonian equations without any reference to a mechanical system seem to appear for the first time. The Hamiltonian is denoted by $Q$ and of course he mentions the special case $H(q,p)=\frac{1}{2}|p|^2-V(q)$ (in fact for $V$ the Newtonian potential). While a general Hamiltonian formulation (compared to the Lagrangian formulation or the Kinetic + Potential formulation) does not add much in most situations, this will become crucial later on, when dealing with normal forms, as these require to make changes of variables mixing $q$ and $p$ coordinates. The same Memoir of Cauchy revisits the Lagrange parenthesis approach, shows that it is dual to the Poisson brackets and applies these to practical computation.

Shortly after and probably independently, Hamilton [Ham34] introduces again the Hamiltonian and comments : *Lagrange and, after him, Laplace and others, have employed a single function to express the different forces of a system, and so to form in an elegant manner the differential equations of its motion. By this conception, great simplicity has been given to the statement of the problem of dynamics; but the solution of that problem, or the expression of the motions themselves, and of their integrals, depends on a very different and hitherto unimagined function, as it is the purpose of this essay to show* (see [Sou86]).

Hamilton introduces on the occasion Hamilton's least action principle which claims that the solutions of Hamilton's equation are critical points of the action $\int_0^T[p(t)\dot q(t)-H(t,q(t),p(t)]\,dt$. Maupertuis's least action principle is different in that would state for mechanical systems that trajectories minimize the integral of the kinetic energy[12] but

[11] I owe this reference to Jacques Féjoz.

[12] In his paper [Mau46], Maupertuis considers that this principle proves the existence of God. His paper appears in the records of the Berlin Academy not in the "Mathematical Sciences" section, but in the "Speculative Philosophy section".

were variations are among the curves conserving energy[13]. The first one deals with the time evolution of a trajectory, the second one only the geometrical trajectory ( a priori it will not say how the particle travels on its trajectory, even though this can be recovered). For the history of the Calculus of Variations see [Gol80] and also [Lüt05], chapter 2.

That the Hamiltonian flows preserve the symplectic form was implicit in [Lag11] and his successors (see for example [Jac66]), formulated in modern form by Poincaré in [Poi90], chapter II: "Théorie des invariants intégraux". He notices that this implies that the flow also preserves the exterior powers of the symplectic form, and in particular the volume, whence the recurrence theorem 3.89.

The importance of Lagrangian submanifolds was probably introduced by Einstein in [Ein17], when looking for higher dimensional classical objects that woud lead to quantized objects. In dimension 2, these were known to be energy levels (i.e. curves) enclosing an area that is an integral multiple of $\frac{h}{2\pi}$, but Einstein was looking for a higher dimensional version and found that they should be Lagrangians having periods integral multiples of $\frac{h}{2\pi}$. This was more systematically introduced in [Mas72], and the Maslov index was introduced by [Arn67].

Darboux's theorem can be first found in [Dar82a; Dar82b] and was generalized by A. Weinstein essentially to Lagrangian submanifolds, and then by Givental in the general case (see Theorem 3.29). The study of Poisson manifolds has gained importance in the last 50 years. Beyond Lie algebras, Poisson manifolds appear as singular reduction of symplectic manifolds, in particular in the study of integrable systems.

Moser's lemma appeared in [Mos65] which was primarily concerned with volume forms. The idea of the lemma (the isotopy method) has many applications going much beyond symplectic geometry for example Morse's lemma and other applications to singularities are some of them. The question of the optimal regularity of the conjugating diffeomorphism including in the Sobolev or $C^{k,\alpha}$ class has been thoroughly studied in particular in the case of volume form (see [DM90; CDK12]). While the density of smooth symplectic maps in the set of $C^1$ symplectic maps is easy (see Exercise 4 in Chapter 4), the volume case is more involved (see [Zeh77] in the $C^{1,\alpha}$ case ($\alpha > 0$) and [Avi10] for the $C^1$ case).

The chaotic properties of Hamiltonian systems probably originate in Poincaré's paper on the three body problem ([Poi90]) where the intersection of heteroclinic orbits yields the non-integrability of the system. The Kolmogorov-Arnold-Moser theorem was discovered by Kolomogorov in the 50's (see [Kol54; Arn63], [Mos65] and [Bos86]) and has been an intense domain of study both from the theoretical or the practical point of view (see also [CG82] and [WR92]).

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[13]There was a controversy between Maupertuis and the German mathematician, König, who attributed the paternity of the least action principle to Leibniz. Voltaire in " Histoire du docteur Akakia et du natif de Saint-Malo" (1752) supports König and makes fun of Maupertuis, in particular of his least action principle. Voltaire will similarly deride Leibniz in "Candide" in 1759.

The chaos was revived in the 60's as computers could visually display what Poincaré could only imagine[14]. This started with the Lorentz attractor, then the work of Smale ([Sma67; Ano67]) and many others.

A huge literature is devoted to the variational study of Lagrangian systems. The case of celestial mechanics is historically among the oldest and most important, and is for example the source of Poincaré's last geometric theorem (see the Comments section (i.e. 6) in Chapter 6). The problem of celestial mechanics is of course that the system can have singularities, in particular collisions. This interest has been revived in recent years, with particular interest for special trajectories of $n$-body problems, whether from the theoretical viewpoint, as for example the so-called choreographies (see [CM00] and [FT04]) or the practical viewpoint as in the Low Energy Transfer for satellites ([Bel04])

Contact geometry is very closely related to symplectic geometry and we choose to stress this connection. We refer to [Gei08] for an introduction (and more) to the subject. An open question was for a long time which manifolds do admit a contact structure. According to Lutz and Martinet, all 3-manifolds do. It is easy to see that in general, the tangent bundle of the manifold must contain a codimension 1 complex subbundle. This happens to be equivalent to the existence of a complex structure on $TM\oplus\varepsilon^1$. Only in 2014 did Borman, Eliashberg and Murphy prove that conversely this "formal" contact structure implies the existence of a genuine contact structure on $M$ (see [BEM15] also for the history of the question). Since the work of Bennequin (see [Ben83]) in dimension 3 and then [Nie06] in all dimensions proved that contact structures exist in two very different flavors : the"tight contact structures" and the "overtwisted contact structures". The overtwisted ones cannot bound a symplectic manifold and moreover the property of being overtwisted has some bearing on the Weinstein conjecture (see [AH09].

[14] In [Poi92] he wrote: On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complexité du problème des trois corps, et, en général, de tous les problèmes de la dynamique où il n'y a pas d'intégrale uniforme...

CHAPTER 4

# More Symplectic differential Geometry: Reduction and Generating functions

## 1. The Metasymplectic principle

This section and the next two illustrate the following

METASYMPLECTIC PRINCIPLE. *(A. Weinstein, see [Wei73a; Wei77])*

> *Everything important is a Lagrangian submanifold.*

EXAMPLES 4.1. (1) Let $(M_i,\omega_i), i=1,2$ be symplectic manifolds and $\varphi$ a smooth map between them. Consider the graph of $\varphi$,

$$\Gamma(\varphi)=\{(x,\varphi(x))\mid x\in M_1\}\subset M_1\times M_2$$

This is a Lagrangian submanifold of $M_1\times\overline{M_2}$ where we defined $\overline{M}_2$ as the manifold $\overline{M}_2$ with the symplectic form $-\omega_2$ and the symplectic form on $M_1\times\overline{M_2}$ is given by

$$(\omega_1\ominus\omega_2)((\xi_1,\xi_2),(\eta_1,\eta_2))=\omega_1(\xi_1,\eta_1)-\omega_2(\xi_2,\eta_2)$$

It is easy to see that $\Gamma(\varphi)$ is a Lagrangian submanifold if and only if $\varphi^*\omega_2=\omega_1$. Note that if $M_1=M_2$, then $\Gamma(\varphi)\cap\triangle_M=Fix(\varphi)$.

(2) Let $(M_{\mathbb{C}},J,\omega)$ be a smooth projective manifold, i.e. a smooth manifold given by the equations

$$M=\{[z_0,\dots,z_N]\in\mathbb{C}P^N\mid P_1(z_0,\cdots,z_N)=\cdots=P_r(z_0,\cdots,z_N)=0\}$$

where $P_j$ are homogeneous complex polynomials. We shall assume the map from $\mathbb{C}^n\setminus\{0\}$ to $\mathbb{C}^r$

$$(z_0,\dots,z_n)\mapsto(P_1(z_0,\dots,z_n),\dots,P_r(z_0,\dots,z_n))$$

has zero as a regular value, so that $M$ is a smooth manifold.

real algebraic geometry is concerned with the case when the $P_j$'s have real coefficients. We then set

$$\begin{aligned}M_{\mathbb{R}}&=\{[x_0,\cdots,x_N]\in\mathbb{R}P^N\mid P_1(z_0,\cdots,z_N)=\cdots=P_r(z_0,\cdots,z_N)=0\}\\&=M\cap\mathbb{R}P^N.\end{aligned}$$

One of the problems in real algebraic geometry is to "determine the relation" between $M_{\mathbb{C}}$ and $M_{\mathbb{R}}$". It is easy to see that $M_{\mathbb{R}}$ is a Lagrangian submanifold of $(M,\omega)$ (of course, possibly empty). Notice that the same complex projective manifold can have different real parts: the simplest example is $M_{\mathbb{C}} = S^2 = \mathbb{C}P^1$, which can be embedded in $\mathbb{C}P^2$ either as the set of $[z_0, z_1, z_2]$ such that $z_0 = 0$ and then $M(\mathbb{R}) = S^1$ or as $z_1 + iz_2 = 0$ and then $M(\mathbb{R}) = \emptyset$.

(3) Lagrangian submanifold appear as generalized solutions of Hamilton-Jacobi equations (see Section 1 of Chapter 8)
(4) Lagrangian submanifolds appear as wave front set of Fourier integral operators (see [Hör71]), or singular supports of (constructible) sheaves ([KS90]).
(5) Let us consider a smooth curve on the plane. We are looking for four points on the curve forming a rectangle of given shape that is with given ratio for the length of its sides (e.g. if the ratio is 1, we are looking for a square inscribed in the curve). This is called the rectangular peg problem and was formulated[1] by O. Toeplitz in 1911 (in [Toe11]). It was solved by J.E. Green and A. Lobb in 2020 ([GL]) by showing that it is equivalent to the non-existence of a Lagrangian Klein bottle in $(\mathbb{R}^4, \sigma_4)$ a result that had been proved by V. Shevchishin in 2009 ([She09a]).

## 2. Symplectic Reduction

Let $(M,\omega)$ be a symplectic manifold and $K$ a submanifold. $K$ is said to be *coisotropic* if we have $T_xK \supset (T_xK)^\omega$ for all $x$ in $K$, i.e. $T_xK$ is coisotropic in $T_xM$. As $x$ varies in $K$, $(T_xK)^\omega$ defines a sub-bundle of $TK$ hence of $TM$. A sub-bundle of a tangent bundle is said to be integrable if there is a foliation of $K$ such that the sub-bundle is equal to the tangent bundle to the leaves and Frobenius's theorem (Theorem 3.6) states that a subbundle is integrable if and only if the Lie bracket of two vector fields tangent to the sub-bindle is tangent to the sub-bundle. .

LEMMA 4.2. *The sub-bundle $(T_xK)^\omega$ of the tangent bundle of $K$ is integrable.*

PROOF. Let $K$ be coisotropic in the symplectic manifold $(M,\omega)$, and $\mathcal{K}(x) = (T_xK)^\omega$. We want to prove that the distribution $\mathcal{K}$ is integrable. Let $X$ be a vector field tangent to $(T_xK)^\omega$, hence to $K$, and denote by $\tilde{\omega}$ the restriction of $\omega$ to $K$. Applying Cartan's formula (see page 52) we have

$$L_X\widetilde{\omega} = di_X\widetilde{\omega} + i_Xd\widetilde{\omega} = d(i_X\widetilde{\omega}) = 0$$

As a result the flow $\varphi_X^t$ preserves $\widetilde{\omega}$, hence preserves $\mathcal{K}$, the kernel of $\widetilde{\omega}$. Therefore we prove that if $X, Y$ are vector fields tangent to $\mathcal{K}$, we have $(\varphi_X)_*(Y)$ is tangent to $\mathcal{K}$. Taking the derivative at $t = 0$ we get $\frac{d}{dt}(\varphi_X^t)_* Y_{|t=0} = [X, Y] \in \mathcal{K}$. We thus proved that the Lie brackets of two vector fields tangent to $\mathcal{K}$ is tangent to $\mathcal{K}$, hence $\mathcal{K}$ is an integrable distribution.

[1] Toeplitz question was slightly different, he only looked for a square, but did not assume the curve is smooth. Without the smoothness assumption, the question is still open.

□

This integrable distribution yields a foliation of $K$, denoted by $\mathscr{C}_K$. We can check that $\omega$ induces a symplectic form (we only need to check it is non-degenerate) on the quotient space $(T_xK)/(T_xK)^\omega$. One might expect $K/\mathscr{C}_K$ to be a a "symplectic something".

Unfortunately, there is usually no quotient, the topological quotient is not even Hausdorff., so there is no manifold structure on the quotient. However, as we shall see at the end of this section, there are certain special cases, when $K/\mathscr{C}_K$ is a manifold, and therefore a symplectic manifold.

Let us now see the effect of the above operation on symplectic submanifolds. We shall need

LEMMA 4.3. *(Automatic Transversality) If $L$ is a Lagrangian in $M$ and $L$ intersects the coisotropic submanifold $K$ transversally, i.e. $T_xL + T_xK = T_xM$ for $x \in K \cap L$, then $L$ intersects the leaves of $C_K$ transversally, $T_xL \cap T_x\mathscr{C}_K = \{0\}$, for $x \in K \cap L$.*

PROOF. Recall from symplectic linear algebra that if $F_i$ are subspaces of a symplectic vector space, then

$$(F_1 + F_2)^\omega = F_1^\omega \cap F_2^\omega.$$

We know $(T_xL)^\omega = T_xL$ and $(T_xM)^\omega = \{0\}$, then the lemma follows from $T_xL + T_xK = T_xM$. □

Now, let's assume $K/\mathscr{C}_K$ is a manifold and denote the projection by $\pi : K \to K/\mathscr{C}_K$ and suppose we have a Lagrangian submanifold $L$ of $M$ such that $K$ and $L$ intersect transversally, so in particular $L \cap K$ is a manifold. Then by the above Lemma, the projection $\pi : (L \cap K) \to K/\mathscr{C}_K$ is an immersion. Indeed

$$\ker d\pi(x) = T_x\mathscr{C}_K = (T_xK)^\omega$$

and

$$\begin{aligned}\ker d\pi(x)|_{T_x(L\cap K)} &\subset \ker d\pi(x) \cap T_xL \\ &\subset (T_xK)^\omega \cap T_xL = \{0\}.\end{aligned}$$

Therefore $d\pi(x)|_{L\cap K}$ is injective and $\pi|_{L\cap K}$ is immersion.

DEFINITION 4.4. *Let $K$ be a coisotropic submanifold in $(M,\omega)$ such that $K/\mathscr{C}_K$ is a smooth manifold. Let $L$ be a Lagrangian in $(M,\omega)$ such that $L$ is transverse to $K$. Then $L_K = (L \cap K)/\mathscr{C}_K$ is called the symplectic reduction of $L$ by $K$.*

and we have

PROPOSITION 4.5. *Under the assumptions of the definition, the symplectic reduction of a Lagrangian submanifold is an immersed Lagrangian in $(K/\mathscr{C}_K, \omega_K)$.*

PROOF. By automatic transversality (Lemma 4.3) $L \cap (T_x K)\omega = \{0\}$ and this implies that the differential of the projection from $L \cap K$ to $K/\mathscr{C}_K$ is injective, i.e. the map is an immersion. The only thing left to check is that $L_K$ is Lagrangian. Let $\tilde{\omega}$ be the induced symplectic form on $K/\mathscr{C}_K$ and $\tilde{v}$ a tangent vector to $L_K$. Assume the preimage of $\tilde{v}$ is $v$, a tangent vector to $L$. Since $L$ is Lagrangian and $\tilde{\omega}$ is induced from $\omega$, we know $L_K$ is isotropic. It's Lagrangian by a dimension count. □

The same argument shows that the reduction of an isotropic submanifold (resp. coisotropic submanifold) is isotropic (resp. coisotropic).

The next proposition tells us that conversely, if $K/\mathscr{C}_K$ is a manifold, Hamiltonians on $(K/\mathscr{C}_K, \omega_K)$ can be lifted to $(M,\omega)$.

PROPOSITION 4.6. *Let $\varphi_H^t$ be a Hamiltonian flow on $(K/\mathscr{C}_K, \omega_K)$. Then there is a Hamiltonian $\widetilde{H}$ on $(M,\omega)$ such that $\varphi_{\widetilde{H}}^t$ induces $\varphi_H^t$. In other words $\varphi_{\widetilde{H}}^t$ preserves $K$ and sends leaves of $\mathscr{C}_K$ to leaves of $\mathscr{C}_K$, and the induced map on $(K/\mathscr{C}_K, \omega_K)$ is $\varphi_H^t$.*

PROOF. Let $\widetilde{H}$ be any function on $[0,1] \times M$ such that $\widetilde{H}_{|K} = H \circ \pi$ where $\pi : K \longrightarrow K/\mathscr{C}_K$ is the projection. Then $d\widetilde{H}$ vanishes on the leaves of $\mathscr{C}_K$ that is on $(T_x K)^\omega$. Thus $X_{\widetilde{H}}$ is orthogonal to $(T_x K)^\omega$, i.e. tangent to $T_x K$, hence the flow preserves $K$. Then since $d\varphi_{\widetilde{H}}$ is symplectic and preserves $K$, it will preserve $(TK)^\omega$, i.e. sends $(T_x K)^\omega$ to $(T_z K)^\omega$ where $z = \varphi_{\widetilde{H}}^t(x)$, and this implies that $\varphi_{\widetilde{H}}^t$ preserves the leaves of $\mathscr{C}_K$. □

EXAMPLES 4.7.

(1) Let $N$ be a symplectic manifold, and $V$ be any smooth submanifold. Define
$$K = T_V^* N = \{(x,p) | x \in V, p \in T_x^* N\}.$$
This is a coisotropic submanifold, and its coisotropic foliation $\mathscr{C}_K$ is given by specifying the leaf through $(x,p) \in K$ to be
$$\mathscr{C}_K(x,p) = \{(x,\tilde{p}) \in K \mid \tilde{p} - p \text{ vanishes on } T_x V\}.$$
It is natural to identify $K/\mathscr{C}_K$ with $T^*V$.

Symplectic reduction in this case, sends a Lagrangian in $T^*N$ to a Lagrangian in $T^*V$.

(2) Let $N_1$, $N_2$ be smooth manifolds and $N = N_1 \times N_2$. We write local coordinates near a point in $T^*N$ as
$$(x_1, p_1, x_2, p_2).$$
where $(x_1, p_1) \in T^*N_1, (x_2, p_2) \in T^*N_2$. Define $K = \{(x_1, p_1, x_2, p_2) | p_2 = 0\}$. The tangent space of $K$ at a point $z = (x_1, p_1, x_2, p_2)$ is given by
$$\{(x_1, p_1, x_2, 0) \mid (x_1, p_1) \in T^*N_1, x_2 \in N_2\}$$
and
$$(T_{(x_1,p_1,x_2,0)}K)^\omega = \{(x_1, p_1, u_2, 0) \mid u_2 \in N_2\}$$
Then we can identify $K/\mathscr{C}_K$ with $T^*N_1$.
Symplectic reduction sends a Lagrangian in $T^*N$ to a Lagrangian in $T^*N_1$.

(3) Let us generalize the above situation to the case of a fibration $\pi : P \longrightarrow N$. Then

$$K_\pi = \{(y,p_y) \mid p_y = 0 \text{ on } \pi^{-1}(\pi(y))\}$$

the the leaf of the coisotropic foliation through $(y_0, p_0)$ is

$$\mathscr{C}_\pi = \{(y,p_y) \in K_\pi \mid y \in \pi^{-1}(\pi(y_0)\}$$

and $K_\pi/\mathscr{C}_\pi$ can be identified to $T^*N$.

(4) Let us consider $S^{2n+1}$ in $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$. Then as a hypersurface $S^{2n+1}$ is automatically coisotropic, and the kernel of $\sigma_{n+1}$ is given by the vector field $X(z_0,...,z_n) = i(z_0,....,z_n)$, so that it is the tangent vector field to the $S^1$-action given by $\theta \mapsto (e^{i\theta}z_0,...,e^{i\theta}z_n)$. The quotient is therefore a symplectic manifold and it is equal to $\mathbb{C}P^n$.

(5) (Marsden-Weinstein reduction) Let us consider a Hamiltonian $G$ action of the connected Lie group $G$ on $M$ with moment map $\mu : M \longrightarrow \mathfrak{g}^*$. This means that for each $v \in \mathfrak{g}$ the one-parameter subgroup of Hamiltonian maps given by $(t,x) \mapsto \exp(tv)\cdot x$ is the flow of the Hamiltonian $H_v(x) = \mu(x)\cdot v$. For $\mu$ to be a moment map, we also require that $\{H_v, H_w\} = H_{[v,w]}$. Note that there is a coadjoint action of $G$ on $\mathfrak{g}^*$ as the dual of the linearization of conjugation : i.e. of $(g,X) \mapsto \mathrm{ad}(g)X$ defined by $\exp(t\mathrm{ad}(g)X) = g^{-1}\exp(tX)g$. We claim that the moment map is $G$ equivariant, or equivalently that the comoment map $\mu^* : \mathfrak{g} \longrightarrow C^\infty(M)$ defined by $\langle \mu(x), Y\rangle = \mu^*(Y)(x)$ is equivariant for the action of $G$ on $M$ and the action ad on $\mathfrak{g}$. Indeed, the linearized statement of this would be that

$$\langle d\mu(x)Z, Y\rangle = \langle \mu(x), [Z,Y]\rangle$$

since the linearization of $\mathrm{ad}(g)$ is the map $Y \mapsto [Z,Y]$. But $Z$ is by assumption the Hamiltonian vector field of $H_Z$, so

$$\begin{aligned}\langle d\mu(x)Z, Y\rangle = dH_Z\cdot Y = dH_Z(X_{H_Y}) = \{H_Z, H_Y\} = H_{[Z,Y]}\\ = \langle \mu(x), [Z,Y]\rangle\end{aligned}$$

It is easy to see that the converse holds: we have a moment map, if and only if the map $v \longrightarrow H_v$ is $G$-equivariant. Now let $\xi \in \mathfrak{g}^*$ be an element fixed by the coadjoint action ($\xi = 0$ will always do !) and consider $\mu^{-1}(\xi)$. Then $G$ acts on $\mu^{-1}(\xi)$ and if $\xi$ is a regular value of $\mu$, then $\mu^{-1}(\xi)$ is a submanifold, and the $G$ action is locally free[2] . Then we claim that $\mu^{-1}(\xi)$ is coisotropic and the coisotropic leaves are the orbits of $G$ : by definition

$$T_x\mu^{-1}(\xi) = \bigcap_{Z\in\mathfrak{g}} \mathrm{Ker}(\langle d\mu(x), Z\rangle)$$

where $\mathrm{Ker}(\langle d\mu(x), Z\rangle)$ means $\{v \mid \langle d\mu(x)v, Z\rangle = 0\}$ and its $\omega$-orthogonal is generated by the $(\mathrm{Ker}dH_Z(x))^\omega = X_{H_Z}(x) = Z(x)$ and this describes the tangent

[2]Indeed, the action of $\exp(tv)$ locally fixes $x$ if $dH_v(x) = 0$ that is $\Im(d\mu(x)) \subset v^\perp$, where $v^\perp = \{\xi \in \mathfrak{g}^* \mid \langle \xi, v\rangle = 0\}$

space to the orbit at $x$. Now if we assume the $G$-action is **free and proper** on $\mu^{-1}(\xi)$, then $M_G^\xi = \mu^{-1}(\xi)/G$ is a manifold and it inherits by our construction a symplectic form, $\omega_G$. The symplectic manifold $(M_G^\xi, \omega_G)$ is called the Marsden-Weinstein reduction of $M$ by $G$ (at $\xi$). For example for the $S^1$ action on $\mathbb{C}^{n+1}$ we have $\mu(z_0, ..., z_n) = \sum_{j=0}^n |z_j|^2$ and since $1 \in \mathbb{R}$ is $ad$ invariant (trivially so, since $S^1$ is abelian), we get $\mu^{-1}(1)/S^1$ that is $\mathbb{C}P^n, \sigma_{FS})$.

**2.1. Lagrangian correspondences.** Let $\Lambda$ be a Lagrangian submanifold in $\overline{T^*X} \times T^*Y$. Then it induces a correspondence from $T^*X$ to $T^*Y$ (or should we say, between $T^*X$ and $T^*Y$) as follows: consider a set $A \subset T^*X$, and $A \times \Lambda \subset T^*X \times \overline{T^*X} \times T^*Y$. Now, denote by $\Delta_{T^*X}$ the diagonal in $T^*X \times \overline{T^*X}$. The submanifold $K = \Delta_{T^*X} \times T^*Y$ is coisotropic, and we define $\Lambda \circ A$ as $(A \times \Lambda) \cap K/\mathscr{K} \subset K/\mathscr{K} = T^*Y$. When $A$ is a submanifold, then $\Lambda \circ A$ is an immersed submanifold provided $A \times T^*Y$ is transverse to $\Lambda$.

If $A$ is isotropic or coisotropic, it is easy to check that the same will hold for $\Lambda \circ A$. In particular if $L$ is a Lagrangian submanifold, then so is $\Lambda \circ L$ and can alternatively be defined as follows : take the symplectic reduction of $\Lambda$ by $L \times \overline{T^*Y}$. This is well defined at least when $L$ is generic. In particular if $\Lambda_1$ is a correspondence from $T^*X$ to $T^*Y$ and $\Lambda_2$ a correspondence from $T^*Y$ to $T^*Z$ then

$$\Lambda_2 \circ \Lambda_1 = \{(x, \xi, z, \zeta) \mid \exists (y, \eta), (x, \xi, y, \eta) \in \Lambda_1, (y, \eta, z, \zeta) \in \Lambda_2\}$$

Note that $\Lambda^a$ (sometimes denoted as $\Lambda^{-1}$) is defined as $\Lambda^a = \{(x, \xi, y, \eta) \mid (y, \eta, x, \xi) \in \Lambda\}$. This is a Lagrangian correspondence from $T^*Y$ to $T^*X$. The composition $\Lambda \circ \Lambda^a \subset T^*X \times \overline{T^*X}$ is, in general, not equal to the identity (i.e. to $\Delta_{T^*X}$, the diagonal in $T^*X$), even though this is the case if $\Lambda$ is the graph of a symplectomorphism. A fundamental example is the correspondence associated to a smooth map $f : X \longrightarrow Y$. Then

$$\Lambda_f = \{(x, \xi, y, \eta) \in T^*X \times T^*Y \mid f(x) = y, \eta \circ df(x) = \xi\}$$

Then if $g : Y \longrightarrow Z$, we have $\Lambda_{g \circ f} = \Lambda_g \circ \Lambda_f$. A more general example is obtained from $C$ a correspondence from $X$ to $Y$, that is just a submanifold in $X \times Y$. Then

$$\Lambda_C = \{(x, \xi, y, \eta) \mid (x, y) \in C,\ \xi dx + \eta dy = 0 \text{ on } T_{(x,y)}C\}$$

is a Lagrangian correspondence. Note that $\Lambda_C = \overline{\nu^* C}$ is the opposite of the conormal of $C$ in $T^*(X \times Y)$.

EXERCISES 4.8. (1) Explicit $\Lambda \circ \Lambda^a$ and compute $\Lambda \circ \Lambda^a$ for $\Lambda = V_x \times V_y$, where $V_x$ is the cotangent fiber over $x$.

(2) Let $f : X \longrightarrow Y$ be a smooth map. For $L$ a Lagrangian in $T^*Y$, define $L_f = \Lambda_f^{-1} \circ L$ that is

$$L_f = \left\{(y, p_y) \mid y = f(x), p_y \circ df(x) = p_x \text{ where } (x, p_x) \in L\right\}$$

(a) Prove that if $L_f$ is smooth and of dimension $\dim(X)$ then it is an exact Lagrangian

(b) Prove that if $f$ is the injection of a submanifold $V$ in $M$, then $L_f = L_V$, as defined in Example 4.7 (1)
(c) Prove that if $f$ is a fibration from $X$ to $Y$ then $L_f$ coincides with the Lagrangian defined in Example 4.7 (3).

## 3. Spaces of Lagrangians

Because we shall mainly deal with exact Lagrangians, and we would like to have a grading, we shall define a space of exact graded Lagrangians as follows.

**3.1. Lagrangian branes.** For $M$ a symplectic manifold, (here we only need $M = T^*N$), if $\Lambda(M)$ is the bundle of Lagrangians subspaces of the tangent bundle to $M$, with fiber the Lagrangian Grassmannian $\Lambda(T_zM) \simeq \Lambda(n)$, we denote by $\widetilde{\Lambda}(M)$ the bundle induced by the universal cover $\widetilde{\Lambda}(n) \longrightarrow \Lambda(n)$.

Given a Lagrangian $L$, we assume we have a lifting of the Gauss map $G_L : L \longrightarrow \Lambda(T^*N)$ given by $x \mapsto T_xL$ to a map $\widetilde{G}_L : L \longrightarrow \widetilde{\Lambda}_p(T^*N)$. This is called a **grading of** $L$ (see [Sei00]). Given a graded $L$, the canonical automorphism of the covering induces a new grading and we denote it as $T(L)$ or $L[1]$, and its $q$-th iteration as $T^q(L)$ or $L[q]$. The grading will yield an absolute grading for the Floer homology or its analogue the Generating function homology (see Proposition and Defintion 7.7) of a pair $(L_1, L_2)$ and hence for the complex of sheaves in the Theorem stated below. We shall seldom mention explicitly the grading, but notice that for exact Lagrangians in $T^*N$, a grading always exists since the obstruction to its existence is given by the Maslov class (see Chapter **??**, Subsection **??**) , and for exact Lagrangians in $T^*N$ the Maslov class vanishes, as was proved by Kragh and Abouzaid (see [Kra13], and also the sheaf-theoretic proof by [Gui12]).

DEFINITION 4.9. *The set $\mathscr{L}(M, d\lambda)$ is the set of Lagrangian branes, that is triples $\widetilde{L} = (L, f_L, \widetilde{G}_L)$ where L is a compact connected*[3] *exact graded Lagrangian, $f_L$ a primitive of $\lambda_{|L}$ and $G_L$ a grading. We sometimes talk about an exact Lagrangian, and this is just the pair $(L, f_L)$. We write $\mathfrak{L}(T^*N)$ for the set of L such that for some function $f_L$, $(L, f_L)$ is an exact Lagrangian. Forgetting about $f_L$ we get the forgetful functor $\mathscr{L}(M, d\lambda) \longrightarrow \mathfrak{L}(M, d\lambda)$.*

When $f_L$ is implicit we only write $L$, for example $0_N$ means $(0_N, 0)$ and graph$(df)$ means (graph$(df), f)$. For $\widetilde{L} = (L, f_L)$ and $c$ a real constant, we write $\widetilde{L} + c$ for $(L, f_L + c)$. Considering Hamiltonian diffeomorphisms as special correspondences, that is Lagrangians in $\overline{T^*N} \times T^*N$ we can consider the corresponding branes, and denote this

[3]It is often the case -for example in the cotangent bundle of a compact connected manifold- that exact Lagrangians must in fact be connected (this follows from [Kra13] who proved that an exact Lagrangian with vanishing Maslov class must intersect). Here we shall assume this for simplicity.

space by $\mathcal{DH}am(T^*N)$. In particular an isotopy in $\mathcal{DH}am(T^*N)$ specifies the Hamiltonian generating the isotopy (and not just a Hamiltonian defined up to adding a constant). In the compact supported case, we may impose that the Hamiltonian is compact supported, so there is no constant to add.

REMARK 4.10. One may check that the image of the forgetful functor $\mathscr{L}(M,d\lambda) \longrightarrow \mathfrak{L}(M,d\lambda)$ is the set of exact Lagrangians with vanishing Maslov class.

Our goal is to describe Lagrangian submanifolds in $T^*N$. Let $\lambda = pdq$ be the Liouville form of $T^*N$. Given any 1-form $\alpha$ on $N$, we can define a smooth manifold

$$G_\alpha = \{(q,\alpha(q)) | x \in N, \alpha(x) \in T_q^*N\} \subset T^*N.$$

recall from Example 3.17 (2) that $G_\alpha$ is Lagrangian if and only if $\alpha$ is closed.

This is equivalent to specifying a lift of $L$ to a Legendrian $\widetilde{L}$ in the contactization of $(M,d\lambda)$ (see Definition 3.76). In particular, $G_\alpha$ is exact if and only if $\alpha = df$ for some function $f$ on $N$. In this case,

$$G_\alpha \cap 0_N = \{q | \alpha(q) = df(q) = 0\} = Crit(f),$$

where $0_N$ is the zero section of $TN$. Quite often $f_L$ is implicit and we just write $L$ instead of $(L, f_L)$. For example we write $0_N$ instead of $(0_N, 0)$ for the zero section, or $G_{df}$ for $(G_{df}, f)$.

Finally, since most of the time we deal with exact Lagrangian Hamiltonianly isotopic to the zero section we set

DEFINITION 4.11. *We denote by $\mathscr{L}_0(T^*N,\sigma)$ the connected component of the zero section in $\mathscr{L}(T^*N,\sigma)$ and by $\mathfrak{L}_0(T^*N,\sigma)$ the connected component of the zero section in $\mathfrak{L}(T^*N,\sigma)$.*

REMARK 4.12. Note that the existence of a grading can be deduced from the existence of a G.F.Q.I. (see Exercice **??**).

### 3.2. The effect of Hamiltonian isotopies.

PROPOSITION 4.13. *The set of Hamiltonians acts on the set $\mathscr{L}(M,d\lambda)$.*

PROOF. The proposition claims that if $H$ is a Hamiltonian and $L_0$ an exact Lagrangian, then $L_1 = \varphi_H(L_0)$ is an exact Lagrangian. Moreover $f_{L_1}$ is determined by $L_0$ and $H$ (but not just by the Hamiltonian map). Indeed, let $i_{L_0} : L_0 \longrightarrow T^*N$ be the inclusion of $L_0$, so that $i_{L_0}^*(\lambda_N) = df_{L_0}$, then $(\varphi_H^t \circ i_{L_0})^*\lambda_N$ satisfies

$$\frac{d}{dt}(\varphi_H^t \circ i_{L_0})^*\lambda_N = i_{L_0}^*(\varphi_H^t)^*(L_{X_H}\lambda_N) = i_{L_0}^*(\varphi_H^t)^*(d(i_{X_H}\lambda_N) + i_{X_H}d\lambda) =$$
$$i_{L_0}^*(\varphi_H^t)^*[d(i_{X_H}\lambda_N - H)]$$

We thus see that $(\varphi_H^t \circ i_{L_0})^*\lambda_N$ is exact and there is a natural choice for the primitive, given by

$$f_{L_1}(i_1(x)) = f_{L_0}(i_0(x)) + \int_0^1 i_{L_0}^*(\varphi_H^t)^*[(i_{X_H}\lambda_N - H)]dt$$

□

Again by abuse of language, we often just write $\varphi_H^1(L_0)$ for the action of $H$ on $(L_0, f_{L_0})$.

The next Proposition shows that conversely, families of exact Lagrangians are induced by Hamiltonian isotopies

PROPOSITION 4.14. *Let $(M, d\lambda)$ be an exact symplectic manifold*

*(1) Let $t \mapsto L_t$ be a smooth family of exact Lagrangians. Then there exists a Hamiltonian isotopy $\varphi_H^t$ such that $\varphi_H^t(L_0) = L_t$.*
*(2) If $j : D^k \times L_0 \longrightarrow \mathscr{L}(M,\omega)$ is smooth, then there is a smooth family of Hamiltonian maps $\varphi_u$ such that $\varphi_u(L_0) = j(u, L_0)$.*
*(3) (Serre fibration property) Let $j : D^k \times I \longrightarrow \mathscr{L}(M,\omega)$ is a family of exact Lagrange embeddings and $\varphi : D^k \times \{0\} \longrightarrow Ham(M,\omega)$ such that $\varphi_u|\, L_0 = j_u$, then there exits an extension $\varphi_{u,t}$ of $\varphi$ to $D^k \times I$ such that $\varphi_{u,t}(L_0) = j_{(u,t)}(L_0)$.*

PROOF. For the first result we first use Weinstein's theorem to extend the embedding of $L$ as an embedding $\Phi$ of a neighbourhood of the zero section of $T^*L$. Now $L'_t = \Phi^{-1}(L_t)$ is well defined for $t$ small enough, and is a graph over the zero section. It is of course Lagrangian, so of the form $G_{\alpha_t}$ , and the exactness assumption implies that $\alpha_t$ is exact hence of the from $df_t$. But then $L'_t$ is the image of $0_N$ by the flow of $H(t,q,p) = \frac{d}{dt} f_t(q)$. Indeed, $\varphi_H^t(q,p) = (q, p + df_t(q)$. To extend $H \circ \Phi^{-1}$ to $M$ we must first truncate $H$ near the boundary of the Weinstein neighbourhood using a cut-off function. This does not change anything as long as we do not modify $H$ on

$$\bigcup_{t\in[0,\varepsilon]} L'_t = \bigcup_{t\in[0,\varepsilon]} \varphi_H^t(0_N)$$

Now $H \circ \Phi^{-1}$ yields a Hamiltonian that will send $L_0$ to $L_t$. By the same argument for each $s$ there exists $\varepsilon > 0$ and a Hamiltonian $H_s$ such that for $t \in ]s-\varepsilon, t+\varepsilon[$ we have $\varphi_{H_s}^t(L_s) = L_t$. By compactness we can divide $[0,1]$ in intervals of size $1/N$ so that for $H_k = H_{k/N}$ we have $\varphi_{H_k}^{[k/N,t]}(L_{k/N}) = L_t$ for all $t \in ]\frac{k-1}{N}, \frac{k+1}{N}[$ (here $\varphi_{H_k}^{k/N} = \mathrm{Id}$). Writing as usual $\varphi_H^{[a,b]}$ for the flow of $X_H$ between times $a$ and $b$, we set

$$\varphi^t = \varphi_N^{[\tau_{N-1}(t),\tau_N(t)]} \circ \varphi_{N-1}^{[\tau_{N-2}(t),\tau_{N-1}(t)]} \circ \dots \circ \varphi_2^{[\tau_1(t),\tau_2(t)]} \circ \varphi_1^{[0,\tau_1(t)]}$$

This is well defined since either $\tau_{k-1}(t) = \tau_k(t)$ (and then by convention $\varphi_k^{[\tau_{k-1}(t)=\tau_k(t)]} = \mathrm{Id}$ or $\frac{k-1}{N} \leq \tau_{j-1}(t) \leq \tau_j(t) \leq \frac{k+1}{N}$.

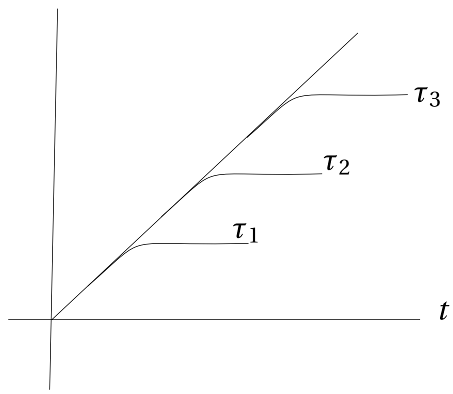


FIGURE 1. The functions $\tau_j$

One can readily check that for if the $\tau_k$ are suitably chosen as on the above figure, then $\varphi^t$ is the required Hamiltonian isotopy.

The same argument works on $I^k$, arguing by induction and using a parametrized version of Weinstein's theorem. Let $v \in I^{k-1}$ and $\Phi_v$ a symplectic embedding of a neighbourhood of $L_0$ in $T^*L_0$ in $(M,\omega)$ such that $\Phi_v(L_0) = j_{(v,0)}(L_0)$. Then by induction $\Phi_v(L_0) = \varphi^1_{H_v}(L_0)$ and for $t$ small enough $j(v,t)(L_0) = \varphi^t(j(v,0)(L_0)) = \varphi^t \circ \varphi^1_{H_v}(L_0) = \varphi^1_{H_{(v,t)}}(L_0)$. As above, gluing these together we get a Hamiltonian family parametrized by $I^{k-1} \times I$ such that $\varphi^1_{H_{v,t}}(L_0) = j(v,t)(L_0)$.

Finally if we have a family $j(v,t)$ of Lagrangian embeddings and a family $H_v$ such that $\varphi^1_{H_v}(L_0) = j(v,0)(L_0)$ then if $K_{v,t}$ is given by the previous lift, we set

$$\varphi_{(v,t)} = \varphi_{K_{v,t}} \circ \varphi^{-1}_{K_{v,0}} \circ \varphi^1_{H_v}(L_0)$$

□

REMARK 4.15. (1) The above proposition means that the map $\varphi \mapsto \varphi(L_0)$ from $\mathrm{DHam}_c(M,\omega)$ to $\mathscr{L}(M,\omega)$ the space of exact Lagrangians is a Serre fibration. This implies that for any CW pair[4] $(X,A)$ a map $j : X \times I \longrightarrow \mathscr{L}(M,\omega)$ such that it has a lift $\varphi^1_{H_{(u,t)}}$ for $u \in X \times \{0\} \cup A \times I$, to $\mathrm{Ham}(M,\omega)$, then there is a lift of $j$ to $\varphi^1_{H_{(u,t)}}$ on $X \times I$ such that $\varphi^1_{H_{(u,t)}}(j(u,0)(L_0)) = L_{(u,t)}$. This follows immediately from the following remarks
   (a) a lifting property only depends on the homotoy type of the pair
   (b) it holds for $D^k \times I \cup S^{k-1} \times I$ since$(D^k \times I, D^k \times I \cup S^{k-1} \times I)$ has the same homotopy type as $(D^k \times I, D^k \times \{0\})$
   (c) an induction argument, obtaining $X$ from $A$ by gluing discs attached by their boundary

[4]i.e. such that up to a homotopy equivalence, $X$ is obtained from $A$ by inductively gluing discs of non-decreasing dimensions along their boundary. This includes any pair of compact manifolds.

(2) If we replace $D^k$ by a compact polyhedron $K$ in (2), we do not know if we can find a family $u \mapsto H_u$. This is equivalent to stating that $\Omega\mathrm{DHam}_c(T^*N)$ is contractible, since we have a fibration

$$\Omega\mathrm{DHam}_c(T^*N) \longrightarrow \mathscr{P}\mathrm{DHam}_c(T^*N) \longrightarrow \mathrm{DHam}_c(T^*N)$$

where $\mathscr{P}\mathrm{DHam}_c(T^*N)$ is the set of paths starting from the identity in $\mathrm{Ham}(T^*N)$, and the map to $\mathrm{DHam}_c(T^*N)$ is the projection to the endpoint. In particular $\mathscr{P}\mathrm{DHam}_c(T^*N)$ is the same as the space of Hamiltonians (up to functions depending only on $t$). So if we want that any map from $K$ to $\mathrm{Ham}(T^*N)$ lifts to $\mathscr{P}\mathrm{DHam}_c(T^*N)$ we must have that $\Omega\mathrm{DHam}_c(T^*N)$ is contractible. We have no idea whether this holds -even understanding $\pi_1(\mathrm{DHam}_c(T^*N))$ is quite open except in very special cases (see however [Sei97; McD10; LDP99; AL17]). Note that if we are only interested in the image of $K$ from the geometric point of view (i.e. the set of $\varphi_u$for $u \in K$ and not the way it is parametrized), it is worth mentioning there is another polyhedron $PK$ (the path space of $K$, which is contractible) and a map $\pi : PK \longrightarrow K$ such that $\jmath \circ \pi$ has a lift.

Note that if $\omega = d\lambda$ is exact, an element in $\varphi_H \in \mathrm{DHam}_c(M, d\lambda)$ defines an element in $\mathscr{L}(M \times M, d\lambda \ominus d\lambda)$ as the correspondence $\Gamma(\varphi)$ obtained from $(\Gamma(\mathrm{Id}), 0) = (\Delta_M, 0)$ by applying $\mathrm{id} \times \varphi_H$. In other words we define

$$f_H(z, \varphi_H(z)) = -\int_0^1 \left[z^*\lambda - H(s, z(s))\right] ds$$

where $z(s) = \varphi_H^s(z)$ and we define $\Gamma(\varphi_H^1, f_H)$ to be the graph of $\varphi_H$ in $\mathscr{L}(M \times M, d\lambda \ominus d\lambda)$.

## 4. Generating functions

One of the main motivating questions in symplectic topology has been the

Arnold conjecture. If $N$ is a closed manifold, $\varphi \in \mathrm{DHam}(T^*N, \sigma_N)$ and $L = \varphi(0_N)$, then $\#(L \cap 0_N) \geq \mathrm{cat}_{\mathrm{LS}}(N)$, where $\mathrm{cat}_{\mathrm{LS}}(N)$ is the minimal number of critical points for a function on $N$.

Remark 4.16. If $L$ is exact, $C^1$ close to $0_N$, then $L = L_{df}$. Therefore, $\#(L \cap 0_N) \geq 2$, if we assume $N$ is compact. ($f$ has at least two critical points, corresponding to maximum and minimum, and we shall find more with Lusternik-Schnirelman's theory, see [LS29; LS34].)

We shall see how the use of generating functions allows us to prove the conjecture in some cases.

Definition 4.17. *A* **generating function with general fiber** *over $N$ is a smooth function $S : E \to \mathbb{R}$ where $E$ is a fiber bundle over $N$, such that*

*1) The map*

$$(q, \xi) \quad \mapsto \quad \frac{\partial S}{\partial \xi}(q, \xi)$$

*has zero as a regular value. As a result the* **fiberwise critical locus**, $\Sigma_S = \{(q,\xi)|\frac{\partial S}{\partial \xi}(q,\xi) = 0\}$ *is a submanifold. (Note that* $\partial S/\partial\xi$ *is a vector of dimension* $k$*, so* $\Sigma_S$ *is a manifold with the same dimension as* $N$*, but may have a different topology.)*

*2)*

$$\begin{array}{rccc} i_S: & \Sigma_S & \to & T^*N \\ & (q,\xi) & \mapsto & (q, \frac{\partial S}{\partial q}(q,\xi)) \end{array}$$

*has image* $L = L_S$*. Then* $(L_S, S\circ i_S)$ *is the exact Lagrangian generated by* $S$*.*

*We reserve the term* **generating function** *for the case where* $E$ *is a vector bundle.*

LEMMA 4.18. *Given* $S$ *satisfying 1) of the definition then* $L_S$ *defined by 2) is an immersed Lagrangian in* $T^*N$*.*

PROOF. Since $S$ is a function from $E = N\times\mathbb{R}^k$ to $\mathbb{R}$, the graph of $dS$ in $T^*(N\times\mathbb{R}^k)$ is a Lagrangian in $T^*(N\times\mathbb{R}^k)$. We will use the symplectic reduction as in the Example 4.7(2) in the previous Section. Define $K$ as a submanifold in $T^*(N\times\mathbb{R}^k)$,

$$K = T^*N\times\mathbb{R}^k\times\{0\}.$$

$K$ is coisotropic as shown in Example 4.7(2). Locally, the graph of $dS$ is given by

$$G_{dS} = \left\{(q,\xi,\frac{\partial S}{\partial q}(q,\xi),\frac{\partial S}{\partial \xi}(q,\xi))\right\}.$$

Then

$$\Sigma_S = G_{dS}\cap K.$$

The regular value condition in 1) ensures that $G_{dS}$ intersects $K$ transversally. By symplectic reduction, we know $i_S$ is an immersion and $L_S$ is a Lagrangian in $T^*N$ because $G_{dS}$ is Lagrangian in $T^*(N\times\mathbb{R}^k)$. □

REMARK 4.19. If $L_S$ is embedded, we have a 1-to-1 correspondence

$$L_S\cap 0_N \simeq Crit(S).$$

This is why it is important to find Generating functions for Lagrangians, as it reduces the Arnold conjecture to a problem about critical points.

**Question:** Which $L$ have a generating function?

**Answer:** (see [Gir90]) $L$ has a generating function if and only if it satisfies the following condition : the tangent bundle $TL$ must be stably homotopic to the vertical bundle of $T^*N$.

For local existence, we have

PROPOSITION 4.20. *(Hamilton, Jacobi, [Ham34; Ham35; Jac66], Maslov, Hörmander [Mas72; Hör71])*

*Let* $L$ *be a Lagrangian submanifold and* $(q_0,p_0)\in L$*. Then there is a generating function* $S$*, such that* $L_S = L$ *in a neighbourhood of* $(q_0,p_0)$*.*

PROOF. Note that if $T_{(q_0,p_0)}L$ is transverse to the vertical, $L$ is locally a graph of a closed form, hence locally the graph of an exact form. In the general case, up to a permutation of variables, $T_{(q_0,p_0)}L$ is transverse to $p_1 = ...., p_r = q_{r+1} = ... = q_n = 0$. Indeed given an injective map from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$, we can find $n$ coordinates in $\mathbb{R}^{2n}$ such that composing with the projection we get an isomorphism[5], hence $L$ is locally the graph of $f(q_1,....q_r,p_{r+1},...,p_n)$. Setting $\overline{q} = (q_1,...,q_r), \overline{p} = (p_1,...,p_r), \widehat{q} = (q_{r+1},....,q_n), \widehat{p} = (p_{r+1},....,p_n)$ we have that $L$ is given by

$$p_j = \frac{\partial f}{\partial q_j}(\bar{q},\bar{p}), q_k = \frac{\partial f}{\partial p_k}(\overline{q},\widehat{p})$$

for $1 \leq j \leq r, r+1 \leq k \leq n$. We then set

$$S(\bar{q},\hat{q},\xi) = f(\bar{q},\xi) - \langle \xi, \hat{q}\rangle$$

and this is the generating function we are looking for. □

**4.1. Transversality for Lagrangian submanifolds.** We here sketch the fact that transversality holds for Lagrangian submanifolds the same way it holds for standard ones. This follows by applying transversality to the generating function, which locally always exist.

To be completed

## 5. G.F.Q.I. existence and uniqueness results

DEFINITION 4.21. *Let $N$ be a closed manifold and $S$ be a generating function on $N \times \mathbb{R}^k$. We say that $S$ is* **quadratic at infinity** *if there exists a non-degenerate quadratic form $Q$ on $\mathbb{R}^k$ such that*

$$S(q,\xi) = Q(\xi) \quad \textit{for } |\xi| >> 0.$$

For simplicity, we will use G.F.Q.I. to mean Generating Function Quadratic at Infinity.. Unfortunately many constructions starting with G.F.Q.I. yield generating functions that are only asymptotically quadratic. For example if $S_1$ is a G.F.Q.I. for $L_1$ and $S_2$ a G.F.Q.I. for $L_2$ then $S(x,y,\xi,\eta) = S_1(x,\xi) + S_2(y,\eta)$ is a generating function for $L_1 \times L_2$, but is not quadratic at infinity. The following Proposition allows us to recover a genuine G.F.Q.I. .

PROPOSITION 4.22. *Let $S$ be a generating function of $L$ such that $\|\nabla(S-Q)\|_{C^0} \leq C$ We say that such a function is* **asymptotically quadratic at infinity**. *Then if $L$ has a generating function asymptotically quadratic at infinity then there exists $\tilde{S}$, a G.F.Q.I. for $L$.*

PROOF. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that

(1) $\rho \equiv 1$ on $[0, A]$
(2) $\rho \equiv 0$ on $[B, +\infty)$

[5]Because for $n \leq m$, an $n \times m$ non-singular matrix has a non zero $n \times n$ minor.

(3) $-\varepsilon \leq \rho' \leq 0$ on $\mathbb{R}_+$

Such a function exists provided $\varepsilon(B-A) > 1$. Define

$$S_1(q,\xi) = \rho(|\xi|)S(q,\xi) + (1-\rho(|\xi|))Q(\xi)$$

and let us prove that

$$\frac{\partial}{\partial \xi} S_1(q,\xi) = 0 \Longleftrightarrow \frac{\partial}{\partial \xi} S(q,\xi) = 0$$

Indeed, setting $Q(\xi) = \frac{1}{2}\left(A_Q \xi, \xi\right)$ we have

$$\frac{\partial}{\partial \xi} S_1(q,\xi) = \frac{\partial}{\partial \xi}\left(\rho(|\xi|)\left(S(q,\xi) - Q(\xi)\right) + Q(\xi)\right) =$$

$$\rho'(|\xi|)\frac{\xi}{|\xi|}\left(S(q,\xi) - Q(\xi)\right) + \rho(|\xi|)\frac{\partial}{\partial \xi}(S-Q)(q,\xi) + A_Q\xi = 0$$

Since for some positive constant $k$ we have $|A\xi| \geq k|\xi|$ and notice that our assumption implies that for some constant $D$, we have $\|S-Q\|_{C^0} \leq C|\xi| + D$.

So we estimate

$$\left|\frac{\partial}{\partial \xi} S_1(q,\xi)\right| \geq k|\xi| - (\varepsilon C|\xi| + D) - C$$

hence for $\varepsilon$ small enough, $\frac{\partial}{\partial \xi} S_1(q,\xi) = 0$ implies

$$|\xi| \leq \frac{\varepsilon D + C}{k - \varepsilon C}$$

and this remains bounded for $\varepsilon$ small enough. We thus choose $0 < \varepsilon < \frac{k}{C}$, then choose $A = \frac{\varepsilon D + C}{k - \varepsilon C}$ then $B > A + \frac{1}{\varepsilon}$. >but now on $|\xi| \leq A$ we have $S_1(q,\xi) = S(q,\xi)$ so obviously $L_{S_1} = L_S$. $\square$

REMARKS 4.23. (1) By abuse of language, we still call $S$ a G.F.Q.I. the replacement of $S$ by $\widetilde{S}$ being understood.

(2) Usually Generating functions quadratic at infinity are defined on a vector bundle $E$ over $N$. However these can be reduced to the standard form we introduced above as follows: let us decompose $E$ as $E^+ \oplus E^-$ where $E_x^+$ (resp. $E_x^-$) is the sum of the positive (resp. negative) eigenspaces for $Q(x)$. Let $F^\pm$ be vector bundles such that $E^+ \oplus F^+$ and $E^- \oplus F^-$ are trivial bundles. Then the G.F.Q.I.

$$\widehat{S}(x,\xi,\eta^+,\eta^-) = S(x,\xi) + |\eta^+|^2 - |\eta^-|^2$$

generates the same Lagrangian as $S$ and the positive and negative bundles are trivial. Now if $Q(x)(\xi,\xi)$ is positive definite on $N \times \mathbb{R}^k$ we can write $Q(x) = A(x)^2$ with $A(x)$ positive definite (see Exercise 3), and then the change of variables $(x,\xi) \longrightarrow (x, A(x)\xi)$ sends $Q$ to $|\xi|^2$. The same holds for the negative part, so by a fiberwise diffeomorphism linear on the fiber we reduced ourselves to the case where $Q(x)(\zeta) = |\zeta^+|^2 - |\zeta^-|^2$ where $\zeta^\pm \in \mathbb{R}^{k_\pm}$.

Note that there are two operations on a G.F.Q.I. that do not affect the generated Lagrangian :

(1) (Fiberwise diffeomorphism) Let $(q;\xi) \mapsto (q;\eta(q,\xi)$ be a fiberwise diffeomorphism. Then
$$\widetilde{S}(q;\xi) = S(q;\eta(q;\xi))$$
generates the same Lagrangian as $S$

(2) (Stabilization) Let $Q'$ be a non degenerate quadratic form on $\mathbb{R}^l$ and set
$$\widetilde{S}(q;\xi,\eta) = S(q;\xi) + Q'(q)(\eta)$$
generates the same Lagrangian as $S$

THEOREM 4.24. *(Sikorav, [Sik87]) Let us assume L is a closed immersed Lagrangian having a G.F.Q.I. and $\varphi \in Ham(T^*N)$, then so does $\varphi(L)$. The same holds if N is a compact manifold with boundary, $\partial L \subset T^*_{\partial N}N$ and $\varphi^t_H(T^*_{\partial N}N) = T^*_{\partial N}N$.*

Sikorav's Theorem is essentially equivalent to a slightly more general fibration theorem that we shall now state.

Let $\mathscr{F}_k$ be the set of $G.F.Q.I.$ generating immersed Lagrangians and defined on $N \times \mathbb{R}^k$. A map from a compact space $X$ to $\mathscr{F}_k$ is said to be smooth if there is a smooth map $\widetilde{S}: X \times N \times \mathbb{R}^k \longrightarrow \mathbb{R}$ such that $\widetilde{S}(x,q;\xi) = S_x(q,\xi)$ and $\widetilde{S}(x,q;\xi) = Q(\xi)$ outside a compact set.

We shall omit the index to indicate the union of the $\mathscr{F}_k$ and a map from $X$ to $\mathscr{F}$ just means[6] a map to one of the $\mathscr{F}_k$. Note that the same can be defined for $N$ non-compact, provided we impose $\widetilde{S}(x,q;\xi) = Q(\xi)$ outside a compact set (so this is true also at infinity in $N$).

Now we have a projection map $\pi : \mathscr{F} \longrightarrow \mathscr{L}(T^*N)$. Note that there is an equivalence relation by stabilization and fiberwise diffeomorphism defined above (i.e. the operations (1), (2) )and we denote by $\overline{\mathscr{F}}$ the quotient quasi-topological space defined as follows: $\overline{S}: Y \longrightarrow \overline{\mathscr{F}}$ is smooth if there exists a smooth map $\widetilde{S}: Y \longrightarrow \mathscr{F}$ such that for all $y \in Y$ we have $\widetilde{S}(y,\bullet) \in \overline{S}(y,\bullet)$. By abuse of language we still denote by $\pi$ the obvious map from $\overline{\mathscr{F}} \longrightarrow \mathscr{L}(T^*N)$.

We now have

THEOREM 4.25 (Théret's theorem (see [Thé99])). *The map $\pi : \overline{\mathscr{F}} \longrightarrow \mathscr{L}(T^*N)$ is a smooth Serre fibration. More precisely given a smooth map $j : D^k \times [0,1] \longrightarrow \mathscr{L}(T^*N)$ and a lift $\overline{S}_0 : D^k \times \{0\} \longrightarrow \overline{\mathscr{F}}$ such that $\pi \circ \overline{S}_0 = j_{|D^k \times \{0\}}$, then there is an extension $\overline{S}$ : $D^k \times [0,1] \longrightarrow \overline{\mathscr{F}}$ such that $\pi \circ \overline{S} = j$.*

First of all we shall see according to Lemma 4.26, it is enough to deal with the case $k = 0$. We consider $X$ to be a smooth manifold with boundary and we shall us the following Lemma for $X = D^k$.

[6]This defines a "quasi-topology" in the terminology of [Spa63].

LEMMA 4.26. *Let $j : X\times L_0 \longrightarrow T^*N$ be a smooth family of exact Lagrangians parametrized by $X$. Then there is a map $v : X\times L_0 \longrightarrow (\mathbb{R}^k)^*$ such that $J : (u,x)\mapsto (u, v(u,x), j(u,x))$ is an exact Lagrangian embedding into $T^*(X\times N)$ and $J(\partial X\times L_0)\subset T^*_{\partial X\times N}(X\times N)$.*

PROOF. We shall occasionally write $j_u(x)$ for $j(u,x)$. Now the Liouville form on $T^*(X\times N)$ is $vdu+\lambda_N$ so for $J$ of the form $(u, v(u,x), j(u,x))$ we have

$$J^*(vdu+\lambda_N) = vdu + j_u^*\lambda_N + \lambda_N(j_u(x))\frac{\partial j}{\partial u}(u,x)du$$

Since by assumption there is a smooth family $f_u$ such that $df_u(x) = j_u^*(\lambda_N)$ we may set

$$v(u,x) = -\lambda_N(j_u(x))\frac{\partial j}{\partial u}(u,x)du + \frac{\partial f}{\partial u}(u,x)du$$

Then $J^*(vdu+\lambda_N) = df$ and $J$ is an exact embedding. It is obvious that $J$ is an embedding, since for fixed $u$, $j_u$ is an embedding, and $J(u,x) = J(u',x)$ obviously implies $u = u'$. □

Note that $f$ is well-defined up to a function of $u$, so $J$ is well defined up to changing $v$ to $v+c(u)$ where $c$ is any smooth function depending only on $u$. Note also that if $S_u$ is a family of G.F.Q.I. for $j_u(L_0)$ defined on a fixed space $N\times\mathbb{R}^l$ , then $\widetilde{S}(u,x,\xi)$ is a G.F.Q.I. for $J$. The converse also holds, since $j_{u_0}(L_0)$ is the symplectic reduction of $J(D^k\times L_0)$ over $u = u_0$.

Using the Lemma, we can find $J_0 : D^k\times L_0 \longrightarrow T^*(D^k\times N)$ and setting $j_t(u,x) = j(u,t,x)$, we get a smooth family $J_t : D^k\times L_0 \longrightarrow T^*(D^k,N)$. Now if $S_0$ is a G.F.Q.I. for $J_0$ and $S_t$ for $J_t$ we get that $S(u,t,\bullet)$ is a G.F.Q.I. for $j_{u,t}(L_0)$ and this proves the theorem. We are thus reduced to the case $k = 0$, that is to Sikorav's Theorem, that we now prove using Brunella's method.

PROOF OF THERÉT'S THEOREM. (Brunella, [Bru91]) Consider first the "special" case $N = \mathbb{R}^n$ and $\varphi\in Ham(T^*\mathbb{R}^n)$ (note that $\mathbb{R}^n$ is non-compact !). We assume $L_0$ coincides with $0_N$ outside a compact set. Let us first assume $\varphi = \varphi^1$ is defined by a generating function in the sense of Jacobi, i.e. a function $h : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ and maps $\varphi_h : T^*N \to T^*N$ given by

$$\varphi_h(q_1,p_1) = (q_2,p_2) \iff \begin{cases} p_1 = \frac{\partial}{\partial q_1}h(q_1,q_2) \\ p_2 = -\frac{\partial}{\partial q_2}h(q_1,q_2) \end{cases}$$

The graph of $\varphi_h$ is a submanifold in $T^*\mathbb{R}^n\times T^*\mathbb{R}^n$ with symplectic form given by $\omega = dp_1\wedge dq_1 - dp_2\wedge dq_2$. It's a Lagrangian if and only if $\varphi_h$ is a symplectic diffeomorphism.

Conversely the graph of $dh$ is a submanifold in $T^*(\mathbb{R}^n\times\mathbb{R}^n)$ with the natural symplectic structure and it's Lagrangian. It is the graph of a symplectic map if and only if it is transverse to both foliations $T^*\mathbb{R}^n\times\{(q_1,p_1)\}$ and $\{(q_0,p_0)\}\times T^*\mathbb{R}^n$. This is just to point out that the existence of $h$ is an $C^1$-open condition on $\varphi$ and that $\varphi_h$ defines a diffeomorphism is a $C^2$-open condition on $h$. Moreover, up to the addition of a constant to $h$, $\varphi_h$ uniquely determines $h$. Finally if the transversality assumptions are satisfied by all maps in the Hamiltonian path $\varphi^t$, then there is a continuous path $h_t$ such that $\varphi^t = \varphi_{h_t}$.

Set $h_0(q_1, q_2) = \frac{1}{2}\left|q_1 - q_2\right|^2$, then

$$\varphi_{h_0}(q_1, p_1) = (q_1 - p_1, p_1).$$

and any symplectic map $\varphi$ $C^1$ close to $\varphi_{h_0}$ is of the form $\varphi_h$. Notice that $\varphi_{h_0}$ preserves the zero section. The same holds for $\varphi_{-h_0}$ given by $\varphi_{-h_0}(q_1, p_1) = (q_1 + p_1, p_1)$ and we have $\varphi_{-h_0} = \varphi_{h_0}^{-1}$.

We shall now need to slightly extend our defintion of G.F.Q.I. when the base is $\mathbb{R}^n$ (therefore non-compact !)

DEFINITION 4.27. *We say that $S : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ is a G.F.Q.I. if we have*

$$S(x,\xi) = Q(\xi)$$

*for $(x,\xi)$ outside a compact set, where $Q$ is a non-degenerate quadratic form on $\mathbb{R}^k$*

Note that then $L_S$ will coincide with the zero section $0_{\mathbb{R}^n}$ outside a compact set.

LEMMA 4.28 (Chekanov's composition formula, see [Che96]). *Let $L$ be a Lagrangian in $T^*\mathbb{R}^n$ such that $L$ coincides with $0_N$ outside a compact set and has a G.F.Q.I. $S(q,\xi)$. If $h = h_0$ near infinity, then $\varphi_h(L)$ has G.F.Q.I. .*

$$\tilde{S}(q;\xi, y) = h(q, y) + S(y;\xi).$$

REMARKS 4.29. (1) This $\tilde{S}$ is only approximately quadratic at infinity. We use Proposition 4.22 to turn it into a real G.F.Q.I. .
(2) We do not assume $L$ is embedded but only immersed. This will be useful later.

For the proof of the claim, we check that $L_{\tilde{S}}$ is $\varphi_h(L_S)$.

$$\frac{\partial \tilde{S}}{\partial \xi}(q;\xi, y) = 0 \Longleftrightarrow \frac{\partial S}{\partial \xi}(y;\xi) = 0.$$

$$\frac{\partial \tilde{S}}{\partial y}(q;\xi, y) = 0 \Longleftrightarrow \frac{\partial h}{\partial y}(x, y) + \frac{\partial S}{\partial y}(y;\xi) = 0.$$

A point in $L_{\tilde{S}}$ is

$$\begin{aligned}
(x, \frac{\partial \tilde{S}}{\partial x}(x;\xi, y)) &= (x, \frac{\partial h}{\partial x}(x, y)) \\
&= \varphi_h(y, -\frac{\partial h}{\partial y}(x, y)) \\
&= \varphi_h(y, \frac{\partial S}{\partial y}(y, \xi)).
\end{aligned}$$

so $(y, \frac{\partial S}{\partial y}(y;\xi))$ belongs to $L_S$. $\square$

We continue the proof of Sikorav's theorem in the case $N=\mathbb{R}^n$.

So if $h$ is $C^2$ close to $h_0$, we proved that $\varphi_h(L)$ has a G.F.Q.I. and then

$$(\varphi_{-h_0}\circ\varphi_h)(L)(\varphi_{h_0}^{-1}\circ\varphi_h)(L)$$

has G.F.Q.I. . Now any compact supported $C^1$ small symplectic map $\psi$ can be written as

$$\psi=\varphi_{h_0}^{-1}\circ\varphi_h$$

where $h=h_0$ outside a compact set.

We may the conclude that for any $\psi$, $C^1$ close to the identity, if $L$ has G.F.Q.I. then $\psi(L)$ has G.F.Q.I. .

Now take $\varphi^t\in Ham(T^*N)$. Write

$$\varphi^t=\varphi^t_{\frac{t(N-1)}{N}}\circ\varphi^{\frac{t(N-1)}{N}}_{\frac{t(N-2)}{N}}\cdots\varphi_0^{\frac{t}{N}}.$$

then each factor is $C^1$ small, so if $L$ has G.F.Q.I. , then $\varphi^t(L)$ has G.F.Q.I. depending smoothly on $t$. This concludes our proof for $N=\mathbb{R}^n$.

We now prove Théret's theorem in the general case. Let us consider an embedding of $N$ into $\mathbb{R}^d$. Then $T_N\mathbb{R}^d$ is coisotropic and its reduction is $T^*N$. So if $S:\mathbb{R}^d\times\mathbb{R}^l$ is a G.F.Q.I. for $L$, then $S_{N\times\mathbb{R}^l}$ is a G.F.Q.I. for $L_N$ the symplectic reduction of $L$ by $T_N^*\mathbb{R}^d$ (that we should denote, in principle by $L_{T_N^*\mathbb{R}^d}$). Conversely let $L_0\in T^*N$ having the G.F.Q.I. $S_0:N\times\mathbb{R}^l\longrightarrow\mathbb{R}$ and consider $\widetilde{S}_0$ an extension of $S_0$ to $\mathbb{R}^d\times\mathbb{R}^l$ such that

(1) $\widetilde{S}_0(q,\xi)=Q(\xi)$ for $(q,\xi)$ outside a compact set
(2) 0 is a regular value of $(q,\xi)\longrightarrow\frac{\partial\widetilde{S}_0}{\partial\xi}(q,\xi)$

The first property is just extension of smooth functions by partition of unity, and the second an application of Sard's theorem. Then the reduction of $L_{\widetilde{S}_0}$ by $T_N^*\mathbb{R}^d$ is $L_0$ and $L_{\widetilde{S}_0}$ coincides with $0_{\mathbb{R}^d}$ outside a compact set. Let $\varphi_H^t$ be a Hamiltonian flow on $T^*N$, it has an "extension" $\varphi^t_{\widetilde{H}}$ to $T^*\mathbb{R}^d$ (see Proposition 4.6) so that the reduction of $\varphi^t_{\widetilde{H}}(L_{\widetilde{S}_0})$ is $\varphi_H^t(L_0)$. Since $\widetilde{S}_0$ is a G.F.Q.I. for $L_{\widetilde{S}_0}$, we proved that $\varphi^t_{\widetilde{H}}(L_{\widetilde{S}_0})$ has a G.F.Q.I. , so does $\varphi_H^t(L_0)$.

Finally we deal with the case where $N$ has boundary. Let $\widehat{N}$ be its double, that is $N\sqcup_{\partial N}N$, where $N$ is identified with the first copy of $N$ in the double. Then $S_0$ can be extended to $\widehat{N}\times\mathbb{R}^k$ and it is the G.F.Q.I. for $\widehat{L}_0$ such that $\widehat{L}_0\cap T_N^*\widehat{N}=L_0$. Similarly if $H$ is a Hamiltonian with flow preserving $T^*_{\partial N}N$ it can be extended to $\widehat{H}$ with flow satisfying the same condition[7] so that $\varphi^t_{\widehat{H}}=\varphi^t_H$ on $T^*N$. So finding a G.F.Q.I. for $\varphi^t_{\widehat{H}}(\widehat{L}_0)$ yields a G.F.Q.I. for $\varphi^t_H(L_0)$ and this concludes the proof of Théret's Theorem, hence of Sikorav's Theorem.

[7] In local coordinates such that $\partial N$ is given by $p_n=0$, the condition on $H$ is $\frac{\partial H}{\partial q_n}(\bar{q},\bar{p},q_n,0)=0$.

REMARK 4.30. (due to A. Weinstein) There are many ways to construct generating functions quadratic at infinity. It is useful to realize that there is a formal generating function that is "the mother of all generating functions" : the action functional. Let $\mathscr{P} = \{\gamma : [0,1] \longrightarrow T^*N \mid \gamma(0) \in 0_N\}$ and the fibration $\mathscr{P} \longrightarrow N$ given by $\gamma \mapsto q(1)$. Consider the function $A_H(q,p) = \int_0^1 (p(t)\dot{q}(t) - H(t,q(t),p(t)))dt$. We claim that "formally", $A_H$ is a G.F.Q.I. for $\varphi_H^1(0_N)$. Indeed, we have

$$DA_H(\gamma)\delta\gamma =$$
$$\int_0^1 \left\{\left(\dot{q}(t) - \tfrac{\partial H}{\partial p}(t,q(t),p(t))\right)\delta p(t) - \left(\dot{p}(t) + \tfrac{\partial H}{\partial q}(t,q(t),p(t))\right)\delta q(t)\right\} dt +$$
$$p(1)\delta q(1) - p(0)\delta q(0)$$

As a result if we denote by $\xi$ the fiber direction, we have

$$\frac{\partial}{\partial \xi} A_H(\gamma) = 0 \Leftrightarrow \dot{q}(t) = \frac{\partial H}{\partial p}(t,q(t),p(t))\ \dot{p}(t) = -\frac{\partial H}{\partial q}(t,q(t),p(t))$$

and then $(q(1), \frac{\partial}{\partial q(1)} A_H(\gamma)) = (q(1),p(1)) = \varphi_H^1(q(0),0)$, so $A_H$ generates $L = \varphi_H^1(0_N)$. Of course this is formal, since we are in an infinite dimensional space, we did not specify which actual topological space we are considering $\gamma$. However this can be made rigorous by using finite dimensional reductions (see [Vit87b], and Exercise 5). The infinite dimensional case is actually rather helpful in computations (see for example Proposition 7.19)

As another example, let $\mathscr{P} = \{\gamma : [0,1] \longrightarrow N\}$ with projection $\pi(\gamma) = (\gamma(0);\gamma(1))$, then the Lagrangian energy $S(\gamma) = \int_0^1 (L(t,q(t),\dot{q}(t))dt$ associated to a Tonelli Lagrangian (i.e. satisfying the conditions of Definition 3.50 ) on $\mathscr{P}$ generates the graph of $\varphi_H^1$ in $T^*(N \times N)$ where $H$ is the Legendre dual of $L$. Indeed, the condition $\frac{\partial S}{\partial \xi}(\gamma) = 0$ means that $dS(\gamma)$ vanishes over all deformations of $\gamma$ with fixed end points. In other words, that $\gamma$ satisfies the Euler-Lagrange equation. Then the set of $(q_0, \frac{\partial S}{\partial q_0}(\gamma), q_1, \frac{\partial S}{\partial q_1}(\gamma)$ is given by

$$(q_0, \frac{\partial L}{\partial v}(0, q_0 \dot{q}_0), q_1, -\frac{\partial L}{\partial v}(1, q_1 \dot{q}_1))$$

that is $(q_0, p_0, q_1, -p_1)$ where $\varphi_H^1(q_0,p_0) = (q_1,p_1)$.

## 6. Comments

Proposition 4.20 is due to Hamilton ([Ham34; Ham35]) for Hamiltonian flows. Of course Hamilton did not know what a Lagrangian submanifold was (neither did Lagrange !), and the notion really appeared in a rarely mentioned paper by Souriau [Sou53] in 1953 under the name of “saturated isotropic”. It is however implicit among previous authors, in particular Einstein (in [Ein17]) who realized that quantization conditions in higher dimensions must quantize Lagrangians (described as submanifolds on which $\int_\gamma pdq$ for $\gamma$ a closed curve on the manifold, takes a discrete set of values). The notion became important starting from Maslov's fundamental book( [Mas72]) on asymptotics of PDE.

That variational problems (at least for Lagrangian convex in the $v$ direction) can be reduced to Hamiltonian systems is rather classical as we saw in Section 5 of Chapter 3. That a Hamiltonian system can be reduced to a variational problem has been known as the least action principle : the flow of $X_H$ is obtained by looking for critical points of $A_H(q,p) = \int_0^1 [p(t)\dot{q}(t) - H(t,q(t),p(t))]dt$ on the set of maps $(q,p):[0,1] \longrightarrow T^*M$. Unfortunately for a long time this was considered useless, since the function $A_H$ has no minima, or even critical points of finite Morse index. This changed with P. Rabinowitz's paper [Rab78] who used a finite dimensional reduction (Galerkin method) to find periodic orbits of such systems. Shortly after, a duality method due to Clarke and Ekeland [CE80; Eke79] gave a different approach, first in the case of convex Hamiltonians in $\mathbb{R}^{2n}$ and then for more general situations (see [EL80b; EL80a; Ber+85]). Other finite dimensional reductions of the action functional were used by Conley and Zehnder (see [CZ83; CZ84]) using the Lyapounov-Schmidt reduction on Fourier decomposition of the path space and Chaperon (see [Cha84b]) using broken-geodesic methods.

These variational formulations all happen to be instances of generating functions as was pointed out by A. Weinstein for the standard action functional and in [Vit87b] for more general cases. The advent of Floer theory (see [Flo88a; Flo89]) putting together the variational approach and Gromov's pseudo-holomorphic curves from [Gro85] allowed one to work in the most general symplectic manifold. Here again there is a connection between the Floer functional and the generating functions (see [Vit95]). These methods go beyond the mere search for periodic solutions. For example homoclinic solutions can be studied using either these classical variational methods ([CES90] or Floer-type approach ([CS95]). These homoclinic orbits give information about the entropy of the system (see[Sér93]).

## 7. Exercises and Problems

(1) (Alternative -geometric- proof for the integrability of the isotropic foliation). Let $K$ be coisotropic in the symplectic manifold $(M,\omega)$, and $\mathcal{K}(x) = (T_xK)^{\omega}$. We want to prove that the distribution $\mathcal{K}$ is integrable.
  (a) Prove the Palais formula :
  (b) Prove that this implies that $\varphi_X^t$ preserves $\mathcal{K}$
  (c) Prove that this implies that the bracket of two vector fields tangent to $\mathcal{K}$ is tangent to $\mathcal{K}$ (use the formula $\frac{d}{dt}(\varphi_X^t)_* Y_{|t=0} = [X,Y]$

(2) (a) Prove that if $\omega$ is a closed 2-form (not necessarily non-degenerate) on the manifold $S$ and if $\ker(\omega_{|S})$ has constant rank then it defines a foliation.
  (b) Use Stefan and Sussman's theorem (see [Ste74; Sus73]) to get a foliation with singularities (that is a partition of the space into immersed submanifolds) without any assumption on the rank of the kernel of $\omega$.

  HINT. *Consider the set of endpoints of paths $\gamma(t)$ such that $\dot{\gamma}(t) \in \ker(\omega(\gamma(t)))$*

(3) (see [Hör71] prop. 2.5.7 p. 123 and cor. 3.1.8, page 141 for the homogeneous case) Let $L$ be a germ of Lagrangian submanifold near 0 in $\mathbb{R}^{2n}$. We assume 0 is an isolated intersection point of $L$ and $\mathbb{R}^n \times \{0\}$.
  (a) Prove that we can choose a linear subspace transverse to both $\mathbb{R}^n \times \{0\}$ and $T_0L$.
  (b) Prove that there exists $S(x,\xi)$ such that $(0,0)$ is an isolated critical point of $S$, and $S$ is a generating function of $L$ in a neighborhood of 0.
  (c) Prove that any two such functions, $S_1, S_2$ are equivalent, that is after replacing $S_j$ by $S'_j(x, \varphi_j(x,\zeta))$ where $\zeta = (\xi,\eta)$ and $S'_j(x,\xi,\eta) = S_j(x,\xi) + Q(\eta)$ where $Q$ is a non-degenerate quadratic form.
  (d) Let $U$ be a neighbourhood of $(0,0)$ and $S_U$ de note the restriction of $S$ to $U$. Prove that $\lim_{0\in U} H_*^{loc}(S_U^{\varepsilon}, S_U^{-\varepsilon}) = H_*^{loc}(S,0)$ does not depend on the choice of $S$ but only on $L$ up to a global shift in grading (i.e. replacing $*$ by $*+d$).

(4) (Density of smooth symplectic maps, see [Zeh77])
  (a) Prove that the set of smooth Lagrangian submanifolds is dense in the set of $C^1$ Lagrangian submanifolds.

  HINT. *Use the fact that L has locally a $C^2$ generating function and smooth the function in a slightly smaller domain.*

  (b) Deduce that the set of smooth symplectic maps is dense in the set of $C^1$ symplectic maps.

  HINT. *Use the fact that the graph of $\varphi$ is Lagrangian and that after a $C^1$ perturbation, a graph remains a graph.*

(5) (The Amann-Zehnder Generating function) Let $H(t,z)$ be a Hamiltonian on the $2n$-torus identified to $\mathbb{C}^n/\mathbb{Z}^{2n}$. We look for $2\pi$-periodic solutions of the $\dot z(t) = J\nabla H(t,z(t))$. We set $A_H(z) = \int_0^{2\pi}[\frac{1}{2}(J\dot z(t), z(t)) - H(t,z(t))]dt$ where $J$ is multiplication by $i$
  (a) For $z \in H^1(S^1,\mathbb{C}^n) = E$ write $z = \sum_{j\in\mathbb{Z}} e^{kJt}z_k$, $P_N z = \sum_{|j|\leq N} e^{kJt}z_k$, $Q_N z = \sum_{|j|>N} e^{kJt}z_k$ where $z_0 \in T^{2n}$ and $z_k \in \mathbb{C}^n$. Prove that $P_N, Q_N$ are orthogonal projectors on spaces $E_N, F_N$ such that $E_N \overset{\perp}{\oplus} F_N = E$
  (b) Set $D$ to be the operator sending $z$ to $J\dot z$. prove that

$$Q_N D Q_N = D_N z = \sum_{j\in\mathbb{Z}} kJe^{kJt}z_k$$

  Prove that $D_N$ is invertible on $F_N$ and that $\|D_N^{-1}\| \leq \frac{1}{N}$
  (c) Let $F(x,y)$ be a function on $X \times Y$ and assume the equation $\frac{\partial F}{\partial y}(x,y) = 0$ is equivalent to $y = \varphi(x)$ for a smooth function $f : X \longrightarrow Y$. Set $f(x) = F(x,\varphi(x))$. Prove that there is a one-to-one correspondence between critical points of $f$ and critical points of $F$.

(d) Prove that for $z \in E_N$ there is a unique $\zeta \in F_N$ such that $Q_N(\dot{\zeta} - \nabla H(z+\zeta)) = 0$, ans the map $z \mapsto \zeta(z)$ is smooth.

HINT. *Rewrite the equation by using $D_N^{-1}$ and apply Banach-Picard's fixed point theorem.*

(e) Prove using 5c that a critical point of $a_H(z) = A_H(z + \zeta(z))$ is a periodic solution of $\dot{z}(t) = \nabla H(t, z)$.

(f)

(6) (See [Dus96], page 83.) Let $L$ be a germ of homogeneous Lagrangian. Then $L$ is locally defined by a homogeneous generating function that is such that $S(q, \lambda \cdot \xi) = \lambda \cdot S(q, \xi)$.

HINT. *Let $S(x, \xi)$ be a generating function for $L$. Then $S_\lambda(x, \xi) = S(x, \lambda \cdot \xi)$ is also a generating function for $L$.*

(7) Let $S$ be a G.F.Q.I. for $L$ in $T^*N$. prove that the degree of the projection $\pi : L \longrightarrow N$ is $\pm 1$. Hint: show that this degree is equal to the intersection number of $L$ and $V_{x_0}$, and that this is equal, for $x_0$ such that $V_{x_0}$ is transverse to $L$, to

$$\sum_{\xi \mid \frac{\partial S}{\partial \xi}(x_0, \xi) = 0} (-1)^{index\left(\frac{\partial^2 S}{\partial \xi^2}(x_0, \xi)\right)}$$

Use then the fact that this quantity is equal to the Euler characteristic of the pair $(E_{x_0}^{+\infty}, E_{x_0}^{-\infty})$ and this is equal to the Euler characteristic of $H^*(Q^{+\infty}, Q^{-\infty})$ that is $\pm 1$ (depending on the parity of the index of $Q$).

(8) (Focal surfaces) Let $S$ be a hypersurface in $(N, g)$ where $g$ is a complete Riemannian metric on $N$. We get from $g$ an isomorphism $\rho_g(x) : T_x^* N \longrightarrow T_x N$, and $\rho : T^*N \longrightarrow TN$ the globally induced diffeomorphism. We denote by $H_g(x, p) = g(x)(\rho_g(x)p, \rho_g(x)p)$ and $\varphi_g^t$ the corresponding Hamiltonian flow and $\psi_g^t : TN \longrightarrow TN$ be given by $\psi_g^t(\rho_g(x, p)) = \rho_g(\varphi_g^t(x, p))$. Note that $\psi_g^t$ is the geodesic flow: $\psi_g^t(x, v) = (y, w)$ where by definition the unique geodesic starting from $x$ with speed $v$ arrives after time $t$ at $y$ with speed $w$. Consider $\nu_S^* = \bigcup_{t \in \mathbb{R}, (x,p) \in \nu^* S} \varphi_g^t(x, p)\} \subset T^*N$ and remember that $\nu_S^*$ is homogeneous Lagrangian. We also denote by $N_S^*$ the intersection $N_S^* = \nu^* S \cap \{H_g = 1\}$.

(a) Prove that the image of $\nu_S^*$ is $\Gamma_S = \{(x, v) \mid v \perp T_x S\}$ and the image of $N_S^*$ is $C_S = \{(x, v) \mid |v|_g = 1, x \perp v\}$.

(b) Prove that if $\varphi^t$ is the flow of a homogeneous Hamiltonian, and $t$ is small enough there exists a hypersurface $S(t)$ such that $\varphi^t(\nu^* S) = \nu^* S(t)$.

HINT. *Prove first that $\varphi^t(\nu^* S)$ is homogeneous, and then that the projection on the base of a homogeneous Lagrangian has rank at most $n - 1$ (where $n = \dim(N)$).*

(c) We now consider the case where $\varphi^t = \varphi_g^t$. Prove that for $t$ small enough, $S(t)$ can also be defined as the set of points $\exp_x(tv)$ for $(x, v) \in C_S$

(d) Prove that, again for $t$ small enough, $\varphi_g^t(N^*S) = N^*S(t)$

(e) Prove that in the generic case, for $t$ large, the projection $\pi : \varphi_g^t(\nu^*S) \longrightarrow N$ has in general singularities and the image of these singularities has codimension 2 in $N$. Points $x$ such that there exists $p$ such that $rkd\pi(x,p) : T_{(x,p)}\varphi_g^t(\nu^*S) \longrightarrow T_xN$ has rank $n-2$ are called focal points. Prove this by using a local generating function for the Lagrangian $\varphi_g^t(\nu^*S)$

(f) Prove that in $\mathbb{R}^n$, focal points correspond to points such that the map $E : (x,t) \longrightarrow x + t\nu(x)$ where $\nu(x)$ is the oriented normal to $S$ at $x$ has rank $< n$. This locus is also called a caustic.

(g) Let $s \mapsto \gamma(s)$ be a smooth curve in $\mathbb{R}^3$. We assume $\gamma$ is parametrized by arc length, that is $|\gamma'(t)| = 1$. Then $\gamma'(t) \perp \gamma(t)$ and $c(t) = \gamma(t) + \gamma'(t)$ is called the center of curvature of the curve at $\gamma(t)$ (and $1/|\gamma'(t)|$ the radius of curvature.

(h) Prove that if a local parametrization of $S$ near $x$ is given by $(s,t) \mapsto x(s,t)$ we have that the standard scalar product of $\mathbb{R}^3$ induces the metric

$$q_S(s,t) = \left|\frac{\partial x}{\partial s}\right|^2 ds^2 + 2\left\langle \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}\right\rangle dsdt + \left|\frac{\partial x}{\partial s}\right|^2 dt^2$$

on $S$ called the first fundamental form, and the form

$$B_S(s,t) =$$

$$\left\langle \frac{\partial^2 x}{\partial s^2}, \nu(x(s,t)\right\rangle ds^2 + 2\left\langle \frac{\partial^2 x}{\partial s \partial t}, \nu(x(s,t))\right\rangle dsdt + \left\langle \frac{\partial^2 x}{\partial t^2}, \nu(x(s,t))\right\rangle dt^2$$

the principal curvatures $\kappa_1(s,t), \kappa_2(s,t)$ at $x(s,t)$ are the eigenvalues of $B_S(s,t)$ with respect to $q_S$, in other words there are vectors $e_1, e_2$ such that $B_S(e_i, v) = \kappa_i q_S(e_i, v)$ for all $v \in T_xS$. We call $c_i(s,t) = x(s,t) + \kappa_i(s,t)\nu(x(s,t))$ the two centers of curvature of $S$ at $x(s,t)$.

(i) Assume $\gamma(t) \in S$ and $\gamma$ is a geodesic on $S$. Check that the center of curvature of a geodesic on $S$ is located between the two centers of curvature of $S$ and conversely, any such point is the center of curvature of some geodesic.

(j) Prove that the $c_i(t)$ are the focal points of the surface.

CHAPTER 5

# Critical point theory according to Conley, Morse and Lusternik-Schnirelman

Après tant de grands hommes qui ont travaillé sur cette matiere, je n'ose presque dire que j'ai découvert le principe universel, sur lequel toutes ces loix sont fondées ; qui s'étend egalement aux Corps durs & aux Corps élastiques ; d'où dépend le Mouvement & le Repos de toutes les substances corporelles.
C'est le principe de la moindre quantité d'action : principe si sage, si digne de l'Etre suprême, & auquel la Nature paroît si constamment attachée ; qu'elle l'observe non seulement dans tous ses changemens, mais que dans sa permanence, elle tend encore à l'observer. Dans le Choc des Corps, le Mouvement se distribue de manière que la quantité d'action, que suppose le changement arrivé, est la plus petite qu'il soit possible. Dans le Repos, les Corps qui se tiennent en équilibre, doivent être tellement situés, que s'il leur arrivoit quelque petit Mouvement, la quantité d'action seroit la moindre.

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Les Loix du mouvement et du repos déduites d'un principe metaphysique, P. L. Moreau de Maupertuis, 1746

*After so many great men worked on this subject, I hardly dare to say that I discovered the universal principle founding these laws; that applies to elastic as well as hard Bodies; whence follows the Motion and Rest of all earthly bodies. It is the principle of least action : principle so wise and worthy of the Supreme Being; and to which Nature is so constantly attached that she observes it not only in all its changes, but she also tends to observe it in Steadiness. In the collision of Bodies, Movement is distributed in such a way that the quantity of action, after the change, is as smal as possible. In Rest, the Bodies in equlibrium must be so placed that if they underwent the slighest movement, the quantity of action would be diminished.*

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(Trans. by the author)

## 1. Basic facts about homotopy theory

Let $X, Y$ be two topological spaces and $f, g$ be two continuous maps from $X$ to $Y$.

DEFINITION 5.1. *We say that $f$ and $g$ are homotopic, and denote it by $f \simeq g$, if there exists a continuous map $F : X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f(x), F(x,1) = g(x)$. This defines an equivalence relation on $C^0(X,Y)$.*

DEFINITION 5.2. *The spaces $X, Y$ are homotopy equivalent if there are maps $f \in C^0(X,Y), g \in C^0(Y,X)$ such that $f \circ g \simeq \mathrm{Id}_Y, g \circ f \simeq \mathrm{Id}_X$.*

For a pair of topological spaces $X, A$ with $A \subset X$ closed, we denote by $X/A$ the pointed topological space $(X \setminus A) \cup \{*\}$, where a neighbourhood in $X/A$ of $x \in X \setminus A$ is just a neighborhood of $x$ in $X \setminus A$ and a neighbourhood of $*$ is the union of $\{*\}$ and a neighborhood of $A$. Equivalently there is an obvious map $X \longrightarrow X/A$ sending $x \in X \setminus A$ to $x$ and any point of $A$ to $*$. The topology on $X/A$ is the quotient topology : a set is open in $X/A$ if and only if its preimage in $X$ is open. Of course we have a distinguished point, $*$ which we often denote by $[A]$ and note that $X/\emptyset = X \cup \{*\}$, $(\emptyset, \emptyset) = (*,*)$. We say that $X/A$ is a pointed space and maps between pointed spaces are usually assumed to send the distinguished point to the distinguished point. For a connected space, the pointed spaces $(X, x)$ for $x \in X$ are all homotopy equivalent, and by abuse of language, we denote them by $X$. A useful property with respect to homology or cohomology is

PROPOSITION 5.3. *If $A$ has a neighbourhood in $A$ that has $A$ as a deformation retract, then we have $H_*(X,A) = H_*(X/A,*)$ and $H^*(X,A) = H^*(X/A,*)$.*

In general if $CA$ is the cone $A \times [0,1]/A \times \{0\}$ and $X \cup CA$ be the space $X \sqcup CA/ \simeq$ where $x \simeq (a,t)$ if and only if $x = a \in A$ and $t = 1$. This is a pointed space, with point given by $A \times \{0\}$. Then $H_*(X,A) = H_*(X \cup CA, *)$ (see [Hat02], pp. 124-125).

For two pairs $(X,A), (Y,B)$ we set $(X,A) \times (Y,B) = (X \times Y, A \times Y \cup X \times B) = (Z,C)$ so that the product corresponds to the smash product of the pointed spaces[1] $(X/A) \wedge (Y/B) = Z/C$. Then $H^*((X,A) \times (Y,B)) = H^*(X \times Y, A \times Y \cup X \times B) = H^*(X,A) \otimes H^*(Y,B)$. Finally an obvious form of excision tells us that for a closed subset $B \subset \mathring{A}$ we have $(X \setminus B)/(A \setminus B) = X/A$.

## 2. The Conley index

In this section we define the Conley index for a dynamical system. In the next section, we shall use it to give simplified approach to Morse theory.

Let $\xi$ be a smooth vector field on a manifold. We want to algebraically compute the size of the invariant set of $\xi$ contained in a domain $U \subset M$. There is a beautiful theory for this, mostly due to Charles Conley (see [Con78], [Eas75], [Fra86], [Sal85]), which simplifies in the case of gradient vector fields[2] since invariant sets are nothing

[1] For $(X,*), (Y,*)$ pointed spaces, the smash product is $X \times Y/(X \times \{*\} \cup \{*\} \times Y)$.

[2] as topologists, we rather use then negative gradient $-\nabla f(x)$ rather than the usual gradient. . .

but critical points of the function and trajectories connecting them (called "heteroclinic" trajectories ). Our first definitions and theorems apply in the general case, but we shall restrict ourselves to gradient-like vector fields in the next section. We assume that all vector fields $\xi$ are smooth and complete, so we can deal with the flow $\varphi_\xi^t$ of the smooth vector field $X$. This is a simplifying assumption, but we will not need anything more general. We denote by $\varphi_\xi^{[a,b]}(x)$ the set of points $\varphi_\xi^t(x)$ for $t \in [a,b]$.

DEFINITION 5.4. *Let $\xi$ be vector field on $M$, and $S$ be a closed subset invariant by the flow. We say that $S$ is isolated if there is a* **compact neighbourhood** *$N$ of $S$ such that $S$ is the largest invariant set contained in $N$. The set $N$ is called an isolating neighbourhood for $S$.*

DEFINITION 5.5. *We say that a subset $A$ in $N$ is positively (resp. negatively) invariant (relative to $N$) if for all $x$ in $A$ and $t \geq 0$ (resp. $t \leq 0$) if $\varphi^{[0,t]}(x) \in N$ then $\varphi^{[0,t]}(x) \in A$.*

Note that given $N$ , there is a maximal invariant set $I(N,\xi)$ (usually we shall omit $\xi$) contained in $N$. Then $N$ is an isolating neighborhood for $I(N)$ if and only if there is no bounded trajectory[3] completely contained in $N$ and touching the boundary of $N$: indeed $S = I(N,\xi)$ should contain this trajectory, but then $N$ would not be a neighbourhood of $S$.

DEFINITION 5.6. *Let $S$ be an isolated invariant set and $N$ be an isolating neighborhood for $S$. A pair $(N_1,N_2)$ with $N_2 \subset N_1 \subset N$ of compact sets is an index pair if*

*(1) ($N_1, N_2$ are positively invariant rel $N_1$)*
*$N_2$ is positively invariant in $N_1$, that is if $x \in N_2$ and for $t \geq 0$ we have $\varphi^{[0,t]}(x) \in N_1$, then $\varphi^{[0,t]}(x) \in N_2$*

*(2) ($N_1 \setminus N_2$ is an isolating neighbourhood for $S$)*
*$S = I(\mathring{N}_1 \setminus N_2)$*

*(3) (exiting $N_1$ has to go through $N_2$)*
*If $x \in N_1$ and for some $t > 0$, $\varphi^t(x) \notin N_1$, then there exists $\tau \in ]0,t[$ such that $\varphi^\tau(x) \in N_2$*

Note that the compactness assumption is not strictly necessary. However there are two possible extensions : first we can assume we are in a locally compact space and add a restriction on the flow (as in the next section with the Palais-Smale condition) or adopt Floer's point of view in which there is no index pair, but the analog of an isolating neighbourhood is well defined, and the homology $H_*(N_1,N_2)$ is well defined (see [Flo88a]), even though $N_1, N_2$ are not defined. Even better, in certain cases the topological spectra corresponding to $N_1/N_2$ is well defined (see an example and application in [Kra18]).

Given $S$, the existence of an index pair is not completely obvious in general (see page 46 of [Con78]). We shall give a proof of this fact. Let us start with the

[3] A bounded trajectory is a trajectory $\varphi^{\mathbb{R}}(x)$ completely contained in a bounded set. Note that the closure of such a trajectory is an invariant set.

DEFINITION 5.7. *Let $S$ be an isolated invariant set with isolating neighbourhood $N$. We set*

$$W_N^+(S) = \left\{x \in N \mid \varphi^{[0,+\infty[}(x) \in N\right\}$$
$$W_N^-(S) = \left\{x \in N \mid \varphi^{]-\infty,0]}(x) \in N\right\}$$

*These are called the "stable" and "unstable" sets of $S$ in $N$.*

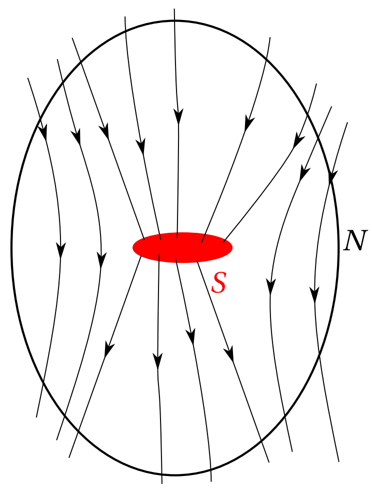


FIGURE 1. The invariant set $S$ and the isolating block $N$

Clearly $W^+(S)$ is positively invariant and $W^-(S)$ is negatively invariant. They are also compact, since they are defined as an intersection of the closed (hence compact) sets $\varphi^{-t}(N)$ for $t \geq 0$. Given $x$ in $W_N^+(S)$ we have that the $\omega$-limit set of $x$, that is the set of accumulation points of $\varphi^t(x)$ as $t$ goes to $+\infty$ is contained in $S$. Similarly for the $\alpha$-limit set of $W^-(S)$.

We summarize this in

PROPOSITION 5.8. *The sets $W^+(S)$ and $W^-(S)$ have the following properties*

*(1) $W^+(S)$ is positively invariant and $W^-(S)$ is negatively invariant.*
*(2) $W^+(S) \cap W^-(S) = S$*
*(3) $W^+(S)$ and $W^-(S)$ are compact*

By definition, points outside $W^-(S)$ eventually exit from $N$. We now prove that points outside a neighbourhood of $W^-(S)$ will exit in bounded time.

LEMMA 5.9. *Let $S$ be an isolated invariant set with isolating neighbourhood $N$. Let $U$ be a neighbourhood of $W^-(S)$ (resp. $W^+(S)$) in $N$. Then there exists $T > 0$ such that if $z \notin U$ and $\varphi^{[-t,0]}(z) \in N$ (resp. $\varphi^{[0,t]}(z) \in N$) then $t < T$.*

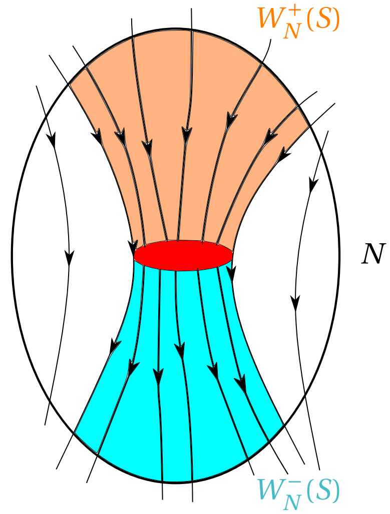


FIGURE 2. The invariant set $S$ and the isolating block $N$

PROOF. By contradiction, there would be a sequence $(t_n)_{n\geq 1}$ going to $+\infty$ and $(z_n)_{n\geq 1}$ outside $U$ such that $\varphi^{[-t_n,0]}(z_n) \in N$. Then if $z_\infty$ is the limit of $z_n$ the continuity of the flow implies that for any finite $t \geq 0$, $\varphi^{[-t,0]}(z_\infty) \in N$, but this implies $\varphi^{]-\infty,0]}(z_\infty) \in N$ hence $z_\infty \in W^-(S)$. This contradicts the assumption that the $z_n$ are outside a neighbourhood of $W^-(S)$. □

DEFINITION 5.10. *Let $K$ be a compact set and for $t \geq 0$, let $P_t(K,N)$ be the set of points at time $t$ of a trajectory starting in $K$ and contained in $N$, that is*

$$P_t(K,N) = \left\{\varphi^t(x) \in N \mid x \in K, \varphi^{[0,t]}(x) \in N\right\}$$

*We also set*

$$P(K,N) = \left\{\varphi^t(x) \in N \mid t \geq 0 x \in K, \varphi^{[0,t]}(x) \in N\right\} = \bigcup_{t\geq 0} P_t(K,N)$$

We shall say that $P(K,N)$ is the union of the positive trajectories starting in $K$ and contained in $N$. We now have

COROLLARY 5.11. *Given neighbourhoods $U^+$ of $W_N^+(S)$ and $U^-$ of $W^-(S)$ in $N$ there exists $T > 0$ such that for any $t \geq T$ we have*

*(1)* $P_t(N,N) \subset U^-$.
*(2)* $P_t(N \setminus U^+, N) = \emptyset$

PROOF. (1) Indeed, take $T_-$ as in Lemma 5.9, associated to $V = V^-$ then $u \in P_t(N,N)$ if and only if $u = \varphi^t(z)$ for $z \in N$ and $\varphi^{]-t,0]}(u) \in N$, but this implies, according to the Lemma, that $u \in V$.

(2) Take $T_+$ as in the same Lemma, associated to $V^+$, then $u \in P_t(N \setminus V^+)$ means $u = \varphi^t(z)$ where $z \notin V^+$ and $\varphi^{[0,t]}(z) \in N$, but this is impossible if $t > T_+$. To conclude we take $T = \max\{T_-, T_+\}$.

□

PROPOSITION 5.12. *Let $S$ be an isolated invariant set with isolating neighbourhood $N$. Then there exists an index pair $(N_1, N_2)$ with $N_1 \subset N$.*

PROOF. In the proof we omit the $N$ subscript as $N$ will be fixed. The idea is that a neighbourhood of $W^-(S)$ can always be used as $N_1$. Indeed, let $K$ be a compact neighbourhood of $W^-(S)$, and $P(K,N)$ be the points on positive trajectories starting in $K$. Clearly $P(K,N)$ is positively invariant and contains $W^-(S)$. We claim that moreover it is compact. Indeed, if $x_n \in K, t_n \geq 0$ satisfy $\varphi^{[0,t_n]}(x_n) \in N$ and $z_n = \varphi^{t_n}(x_n) \in P(K,N)$ and $\lim_n z_n = z_\infty$, then either $t_n$ is bounded and then, up to taking a subsequence we may assume that $\lim_n x_n = x_\infty \in K, \lim_n t_n = t_\infty \geq 0$ and then $z_\infty = \varphi^{t_\infty}(x_\infty) \in P(K,N)$. Otherwise we may assume $\lim_n t_n = +\infty$. We claim that this can only happen when $z_\infty \in W^-(S)$. Otherwise, we could find a neighbourhood $V$ of $W^-(S)$ not containing the $z_n$. But according to Lemma 5.9, since $\varphi^{[-t_n,0]}(z_n) = \varphi^{[-t_n,0]}(\varphi^{t_n}(x_n)) = \varphi^{[0,t_n]}(x_n) \in N$, this implies $t_n \leq T$, which contradicts $\lim_n t_n = +\infty$. So we must have $z_\infty \in W^-(S) \subset P(K,N)$, and this proves compactness.

We may thus set $N_1 = P(K,N)$ which is of course positively invariant in $N$. We then set $N_2 = P(N_1 \setminus V, N_1)$ where $V$ is a neighbourhood of $W^+(S)$ in $N_1$. Then $N_2$ is positively invariant relative[4] to $N_1$, does not intersect $S$, and using again Lemma 5.9, we see that $N_2$ is compact. Moreover if a trajectory exits from $N_1 = P(K,N)$ it must exit from $N$ since $N_1$ is positively invariant in $N$ and hence from $V$, and will thus go through $N_2$. Then $N_1 \setminus N_2$ will contain $S$ and be contained in the neighbourhood $K \cap V$ of $S$.

[4]Because $N_1$ is positively invariant rel $N$, we may infer that $N_2$ is also positively invariant rel. $N$

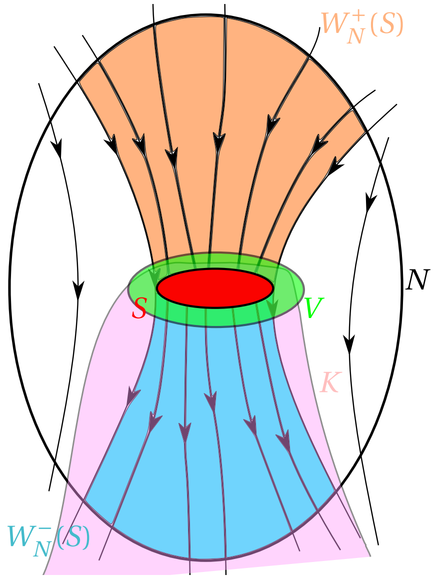


FIGURE 3. The invariant set $S$ (in red), the isolating block $N$ and the sets $K$ (in pink) and $V$ (in green)

□

REMARKS 5.13. In view of the following proposition, we notice the following facts about the index pair constructed above, that we shall denote $(N_1^{st}, N_2^{st})$ and call "the standard pair" associated to the data $N, K, V$:

(1) $N_2$ is positively invariant with respect to $N$, since it is positively invariant with respect to $N_1$ and $N_1$ is positively invariant with respect to $N$.
(2) $(N_1^{st}, N_2^{st})$ is the standard pair associated to $N_1^{st}$ or to $(N_1^{st} \setminus N_2^{st})$.
(3) Let $N \subset \widetilde{N}$ be two isolating neighbourhoods for $S$, and $(N_1^{st}, N_2^{st})$ be the standard index pair we constructed associated to $N, K, V$, and $(\widetilde{N}_1^{st}, \widetilde{N}_1^{st},)$ the standard pair associated to $\widetilde{N}, \widetilde{K}, \widetilde{V}$. We assume $K = N \cap \widetilde{K}$ and $V = \widetilde{V}$. Then a positive trajectory in $N$ starting on $\widetilde{K}$ must start in $K$. So a positive trajectory in $\widetilde{N}$ starting in $K$ remains in $N$ (hence in $N_1^{st}$) until it hits $N_2^{st}$. As a result $N_1^{st} \setminus N_2^{st} = \widetilde{N}_1^{st} \setminus \widetilde{N}_2^{st}$ so $N_1^{st}/N_2^{st} \simeq \widetilde{N}_1^{st}/\widetilde{N}_2^{st}$.

PROPOSITION 5.14. *Let $S$ be an isolated invariant set and $N_1, N_2$ be an index pair as above. Then the pointed homotopy type of the space $N_1/N_2$ only depends on $S$, and is denoted $h(S, \varphi^t)$ (or $h(S)$ if there is no ambiguity). If $h(S) \neq (*,*)$, then $S \neq \emptyset$.*

PROOF. We will first prove that if $(N_1, N_2)$ is an index pair, and $(N_1^{st}, N_2^{st})$ is the standard index pair associated to $N_1 \setminus N_2$ (and some neighbourhoods $K, V$), then $N_1/N_2 \simeq N_1^{st}/N_2^{st}$.

First define

$$f_t(x) = \begin{cases} \varphi^t(x) & \text{if } \varphi^{[0,t]}(x) \in N_1 \setminus N_2 \\ [N_2] & \text{otherwise} \end{cases}$$

By continuity of the flow, $f_t$ defines a continuous map $N_1/N_2 \longrightarrow N_1/N_2$ for all $t \geq 0$. Let $T$ be defined in Corollary 5.11 associated to $(N_1, N_2)$ and $U^- = K$ and $U^+ = N \setminus N_2^{st}$. For $t \geq T$ we have

(a) $P_t(N_1, N_1) \subset K \subset N_1^{st}$ (since all points that do not exit in time less than $T$ end up in $K$)
(b) $P_t(N_2^{st}, N_1) = \emptyset$ (since all points in $N_2^{st} \subset N \setminus U^+$ exit in time less than $T$)
(c) $P_t(N_2, N_1^{st}) = \emptyset$ (since all points in $N_2$ exit from $N_1$ hence from $N_1^{st}$ in time less than $T$)

and for $t \geq T$, $f_t$ defines a map from $N_1^{st}/N_2^{st}$ to $N_1/N_2$, since $f_t(N_1^{st}) \subset N_1$, because $N_1^{st} \subset N_1$ and $f_t(N_2^{st}) \subset N_2$ because of (b).

Similarly for

$$g_t(x) = \begin{cases} \varphi^t(x) & \text{if } \varphi^{[0,t]}(x) \in N_1^{st} \setminus N_2^{st} \\ [N_2^{st}] & \text{otherwise} \end{cases}$$

We claim that by positive invariance, $g_t$ defines a map $N_1^{st}/N_2^{st} \longrightarrow N_1^{st}/N_2^{st}$ and for $t \geq T$ a map from $N_1/N_2$ to $N_1^{st}/N_2^{st}$, since according to (a), $P_t(N_1, N_1) \subset N_1^{st}$ and (c) implies that $g_t(N_2) \subset \{[N_2^{st}]\}$.

Now we claim that $f_T \circ g_T = f_{2T} : N_1/N_2 \longrightarrow N_1/N_2$ and $g_T \circ f_T = g_{2T} : N_1^{st}/N_2^{st} \longrightarrow N_1^{st}/N_2^{st}$. But since $f_{2T} : N_1/N_2 \longrightarrow N_1/N_2$ and $g_{2T} : N_1^{st}/N_2^{st} \longrightarrow N_1^{st}/N_2^{st}$ are homotopic to $f_0 = \mathrm{Id}$ and $g_0 = \mathrm{Id}$, the maps $f_T, g_T$ are homotopy inverse of each other.

Note that the above proof implies that the homotopy type of $N_1^{st}/N_2^{st}$ does not depend on the choice of $K, V$. Note also that according to Remark 5.13 (3), $N_1^{st}/N_2^{st}$ does not depend on $N$. This concludes our proof of the main statement.

That $h(S) \neq *$ implies $S \neq \emptyset$ is clear : if $S = \emptyset$, then $W^{\pm}(S) = \emptyset$, so we can choose $K = \emptyset$, hence $N_1^{st} = N_2^{st} = \emptyset$ and since $(\emptyset, \emptyset) = (*, *)$ we may conclude. ☐

For a shorter proof, we refer to Exercice 2 (or [Sal85]). In fact the proof of the theorem tells us more than stated : given two pairs $N_1, N_2$ and $N_1', N_2'$, there is a well defined (up to homotopy) homotopy equivalence from $N_1/N_2$ to $N_1'/N_2'$. These spaces and maps can be put together, and they define what is called a connected simple system[5] the index of $S$, denoted $I(S)$ or $I(S, \varphi^t)$. It is a more refined invariant than $h(S)$, and allows sometimes to solve some subtle questions, but we shall not use this result here.

One of the important features of the Conley index is the continuity. To be precise, we have

[5]This is a category having a single morphism between any two objects- which is then necessarily an isomorphism. It is the formalization of the sentence "unique up to a unique isomorphism". In our case $h(S)$ is unique, up to a homotopy equivalence which is itself unique up to homotopy...

PROPOSITION 5.15 (Continuation theorem, see [Con78; Sal85]). *Let $\xi_\lambda^t$ be a continuous family of vector fields. Let $N$ be an isolating neighbourhood for all the invariant sets $S_\lambda$ of $\xi_\lambda$. Then $h(S_\lambda)$ does not depend on $\lambda$.*

PROOF. We refer the reader to Conley's book or Salamon's article. □

Note that given a family of vector fields $\xi_\lambda$, the set of $\lambda$ for which $N$ is an isolating neighbourhood for $S_\lambda$ is open. Note that the reader who does not want to bother with homotopy types can consider only the (co)homology of our spaces, and notice that $H^*(X/A,*) = H^*(X,A)$. In practice, this is all we shall use.

The most common and convenient case of index pair is when $B$ is a domain with smooth boundary, $N_1 = \overline{B}, N_2 = \partial^- B$ where $\partial^- B$ is the set of exit points from $B$, that is

$$\partial^- B = \{x \in B \mid \exists \varepsilon > 0, \varphi^{]0,\varepsilon[}(x) \cap B = \emptyset\}$$

We also define $\partial^+ B = \{x \in B \mid \exists \varepsilon > 0, \varphi^{]-\varepsilon,0[}(x) \cap B = \emptyset\}$ and we now set

DEFINITION 5.16. *Let $B$ be a compact codimension $0$ submanifold with boundary, $\partial B$. $B$ is an isolating block if $\partial B = \partial^- B \cup \partial^+ B$ where $\partial^- B = \{x \in B \mid \exists \varepsilon > 0, \varphi^{]0,\varepsilon[}(x) \cap B = \emptyset\}$ and $\partial^+ B = \{x \in B \mid \exists \varepsilon > 0, \varphi^{]-\varepsilon,0[}(x) \cap B = \emptyset\}$.*

REMARK 5.17. Our assumption excludes the case of a set given by $B = \{(x,y) \mid y < x\sin(1/x)) \mid x \in [-1,1]\}$ and $\xi = \frac{\partial}{\partial x}$. In this case, the trajectory exits and reenters infinitely many times as $x$ goes to $0$ so $0 \notin \partial B^+ \cup \partial B^-$.

In concrete terms, this means that each point $x$ of the boundary is either a strict ingress point, if $x \in \partial^+ B, \varphi^{[0,\varepsilon[}(x) \in B$, a strict egress point if $x \in \partial^- B, \varphi^{]-\varepsilon,0]}(x) \in B$ or is touched by a trajectory tangent to $\partial B$ from outside (i.e. if a trajectory has a single point of contact with $\partial B$ ) if $x \in \partial^- B \cap \partial^+ B$.

PROPOSITION 5.18. *Let $B$ be an isolating block, and $S$ the maximal invariant set in $B$. Then $S$ is an isolated invariant set and $(B, \partial^- B)$ is an index pair for $S$.*

PROOF. Indeed, $S$ is isolated since there can be no trajectory contained in $B$ and not contained in $\mathring{B}$. The second statement is then clear. □

Note that for gradient like flows, $S$ is of the form a union of critical points and heteroclinic orbits, and that if $\xi$ is a negative pseudogradient of the function $f$ that is $df(x) \cdot \xi(x) < 0$ outside the set of critical points, then any neighbourhood of a set $S$ consisting of critical points and connecting trajectories which is an isolated set yields an index pair.

REMARK 5.19. We sometimes deal with a degenerate situation, where $\xi$ is tangent to a subset of $\partial B$. We can usually add to the vector field $\varepsilon(x)\nu(x)$ where $\nu(x)$ is normal to $\partial B$ and be reduced to the above case.

EXAMPLES 5.20. (1) Let $f$ be a smooth function on the compact closed manifold $M$. Then $(f^b, f^a)$ is an index pair for $-\nabla f(x)$ and $H^*(f^b/f^a) = H^*(f^b, f^a)$. We can also consider $U = f^b \setminus f^a$ as an isolating block, and $\partial^- U = f^{-1}(a)$.

(2) We may also consider a union $U$ of trajectories contained in $f^b \setminus f^a$. Then the part of $\partial U$ not contained in $f = a$ or $f = b$ is a union of trajectories, so we do not have an isolating block. However we can extend the definition of isolating block to this more general situation.

(3) Let $M$ be a compact manifold with boundary $\partial M$, $f$ be a smooth function on $M$, and consider the vector field $\xi(x) = -\nabla f(x)$. On $\partial M$ the vector filed $\xi$ can either point inward or outward. Set $\partial^- M$ be the region where $\xi$ points outwards. Then $U = f^b \setminus f^a$ is an isolating block, and $\partial^- U = f^{-1}(a) \cup (\partial^- M \cap U)$. As a result $H^*(U, \partial^- U) = H^*(f^b, f^a \cup (\partial^- U \cap f^b))$.

(4) Let $f$ be a function having an isolated non-degenerate critical point. According to Morse's lemma (see exercice 30 in Chapter 3) after applying a local diffeomorphism, we can write $f(x, y) = |x|^2 - |y|^2$ where $x \in \mathbb{R}^{n-i}$, $y \in \mathbb{R}^i$. Consider $B = D^{n-i} \times D^i$, an isolating block for $-\nabla f$, with exit set $D^{n-i} \times \partial D^i$. The Conley index is then $D^{n-i} \times (D^i, \partial D^i)$, that has the homotopy type of $(S^i, *)$.

(5) Let $f$ having a non-degenerate critical manifold $V$, and $d^2 f$ has negative bundle $\nu^-$ and positive bundle $\nu^+$ so that the normal bundle is $\nu = \nu^+ \oplus \nu^-$. Then taking for $B$ a square tubular neighbourhood of $V$, that is on each fiber we take $D^+ \times D^-$ , the product of the unit disc bundles of $\nu^+$ and $\nu^-$, and the exit set will be $\partial D^+ \times D^-$ and the Conley index is homotopic to $D^+/\partial D^+$. This is by definition the Thom space of $\nu^+$.

It is often useful in the end to consider the cohomology of $h(S)$ and we get in Example 1, using excision, that $H^*(B, \partial^- B) = H^*(f^b, f^a)$. We may thus consider that the (homological) contribution of the critical points and connecting trajectories contained in $f^{[a,b]}$ is given by $H^*(f^b, f^a)$. Conley theory for gradient flows can be considered as an extension of this situation : we want to be able to talk about the topological contribution of a set of critical points and connecting trajectories, without assuming that this is the set of all critical points and trajectories contained between two level sets.

For a pair of pointed spaces $X, Y$ we define the sum (or wedge sum) $X \vee Y$ as the union $X \cup Y$ divided by the relation identifying the two base points. Then we claim the following obvious result

PROPOSITION 5.21. *Let $S, S'$ be isolated invariant sets with index pairs $(N_1, N_2), (N_1', N_2')$. Assume that there is an isolating neighbourhood $M$ of $S \cup S'$ containing no trajectory connecting $S$ and $S'$, so that $S \cup S'$ is an isolated invariant set in $M$. Let $(M_1, M_2)$ be an index pair in $M$. Then*

$$h(S \cup S') = h(S) \vee h(S') = (M_1/M_2)$$

This is particularly useful to prove that there must be a trajectory connecting $S$ and $S'$ as in the following

EXAMPLES 5.22. (1) In the first example, we have $h(S) = (*, *), h(S') = (S^1, *)$ and $h(S \cup S')$ is equal to $(*, *)$ which is different from $(* \vee S^1, *)$.

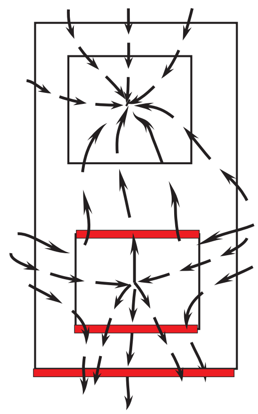

FIGURE 4. A situation where $h(S \cup S') \neq h(S) \vee h(S')$. In red the exit set.

(2) In the next example, we have a connecting trajectory between $S$ and $S'$, but it cannot be detected by the Conley index. This is not surprising since a heteroclinic connection between two hyperbolic points with the same index can be "killed" by a small deformation, which will not modify $h(S \cup S')$.

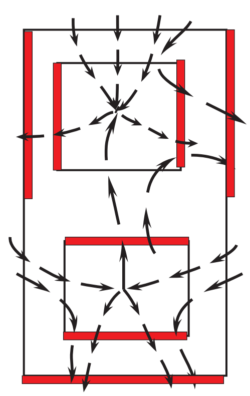

FIGURE 5. A situation where $h(S \cup S') = h(S) \vee h(S')$, in spite of a connection between $S$ and $S'$.

(3) Let $X, Y$ be two manifolds and $\varphi^t, \psi^t$ be flows on each of them respectively. Then $\varphi^t \times \psi^t$ is a flow on $X \times Y$ and if $S, T$ are isolated invariant sets with $(N_1, N_2), (M_1, M_2)$ their respective index pairs in $X$ and $Y$, we have $h(S \times T) = h(S) \wedge h(T)$, since $(N_1 \times M_1, N_2 \times M_1 \cup N_1 \times M_2)$ is an index pair for $S \times T$.

## 3. Conley index for gradient flows and Morse theory

In this section, we assume $M$ is a complete Riemannian manifold and $f \in C^\infty(M, \mathbb{R})$ is a smooth function on a possibly noncompact manifold $M$, that we however assume to be finite-dimensional.

We will here apply the results and methods of the previous section to the vector field $\xi = -\nabla f(x)$ or to similar vector fields called "pseudo-gradients vector fields" for $f$. Remember that an invariant set is a union of critical points and heteroclinic orbits connecting them.

DEFINITION 5.23. *A pseudo-gradient vector field $\xi$ for a smooth function $f$ is a vector field such that there exists a neighbourhood $U$ of the set of critical points such that*

*(1) In $U$ there is a riemannian metric such that $\xi(x) = -\nabla f(x)$*

*(2) There is a positive constant $C$ such that outside of $U$ we have $df(x)\xi(x) \leq -C$*

*(3) If moreover $f$ is a Morse function, we assume that in some chart where $f(u_1, ..., u_n) = u_1^2 + \ldots + u_{n-i}^2 - u_{n-i+1}^2 - \ldots - u_n^2$, the metric is the euclidean metric so that we have $\xi(u) = -\nabla f(u) = (-u_1, \ldots, -u_{n-i}, u_{n-i+1}, \ldots, u_n)$*

A generic function in $C^2(M, \mathbb{R})$ is Morse (see [Mil63]). This is just transversality, a consequence of Sard's theorem (see Exercise 3). We shall sometimes deal with the more general situation of a Morse-Bott function, that is a function having non-degenerate critical submanifolds). We shall usually assume the pseudo-gradient is Morse-Smale, that is the stable and unstable manifolds of two different critical points $x_i, x_j$ (or two critical submanifolds $V_i, V_j$) are transverse. That this can be achieved by an arbitrarily small smooth perturbation is proved for example in Milnor's book ([Mil65], Lemma 5.2). For a Morse or Morse-Bott function, the sets $W^+(S), W^-(S)$ associated to a critical point or critical manifold $S$ are immersed submanifolds (see Exercise 4).

When dealing with a non-compact manifold $M$ we assume it is endowed with a complete riemannian metric and assume the function $f$ satisfies the following Palais-Smale condition (denoted from now on as (PS) condition)

DEFINITION 5.24 (Condition (PS), see [PS64]). *If a sequence $(x_n)_{n\geq 1}$ satisfies*

*(1) $df(x_n) \to 0$*

*(2) $f(x_n) \to c$*

*then $(x_n)_{n\geq 1}$ has a converging subsequence.*

REMARK 5.25. Clearly, the limit of the subsequence is a critical point at level $c$.

The main consequence of the (PS) condition is that the set of critical points in a region where $f$ is bounded is compact (apply the condition to a sequence of critical points). But there is more, that is that topologically nothing happens outside a compact set. In our setting, this means that we can use non-compact index pairs, i.e. we may replace the compactness assumption on $N_1$ by the assumption that $f$ is bounded on $N_1$. This is a consequence of the following fact :

LEMMA 5.26. *Assume $f$ satisifies the Palais -Smale condition. Let $V$ be a neighbourhood of $K_{[a,b]}$ the set of critical points in $f^{[a,b]}$. Let $\varphi^t(x)$ be the flow of $-\nabla f(x)$. Then there exists $T > 0$ such that if $\varphi^{[s,t]}(z) \in f^{[a,b]} \setminus V$ then $t - s < T$.*

PROOF. Indeed,

$$\frac{d}{dt} f(\varphi^t(x)) = df(\varphi^t(x))\nabla f(\varphi^t(x)) = -|df(\varphi^t(x))|^2$$

The Palais -Smale condition implies that there exists $\varepsilon_0 > 0$ such that, as long as we stay outside $V$, we have $|df(\varphi^t(x))|^2 \geq \varepsilon_0$. As a result $f(\varphi^t(x)) - f(x) \leq -\varepsilon_0 t$, so $T = \frac{b-a}{\varepsilon_0}$ will do. □

From this we easily infer

PROPOSITION 5.27. *Let $N_1, N_2$ be a pair satisfying the assumption of an index pair for $-\nabla f(x)$ except that compactness is replaced by $N_1 \subset f^{[a,b]}$. Then there exists $\overline{N}_1, \overline{N}_2$ an index pair (in the usual sense) such that $N_1/N_2 \simeq \overline{N}_1/\overline{N}_2$*

PROOF. We first notice that the compactness of $S$ is an immediate consequence of the previous Lemma. Indeed $V$ is compact, and the set of bounded trajectories in $f^{[a,b]}$ is contained in $\varphi^{[0,T]}(V)$, and is therefore compact. Moreover by the same argument, $W_N^-(S)$ is also compact, since the portion of trajectory outside a compact neighbourhood of $S$, correspond to a bounded time interval. As a result $W_N^-(S)$ is compact, and hence we can construct, as in Proposition 5.12 an index pair $(N_1^{st}, N_2^{st})$ in a neigbourhood of $W_N^-(S)$. Now if we follow the flow of $-\nabla f(x)$, since $N_1^{st}$ is a neighbourhood of $W_N^-(S)$, it sends $N_1$ to $N_1^{st}$ and the proof is the same as for Proposition 5.14.

□

Move or remove the following Proposition ?

Note the following

PROPOSITION 5.28. *Let $B$ be an isolating block for $\xi = -\nabla f$, $a, b$ regular values of $f$. Then $(B \cap f^{[a,b]}, \partial^- B \cap f^{[a,b]} \cup f^{-1}(a))$ is also an index pair (for the invariant set $S \cap f^{[a,b]}$).*

PROOF. The proof is left as as an exercise. □

In the Morse case, the situation is quite simpler. Let $x_j$ be a critical point of index $j$ and consider $(N_j, N_{j-1})$ an index pair with $\{x_j\}$ a maximal invariant set. Note that the $W^{\pm}_{N_j}(x_j)$ are injectively immersed submanifolds (in fact injectively immersed discs) obtained by following the flowlines of $-\nabla f(x)$ starting form the local unstable disc. In the local coordinates given by Definition 5.23 we have

$$W^-_{loc}(x_j) = \left\{ (0,...,0,u_{n+1-i},...,u_n) \mid \sum_{j=1}^{i} u^2_{n+1-j} \leq \varepsilon \right\}$$

We can arbitrarily orient one of the two, and the other one will be automatically oriented if we require that $x_j$ is a positive intersection of $W^+_{N_j}(x)$ and $W^-_{N_j}(x)$. The gradient flow is said to be Morse-Smale if stable and unstable manifolds intersect transversely. In particular a heteroclinic orbit can only go from a point of index $j$ to a point of index less than $j$. According to our Proposition, we can replace the pair $(N_j, N_{j-1})$ by a standard pair, that is here a union of neighbourhoods of the $W^-_{N_j}(x_j)$ modulo their exit set, that is clearly homologous to $W^-_{N_j}(x_j)$ modulo its boundary. Note that an orientation of $W^-_{N_j}(x_j)$ is equivalent to choosing a generator of $H_{d_j}(W^-_{N_j}(x_j), \partial W^-_{N_j}(x_j))$, where $d_j = index(d^2 f(x_j))$.

Now we claim the following

PROPOSITION 5.29 ([Con78; Flo88b]). *Let $f$ be a smooth function satisfying the Palais-Smale condition and $\xi$ a pseudo-gradient vector field. For any pair of real numbers $a < b$, there is a family $N_j$ of compact sets such that $N_{-1} = f^a, N_n \subset f^b$, $(N_j, N_{j-1})$ is an index pair for $\xi$ containing all critical points of index $j$. We have $H^k(N_j, N_{j-1}) = k^{m_j}$ for $k = j$ and $0$ otherwise. Moreover the map*

$$\delta_j : H^j(N_j, N_{j-1}) \longrightarrow H^{j+1}(N_{j+1}, N_j)$$

*has for matrix $(d_j^{k,l})_{\substack{1\leq k\leq m_j \\ 1\leq l\leq m_{j+1}}}$ where $d_j^{k,l}$ counts the number of gradient trajectories of $\xi$ between $x_k$ and $y_l$.*

*Moreover we have the homotopy equivalence*

$$N_n / N_{-1} \simeq f^b / f^a$$

PROOF.

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The construction is done by induction. We set $N_{-1} = f^a$, and assume $N_j$ has been constructed. Consider the set $x_1, ..., x_{m_j}$ of critical points of index $j$, and the neighbourhoods $U_j(x_k)$ of the unstable manifold of $x_k$, $W^-_j(x_k) = W^-_{N_j}(x_k)$. A trajectory in $W^-_j(x_k)$ either ends up in $f^a$ or stops at a critical point of lower index (by the Morse-Smale condition). In both cases for $t$ large enough $\varphi^t(\partial W^-_j(x_k))$ will be in $N_{j-1}$. As in the construction of the standard index pairs, we can replace the pair $(N_j, N_{j-1})$ by the union of the neighbourhoods of $W^-_{N_j}(x_j)$ represented in Figure 6. On this pair, it is clear that $W^-_j(x_k) = W^-_{N_j}(x_k)$ generates the homology $H_*(N_j, N_{j-1})$ and

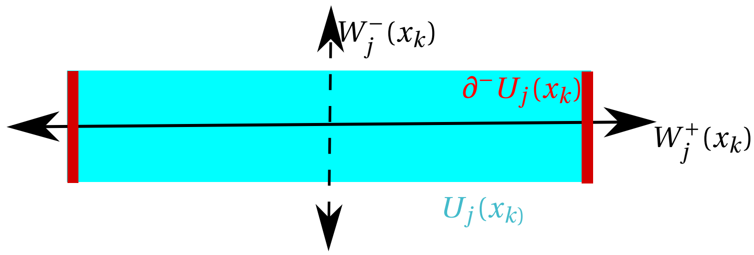


FIGURE 6. The sets $W_{N_j}^+(x_k)$ in solid black, $W_{N_j}^-(x_k)$ is the dotted line. We have $U(x_k)$ in blue and $\partial^- U_{x_k}$ in red.

$W_j^+(x_k) = W_{N_j}^+(x_k)$ is dual to this generator, so represents, by intersection, the generator of the cohomology $H^*(N_j, N_{j-1})$. Let $y_1, ... y_{m_{j+1}}$ be the critical points of index $j+1$. We define $N_{j+1}$ as the union of $N_j$ and the neigbourhoods $U_{j+1}(y_k)$ of the unstable manifolds $W_{j+1}^-(y_k)$. Clearly $H_*(N_{j+1}, N_j)$ is generated by the $\left[W_{j+1}^-(y_l)\right]$, and the map $\partial_{j+1} : H_{j+1}(N_{j+1}, N_j) \longrightarrow H_j(N_j, N_{j-1})$ is given by $\partial_{j+1}(y_j) = \left[\partial W_{j+1}^-(y_l)\right]$ in $H_j(N_j, N_{j-1})$ and the $x_k$ component is given by intersecting with $W_j^+(x_k)$, so is given by $\partial W_{j+1}^-(y_l) \cap W_j^+(x_k)$. Its matrix is then given by $(d_j^{k,l})_{\substack{1\leq k\leq m_j \\ 1\leq l \leq m_{j+1}}}$. Now it is well known that $\partial_j$ is dual to $\delta_j$. ☐

REMARKS 5.30. (1) We used the following facts. If $A \subset B \subset X$ where $X$ is a manifold (not necessarily closed) and $V \cap B$ is a closed submanifold in $B \setminus A$ of codimension $k$, then intersection with $V$ defines a class in $H^k(B, A)$, by sending $\sigma \in C_k(B, A)$ to $\#(\sigma \cap V)$ the intersection number of $V$ and $\sigma$ counted with signs. Indeed $V \cap B$ defines a class in the Borel-Moore homology of $B$ (that is the homology of the chain complex of locally finite chains) $H_*^{BM}(B)$ and the Poincaré duality (which is not an isomorphism, since we do not assume $X$ is a closed manifold), associates to $V$ a class in $H^{n-*}(B, A)$ (for more details see [Hat02], p. 239 and [BM60]).

(2) Note that the inclusion $N_n \subset f^b$ is a homotopy equivalence. Indeed, there is a $t \geq 0$ such that $f(x) \leq b$ implies $\varphi^t(x) \in N_n$, because $N_n$ contains $f^a$ and a neighbourhood of all critical points. But by Lemma 5.26 if $x \notin N_n$ then for $t$ large enough $\varphi^{[0,t]}(x)$ will not be contained in $f^{[a,b]} \setminus N_n$. Since $f$ is decreasing on the flow and $N_n$ is positively invariant, we must have $\varphi^t(x) \in N_n \cup f^a = N^n$. Since the flow preserves $N_{-1} = f^a$, we thus proved that $(N_n, N_{-1})$ is homotopy equivalent to $(f^b, f^{a)}$.

This implies (see [Tho49; Mil65; Wit82; Lau92] and [Flo88b] for this proof)

THEOREM 5.31. *(Thom-Smale-Witten) Let $f$ be a smooth function satisfying the Palais-Smale condition and $\xi$ a pseudo-gradient vector field for $f$. Let $C^j(a,b)$ be the group generated by the critical points of index $j$ contained in $f^{[a,b]}$ and $\delta^j$ be the map defined by the matrix counting (with sign) the number of trajectories of $\xi$ from $x_k$ to $y_l$. Then we have*

$$H^*(C^*([a,b],d^*)) = H^*(N_n, N_{-1}) = H^*(f^b, f^a)$$

PROOF. We set $C^j = H^j(N_j, N_{j-1})$ and note that according to Proposition 5.29, under the identification of $C^j(a,b)$ with $H^j(N_j,N_{j-1})$ the map $\delta^j$ coincides with the coboundary map

$$\delta_j : H^j(N_j, N_{j-1}) \longrightarrow H^{j+1}(N_{j+1}, N_j)$$

We wish tho prove that the cohomology of $(C^*, \delta^*)$ equals $H^*(f^b, f^a)$. We consider the diagram

$$\begin{tikzcd}
 & & 0 & C^{j+1} & \\
H^j(N_{j+1},N_j) \simeq 0 \arrow[r] & H^j(N_{j+1},N_{j-2}) \arrow[r, "\alpha"] & H^j(N_j,N_{j-2}) \arrow[r, "\gamma"] \arrow[u] & H^{j+1}(N_{j+1},N_j) \arrow[r] \arrow[u] & \dots \\
 & & H^j(N_j,N_{j-1}) \arrow[r, "\simeq"] \arrow[u, "\beta"] \arrow[ur, "\delta^j"] & C^j \arrow[u] & \\
 & & H^{j-1}(N_{j-1},N_{j-2}) \arrow[r, "\simeq"] \arrow[u, "\delta^{j-1}"] & C^{j-1} \arrow[u] &
\end{tikzcd}$$

By exactness of the horizontal sequence, we have $H^j(N_{j+1}, N_{j-2}) = \mathrm{Im}(\alpha) = \ker(\gamma)$. Since $\beta$ is onto, $\ker(\gamma) = \ker(\gamma\circ\beta)/\ker(\beta)$. But $\mathrm{Ker}(\gamma\circ\beta) = \mathrm{Ker}(\delta^j)$ and $\mathrm{Ker}(\beta) = \mathrm{Im}(\delta^{j-1})$ because the left-hand side vertical sequence is exact. Finally, we get $H^j(N_{j+1}, N_{j-2}) = \mathrm{Ker}(\delta^j)/\mathrm{Im}(\delta^{j-1})$. Now to conclude the proof we shall notice that replacing $N_{j+1}$ by $N_n$ and $N_{j-2}$ by $N_0$ does not affect the cohomology in degree $j$. The intuition behind this claim is that crossing a level containing a critical point of index $k$ can only modify the cohomology in degrees $k-1, k$ or $k+1$. Since going from $N_{j+1}$ to $N_n$ (resp. from $N_0$ to $N_{j-2}$) we only cross critical points of index greater than $j+2$ (resp. less or equal to $j-2$) this will not modify $H^j$. Now let us give a little bit more detail. First of all $H^{j-1}(N_k, N_0) = 0$ for $k \leq j-2$ is proved by induction, using the long exact sequence

$$0 = H^{j-1}(N_k, N_0) \longrightarrow H^j(N_{k+1}, N_k) \longrightarrow H^j(N_{k+1}, N_0) \longrightarrow H^j(N_k, N_0) = 0 \qquad ..$$

From the exact sequence of the triple $(N_{j+1}, N_{j-2}, N_0)$

$$0 = H^{j-1}(N_{j-2}, N_0) \longrightarrow H^j(N_{j+1}, N_{j-2}) \longrightarrow H^j(N_{j+1}, N_0) \longrightarrow H^j(N_{j-2}, N_0) = 0 \qquad ..$$

we may deduce that $H^j(N_{j+1}, N_0) = H^j(N_{j+1}, N_{j-2})$. A similar argument shows that $H^j(N_{j+1}, N_0) \simeq H^j(N_n, N_0)$. □

COROLLARY 5.32. *(Morse inequalities) Let $f$ be a Morse function on a manifold $M$, satisfying the Palais-Smale condition. Let $m_j(a,b)$ be the number of critical points in $f^{[a,b]}$ of index $j$ and $b_j(a,b) = \dim H^j(f^b, f^a)$. Then we have for all $p$*

$$\sum_{j=0}^{p} (-1)^{p-j} m_j(a,b) \geq \sum_{j=0}^{p} (-1)^{p-j} b_j(a,b)$$

*In particular if $\chi$ is the Euler characteristic, we have $\chi(H^*(f^b, f^a)) = \sum_{j=0}^{n} (-1)^{p-j} m_j(a,b)$ and for all $j$, $m_j(a,b) \geq b_j(a,b)$.*

PROOF. Let us start with the Euler characteristics. If $(C^*, \delta^*)$ is a chain complex, then its homology $H^*(C^*, \delta^*)$ satisfies the equation

$$\sum_j (-1)^j \dim(C^j) = \sum_j (-1)^j \dim(H^j)$$

Indeed we have exact sequences

$$0 \longrightarrow \mathrm{Ker}(\delta^j) \longrightarrow C^j \longrightarrow \mathrm{Im}(\delta^j) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Im}(\delta^{j-1}) \longrightarrow \mathrm{Ker}(\delta^j) \longrightarrow H^j \longrightarrow 0$$

so $\dim(C^j) = \dim(\mathrm{Ker}(\delta^j) + \dim(\mathrm{Im}(\delta^j))$ and $\dim(H^j) = \dim(\mathrm{Ker}(\delta^j) - \dim(\mathrm{Im}(\delta^{j-1}))$ taking the alternating sum, we get

$$\sum_j (-1)^j \dim(C^j) = \sum_j (-1)^j \dim(\mathrm{Ker}(\delta^j) + \sum_j (-1)^j \dim(\mathrm{Im}(\delta^j)) =$$
$$\sum_j (-1)^j \dim(\mathrm{Ker}(\delta^j) - \sum_j (-1)^j \dim(\mathrm{Im}(\delta^{j-1})) = \sum_j (-1)^j \dim(H^j)$$

Consider the truncated complex $\tau_p(C^*)$

$$0 \longrightarrow C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \dots \qquad C^p \xrightarrow{\delta^p} \mathrm{Im}(\delta^p) \longrightarrow 0$$

This is again a chain complex, and it has the same homology as $(C^*, \delta^*)$, up to dimension $p$ and its $p+1$ homology is zero. Therefore

$$\sum_{j=0}^{p} (-1)^j \dim(H^j) = \sum_{j=0}^{p} (-1)^j \dim(C^j) + (-1)^{p+1} \dim(\mathrm{Im}(\delta^p))$$

as a result

$$\sum_{j=0}^{p} \dim(C^j) - \sum_{j=0}^{p} (-1)^j \dim(H^j) = \dim(\mathrm{Im}(\delta^p)) \geq 0$$

□

Theorem 5.31 may be extended without effort as follows

THEOREM 5.33. *Let $f$ be a smooth function satisfying the Palais-Smale condition and $\xi$ a Morse-Smale pseudo-gradient vector field for $f$. Let $S$ be a compact invariant set for $\xi$, that is a union of critical points and heteroclinic orbits connecting them and $(N_1, N_2)$ an index pair for $S$. Let $C^j(N_1, N_2)$ be the complex generated by the critical points in $S$ of index $j$, and $\delta^j : C^j(N_1, N_2) \longrightarrow C^{j+1}(N_1, N_2)$ be given by the matrix counting (with sign) the number of* **trajectories of $\xi$ contained in** $N_1 \setminus N_2$ *and connecting $x_k$ to $y_l$. Then we have*

$$H^*(C^*(N_1, N_2), \delta^*) = H^*(N_1, N_2)$$

A proof will be added here

## 4. Lusternik-Schnirelman and the calculus of critical levels

We now consider a function $f \in C^\infty(M, \mathbb{R})$ but do not make any assumption on the critical points. We still assume in this section that all functions satisfy the Palais-Smale condition. Given $a < b < c$, there is natural embedding

$$(f^b, f^a) \hookrightarrow (f^c, f^a).$$

inducing a morphism

$$H^*(f^c, f^a) \to H^*(f^b, f^a).$$

DEFINITION 5.34 (Cohomological critical value). *Let $\alpha \in H^*(f^c, f^a) \setminus \{0\}$. Define*

$$c(\alpha, f) = \inf\{b \mid \text{image of } \alpha \text{ in } H^*(f^b, f^a) \text{ is not zero}\}.$$

Note that we could define $c(0, f)$ as $+\infty$, but this is not very useful. Since the embedding also induces

$$H_*(f^b, f^a) \hookrightarrow H_*(f^c, f^a),$$

the same can be done for $\omega \in H_*(f^c, f^a) \setminus \{0\}$.

DEFINITION 5.35. *For $\omega \in H_*(f^c, f^a) \setminus \{0\}$, define*

$$c(\omega, f) = \inf\{b \mid \omega \text{ is in the image of } H_*(f^b, f^a)\}.$$

EXAMPLE 5.36. Let $M$ be a compact manifold 1 the generator of $H^0(M)$ and $\mu_M$ the generator of $H^m(M)$, where $m = \dim(M)$. Then $c(1, f) = \min f$, $c(\mu_M, f) = \max(f)$. Indeed, the map $H^0(M) \longrightarrow H^0(f^a)$ is non zero as long as $f^a$ is non-empty, that is as long as $a \geq \min(f)$. So $c(1, f) = \min(f)$. On the other hand $\mu_M$ can be represented by a class supported in an arbitrarily small open set (think of the de Rham cohomology, $\mu_M$ can be represented by a bump function times a volume form), alternatively $H^m(X) = 0$ if $X$ is a manifold with non-empty boundary. Thus if $a < \max f$ the map $H^m(M) \longrightarrow H^m(f^a)$ is zero and $c(\mu_M, f) = \max f$.

PROPOSITION 5.37. *The numbers $c(\alpha, f)$ and $c(\omega, f)$ are critical values of $f$. Moreover if $\alpha \in H_d(f^c, f^a)$ (resp. $\omega \in H^d(f^c, f^a)$), then $K_c$ contains a critical point with index $i$, nullity $\nu$ such that*

$$i \leq d \leq d + \nu$$

PROOF. We prove only the case of $c(\alpha, f)$ since the proof for $c(\omega, f)$ is similar. Let $\gamma = c(\alpha, f)$, assume $\gamma$ is not a critical value. Since $f$ satisfies (PS) condition, we have

$$H^d(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}) = 0$$

Consider the long exact sequence for the triple $(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}, f^a)$,

$$H^d(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}) \to H^d(f^{\gamma+\varepsilon}, f^a) \to H^d(f^{\gamma-\varepsilon}, f^a) \to H^{d+1}(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon})$$

Then $H^d(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}) = 0$ implies that the second arrow above is injective. But by definition of $\gamma$, the image of $\alpha$ in $H^d(f^{\gamma+\varepsilon}, f^a)$ is non-zero while its image in $H^d(f^{\gamma-\varepsilon}, f^a)$ vanishes which contradicts the injectivity. In fact this argument shows that $H^d(f^{c+\varepsilon} f^{c-\varepsilon}) \neq 0$. For the second statement, we make a $C^2$ small perturbation, so that $f$ becomes a Morse function. If a critical point $x_0$ has index $i$ and nullity $\nu$ then $d^2 f(x)$ is positive definite on a space of dimension $n - \nu - i$ and negative definite on as space of dimension $i$. By continuity this still holds after $C^2$-perturbation, so the critical points of the perturbed function that appear near $x_0$ all have index $j$ such that $i \leq j \leq i + \nu$. So if $d \notin [i, i + \nu]$ we have created no critical point of index $d$, but this implies $H^d(f^{c+\varepsilon} f^{c-\varepsilon}) = 0$ contradicting the definition of $c = c(\alpha, f)$. □

We are now going to show that Poincaré dual classes induce corresponding critical values for $f$ and $-f$. Recall Alexander duality (see [Spa66] page 342 theorem 10)

$$AD : H_*(f^c, f^a) \xrightarrow{\simeq} H_c^{n-*}(X - f^a, X - f^c) = H_c^{n-*}((-f)^{-a}, (-f)^{-c})$$

In our situation, the Palais-Smale condition implies that we may forget about the compact support (use Proposition 5.27).

PROPOSITION 5.38. *Assume that $f$ satisfies the Palais-Smale condition. Then for $\omega \in H_*(f^c, f^a) \setminus \{0\}$,*

*1) $c(\omega, f) = -c(AD(\omega), -f)$;*

*2) If $M$ is compact, $a = -\infty$ and $c = +\infty$ we have $c(1, f) = -c(\mu, -f)$ where $1 \in H^0(M)$ and $\mu \in H^n(M)$ are generators.*

PROOF. (1) Using the fact that $X \setminus f^a = (-f)^{-a}$ and that according to Lemma 5.27, $H_c^*((-f)^{-a}, (-f)^{-b}) \simeq H^*((-f)^{-a}, (-f)^{-b})$, the proof reduces to diagram chasing on the following, where vertical arrows are induced by inclusions

$$\begin{array}{ccc}
H_*(f^c, f^b) & \xrightarrow{AD} & H^{n-*}(X \setminus f^a, X \setminus f^c) = H^{n-*}((-f)^{-b}, (-f)^{-c}) \\
\uparrow & & \uparrow \\
H_*(f^c, f^a) & \xrightarrow{AD} & H^{n-*}(X \setminus f^a, X \setminus f^b) = H^{n-*}((-f)^{-a}, (-f)^{-c}) \\
\uparrow & & \uparrow \\
H_*(f^b, f^a) & \xrightarrow{AD} & H^{n-*}(X \setminus f^c, X \setminus f^b) = H^{n-*}((-f)^{-b}, (-f)^{-a})
\end{array}$$

Indeed, $b \geq c(\omega, f)$ if and only if the lower left vertical map $H_*(f^b, f^a) \longrightarrow H_*(f^c, f^a)$ has $\omega$ in its image. Since AD is an isomorphism, this is equivalent to stating that $\alpha = AD(\omega) \in H^{n-*}((-f)^{-a}, (-f)^{-c})$ is contained in the image of $H^{n-*}((-f)^{-a}, (-f)^{-b})$ i.e. that its image in $H^*((-f)^{-b}, (-f)^{-a})$ is zero. As a result $b \geq c(\omega, f)$ if and only if $-b \leq c(\alpha, -f)$ that is $b \geq -c(\alpha, -f)$. This means that $c(\omega, f) = -c(\alpha, -f)$.

(2) It follows from the fact that

$$c(1, f) = \min(f) \text{ and } c(\mu, f) = \max(f).$$

□

We refer to Exercice 7 for a generalization of this result.

The following is a crucial result : it shows that critical values associated to certain pairs of cohomology classes must be distinct (see Exercice 8 for more examples). Beyond multiplicity results, this theorem will also crucially imply the triangle inequality for capacities

THEOREM 5.39. *(Lusternik-Schnirelman[6]) Assume $\alpha \in H^*(f^c, f^a) \setminus \{0\}$ and $\beta \in H^*(M) \setminus \{0\}$, then*

$$c(\alpha \cup \beta, f) \geq c(\alpha, f) \tag{5.1}$$

*If equality holds in equation 5.1 with common value $\gamma$, then for any neighborhood $U$ of $K_c = \{x | f(x) = \gamma, df(x) = 0\}$, we have $\beta \neq 0$ in $H^*(U)$.*

PROOF. Inequality 5.1 is obvious because $\alpha = 0$ in $H^*(f^b, f^a)$ implies $\alpha \cup \beta = 0$ in $H^*(f^b, f^a)$.

Assume equality holds in (5.1). Then to any sufficiently small neighbourhood $U$ of $K_c$ we may associate by Proposition 5.12 an index pair $(N_1, N_2)$ such that $N_1 \subset U$. This follows from the fact that using the (PS) condition, we see that for $\varepsilon$ small enough, the critical points in $f^{c+\varepsilon} \setminus f^{c-\varepsilon}$ are contained in $U$. Indeed, if we had a sequence $(x_n)_{n\geq 1}$ of critical points in $f^{c+\varepsilon_n} \setminus (f^{c-\varepsilon_n} \cup U)$ we would get a limiting point $x_\infty$, a critical point in $f^{=c} \setminus U$, a contradiction. Moreover $(f^{c+\varepsilon}, f^{c-\varepsilon})$ is another index pair, so we must have $H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) = H^*(N_1, N_2)$. Now assume $\beta$ vanishes in $H^*(U)$ hence in $H^*(N_1)$. Then $\alpha$ has non-zero image in $H^*(f^{c+\varepsilon}, f^{c-\varepsilon}) = H^*(N_1, N_2)$, but then $\alpha \cup \beta = 0$ in $H^*(N_1, N_2)$ hence in $H^*(f^{c+\varepsilon}, f^{c-\varepsilon})$. But this implies that $c(\alpha \cup \beta, f) \geq c + \varepsilon$ a contradiction.

□

[6]Schnirelman is also sometimes spelled Shnirelman, both are transcriptions of Шнирелма́н. Lusternik is sometimes spelled Ljusternik or Liusternik, both transcriptions of Люсте́рник. Both were involved in the 1930's in the Luzin affair, (concerning the mathematician Лузин, see [DL16]), in which Luzin was accused in Soviet Union of being an "enemy of the People". One of the main official charges against Luzin was that Luzin published his best work abroad. Ironically, Lusternik and Shnirelman's theorem was published in french in French journals (see [LS29; LS34]).

REMARK 5.40. Note first that in this section, $H^*$ can be any cohomological theory. If $\beta \neq 0$ in $H^p(U)$ for all neighbourhoods $U$ of $K_c$, If equality in (5.1) holds, then $\beta \neq 0$ in $H^p(U)$ for all neighbourhoods $U$ of $K_c$. This implies by definition that the cohomological dimension of $K_c$ is at least $p$. If $p \geq 1$ this implies that the Lebesgue covering dimension of $K_c$ is non-zero, which implies that $K_c$ cannot be totally disconnected[7]. In particular $K_c$ must be uncountable.

We now have

PROPOSITION 5.41. *Let $f$ be a function on $C^\infty(M,\mathbb{R})$ satsifying the Palais-Smale condition. Let $\alpha \in H^*(f^c, f^a)$ and $\beta_j \in H^*(M) \setminus H^0(M)$. If $\alpha \cup \beta_1 \cup \ldots \cup \beta_k \neq 0$ in $H^*(f^c, f^a)$, then $f$ has at least $k+1$ critical points in $f^c$.*

PROOF. Indeed, we have

$$c(\alpha, f) \leq c(\alpha \cup \beta_1, f) \leq \ldots \leq c(\alpha \cup \beta_1 \cup \ldots \cup \beta_k, f) \leq c$$

If all these critical values are distinct, then $f$ has at least $k+1$ critical points below level $c$. If two critical values are equal, then according to to Remark 5.40 there are infinitely many critical points. □

DEFINITION 5.42. *Let $M$ be a topological space. We define $cl(M)$ to be the maximum number of classes of positive degree such that their product is non-zero.*

COROLLARY 5.43. *Let $f \in C^\infty(M,\mathbb{R})$ with $M$ compact, then*

$$\#Crit(f) \geq cl(M) + 1$$

PROOF. Apply the Proposition with $\alpha = 1$. □

We state for future use the following property.

PROPOSITION 5.44. *Let $u : M \longrightarrow N$ be a map, then we have for $\alpha \in H^*(f^c, f^a)$ and $u^*(\alpha) \in H^*((f \circ u)^c, (f \circ u)^a)$ the following inequality*

$$c(\alpha, f) \leq c(u^*(\alpha), f \circ u)$$

PROOF. The map $u$ sends $(f \circ u)^c$ to $f^c$. Then we have an induced map

$$u^* : H^*(f^b, f^a) \longrightarrow H^*((f \circ u)^b, (f \circ u)^a)$$

and a commutative diagram
$$\begin{tikzcd} H^*(f^b, f^a) \arrow[r, "u^*"] & H^*((f \circ u)^b, (f \circ u)^a) \\ H^*(f^c, f^a) \arrow[r, "u^*"] \arrow[u] & H^*((f \circ u)^c, (f \circ u)^a) \arrow[u] \end{tikzcd}$$
so that if $\alpha$ goes to 0 in $H^*(f^b, f^a)$, then $u^*(\alpha)$ goes to 0 in $H^*((f \circ u)^b, (f \circ u)^a)$. As a result we get the inequality. □

[7] see `https://encyclopediaofmath.org/wiki/Zero-dimensional_space`

## 5. Appendix: When is a flow gradient-like ? Conley's Lyapounov functions

This section to be reworked

Until now we only dealt with pseudo-gradient flows on manifolds, that is flows $\varphi^t$ such that there exists a function $f$ on $M$ such that $f(\varphi^t(x))$ is strictly decreasing except at critical points, plus some local condition near the set of zeros of $\xi$. But we have not dealt with the following question :

given a flow, is it the pseudo-gradient of some function ?

The behaviour near critical points is delicate, because a pseudo-gradient is defined as a real gradient near such points, so we shall weaken the question and also localize it. We thus set

DEFINITION 5.45. *Let $\varphi^t$ be a flow in $M$ and $U$ be an open relatively compact set in $M$. We say that a flow is gradient-like on $U$, if there is a function $f \in C^\infty(U,\mathbb{R})$ such that $df(x)\xi(x) \leq 0$ in $U$ and the inequality is strict outside the set of zeroes of $\xi$.*

So our question is:

When is a vector field gradient-like on $U$ ?

Usually we are interested in the case $U = M$, but we shall see an application of the general case. Note that this sections is inspired by [Fat22], [WY73] where stronger statements are proved (in particular $\xi$ is only assumed to be $C^1$, but the function obtained is still smooth).

In the sequel we shall always assume $\overline{U}$ is compact. As a result the reader can check that we do not need the flow to be defined for all $t$, but only to be defined as long as $\varphi^t(x)$ remains in $\overline{U}$, which is automatic by a standard theorem for ODE. We may also replace $\xi(x)$ by $\rho(x)\xi(x)$ where $\rho$ is supported in a compact neighbourhood of $\overline{U}$ and $\rho = 1$ on $\overline{U}$. Then the trajectories of the new vector field coincide with the trajectories of the old one as long as they remain in $\overline{U}$ and the new vector field is complete. It will be convenient to know the following

LEMMA 5.46 (Conley, see [Con78], p.33).

(1) *Let $U$ be a domain and $f$ a function defined on $U$, such that $f(\varphi^t(x)) \leq f(x)$ for all $t > 0$ and such that the inequality is strict for $x \in W$. Then the function*
$$g(x) = \int_0^{+\infty} e^{-t} f(\varphi^t(x))\,dt$$
*satisfies $dg(x)\xi(x) \leq 0$ and the inequality is strict for $x \in W$.*

(2) *Let $f$ be a smooth function such that $f(\varphi^1(x)) \leq f(x)$. Then the function $F(x) = \int_0^1 f(\varphi^u(x))\,du$ satisfies $F(\varphi^s(x)) \leq F(x)$ for all $s \geq 0$. More precisely $\frac{d}{dt}F(\varphi^t(x)) = f(\varphi^1(x)) - f(x)$.*

PROOF. The proof is straightforward :

(1)
$$dg(x)\xi(x) = \int_0^{+\infty} e^{-t} df(\varphi^t(x))d\varphi^t(x)\xi(x)dt = \int_0^{+\infty} e^{-t} df(\varphi^t(x))\xi(\varphi^t(x))dt$$
since $\xi(\varphi^t(x)) = d\varphi^t(x)\xi(x)$. Now the integrand is nonpositive and since
$$\int_0^t df(\varphi^t(x)\xi(\varphi^t(x)))dt = f(\varphi^t(x)) - f(x)$$
it will be negative for $t$ large enough, the integral defining $dg(x)\xi(x)$ must also be strictly negative.

(2) We have
$$\frac{d}{dt}F(\varphi^t(x)) = \frac{d}{dt}F(\varphi^t(x)) = \int_0^1 \frac{d}{dt}f(\varphi^{u+t}(x))du =$$
$$\int_0^1 \frac{d}{du}f(\varphi^{u+t}(x))du = f(\varphi^1(x)) - f(x) \leq 0$$
Therefore $F$ is decreasing along the trajectories and $F(\varphi^s(x)) \leq F(x)$ for all $x$.

□

DEFINITION 5.47 (Pseudo-orbits and chain recurrent set). *Let $\varphi^t$ be a flow on $M$, $U$ an open set in $M$. Let $x, y \in U$ and $\varepsilon, T$ positive numbers. We define an* **$(\varepsilon, T)$ pseudo-orbit of length $T_n$ in $U$ from $x$ to $y$** *a (discontinuous) path $\gamma : [0, T_n] \longrightarrow U$ such that there exist $(T_k)_{0\leq k\leq n}$ with $T_k - T_{k-1} \geq T$ and*

*(1) $\gamma(0) = x, \gamma(T_n) = y$*
*(2) $\gamma(t) = \varphi^{t-T_k}(\gamma(T_k))$ for $T_k \leq t < T_{k+1}$*
*(3) $d(\varphi^{T_{k+1}-T_k}(\gamma(T_k)), \gamma(T_{k+1})) \leq \varepsilon$*

*We say that $x$ is chain recurrent in $U$ if for any $(\varepsilon, T)$ there is an $(\varepsilon, T)$ pseudo-orbit in $U$ from $x$ to itself. We denote by $\mathscr{R}(U)$ the set of chain recurrent points in $U$. The flow is chain recurrent on $U$ if every point is chain recurrent (i.e $\mathscr{R}(U) = U$).*

REMARKS 5.48. (1) An $(\varepsilon, T)$ pseudo-orbit in $U$ is made of pieces of real orbits of length at least $T_k - T_{k-1} \geq T$, and we allow jumps of size at most $\varepsilon$.
(2) It is easy to see that the chain recurrent set is invariant by the flow.
(3) Moreover given $\eta > 0, t_0 > 0$, if $\varphi^{t_0}(x) = z$, we may choose $\varepsilon(t_0, \eta) > 0$ sufficiently small, so that any $(\varepsilon, 1)$ pseudo-orbit of length $t_0$ will pass at distance at most $\eta$ from $z$. This immediately follows from the continuity of the flow. Moreover $\varepsilon(t_0, \eta)$ can be chosen independent from $x \in U$ provided $\overline{U}$ is compact.
(4) The same argument as above implies that any $(\varepsilon, 1)$ orbit is approximated by an $(\eta, T)$ orbit for $\eta < \eta(\varepsilon, T)$. As a result it is enough to only deal with $(\varepsilon, 1)$ pseudo-orbits to simplify notations.

Conley's theorem states that any flow is decomposed into a gradient-like flow and a chain recurrent flow (but theses flows are then on metric spaces, not on the original manifold $M$).

DEFINITION 5.49. *Let $\varphi^t$ be a flow on $M$ and $U$ an open set in $M$. A function $f$ on $U$ is a Lyapounov function if for all $x \in U$ we have $df(x)\xi(x) \le 0$. The critical set for $f$ in $U$ is the set $\mathscr{C}(f,U)$ of critical points of $f$ that is the set of $x \in U$ such that $df(x)\xi(x) = 0$. The Lyapounov functions is said to be strict if $df(x)\xi(x) < 0$ whenever $x$ is not in $\mathscr{R}(U)$ (i.e. $\mathscr{C}(f,U) = \mathscr{R}(U)$).*

Note that if $f(\varphi^t(x)) = f(x)$ for some $t \neq 0$ such that $\varphi^{[0,t]}(x) \subset U$, then $x$ is called a neutral point. Note that if $\xi$ is the vector field generating $\varphi^t$, then the set of neutral points is contained in the set of $x$ such that $df(x)\xi(x) = 0$ (but may be smaller). In the litterature several extra conditions are often imposed to Lyapounov functions.

We first characterize the recurrence set.

DEFINITION 5.50. *An attractor in $U$ is an isolated invariant set $A \subset U$ such that there is an isolating neighbourhood $V \subset U$ of $A$ such that $\varphi^t(\overline{V}) \subset V$ for all $t > 0$. Then $A = \bigcap_{t\ge 0}\varphi^t(V)$. A repeller is the attractor of the flow $\varphi^{-t}$. The repeller associated to $A$ in $U$ is the set $A_U^* = \{x \in U \mid \varphi^{[0,+\infty[}(x) \subset U \setminus V\} = \bigcap_{t\ge 0}\varphi^{-t}(U\setminus V)$, that is the set of orbits that remain in $U \setminus V$ for positive time. We denote by $\mathscr{A}(U)$ the set of attractors in $U$.*

Note that automatically $A$ and $A_U^*$ are indeed invariant sets and a repeller has $U\setminus\overline{V}$ as isolating neighbourhood. It is in fact the set of points which remain in $U$ but do not end up in $A$ in positive time.

EXAMPLES 5.51.

(1) For the vector field $-\nabla f$, the set of minima of $f$ is an attractor $A$, and then $A^*$ is the set of the other critical points and heteroclinic orbits which do not end up in $A$.

(2) If $a$ is a regular value of $f$, and $\xi$ a pseudo-gradient for $f$, then the set $A(\xi,a)$ of critical points with critical value below $a$ and of connecting orbits between them is an attractor. Then $A^*(\xi,a)$ is the set of critical points and heteroclinic orbits contained in $M\setminus A(\xi,a)$. Note that $A^*(\xi,a) = A(-\xi,-a)$.

PROPOSITION 5.52. *Let $\varphi^t$ be a flow on $M$. Then*

$$\mathscr{R}(U) = \bigcap_{A\in\mathscr{A}(U)} (A \cup A_U^*)$$

PROOF. First we prove that if $x \notin A\cup A_U^*$ for some attractor $A$, then $x \notin \mathscr{R}(U)$. Indeed let $V$ be a neighbourhood of $A$ in $U$. We may choose $V$ so that $x \notin V$ and then if $x \notin A_U^*$ then $\varphi^{t_0}(x) \notin \overline{U}\setminus V$ for some $t_0 \ge 0$ and if $\eta < d(\varphi^{t_0}(x), \overline{U}\setminus V)$ there is an $\varepsilon > 0$ such that any $(\varepsilon,1)$ pseudo-orbit passes in the complement of $\overline{U}\setminus V$. If such a pseudo-orbit remains in $U$, then it must pass in $V$. Set $\eta < d(\overline{U}\setminus V, \varphi^1(V))$. Then since $\varphi^{t_0+1}(x) \in \varphi^1(V)$, for $\varepsilon$ small enough, an $(\varepsilon,1)$ pseudo-orbit $\gamma$ will satisfy $d(\gamma(t_0+1), \varphi^{t_0+1}(x)) < \eta$ hence $\gamma(t_0+1) \in V$. But then $d(\gamma(t_0+2), \varphi^1(\gamma(t_0+1))) < \eta$ hence $\gamma(t_0+2) \in V$. By induction, we see that the pseudo-orbit is trapped in $V$ so cannot go back to $x$ and $x \notin \mathscr{R}(U)$

Conversely assume $x \in \bigcap_{A \in \mathscr{A}(U)} (A \cup A_U^*)$ and assume there is no $(\varepsilon, T)$ pseudo-orbit from $x$ to $x$. Let $\Omega_U(x,\varepsilon) \subset U$ be the set of $y$ such that there is an $(\varepsilon, T)$ pseudo-orbit in $U$ from $x$ to $y$. This set is open by definition and $\varphi^t(\overline{\Omega_U(x,\varepsilon)}) \subset \Omega_U(x,\varepsilon)$ for $t > 0$. Indeed, it is enough to prove this for $0 \leq t \leq T$ and if $z \in \overline{\Omega_U(x,\varepsilon)}$ there exists an $(\varepsilon, T)$ pseudo-orbit $\gamma$ from $x$ to $y$ with $d(y,z) < \varepsilon$ and $d(\varphi^t(y), \varphi^t(z)) < \varepsilon$ for $0 \leq t \leq T$. Then concatenating $\gamma(0, T_n)$ with $\varphi^{[0,T]}(z)$ we get an $(\varepsilon, T)$-pseudo-orbit of length $T_n + T$ from $x$ to $\varphi^t(z)$. We thus get an attractor $A = \bigcap_{t \geq 0} \varphi^t(\Omega_U(x,\varepsilon))$. If $x \in A$ then there would be an $(\varepsilon, 1)$ pseudo-orbit in $U$ from $x$ to $x$ contradicting our assumption. So we must have $x \in A_U^*$. Let then $z$ be a limit point of the sequence $\varphi^{nT}(x)$. Since $\varphi^{nT}(x) \in \varphi^{mT}(\overline{\Omega(x,\varepsilon)})$ for all $m < n$, and $\varphi^{mT}(\overline{\Omega(x,\varepsilon)})$ is closed, we may conclude $z \in A$. But then we cannot have $x \in A_U^*$ since by definition the orbit of a point in $A_U^*$ remains in $U$ but never approaches $A$. $\square$

EXERCISE 5.53. Prove for $\xi$ a pseudo-gradient flow for $f$, that $\mathscr{R}$ is the set of critical points of $f$.

LEMMA 5.54. *The number of attractors is countable.*

PROOF. Let $(U_j)_{j \in \mathbb{N}}$ be a countable basis for the topology of $M$. For each $A$ there is a neighbourhood $U$ of $A$ such that $\varphi^1(\overline{U}) \subset U$. Since there is a set $I_A \subset \mathbb{N}$ such that $A \subset \bigcup_{i \in I(A)} U_i \subset U$, we may by compactness of $A$ replace $I(A)$ by a finite set and $U$ by $\bigcup_{i \in I(A)} U_i$. Thus to each $A$ we can associate a finite subset of $\mathbb{N}$, and this determines $A$ since $A = \bigcap_n \varphi^n(U)$. But the set of finite subsets of $\mathbb{N}$ is countable, and since we constructed an injection form the set of attractors to the set of finite subsets of $\mathbb{N}$, so is the set of attractors. $\square$

LEMMA 5.55. *Let $A$ be an attractor in $U$ for the flow $\varphi^t$. Then there exists a smooth function $F : U \longrightarrow [0,1]$ such that $A = F^{-1}(0), A^* = F^{-1}(1)$ and $dF(x)\xi(x) < 0$ in the complement of $A \cup A^*$.*

PROOF. Let $V$ be a neighbourhood of $A$ such that $\varphi^t(\overline{V}) \subset V$ for $t > 0$. In particular $\varphi^1(\overline{V}) \subset V$. Let $W$ be a neighbourhood of $\overline{U}$ and $g$ be smooth function such that

(1) $g = 1$ on $U \setminus V$
(2) $g = 0$ on $\varphi^1(\overline{V}) \cup M \setminus W$
(3) $0 \leq g \leq 1$.

Then if $x, \varphi^1(x) \in U$, we have $g(\varphi^1(x)) \leq g(x)$ since

- either $x \in M \setminus V$ and then $g(x) = 1$ so obviously $g(\varphi^1(x)) \leq g(x)$
- or $x \in V$ and then $\varphi^1(x) \in \varphi^1(V)$ and $f(\varphi^1(x)) = 0$ so obviously $g(\varphi(x)) \leq g(x)$.

According to Lemma 5.46 we have that $f(x) = \int_0^1 g(\varphi^u(x))du$ satisfies $f(\varphi^s(x)) \leq f(x)$ for all $s \geq 0$ and $df(x)\xi(x) < 0$ whenever $g(\varphi^1(x)) < g(x)$. In particular this holds for

$x \in \overline{V} \setminus \varphi^1(\overline{V})$. We set

$$F(x) = \sum_{n\in\mathbb{Z}} \frac{1}{2\cdot 3^{|n|}} g(\varphi^n(x))$$

and the function $F$ is also smooth. Notice that $\sum_{n\in\mathbb{Z}} \frac{1}{2\cdot 3^{|n|}} = 1$, so that $0 \leq F(x) \leq 1$. Moreover if $x \in A$, then $g(\varphi^n(x)) = 0$ and $F = 0$ and if $x \in A_U^*$, then $g(\varphi^n(x)) = 1$ for all $n \in \mathbb{Z}$ Now let $x$ in $U$. If $x \notin A_U^* \cup A$ there exists $n$ such that

(1) if the orbit of $x$ remains in $U$ we have $\varphi^n(x) \in \overline{V} \setminus \varphi^1(\overline{V})$ and such an $n$ is unique
(2) $\varphi^n(x) \in M \setminus U$ and $n$ is minimal.

Moreover we have $\varphi^n(x) \in (\overline{V} \setminus \varphi^1(\overline{V})) \cup (M \setminus U)$ and this defines $n$ uniquely. As a result $0 < F(x) < 1$ and

$$dF(x)\xi(x) = \frac{1}{2\cdot 3^{|n|}} dg(\varphi^n(x))\xi(\varphi^n(x)) < 0$$

□

THEOREM 5.56. *(Existence theorem for strict Lyapounov functions) Let $\varphi^t$ be the flow of the smooth vector field $\xi$ in $M$ and $U$ an open relatively compact subset. Then there is a smooth strict Lyapounov function for $\xi$ on $U$.*

PROOF. Since the number of attractors is countable and $\mathscr{R}$ is the intersection of the $A_n \cup A_n^*$, there is a function $f_n$ such that $f_n = 0$ on $A_n$ $f_n = 1$ on $A_n^*$ and $df_n(x)\xi(x) < 0$ elsewhere, we set

$$L(x) = \sum_{n\in\mathbb{N}} c_n f_n(x)$$

where the $c_n$ are positive and $\sum_{n\in\mathbb{N}} c_n \|f_n\|_{C^n} < +\infty$ (so that for all $k$, we have $\sum_{n\in\mathbb{N}} c_n \|f_n\|_{C^k} < +\infty$, hence the sum converges in $C^k$ for all $k$) . Then $L$ is smooth and $dL(x)\xi(x) < 0$ unless $x$ is in the intersection of the $A_n \cup A_n^*$, that is $x$ is in $\mathscr{R}(U)$. □

COROLLARY 5.57. *If the set of chain recurrent points is equal to the set of fixed points then the flow is gradient like.*

PROOF. Indeed, the Lyapounov functions satisfies $dL(x)\xi(x) < 0$ outside of the set of zeros of $\xi$, so the flow is gradient-like for $L$. □

COROLLARY 5.58 ([DH72] thm. 6.4.1, [Sul76], Thm II.26, and also [LS94; Fat22]). *If $\xi$ is a vector field on a manifold $M$ and $K$ a compact subset, then there is a a function $f$ defined in a neighbourhood of $K$ such that $df(x)\cdot\xi(x) < 0$ on $K$ if and only if there is no orbit completely contained in $K$.*

PROOF. Assume no trajectory is completely contained in $K$. First there is a neighbourhood $U$ of $K$ such that for all points in $K$ and all $T > 0$ there exist $s < -T < T < t$ such that $\varphi^s(x) \notin U, \varphi^t(x) \notin U$. Otherwise there would be points $x$ in $K$ such that either $L_\alpha(x)$ or $L_\omega(x)$ is in $\overline{U}$, so there would be invariant sets arbitrarily close to $K$ hence an invariant set in $K$. This implies that $\mathscr{R}(U) = \emptyset$. But then Theorem 5.56implies the existence of a smooth strict Lyapounov function in $U$. Obviously this implies that $K$ contains no element of $\mathscr{R}$, hence the strict Lyapounov function on $K$ satisfies $dL(x)\xi(x) < 0$ on $K$.

Unfinished

$\square$

## 6. Exercises and Problems

(1) Prove that if $S, N$ are isolated invariant set and corresponding isolating neighbourhood, then there is a neighbourhood of $N$ that is still isolating for $S$, and that $N$ is still isolating for a nearby flow.

(2) (Equivalence of homotopy index for $S$, the fast track, [Sal85]) Let $(N_1, N_2)$ and $(N_1', N_2')$ be index pairs for the isolated invariant set $S$.
  (a) Prove that there exists $T > 0$ such that for $t \geq T$ we have
    - $\varphi^{[-t,t]}(x) \in N_1 \setminus N_2$ implies $x \in N_1' \setminus N_2'$
    - $\varphi^{[-t,t]}(x) \in N_1' \setminus N_2'$ implies $x \in N_1 \setminus N_2$
  (b) choose $t \geq T$. Prove that the map $f : N_1/N_2 \longrightarrow N_1'/N_2'$ given by
    - $f(x) = \varphi^{3t}(x)$ if $\varphi^{[0,2t]}(x) \in N_1 \setminus N_2$ and $\varphi^{[t,3t]}(x) \in N_1' \setminus N_2'$
    - $f(x) = [N_2']$ otherwise

    is a continuous map and that all these maps are homotopic
  (c) Prove that if $g : N_1'/N_2' \longrightarrow N"_1/N"_2$ corresponds as above to the pairs $(N_1', N_2')$ and $(N"_1, N"_2)$, $h$ to the index pairs $(N_1, N_2)$ and $(N"_1, N"_2)$, then $h$ is homotopic to $g \circ f$
  (d) Reprove Proposition 5.14.

(3) (Sard's theorem ) Let $f \in C^k(\mathbb{R}^n, \mathbb{R})$.
  (a) Prove that if $k \geq n$ the set of critical values of $f$ has measure 0.

    HINT. *First prove the result for $n = 1$. Then for $n = 2$, the set $\frac{\partial f}{\partial x_1}(x_1, x_2) = 0$ is a $C^1$ curve outside the set where $(\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_1^2}) = (0,0)$ and we may apply the case $n = 1$. Then the image of the set of points where $D^2 f(x_1, x_2) = 0$ has measure 0. Extend then to any dimension and replace $\mathbb{R}^n$ by any manifold $M$ (see [Mil97]).*

  (b) Prove that there are measure 0 sets that cannot be the set of critical values of a $C^\infty$ function.

    HINT. *Apply Sard's theorem to $(x, y) \mapsto f(x) + f(y)$ to show that the set of critical values, $C_f$, must satisfy $C_f + C_f = \{x + y \mid x \in C_f\}$ has measure 0 (or even $C_f^{(k)} = \{x_1 + ... + x_k \mid x_j \in C_f\}$ has measure 0.*

  (c) Prove that there is a function in $C^1(\mathbb{R}, \mathbb{R})$ having as set of critical values the "middle third" Cantor set (i.e. $C$ is the set of $\sum_{j=0}^{+\infty} \frac{a_n}{3^n}$ with $a_n \in \{0, 2\}$ and also the intersection of the $C_n$ where $C_0 = [0, 1]$ and $C_n$ is written as a union of intervals, $C_{n+1}$ is obtained by removing the "middle third of each interval. ).

    HINT. *(see [Gri85] ) Consider $f_n$ with compact support on $[0, 1] \setminus C_n$ such that on any interval of $[0, 1] \setminus C_n$ we have $\int_I f_n(t)dt = |I|$ where $|I|$ is the lenght of the interval. Show that $f_n$ converges to a continuous function, $f$ and $g(t) = \int_0^t f(s)ds$ answers our requirements.*

(d) Prove that there is a function in $C^1(\mathbb{R}^2,\mathbb{R})$ having as set of critical values an interval.

(4) (The stable and unstable manifolds for a Morse function) Let $f$ be a Morse function and $x$ be a critical point. Prove that the unstable manifold of $-\nabla f(x)$ is the immersed image of a disc. Prove that the same holds if $f$ is Morse-Bott, except that the unstable manifold is now the immersed image of the negative bundle of the critical manifold.

(5) (Local Lefschetz fixed point [McC89]) Let $U$ be an open set with smooth boundary $\partial U$ in a closed manifold $M$, $f$ a smooth function with isolated critical points and $\xi$ be a pseudo-gradient vector field (i.e. $df(x)\xi(x) > 0$ on the complement of the critical points. We denote by $\partial^- U$ the subset of $\partial U$ where $\xi$ points outwards from $\partial U$. Prove that the Euler characteristic of $H^*(U,\partial^- U)$ is equal to the sum of the Lefschetz indices of $\xi$ at the critical points of $f$ inside $U$.

(6) Let $f$ be a function defined on a manifold and having an isolated critical point at $x_0$ with critical value $c$.
  (a) We set $H^*_{loc}(f,x_0)$ to be the limit over the neighbourhoods $U$ of $x_0$ of $H^*(f^{c+\varepsilon}\cap U, f^{c-\varepsilon})\cap U)$ for $\varepsilon$ small enough. Prove that it is well defined.
  (b) Let us consider the gradient vector field $\nabla f(x)$ on $M$, and set $L(\nabla f, x_0)$ to be the index of this vector field near $x_0$ ($L$ is for Lefschetz as this is also called the Lefschetz index). Prove that $L(\nabla f, x_0)$ is equal to the Euler characteristic of $H^*_{loc}(f,x_0)$ (use Exercise 5).
  (c) From now on, we assume the manifold has dimension 2. Using Lusternik-Schnirelman's theory, prove that there is only one degree $d$ for which $H^d_{loc}(f,x_0)$ is non-zero (show first by contradiction that it cannot be nonzero for $d=0,1$, then apply this to $-f$).
  (d) Prove that $L(\nabla f, x_0)\leq 1$.
  (e) Give examples of functions such that $L(\nabla f, x_0) = 1-h$ for all non-negative values of $h$.

(7) Let $V$ be a linear subspace in $H^d(f^c,f^a)$ (resp. $W$ a linear subspace in $H_d(f^c,f^a)$) and define

$$c(V,f)=\inf\{b \mid V \text{ contains the Kernel of the map } H^*(f^c,f^a)\longrightarrow H^*(f^b,f^a)\}$$

$$c(W,f)=\sup\{b \mid W \text{contains the image of } H_*(f^b,f^a)\}$$

Let us remind that for coefficients in a field, the universal coefficient theorem states that $H^*(X,\mathbb{K})\simeq \mathrm{Hom}(H^*(X,\mathbb{K}),\mathbb{K})$ (see [Hat02], p.199)
  (a) Prove that both $c(V,f)$ and $c(W,f)$ are critical values of $f$.
  (b) Prove that if $V_\omega=\{\alpha \mid \langle\alpha,\omega\rangle=0\}$ and $W_\alpha=\{\omega \mid \langle\alpha,\omega\rangle=0\}$ then $c(V_\omega,f)=c(\omega,f)$ and $c(W_\alpha,f)=c(\alpha,f)$

(c) More generally, set $V^{\perp} = \{\omega \in H_*(f^c, f^a) \mid \langle \alpha, \omega\rangle = 0\}$. Then $c(V, f) = -c(AD(V^{\perp}), -f)$ where $AD$ is Alexander duality from $H^*(f^b, f^a)$ to $H_{n-*}((-f)^{-a}, (-f)^{-b})$ and similarly for $W$.

One aspect of Poincaré duality is the statement that the bilinear form defined on $H^k(X) \otimes H^{n-k}(X)$ by $(\alpha, \beta) \longrightarrow \alpha \cup \beta$ defines a non-degenerate pairing. Thus for $\omega \in H_k(X)$, the map $\alpha \mapsto \langle \alpha, \omega\rangle$ defines a linear form and is thus associated to a class $\beta \in H^{n-k}(X)$, called its Poincaré dual and denoted $PD(\omega)$. We denote by $V^{\circ}$ the orthogonal of $V$ for this pairing.

(a) Prove that if $V_1, V_2$ are in duality, then $PD(V_1^{\perp}) = V_2^{\circ}$
(b) Prove that $c(V, f) = -c(V^{\circ}, -f)$

(8) (Massey products and critical levels, see [Vit97]) Let $f$ be a smooth function on the manifold $M$ satisfying the (PS) condition. Let $x, y, z$ be de Rham cohomology classes (see[BT82]) represented by closed forms $\alpha, \beta, \gamma$ respectively. Assume $x \cdot y, y \cdot z = 0$.

Then define $\langle x, y, z\rangle$ as follows : there is a form $\sigma, \tau$ such that

- $\alpha \cup \beta = d\sigma$
- $\beta \cup \gamma = d\tau$

(a) Let $A$ be an affine subspace of $H^(M)$ and we set $c(A, f) = \sup\{c(u, f) \mid u \in A\}$. Prove that provided $A \neq \{0\}$, $c(A, f)$ is a critical value for $f$.
(b) Prove that the form $\alpha \cup \tau + \sigma \cup \gamma$ is closed and represents a cohomology class, well defined in $H^*(M)/(x \cdot H^*(M) + zH^*(M))$ and denoted by $\langle x, y, z\rangle$
(c) Prove that if $\langle x, y, z\rangle \neq 0$, then we can define $c(\langle x, y, z\rangle, f)$ and we have

$$c(z, f) \geq \min\{c(x, f), c(z, f)\}$$

and equality can only occur if $\check{H}^j(K_c) \neq 0$ for some $j$ in $\{\deg(z), \deg(x) + deg(y) - 1, deg(z) + \deg(z) - 1\}$.

HINT. *Imitate the proof of Lusternik-Shnirelman's theorem.*

## 7. Comments

The search for maxima and minima of functions is as old (or even older) as Calculus: Fermat, Pascal, Leibniz, Newton all searched for some maxima or minima. However the notion that the minimum is sometimes not achieved was not clarified until quite late : it is well known that Dirichlet principle about existence of a harmonic function on the disc with prescribed value on the boundary originally was "proved" by stating that such a function realizes the minimum of $\int_D |\nabla u|^2 dx$ among the function such that $u = f$ on $\partial D$. Weierstrass gave a counterexample of a functional having no minimum. As a result other methods were used to prove existence of harmonic functions with prescribed values on the boundary, notably Poincaré's "balayage" method ( [Poi99]) before Hilbert made the minimization method rigorous. As for finding critical points which were neither maxima nor minima, they were usually reduced to maxima or minima by adding some constrain. For example Poincaré's method to prove the existence of a closed geodesic on a convex 2-sphere was to look for the shortest curve such that the integral of the curvature enclosed by the curve is half the value of the integral of the total curvature (that is equal to $2\pi$ by Gauss-Bonnet). The minmax method introduced by Birkhoff in [Bir17] corresponds in modern language to finding the $c(\alpha, f)$ for some class $\alpha$ and proving that it is a critical value of $f$. This method was then developed by Lusternik-Shnirelman ([LS29; LS34] and Morse [Mor34]. The postwar approach is set in the framework of Hilbert or Banach spaces in [PS64] and Birkhoff and Lusternik-Shnirelman's ideas were rediscovered in different guises (under the name of Mountain pass for example in [AR73], etc) and used in particular for finding solutions of ODE, of PDE (see [Nir81] for a survey), in the framework of Geometric measure theory, etc.

The Thom-Smale-Witten theorem (Theorem 5.31) was proved in different guises, first partially by Thom in [Tho49], where he uses a Morse function to construct a celle decomposition of the manifold. However the identification of the boundary maps seems to be missing. Smale's proof of the h-cobordism theorem, and in particular the proof presented by Milnor in [Mil65] identifies the boundary maps as the number of trajectories connecting two critical points (of consecutive indices). However the proof is only completely done for an ordered Morse function, but a posteriori this is irrelevant. Witten in [Wit82] gives a proof of the theorem connecting this to Hodge theory. The small eigenvalues of the Witten Laplacian are in one-to-one correspondence with critical points, and the non-zero (but small) eigenvalues are paired by the instantons, that correspond to the trajectories connecting the critical points. This point of view was made rigorous in particular in the work of Helffer-Sjöstrand ([HS84; HS85b; HS85a; HS85c]) and Bismut ([Bis86]. Laudenbach ([Lau92]) explores the fact that the homology classes are represented by the unstable manifolds of the critical points. Our proof follows Floer in [Flo88b] and is shows the advantages of Conley's theory even in the simple framework of gradient flows.

The existence of Lyapounov functions is due to Conley. It has been revisited many times including in recent years, in particular to extend its setting from compact to non-compact metric spaces, to improve on the regularity of the Lyapounov function, and to apply it in the case of homeomorphisms and not flows. This section borrows from [Aki93; Fra17; FP19].

The last chapter of this topological theory of critical points should have been the persistence homology point of view but we had no space to develop here. This originates from ideas due to Barannikov ( [Bar94a]), rediscovered some years later and became a central tool in topological data analysis [ELZ02; ZC05]. This was also applied to PDE (see [LNV13]) and then symplectic topology starting from [PS16].

CHAPTER 6

# Generating functions for Lagrangians on cotangent bundles of compact manifolds.

The goal of this chapter is to prove the Arnold conjecture and some applications, using the results of Chapter 5 and to prove the uniquenes theorem for G.F.Q.I. . As an application we may define Generating function homology for Lagrangians and their spectral invariants.

## 1. Applications of the Existence theorem for G.F.Q.I. and Morse-Conley theory

We first need to show that a G.F.Q.I. has critical points. Indeed, a G.F.Q.I. satisfies the (PS) condition. It suffices to check this for a non-degenerate quadratic form $Q$. Let $Q(x) = \frac{1}{2}(A_Q x, x)$, then $dQ(x) = A_Q(x)$. Since $Q$ is nondegenerate, we know $A_Q$ is invertible and

$$dQ(x_n) \to 0 \Longrightarrow A_Q x_n \to 0 \Longrightarrow x_n \to 0.$$

We now have

PROPOSITION 6.1. *For $b >> 0$ and $a << 0$ we have*

$$H^*(S^b, S^a) \cong H^{*-i}(N).$$

PROOF. One can replace $S$ by $Q$ since $S = Q$ at infinity. Define

$$Q^\lambda = \{\xi | Q(\xi) \leq \lambda\}.$$

$$\begin{aligned} H^*(S^b, S^a) &= H^*(N \times Q^b, N \times Q^a) \\ &= H^*(N) \times H^*(Q^b, Q^a). \end{aligned}$$

Since $Q$ is a quadratic form, it's easy to see $H^*(Q^b, Q^a)$ is the same as $H^*(D^-, \partial D^-)$ where $D^-$ is the disk in the negative eigenspace of $Q$ (hence has dimension $index(Q)$, the number of negative eigenvalues). □

Due to this isomorphism, to each $\alpha \in H^*(N)$, we associate $\tilde{\alpha} \in H^*(S^\infty, S^{-\infty})$ corresponding to $\alpha \otimes T$, where $T$ is the generator of $H^*(D(E^-), S(E^-))$. We shall call this map the Thom isomorphism[1]. Define

$$c(\alpha, S) = c(\tilde{\alpha}, S).$$

[1] even though the whole point of Thom's isomorphism is that it holds for non-trivial bundles.

Since Alexander duality induces Poincaré duality via the Thom isomorphism, i.e. $AD(\alpha \otimes T) = PD(\alpha) \otimes T$, we have

PROPOSITION 6.2. *Let $\alpha \in H^d(N)$ and $PD(\alpha) \in H_{n-d}(N)$ be Poincaré dual classes. Then we have*

$$c(\alpha, S) = -c(PD(\alpha), -S)$$

We claim the next result will be crucial in the sequel.

PROPOSITION 6.3. *(Triangle inequality for G.F.Q.I. ) For $\alpha_1, \alpha_2 \in H^*(N)$,*

$$c(\alpha_1 \cup \alpha_2, S_1 \oplus S_2) \geq c(\alpha_1, S_1) + c(\alpha_2, S_2),$$

*where*

$$(S_1 \oplus S_2)(x, \xi_1, \xi_2) = S_1(x, \xi_1) + S_2(x, \xi_2).$$

PROOF. First let us consider $(S_1 \boxtimes S_2)(x, y; \xi, \eta) = S(x, \xi) + S(y, \eta)$. We have $(S_1 \boxtimes S_2)^c = \bigcup_{t \in \mathbb{R}} (S_1^{c-t} \times S_2^t)$. Now if $[\alpha]$ is a cohomology class vanishing on $S_1^{c_1-\varepsilon}$ it can be represented by a cocycle vanishing on $S_1^{c_1-\varepsilon}$. Similarly for $[\beta]$ a class vanishing on $S_2^{c_2-\varepsilon}$, we may assume the cocycle $\beta$ vanishes on $S_2^{c_2-\varepsilon}$. We may then conclude that $\alpha \boxtimes \beta$ vanishes on $\bigcup_{t \in \mathbb{R}} (S_1^{c-t} \times S_2^t)$ provided $c \leq c_1 + c_2 - 2\varepsilon$. As a result

$$c(\alpha \boxtimes \beta, S_1 \boxtimes S_2) \geq c(\alpha, S_1) + c(\beta, S_2)$$

Now applying Proposition 5.44 to the diagonal map $d : X \longrightarrow X \times X$ and noticing that $(S_1 \boxtimes S_2) \circ d = S_1 \oplus S_2$ we get the triangle inequality. □

REMARK 6.4. The isomorphism mentioned above is precisely

$$\begin{array}{ccc} H^*(N) \otimes H^*(D^-, \partial D^-) & = & H^*(S^\infty, S^{-\infty}) \\ \alpha \otimes T & \mapsto & T \cup p^* \alpha \end{array}$$

where $p : E = N \times \mathbb{R}^k \to N$ is the projection. Now for $E = E_1 \times_N E_2 = \{(z_1, z_2) \in E_1 \times E_2 \mid p_1(z_1) = p_2(z_2)\}$ we denote by $\pi_1 : E \longrightarrow E_1, \pi_2 : E \longrightarrow E_2$ the projections.

$$\begin{array}{ccccccc} H^*((S_1 \oplus S_2)^\infty, (S_1 \oplus S_2)^{-\infty}) & \cong & H^*(N) & \otimes & H^*(D_1^-, \partial D_1^-) & \otimes & H^*(D_2^-, \partial D_2^-) \\ T \cup p^* \alpha & & \alpha & & T_1 & & T_2 \end{array}$$

So for $\alpha = \alpha_1 \cup \alpha_2$

$$\begin{aligned} T \cup p^* \alpha &= (\pi_1)^* T_1 \cup (\pi_2)^* T_2 \cup p^*(\alpha_1 \cup \alpha_2) \\ &= (\pi_1)^*(T_1 \cup p_1^* \alpha_1) \cup (\pi_2)^*(T_2 \cup p_2^* \alpha_2) \end{aligned}$$

As a result we get

PROPOSITION 6.5. *Let $S$ be a G.F.Q.I. on $N \times \mathbb{R}^k$. Then $S$ has at least $cl(N) + 1$ critical points. If they are all non degenerate we have the Morse inequalities*

$$\sum_{j=1}^{p} (-1)^{p-j} m_j(S) \geq \sum_{j=1}^{p} (-1)^{p-j} b_j(N)$$

*where $m_j(S)$ is the number of critical points of index $j$ + index$(Q)$ of $S$ (here $Q$ is the quadratic form asymptotic to $S$ at infinity).*

PROOF. The first statement follows from Corollary 5.43. The second is the Morse inequality, using the fact that $H^*(S^\infty, S^{-\infty}) = H^{*-i}(N)$. □

We finally have, for a G.F.Q.I. $S$ defined on a bundle $E$ over $Y$, and for $f : X \longrightarrow Y$ a smooth map a map $\widetilde{f} : f^*E \longrightarrow E$ living over $f$, in other words the following is a commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "\widetilde{f}"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We

REMARK 6.6. $L_S$ is always exact since $\lambda|_{L_S} = dS|_{\Sigma_S}$.

$$L_S = \{(x, \frac{\partial S}{\partial x}(x,\xi)) | \frac{\partial S}{\partial \xi}(x,\xi) = 0\}.$$

$$\lambda|_{L_S} = p dx = \frac{\partial S}{\partial x}(x,\xi) dx = dS,$$

since for points on $L_S$, $\frac{\partial S}{\partial \xi} = 0$.

A result by Fukaya, Seidel and Smith ([FSS08]) and then by [Kra13] grants that under quite general assumptions, the projection $\pi : L \to N$ of an exact Lagrangian submanifold is a homotopy equivalence.

EXERCISE 6.7. Prove that if $L$ has G.F.Q.I. $S$, then $\deg(\pi : L \to N) = \pm 1$.

HINT. *Choose a generic point $x_0 \in N$. The degree is the multiplicity with sign of the intersection of L and the fiber over $x_0$. That is counting the number of $\xi$ with $\frac{\partial S}{\partial \xi}(x_0,\xi) = 0$, i.e. the number of critical points of function $\xi \mapsto S(x_0,\xi)$ with sign*

$$(-1)^{index(\frac{d^2S}{d\xi^2}(x_0,\xi))}.$$

*Therefore*

$$\deg(\pi : L \to N) = \sum_{\xi_j} (-1)^{index(\frac{d^2S}{d\xi^2}(x_0,x_j))}$$

*where the summation is over all $\xi_j$ with $\frac{\partial S}{\partial \xi}(x_0,\xi_j) = 0$. The summation is finite since S has quadratic infinity and the sum is the Euler number of the pair $(S^b, S^a)$ for large b and small a. Finally, check that for all quadratic form Q, the Euler number of $(Q^b, Q^a)$ is $\pm 1$.*

By the previous claim, for large $b$ and small $a$

$$H^*(S^b, S^a) \cong H^{*-i}(N).$$

Since $N$ is compact, we know $H^*(N) \neq 0$. This implies that $S$ has at least one critical point and $(L_S \cap 0_N) \neq \emptyset$.

THEOREM 6.8 (Hofer's Lagrangian intersection theorem ([Hof85]). *Let $N$ be a compact manifold and $L=\varphi(0_N)$ for some $\varphi \in Ham(T^*N)$, then*

$$\#(L\cap 0_N) \geq cl(N)+1$$

*If all intersection points are transverse, then*

$$\#(L\cap 0_N) \geq \sum b_j(N).$$

*Here*

$$cl(N) = \max\{k \mid \exists \alpha_1,\cdots,\alpha_{k-1} \in H^*(N)\setminus H^0(N) \text{ such that } \alpha_1\cup\cdots\cup\alpha_{k-1} \neq 0\}$$

*and*

$$b_j(N) = \dim H^j(N).$$

Note that for $N=T^n$ this had been proved by M. Chaperon in [Cha84a].

PROOF. We know that $L$ has a G.F.Q.I. by Sikorav's Theorem (Theorem 4.24) and according to Proposition 6.5 the G.F.Q.I. has the announced number of critical points. Since each critical point corresponds to a point in $L\cap 0_N$, this concludes the proof. □

COROLLARY 6.9. *if $L=\varphi^1(0_N)$, then*

$$\#(L\cap 0_N) \geq 2$$

THEOREM 6.10. *(Conley-Zehnder[CZ83]) Let $\varphi \in Ham(T^{2n})$, then*

$$\#Fix(\varphi) \geq 2n+1.$$

*If all fixed points are non-degenerate, then*

$$\#Fix(\varphi) \geq 2^{2n}.$$

REMARK 6.11. $2n$ is the cup-length of $T^{2n}$ and $2^{2n}$ is the sum of Betti numbers of $T^{2n}$.

PROOF. Let $(x_i,y_i)$ be coordinates of $T^{2n}$. We will write $(x,y)$ for simplicity. The symplectic form is given by $\omega = dy\wedge dx$. Consider $T^{2n}\times\overline{T^{2n}}$ with coordinates $(x,y,X,Y)$, whose symplectic form is given by

$$\omega = dy\wedge dx - dY\wedge dX.$$

With this $\omega$, the graph of $\varphi$, $\Gamma(\varphi)$ is a Lagrangian. Consider another symplectic manifold $T^*T^{2n}$, denote the coordinates by $(a,b,A,B)$. Note that $x,y,X,Y,a,b$ take value in $T^n=\mathbb{R}^n/\mathbb{Z}^n$ and $A,B$ takes value in $\mathbb{R}^n$.

It has the natural symplectic form as a cotangent bundle

$$\omega = dA\wedge da + dB\wedge db.$$

Define a map $F: T^*T^{2n}\to T^{2n}\times\overline{T^{2n}}$

$$F(a,b,A,B) = \left(\frac{2a-B}{2},\frac{2b+A}{2},\frac{2a+B}{2},\frac{2b-A}{2}\right) \bmod \mathbb{Z}^n.$$

It's straightforward to check that $F$ is a symplectic covering.

Let $\triangle_{T^{2n}}$ be the diagonal in $T^{2n}\times\overline{T^{2n}}$. It lifts to $0_{T^{2n}}\subset T^*T^{2n}$ and the projection $\pi$ induces a bijection between $0_{T^{2n}}$ and $\triangle_{T^{2n}}$. Of course $0_{T^{2n}}$ is only one component in the preimage of $\triangle_{T^{2n}}$ corresponding to $A=B=0$ (other components are given by $A=A_0, B=B_0$ where $A_0,B_0\in\mathbb{Z}^n$. Now assume $\varphi$ is the time one map of $\varphi^t\in Ham(T^{2n})$.

$$\Gamma(\varphi^t)=(id\times\varphi^t)(\triangle_{T^{2n}}).$$

This Hamiltonian isotopy lifts to a Hamiltonian isotopy $\Phi^t$ of $T^*T^{2n}$ such that

$$\pi\circ\Phi^t=\phi^t\circ\pi.$$

Then the restriction of the projection to $\Phi^t(0_{T^{2n}})$ remains injective, since

$$\pi(\Phi^t(u))=\pi(\Phi^t(v))$$

implies

$$\phi^t(\pi(u))=\phi^t(\pi(v))$$

but since $\pi$ is injective on $0_{T^{2n}}$ and $\phi^t$ is injective, this implies $u=v$.

Therefore to distinct points in $\Phi^t(0_{T^{2n}})\cap 0_{T^{2n}}$ correspond distinct points in $\Gamma(\varphi)\cap\triangle_{T^{2n}}=Fix(\varphi)$.

According to Hofer's theorem, the first set has at least $2n+1$ points, so the same holds for the latter. □

REMARK 6.12. The theorem doesn't include all fixed point $\varphi$. Indeed, we could have done the same with any other component of $\pi^{-1}(\triangle_{T^{2n}})$ (they are parametrized by pairs of vectors $(A_0,B_0)\in\mathbb{Z}^n\times\mathbb{Z}^n$), and possibly obtained other fixed points. What is so special about those we obtained ? It is not hard to check that they correspond to periodic contractible trajectories on the torus. Indeed, a closed curve on the torus is contractible if and only if it lifts to a closed curve on $\mathbb{R}^{2n}$. Now, our curve is $\Phi^t(a,b,0,0)$ and projects on $(id\times\varphi^t)(x,y,x,y)=(x,y,\phi^t(x,y))$. Since $\Phi^1(a,b,0,0)\in 0_{T^{2n}}$, we may denote $\Phi^1(a,b,0,0)=(a',b',0,0)$, and since $\phi^1(x,y))=(x,y)$, we have $a'=x=a, b'=y=b$. Thus $\Phi^t(a,b,0,0)$ is a closed loop projecting on $(id\times\varphi^t)(x,y,x,y)$, this last loop is therefore contractible, hence the loop $\varphi^t(x,y)$ is also contractible.

THEOREM 6.13. *(Poincaré and Birkhoff) Let $\varphi$ be an area preserving map of the annulus, shifting each circle (boundary) in opposite direction, then $\#Fix(\varphi)\geq 2$.*

PROOF. Assume $\varphi$ is the time one map of a Hamiltonian flow $\varphi^t$ associated to $H=H(t,r,\theta)$, where $(r,\theta)$ is the polar coordinates of the annulus($1\leq r\leq 2$). Assume without loss of generality

$$\frac{\partial H}{\partial r}>0 \text{ for } r=2$$

and

$$\frac{\partial H}{\partial r}<0 \text{ for } r=1.$$

One can extend $H$ to $[\frac{1}{2},\frac{5}{2}]\times S^1$ such that $\frac{\partial H}{\partial r}(r,\theta)<0$ for $r<1$, $\frac{\partial H}{\partial r}(r,\theta)>0$ for $r>2$ and

$$H(t,r,\theta)=-r \text{ on } [\frac{1}{2},\frac{2}{3}]$$

and

$$H(t,r,\theta)=r \text{ on } [\frac{7}{3},\frac{5}{2}].$$

Take two copies of this enlarged annulus and glue them together to make a torus. Then $\#Fix(\varphi)\geq 3$. At least one copy has two fixed points. □

## 2. First proof of the Arnold Conjecture

Let $N$ be a compact manifold and $\varphi\in Ham(T^*N)$, then $L=\varphi(0_N)$ is a Lagrangian. We have proved the following

THEOREM 6.14. *Let $N$ be a compact manifold and $\varphi\in Ham(T^*N)$, then $L=\varphi(0_N)$ is a Lagrangian having a G.F.Q.I. .*

There are several consequences

- Hofer's theorem: $\#(\varphi(0_N)\cap 0_N)\geq 2$; (In fact Hofer's theorem says more.)
- Conley-Zehnder theorem: $\#Fix(\varphi)\geq 2n+1$ for $\varphi\in Ham(T^{2n})$;
- Poincaré-Birkhoff Theorem.

We are going to deal with
1) Uniqueness of G.F.Q.I. of $L$
2) Calculus of critical levels.

REMARK 6.15. Theorem 6.14 extends to continuous family, i.e. if $\varphi_\lambda$ is a continuous family of Hamiltonian diffeomorphisms and $L_\lambda=\varphi_\lambda(0_N)$, then there exists a continuous family of G.F.Q.I. $S_\lambda$.

REMARK 6.16. Theorem 6.14 holds also for Legendrian isotopies as we shall see in Appendix 4 and [Che96]). Let $J^1(N,\mathbb{R})\equiv T^*N\times\mathbb{R}$ and define

$$\alpha=dz-pdq.$$

DEFINITION 6.17. *$\Lambda$ is called a Legendrian if and only if $\alpha|_\Lambda=0$.*

Given a smooth function $f\in C^\infty(N,\mathbb{R})$, the submanifold defined by

$$z=f(x), p=df, q=x$$

is a Legendrian. One similarly associates to a generating function, $S:N\times\mathbb{R}^k\longrightarrow\mathbb{R}$ a legendrian submanifold (under the same transversality assumptions as for the Legendrian case)

$$\Lambda_S=\{(x,\frac{\partial S}{\partial x}(x,\xi),S(x,\xi))\mid \frac{\partial S}{\partial \xi}=0\}$$

Denote the projection from $T^*N \times \mathbb{R}$ to $T^*N$ by $\pi$. Then any Legendrian submanifold projects down to an (exact) Lagrangian. Moreover, any exact Lagrangian can be lifted to a Legendrian. Note however that there are Legendrian isotopies that do not project to Lagrangian ones. So Chekanov's theorem is in fact stronger than Sikorav's theorem, even though the proof is the same.

## 3. Uniqueness of G.F.Q.I.

Let $\varphi \in Ham(T^*N)$ and $L = \varphi(0_N)$. Denote a G.F.Q.I. for $L$ by $S$. We will show that we can obtain different G.F.Q.I. by the following three operations.

(1) (Conjugation) If smooth map $\xi : N \times \mathbb{R}^k \to \mathbb{R}^k$ satisfies that for each $x \in N$, $\xi(x,\cdot) : \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism and linear at infinity, then we claim:

$$\tilde{S}(x,\eta) = S(x,\xi(x,\eta))$$

is again G.F.Q.I. for $L$. Recall from the definition of generating function

$$L_{\tilde{S}} = \{(x, \frac{\partial \tilde{S}}{\partial x}(x,\eta)) | \frac{\partial \tilde{S}}{\partial \eta}(x,\eta) = 0\}$$

and

$$L_S = \{(x, \frac{\partial S}{\partial x}(x,\xi)) | \frac{\partial S}{\partial \xi}(x,\xi) = 0\}.$$

Since $\frac{\partial \xi}{\partial \eta}$ is invertible, the chain rule says $\frac{\partial \tilde{S}}{\partial \eta}(x,\eta)$ and $\frac{\partial S}{\partial \xi}(x,\xi(x,\eta))$ simultaneously. On such points,

$$\frac{\partial \tilde{S}}{\partial x}(x,\eta) = \frac{\partial S}{\partial x}(x,\xi(x,\eta)) + \frac{\partial S}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \frac{\partial S}{\partial x}(x,\xi(x,\eta)).$$

(2) (Stabilization) If $q$ is a non-degenerate quadratic form, then

$$\tilde{S}(x,\xi,\eta) = S(x,\xi) + q(\eta)$$

is a G.F.Q.I. for $L$, the reason being that

$$\frac{\partial \tilde{S}}{\partial x}(x,\xi,\eta) = \frac{\partial S}{\partial x}(x,\xi)$$

and

$$\frac{\partial \tilde{S}}{\partial \xi} = \frac{\partial \tilde{S}}{\partial \eta} = 0 \quad \Longleftrightarrow \quad \begin{cases} A_q \eta = 0 \Longrightarrow \eta = 0 \\ \frac{\partial S}{\partial \xi}(x,\xi) = 0 \end{cases}$$

where $A_q$ is given by $(A_q\eta, \eta) = q(\eta)$ for all $\eta$ and is invertible since $q$ is non-degenerate.

(3) (Shift) By adding a constant,

$$\tilde{S}(x,\xi) = S(x,\xi) + c.$$

DEFINITION 6.18. *Two G.F.Q.I. , $S_1, S_2$ are said to be equivalent if by applying to each of them some of the operations (1), (2),(3) we get G.F.Q.I. $\widetilde{S}_1, \widetilde{S}_2$ such that $\widetilde{S}_1 = \widetilde{S}_2$.*

The G.F.Q.I. is unique up to the above operations in the sense that

THEOREM 6.19 (Uniqueness theorem for G.F.Q.I. ([Vit92; Thé99]). *If $S_1, S_2$ are G.F.Q.I. for $L = \varphi(0_N)$, then they are equivalent.*

PROOF. First of all we want to prove

LEMMA 6.20. *The set of Lagrangians satisfying the uniqueness property is stable by Hamiltonian isotopy.*

PROOF. This is just the rephrasing of the standard fact that for a Serre fibration all the fibers are homotopy equivalent (hence have the same number of connected components). This will reduce our argument to dealing with the case of the zero section. So consider $\pi : \mathscr{F} \longrightarrow \mathscr{L}(T^*N)$ be the fibration defined in Chapter 4, Theorem 4.25. Then let $L_1 = \varphi(L_0)$ and assume uniqueness holds for $L$. We claim that if $S_1, S_1'$ are G.F.Q.I. for $L_1$, they can be connected by a path in $\pi^{-1}(L_1)$. Indeed, let $L_t$ be a path connecting $L_0$ and $L_1$. Then by the path lifting property, up to equivalence, there is a path $S_t, S_t'$, such that they are both G.F.Q.I. for $L_t$. Now $S_0, S_0'$ are G.F.Q.I. for $L_0$ and since we assume uniqueness for $L_0$, they are equivalent, so doing the same operations on the whole path, we may assume $S_0 = S_0'$. Now consider the loop in $\mathscr{L}(t^*N)$ going from $L_1$ to $L_0$ and then back. It is of course contractible and we can deform it to the constant path at $L_1$. On the other hand following $S_t$ from $S_1$ to $S_0 = S_0'$ and then $S_t'$ from $S_0'$ to $S_1'$ is a lift of this loop. Now the deformation of the loop in $\mathscr{L}(T^*N)$ to the constant path at $L_1$ can be lifted to $\mathscr{F}$, and at the end of this deformation, we obtain a path over $\pi^{-1}(L_1)$ connecting $S_1$ to $S_1'$.

Now we need to prove that if there is a path of G.F.Q.I. $S_t$ generating $L_1$ then $S_0$ and $S_1$ are equivalent. This will use the idea of Moser's lemma, but in a slightly more complicated situation. We look for a fiber preserving isotopy $(x,\xi) \mapsto (x, \varphi_t(x,\xi))$ such that $S_t(x, \varphi_t(x,\xi)) = S_0(x,\xi)$. This is given by a family of time-dependent vector fields $X_t(x,\xi)$ defined on $\mathbb{R}^k$ parametrized by $x \in N$ : we have $\frac{d}{dt}\varphi_t(x,\xi) = X_x(t, \varphi_t(x,\xi))$. Then, taking the time derivative of $S_t(x, \varphi_t(x,\xi)) = S_0(x,\xi)$ we get

$$\frac{\partial}{\partial \xi} S_t(x,\xi) X_t(x,\xi) + \frac{\partial}{\partial t} S_t(x, \varphi_t(x,\xi)) = 0$$

Note that if $\Sigma_t = \{(x,\xi) \mid \frac{\partial}{\partial \xi} S_t(x,\xi) = 0\}$, the above equation can be trivially solved in the complement of a neighbourhood of $\Sigma_t$ : set

$$X_t(x,\xi) = -\frac{\frac{\partial}{\partial t} S_t(x, \varphi_t(x,\xi))}{\|\frac{\partial}{\partial \xi} S_t(x,\xi)\|^2} \nabla_\xi S_t(x,\xi)$$

Since the set of solutions is $C^\infty$ linear, we are reduced to finding a solution in a neighbourhood of $\Sigma_t$.

We claim that we may assume $\Sigma_t = \Sigma_0$ and $i_t = i_0$. Indeed the $\sigma_t$ are fiberwise diffeomorphic by $i_t^{-1} \circ i_0$ since $i_t$ and $i_0$ preserve the fibers over $N$. This extends to a fiberwise diffeomorphism of the ambient space $N \times \mathbb{R}^k$. Once this is done, we must have $i_t = i_0$ on $\Sigma = \Sigma_0$, since $f_L = S_t \circ i_t$ on $\Sigma$, we have that $S_t = S_0$ on $\Sigma$. We now need a parametrized version of Hadamard's lemma, there is a vector valued function defined on $\Sigma$ such that

$$\frac{\partial}{\partial \xi} S_t(x,\xi) X_t(t,\xi) + \frac{\partial}{\partial t} S_t(x, \varphi_t(x,\xi)) = 0$$

□

LEMMA 6.21 (Hadamard's lemma). *Let $(t,x,\xi) \mapsto F_t(x,\xi)$ be a smooth map from $[0,1] \times N \times \mathbb{R}^k$ to $\mathbb{R}^k$ such that $0$ is a regular value and $(F_t)^{-1}(0) = \Sigma$ is fixed. Let $G_t(x,\xi)$ be a function from $[0,1] \times N \times \mathbb{R}^k$ to $\mathbb{R}$ vanishing on $\Sigma$. Then there exists $V_t(x,\xi)$ from $[0,1] \times N \times \mathbb{R}^k$ to $\mathbb{R}^k$ such that in a neighbourhood of $\Sigma$*

$$\langle F_t(x,\xi), V_t(x,\xi) \rangle = G_t(x,\xi)$$

PROOF. First of all this reduces to a local statement: indeed if we cover $\Sigma$ by open sets $U_j$, $\rho_j$ is a corresponding partition of unity, and we solve the above equation by $V_t^j(x,\xi)$ in $U_j$, then $V_t(x,\xi) = \sum_j \rho_j(x,\xi) V_t^j(x,\xi)$ solves the above equation in the union of the $U_j$.

Now let $U_j$ be such that there is a chart $\varphi_j : (\Sigma \times \mathbb{R}^k) \cap U_j \longrightarrow V_j \times W_j$ where $V_j \subset \mathbb{R}^n$, $W_j \subset \mathbb{R}^k$ and $\varphi_j(x,0) = (\psi_j(x),0)$. We are thus reduced to $F_t(v,w)$ and $G_t(v,w)$ such that $G_t(v,0) = 0$ and $dF_t(v,0)$ is onto. Then a further change of coordinates reduces to the case $F_t(v,w) = w$, so we must write

$$G_t(v,w) = \sum_{i=1}^{n} w_i G_t^i(v,w)$$

But this is just the fundamental theorem of calculus

$$G_t(v,w) = G_t(v,0) + \sum_{i=1}^{n} w_i \int_0^1 \frac{\partial G}{\partial w_i}(sv,w)\,ds$$

and since $G_s(v,0) = 0$, this concludes our proof by setting

$$V_t^i(x,\xi) = \int_0^1 \frac{\partial G}{\partial w_i}(sv,w)\,ds$$

□

To conclude the proof of our theorem, we must prove that uniqueness holds for $0_N$. So consider $S$ a G.F.Q.I. for $0_N$. Clearly the only critical points of $S$ is a Morse-Bott critical manifold $Z$ projecting diffeomorphically on $N$ since the projection of $\Sigma_S$ to $N$ identifies with the projection of $L_S$ on the base $N$ of $T^*N$. So by a fiber preserving diffeomorphism, we may assume $Z = N \times \{0\}$. Then set $S_x(\xi) = S(x,\xi)$, we may assume $S_x$ is a quadratic form $q_x$ at 0, and coincides with $Q_x = Q$ at infinity.

PROPOSITION 6.22. *Let $S$ be a G.F.Q.I. equal to $Q$ at infinity, and coinciding with $q$ near $N \times \{0\}$ and having no other fiberwise critical point. Then $S$ is fiberwise diffeomorphic to $q$.*

PROOF. □

□

The main consequence of this theorem is that given $L = \varphi(0_N)$, for different choices of G.F.Q.I. , we know the relation between $H^*(S^b, S^a)$. It suffices to trace how $H^*(S^b, S^a)$ changes by operation 1,2,3.

It's easy to see that $H^*(S^b, S^a)$ is left invariant by operation 1, because the pair $(S^b, S^a)$ is diffeomorphic to $(\tilde{S}^b, \tilde{S}^a)$.

For operation 3,

$$H^*(\tilde{S}^b, \tilde{S}^a) = H^*(S^{b-c}, S^{a-c}).$$

For operation 2, we claim that for $b > a$

$$H^*(\tilde{S}^b, \tilde{S}^a) = H^{*-i}(S^b, S^a)$$

where $i$ is the index of $q$. We have two cases to check: replacing $S(x,\xi)$ by $S(x,\xi) + \eta^2$ and by $S(x,\xi) - \eta^2$, where $\eta \in \mathbb{R}$.

LEMMA 6.23. *Let $\tilde{S}(x,\xi,\eta) = S(x,\xi) + \eta^2$. Then $H^*(\tilde{S}^b, \tilde{S}^a) = H^*(S^b, S^a)$*

PROOF. Indeed, we have a retraction by deformation $\tilde{S}^t \longrightarrow S^t$ given by $p : (x,\xi,\eta) \mapsto (x,\xi)$ and having as homotopy inverse $j : (x,\xi) \mapsto (x,\xi,0)$. We have $p \circ j = \mathrm{id}$ and $j \circ p$ is the map $(x,\xi,\eta) \mapsto (x,\xi,0)$ and this is homotopic to the identity by $r_t(x,\xi,\eta) = (x,\xi,t\eta)$. □

LEMMA 6.24. *Let $\tilde{S}(x,\xi,\eta) = S(x,\xi) - \eta^2$. Then $H^*(\tilde{S}^b, \tilde{S}^a) = H^{*-1}(S^b, S^a)$*

PROOF. Note that $(-\tilde{S})(x,\xi,\eta) = (-S)(x,\xi) + \eta^2$. We shall reduce this Lemma to the previous one, using Alexander duality ([Spa66], page 296, theorem 17). Note that if $S$ is defined on $E$ with $\dim(E) = d$, then $\tilde{S}$ is defined on $\tilde{E} = E \oplus \mathbb{R}$, of dimension $d+1$.

$$AD: \quad H_*(S^b, S^a) \quad \to \quad H_c^{d-*}(E - S^a, E - S^b) = H_c^{d-*}((-S)^{-a}, (-S)^{-b}).$$

According to Proposition 5.27, $H_c^*((-S)^{-a}, (-S)^{-b}) \simeq H^*((-S)^{-a}, (-S)^{-b})$. Thus we have the following diagram,

$$\begin{array}{ccc} H^*(S^b, S^a) & \xrightarrow{AD^{-1}} & H_{d-*}(E - S^a, E - S^b) = H^{d-*}((-S)^{-a}, (-S)^{-b}) \\ & & \Big\downarrow \simeq (by Lemma\ 6.23) \\ H^{*+1}(\tilde{S}^b, \tilde{S}^a) & \xleftarrow[AD]{} & H_{d-*}(\tilde{E} - \tilde{S}^a, \tilde{E} - \tilde{S}^b) = H_{d-*}((-\tilde{S})^{-a}, (-\tilde{S})^{-b}). \end{array}$$

As a result $H^*(\tilde{S}^b, \tilde{S}^a) = H^{*-1}(S^b, S^a)$. □

REMARK 6.25. The theorem holds for $L = \varphi(0_N)$ only, no result is known for general $L$. However we shall see in the second part using sheaves, that for any exact embedded $L$ having a G.F.Q.I. , $S$ (whether or not $L$ is Hamiltonianly isotopic to the zero section[2]), $H^*(S^b, S^a)$ does not depend on the choice of $S$.

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[2] even though Arnold nearby Lagrangian conjecture implies it has one.

Let us however mention

**Conjecture.** *(Arnold's nearby Lagrangian conjecture, see [Arn87; LS91]) Let $L \subset T^*N$ be an exact Lagrangian, then there exists $\varphi \in Ham(T^*N)$ such that $L = \varphi(0_N)$.*

### 4. Appendix : contact topology and Linear at infinity generating functions

We now deal with the case of a Legendrian submanifold in $J^1(N,\mathbb{R})$. We have a projection $J^1(N,\mathbb{R}) \longrightarrow T^*N$. This map takes the one-jet of a function tand forgets about its value, to only remember its derivative. First of all, according to Proposition 3.86, an exact Lagrangian in $T^*N$ has a unique - up to translation- lift to a Legendrian in $J^1(N,\mathbb{R})$. We shall deal with two extensions of the previous results

(1) We prove that Théret's theorem extends to embedded Legendrians : we only need a regular Lagrangian isotopy lifting to a Legendrian isotopy to guarantee the existence of G.F.Q.I.
(2) For $N = \mathbb{R}^n$, we replace G.F.Q.I. by GFLI, that is a Generating Function Linear at Infinity. When studying compact Legendrian submanifolds in $J^1(\mathbb{R}^n,\mathbb{R})$ or compact Lagrangians in $T^*(\mathbb{R}^n)$ we cannot use GFQI since for a Legendrian (or Legendrian) having a G.F.Q.I. , the projection on the base must be onto. This is unsuitable for studying knots. However a variation of G.F.Q.I. , the Linear at Infinity Generating functions (GFLI) is available, and has been developed in [JT06; ST13; BST15; SS16]. They satisfies the analogue of Théret's theorem.

DEFINITION 6.26. *A smooth function $S : N \times \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ is a Generating function Linear at Infinity if it is a Generating function and outside a compact set we have $S(x,\eta,v) = L(\eta)$, where $L$ is a linear form. The associated Legendrian is given by*

$$\Lambda_S = \left\{(x, \frac{\partial S}{\partial x}(x,\eta), S(x,\eta)) \mid \frac{\partial S}{\partial \eta}(x,\eta) = 0, \frac{\partial S}{\partial v}(x,\eta) = 0\right\}$$

Note that an embedded Legendrian will project on an immersed Lagrangian, and the double points of the immersion corresponds to the so-called chords of the Legendrian Let $\Lambda$ be a Legendrian in $J^1(N,\mathbb{R})$. A chord of $\Lambda$ is an (unordered) pair of points $(x,p,z),(x,p,z')$ in $\Lambda$. The difference $|z'-z|$ is called the height of the chord.

As a result, if $S$ is a GFLI for $\Lambda$, we are not so much interested in critical points of $S$, corresponding to intersection points of $\Lambda$ with $M \times \{0\} \times \mathbb{R}$ but rather critical points of the **difference function**.

DEFINITION 6.27. *Let $S : N \times \mathbb{R}^k \longrightarrow \mathbb{R}$ be a GFLI for the Legendrian $\Lambda$ in $J^1(N,\mathbb{R})$. Then $\mathscr{D}_S(x,\xi,\eta) = S(x,\xi) - S(x,\eta)$ is called the difference function associated to $S$. We set $GH^*(\Lambda;a,b) = H^*(\mathscr{D}_S^b, \mathscr{D}_S^a)$ and $GH^*(\Lambda) = H^*(\mathscr{D}_S^{+\infty}, \mathscr{D}_S^{\varepsilon})$ where $\varepsilon$ is positive, small enough so that there are no chords of height less than $\varepsilon$.*

The fibration theorem hold in our setting. Let $\mathscr{F}_l$ be the set of Generating functions linear at infinity.

THEOREM 6.28 (Théret's theorem (see [Thé99])). *The map* $\pi : \overline{\mathscr{F}}_l \longrightarrow \mathscr{L}]\}(J^1N)$ *is a smooth Serre fibration. More precisely given a smooth map* $j : D^k \times [0,1] \longrightarrow \mathscr{L}eg(J^1N)$ *and a lift* $\overline{S}_0 : D^k \times \{0\} \longrightarrow \overline{\mathscr{F}}_l$ *such that* $\pi \circ \overline{S}_0 = j_{|D^k \times \{0\}}$, *then there is an extension* $\overline{S} : D^k \times [0,1] \longrightarrow \overline{\mathscr{F}}_l$ *such that* $\pi \circ \overline{S} = j$.

PROPOSITION 6.29. *Chords are in one-to-one correspondence with critical points of* $\mathscr{D}$ *and the critical values correspond to the height. We say that a chord is non-degenerate if it corresponds to a non-degenerate critical point of* $\mathscr{D}$. *As a result if all the chords are non-degenerate we have*

$$\#(\mathrm{Chords}(\Lambda)) \geq \sum_j \dim GH^j(\Lambda)$$

Contrary to the Lagrangian case, where the total Generating Function cohomology equals the cohomology of the base, we may here have different values for $GH^*(\Lambda)$. However this only depends on the Legendrian isotopy class of $\Lambda$.

PROPOSITION 6.30 ([Tra01]). *Let* $t \mapsto \Lambda_t$ *be an isotopy of Legendrian submanifolds. Then* $GH^*(\Lambda_0) = GH^*(\Lambda_1)$.

PROOF. We know by Théret's theorem that we may find $S_t(x,\xi)$ a GFLI defined on $[0,1] \times N \times \mathbb{R}^k$ such that $S_t$ generates $\Lambda_t$. We then get for $0 < \varepsilon < c$ with $\varepsilon$ small enough, and $c$ large enough, that no critical value of $\mathscr{D}_t = \mathscr{D}_{S_t}$ crosses the levels $\varepsilon$ or $c$. Then $\mathscr{D}_{t_0}$ satisfies the (PS) condition and $(\mathscr{D}{t_0}^c, \mathscr{D}{t_0}^\varepsilon)$ is an index pair for any pseudo-gradient vector field of $\mathscr{D}_{t_0}$ associated to the invariant set made of critical points in $\mathscr{D}{t_0}^c \setminus \mathscr{D}{t_0}^\varepsilon$ and heteroclinic trajectories connecting them. For $t$ close to $t_0$, since $t \mapsto \mathscr{D}_t$ is continuous for the $C^2$ topology, we have that for $t$ close to $t_0$,

(1) The set $\mathscr{D}_{t_0}^c \setminus \mathscr{D}_{t_0}^\varepsilon$ is an isolating block for $\nabla \mathscr{D}_t$
(2) The isolated invariant set for $\nabla \mathscr{D}_t$ is contained in a neighbourhood of the isolated invariant set for $\nabla \mathscr{D}_{t_0}$ and

As a result the maximal invariant set for $\nabla \mathscr{D}_t$ contained in $\mathscr{D}_{t_0}^c \setminus \mathscr{D}_{t_0}^\varepsilon$ coincides with the maximal invariant set for $\nabla \mathscr{D}_t$ contained in $\mathscr{D}_t^c \setminus \mathscr{D}_t^\varepsilon$. This implies

$$H^*(\mathscr{D}_t^c, \mathscr{D}_t^\varepsilon) = H^*(\mathscr{D}_{t_0}^c, \mathscr{D}_{t_0}^\varepsilon)$$

for $t$ close to 0. So for each $t_0 \in [0,1]$ we can find $\delta > 0$ such that for $t \in ]t_0 - \delta, t_0 + \delta[$ we have $GH^*(\Lambda_t) = GH^*(\Lambda_{t_0})$ By a compactness argument we may cover $[0,1]$ by finitely many such intervals $]t_0 - \delta, t_0 + \delta[$. But then $GH^*(\Lambda_0) = GH^*(\Lambda_1)$. □

REMARK 6.31. For an immersed generic Lagrangian $L$ , there is an embedded Legendrian lift $\Lambda$ and $GH^*(\Lambda)$ counts the number of chords of $\Lambda$, which are in one-to-one correspondence with the double points of $L$. One could think that this gives an easy way to estimate the number of such double points, but there is no simple way to determine the Legendrian isotopy class of $\Lambda$ from inspecting $L$. The only easy result is that if $\Lambda$ is Legendrian isotopic to the lift of an embedded Lagrangian, then $GH^*(\Lambda) = 0$.

## 5. Notations and conventions

Let $M$ be a closed manifold and $\alpha, \beta$ be cohomology classes in $H^d(M), H^{n-d}(M)$, $a, b$ be homology classes in $H_d(M), H_{n-d}(M)$. We have Poincaré duality

$$PD : H_d(M) \longrightarrow H^{n-d}(M)$$

sending $a$ to $\beta$ if and only if

$$\forall \alpha \in H^d(M) \ \langle \alpha, a \rangle = \langle \alpha \cup PD(a), [M] \rangle$$

## 6. Comments

The Poincaré-Birkhoff theorem also known as "Poincaré last geometric theorem" was stated by Poincaré in 1912 in [Poi12]. Poincaré could only deal with some very special cases, and the theorem was then proved by Birkhoff in 1913 in [Bir13]. We refer to [Bir27; BN77] for more recent expositions. Birkhoff himself complained in his 1926 book ([Bir27]), that

*"Poincaré's last geometric theorem and modifications thereof yield an additional instrument for establishing the existence of periodic motions. Up to the present time no proper generalization of this theorem to higher dimensions has been found, so that its application remains limited to dynamical systems with two degrees of freedom."* This was revived tirelessly by Arnold (see [Arn65; Arn97]) starting from the 60's. In the 70's some extension of Birkhoff's results to surfaces was proved by Nikishin ([Nik74], see Exercise 1) and then Y. Eliashberg ([Eli78]), and a contact version in dimension 3 by D. Bennequin in, his thesis ([Ben83]). Higher dimensional results started with Conley and Zehnder [CZ83], and then Gromov [Gro85]. This brought together Hamiltonian dynamics and the then emerging field of symplectic topology. In Gromov's terminology ([Gro86]) this are illustrations of symplectic rigidity. The idea of a generating functions goes back to Jacobi (see [Jac66]) and the version with extra variables to Hörmander (see [Hör71]) and it was pointed out by A. Weinstein that the action functional was an infinite dimensional generating function and it was proved in [Vit87b] that their finite dimensional-reductions shared this property. The importance of the quadratic at infinity property goes back to Laudenbach and Sikorav ([LS85]).

While Theorem 6.8 or Corollary 6.9 look like topological theorems, their straightforward extension to symplectic continuous maps (or rather "Hamiltonian continuous maps") holds in dimension 2 (see [Mat00]), but, surprisingly, does not hold in dimension greater than 4 (see [BHS18]): there are symplectic homeomorphisms of any symplectic manifold having only one fixed point.

The description of Lagrangians through generating functions in $T^*N$ tends to be superseded by a description using sheaf theory (see [Tam08; GKS12; Gui12; Vit19]) which allows to use the same ideas for general exact Lagrangians, without assuming they are Hamiltonianly isotopic to the zero section. We shall use this approach in the second volume of these notes. Note however that in some instances, it seems that Generating function still give more information (see for example [Abo+]).

Generating functions are also used in the contact setting, see [Tra01; JT06; BST15; CS15; SS16]

Numerical integration of Hamiltonian systems faces two difficulties both related to the fact that their trajectories cannot be asymptotically stable: since the linearized map is volume preserving, it cannot have its eigenvalues of module less than 1, while one is interested in computations to predict the trajectories of particles in high powered accelerators or the behaviour of the solar system over periods of several hundred millions years. These computations are very sensitive to very small inaccuracies in the initial conditions, the parameter of the equations and the choice of the discretization methods.

At least one would like the discretized map to be symplectic as well. This was first proposed by DeVogelaere ([De 56]) in 1956 (see also the survey in [CS90]). It was later discovered by Lasagni in [Las88], that certain Runge-Kutta methods are in fact symplectic. The second difficulty stems from the fact that even when the Hamiltonian is time-independent, so that the flow preserves the levels of the Hamiltonian, the approximating scheme does not preserve any level set : even if the original system lives on a bounded enegy level, the discretized system can very well diverge to infinity. In fact both questions are somehow related (see Exercises 8 and 9).

## 7. Exercises and Problems

(1) (Nikishin's index theorem) Let $F$ be an area preserving diffeomorphism of the plane having an isolated fixed point at 0. We write $F(q,p) = (Q(q,p), P(q,p))$. The index of the fixed point 0 is the degree of the map $f(q,p) = (Q(q,p) - q, P(q,p) - p)$ defined from $D(\varepsilon) \setminus \{0\}$ to $\mathbb{R}^2 \setminus \{0\}$ (the index of such a map $f$ is the index of the restriction to the circle of radius $r < \varepsilon$ of $z \mapsto \frac{f(z)}{|f(z)|}$. We denote this number $L(F,0)$ ($L$ is for Lefschetz as this is also called the Lefschetz index). We want to prove that $L(F,0) \leq 1$.
   (a) Show that $L(F,0)$ does not depend on the choice of $r$
   (b) Show that if 0 is a non-degenerate fixed point of $F$ (that is $\det(dF(0) - \mathrm{Id}) \neq 0$), then $L(F,0) \in \{\pm 1\}$
   (c) We assume $dF(0)$ is degenerate. Show that after a suitable (symplectic!) change of coordinates, we may assume $dF(0) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ where $a \in \{0, \varepsilon\}$ where $\varepsilon > 0$.
   (d) Prove that the graph of $F$ that is
   $$\Gamma(F) = \{(x,y,X,Y) = (Q,p,P-p,q-Q) \mid F(q,p) = (Q,P)\}$$
   is Lagrangian in $\mathbb{R}^4$ with the symplectic form $dX \wedge dx + dY \wedge dy$ and that it is (locally) a graph over $(x,y)$.
   (e) Prove that $\Gamma(F)$ is the graph of the differential of a function $S(x,y) = S(Q,p)$ and that
   $$F(q,p) = (Q,P) \Longleftrightarrow P - p = \frac{\partial S}{\partial Q}(Q,p), q - Q = \frac{\partial S}{\partial p}(Q,p)$$
   (f) Using Exercise 6 of Chapter 5 conclude that $L(F,0) \leq 1$.

(2) (Characterization of exact Lagrangians, see [Sik91]) Let $\Gamma_\alpha$ be the graph of a non-closed one form in $T^*N$ for $N$ compact. Prove that there is a neighbourhood of $\Gamma_\alpha$ containing no Lagrangian submanifold Hamiltonianly isotopic to the zero section. Consider $L \cap \nu^*\gamma$ where $\gamma$ is a smooth loop in $N$ and $\nu^*\gamma = \{(\gamma(t), p) \mid \langle p, \dot\gamma(t)\rangle = 0\}$. Prove that
   (a) If $L$ is Hamiltonianly isotopic to $0_N$ we have $L \cap \nu^*\gamma \neq \emptyset$
   (b) Prove that if $\alpha$ is a one-form on $S^1$, there is a function $f$ on $S^1$ such that $\alpha - df = c\,d\theta$ where $\int_{S^1} \alpha = 2\pi c$.
   (c) Prove that if $\alpha$ is not closed there is a Hamiltonian isotopy of the type $(x,p) \mapsto (x, p - t\,df(x))$ moving $\Gamma_\alpha$ away from $\nu^*\gamma$
   (d) Conclude

(3) Prove that the map $\pi_1(\mathrm{Ham}(T^2n)) \longrightarrow \pi_1(T^{2n})$ given by $(\varphi^t)_{t\in[0,1]} \longrightarrow (\varphi^t(x))_{t\in[0,1]}$ is zero. For this notice that we may reduce to the case where $(\varphi^t)_{t\in[0,1]}$ is generated by some Hamiltonian $H$, then use Remark 6.12 to prove that one of the orbits must be contractible and finally that they are all homotopic, hence all contractible.

(4) We want to prove a homotopical version of Lemma 6.24, that is that the pair $(\tilde{S}^b, \tilde{S}^a)$ is homotopy equivalent to the suspension of $(S^b, S^a)$. The suspension of the pair $(X, A)$ is given by $\Sigma(X, A) = (X \times I, X \times \{0, 1\} \cup A \times I)$. We look at the set $\tilde{S}^b = \{S(x,\xi) - \eta^2 \leq b\}$ and look at the projection $(x, \xi, \eta) \mapsto (x, \xi)$. So for fixed $(x, \xi)$ the set of $\eta$ such that $\tilde{S}(x, \xi, \eta) \in \tilde{S}^b$ is given by
- $\mathbb{R}$ if $S(x,\xi) \leq b$
- $\mathbb{R} \setminus ] - r_b(x,\xi), r_b(x,\xi)[$ where $r_b(x,\xi)^2 = S(x,\xi) - b$ if $(x,\xi) \notin S^b$

(a) Show that $\tilde{S}^b$ is homotopy equivalent to $(S^b \times I) \cup ((E \setminus S^b) \times \partial I)$,

(b) Let $(X, A)$ be a pair. Show that the suspension $\Sigma(X, A)$ of $(X, A)$ is homotopy equivalent to

$$(X \times I \cup (E \setminus X) \times \partial I, A \times I \cup (E \setminus A) \times \partial I)$$

(c) Prove that $(\tilde{S}^b, \tilde{S}^a)$ is homotopy equivalent to $\Sigma(S^b, S^a)$.

CHAPTER 7

# Spectral invariants in symplectic topology

## 1. Actions and indices for intersection points

Let $(M, d\lambda)$ be an exact symplectic manifold $(M, d\lambda)$. We assume that for all Lagrangians, the relative Chern class vanishes on discs, that is $c_1(M)\pi_2(M, L) = 0$. Note that this condition is automatically satisfied for a cotangent bundle. It then follows from Corollary **??** that the Maslov class of the Lagrangian is well defined.

Remember that in Definition 4.9, we defined the Lagrangian branes as the triples $(L, f_L, \widetilde{G})$ such that $L$ is a connected exact Lagrangian, $df_L = \lambda_L$ and $\widetilde{G}$ is a grading, assuming that the Maslov class of $L$ vanishes. Also that $\mathfrak{L}(M, d\lambda)$ is just the set of exact Lagrangians and the image of $\mathscr{L}(M, d\lambda)$ by the forgetful functor $(L, f_L, \widetilde{G}) \mapsto L$ is the set of exact Lagrangians with vanishing Maslov class. We often denote by $\widetilde{L}$ an element in $\mathscr{L}(M, d\lambda)$ and $L$ its image in $\mathfrak{L}(M, d\lambda)$. For the definition of the index $i$ of a path in the Lagrangian Grassmannian we refer to Definition **??**

DEFINITION 7.1. *Let $\widetilde{L}_1, \widetilde{L}_2 \in \mathscr{L}(M, d\lambda)$. Let $x, y \in \widetilde{L}_1 \cap \widetilde{L}_2$ . We denote by $\ell_{\widetilde{L}_1,\widetilde{L}_2}(x)$ the quantity $f_{L_1}(x) - f_{L_2}(x)$ and by $\ell_{L_1,L_2}(x, y) = \ell_{\widetilde{L}_1,\widetilde{L}_2}(x) - \ell_{\widetilde{L}_1,\widetilde{L}_2}(y)$. If there is no ambiguity, we just write $\ell(x, y)$. Since $T_x\widetilde{L}_1, T_x\widetilde{L}_2$ belong to $\widetilde{\Lambda}(T_xM)$ they define a unique path -up-to homotopy- connecting $T_xL_1$ to $T_xL_2$, hence an index $i(T_x\widetilde{L}_1, T_x\widetilde{L}_2)$. We set $m_{L_1,L_2}(x) = i(T_x\widetilde{L}_1, T_x\widetilde{L}_2)$ and*

$$m_{L_1,L_2}(x, y) = i(T_x\widetilde{L}_1, T_x\widetilde{L}_2) - i(T_y\widetilde{L}_1, T_y\widetilde{L}_2)$$

It is easy to check that $\ell_{L_1,L_2}(x, y)$ does not depend on the choice of $f_{L_1}, f_{L_2}$ since replacing$f_{L_j}$ by $f_{L_j} + c_j$ where $c_j$ is a constant does not change the value of

$$\ell_{\widetilde{L}_1,\widetilde{L}_2}(x) - \ell_{\widetilde{L}_1,\widetilde{L}_2}(y) = (f_{L_1}(x) - f_{L_2}(x)) - (f_{L_1}(y) - f_{L_2}(y))$$

Similarly $m_{L_1,L_2}(x, y)$ does not depend on the choice of the gradings $\widetilde{G}_1, \widetilde{G}_2$ of $\widetilde{L}_1, \widetilde{L}_2$ since Note that the assumption on the vanishing of the Maslov class and Proposition **??** imply that the number $i(T_{\gamma(t)}L_1, T_{\gamma(t)}L_2)$ is well defined.

## 2. Lagrangians in $T^*N$ and their spectral invariants

In this section $N$ will be a smooth connected manifold. If $N$ is not compact all isotopies, Hamiltonians, diffeomorphisms will be assumed to be compact supported unless otherwise stated.

DEFINITION 7.2. *We denote by $\mathscr{L}_0(T^*N)$ and $\mathfrak{L}_0(T^*N)$ the set of elements $\widetilde{L}, L$ in $\mathscr{L}(T^*N, d\lambda)$ and $\mathfrak{L}(T^*N, d\lambda)$ Hamiltonianly isotopic to the zero section.*

If $L\in\mathscr{L}(T^*N)$ let $f_L$ be a primitive of $\lambda_{|L}$. For any path $\gamma$ in $L$ connecting $x$ to $y$, we have $\int_\gamma\lambda=f_L(x)-f_L(y)$.

PROPOSITION 7.3. *Let $L_1,L_2$ be exact Lagrangians with vanishing Maslov class, and $x,y\in L_1\cap L_2$. Let $\gamma_1$ (resp. $\gamma_2$) be a path in $L_1$ (resp. $L_2$) connecting $x$ to $y$. Then*

$$\ell_{L_1,L_2}(x,y)=\int_{\gamma_1}\lambda-\int_{\gamma_2}\lambda$$

*and*

$$m_{L_1,L_2}(x,y)=i(T_{\gamma(t)}L_1,T_{\gamma(t)}L_2)$$

We sometimes use the notation $f_{L_1,L_2}(x)=f_{L_1}(x)-f_{L_2}(x)$. When $L_1,L_2$ have a G.F.Q.I. we may state

PROPOSITION 7.4. *Let $S_i$ be G.F.Q.I. for $L_i\in\mathscr{L}(T^*N)$ ($i=1,2$). Then intersection points of $L_1\cap L_2$ are in one-to-one correspondence with critical points of $S(x,\xi,\eta)=S_1(x,\xi)-S_2(x,\eta)$. Moreover if*

$$x=(q,p)=(q,\frac{\partial}{\partial q}S_1(q,\xi))=(q,\frac{\partial}{\partial q}S_2(q,\eta))$$

*are such intersection points, we have*

$$\ell_{L_1,L_2}(x)=S_1(q,\xi)-S_2(q,\eta)$$
$$m_{L_1,L_2}(x,x')=index(d^2S(x,\xi,\eta))-index(d^2S(x',\xi',\eta'))$$

PROOF. If $\Sigma_{S_1}$ is the fiber-critical locus of $S_1$ that is $\{(q,\xi)\mid\frac{\partial S_1}{\partial\xi}(q,\xi)=0\}$, then $\lambda_L$ corresponds (via $i_S$) to $dS_{\Sigma_S}$. Therefore $f_{L_j}\circ i_{S_j}=S_j$ if $x=i_{S_j}(q,\xi_j)$ we have $f_{L_1}(x)-f_{L_2}(x)=S_1(q,\xi_1)-S_2(q,\xi_2)$ □

If $\varphi_H^t$ is the flow of the Hamiltonian $H(t,z)$, there is another formula for $\ell(x,y)$ from Proposition 4.13.

PROPOSITION AND DEFINITION 7.5. *Let $L\in\mathscr{L}(T^*N)$.*

*(1) We can normalize the G.F.Q.I. of $L$ so that $S=f_L\circ i_S$ on $\Sigma_S$.*
*(2) Let $L_1=\varphi_H^1(L)$ and $\gamma(t)=\varphi_H^t(z)$. Then for $z\in L$ we have*

$$f_{L_1}(\varphi_H^1(z))=f_L(z)+\int_0^1[\gamma^*\lambda-H(t,\gamma(t))]dt$$

*defines a function on $L_1$ satisfying $df_{L_1}=\lambda_{|L_1}$. Therefore we shall write $\varphi_H^1(L,f_L)=(\varphi_H^1(L),f_{L_1})$, but keeping in mind that the operation on the left hand side depends on the choice of $H$ and not just on the Hamiltonian isotopy (so changing $H$ to $H+c$ changes $f_{L_1}$ to $f_{L_1}-c$)*

PROOF. Set $\gamma$ to be a path from $x$ to $y$ in $L$ and $u(s,t) = (\varphi_H^t(\gamma(s),t) \in M\times\mathbb{R}$. Consider the form $\lambda - Hdt$ on $M\times\mathbb{R}$ and let us compute

$$\int_{[0,1]^2} u^*(d(\lambda - Hdt)) = \int_{[0,1]^2} u^*(\omega - dH\wedge dt)$$

We claim that this integral vanishes, because $X_H - \frac{\partial}{\partial t}$ is in the kernel of $\omega - dH\wedge dt$. As a result applying Stoke's formula we get, setting $x(t) = \varphi^t(x), y(t) = \varphi^t(y)$

$$0 = \int_{\partial[0,1]^2} u^*(\lambda - Hdt) =$$

$$\int_0^1 \gamma^*\lambda - \gamma^*(\varphi^1)^*\lambda + \int_0^1 x^*\lambda - H(t,x(t))dt - \int_0^1 y^*\lambda - H(t,y(t))dt$$

Now since if $\gamma$ is a path in $L$ from $x$ to $y$, $\varphi^1\gamma$ is a path in $L_1$ from $\varphi^1(x)$ to $\varphi^1(y)$, we have

$$f_L(x) - f_L(y) - f_{L_1}(\varphi^1(x)) + f_{L_1}(\varphi^1(y)) = \int_0^1 x^*\lambda - H(t,x(t))dt - \int_0^1 y^*\lambda - H(t,y(t))dt$$

In other words up to constant, $f_{L_1}(x) = f_L(x) + \int_0^1 x^*\lambda - H(t,x(t))dt$. We can choose the constant so that we have an equality. $\square$

REMARK 7.6.

(1) One should be careful: as an exact Lagrangian, $\varphi_H^1(L)$ (we should maybe write $\varphi_H(L,f_L)$) does not depend only on the flow, but on the choice of $H$.
(2) It will be sometimes more convenient to rewrite the formula "backwards" that is, setting for $z\in L_t = \varphi_H^t(L)$ and $\rho(s) = \varphi_H^{-s}(z)$ so that $\rho(t)\in L$

$$f_{L_t}(z) = f_L(\rho(t)) - \int_0^t \left[\rho^*\lambda + H(t-s,\rho(s))\right] ds$$

(3) According to Corollary 23 there is a symplectomorphism $\psi$ defined in a neighborhood of $0_L$ in $T^*L$ such that $\psi(0_L) = L$ and $\psi^*(\lambda_N) = \lambda_L + \pi_L^*(df_L\circ\psi)$, where $\pi_L : T^*L \longrightarrow L$ is the canonical projection. Then if $\psi^{-1}(L_1)$ is the graph of $dg_1$, then $f_{L_1} = f_L + g_1\circ\psi$. So we can look at a Hamiltonian $H$ defined in a neighborhood of the zero section of $L$. For small time $t$ we have $\varphi_H^t(0_L) = gr(-t\cdot H_{0_L}) + o(t)$. Indeed, up to higher order terms in $t$, we have $q(t) = q(0) + t\frac{\partial H}{\partial p}(q(0),0), p(t) = -t\frac{\partial H}{\partial q}(q(0),0)$. Then to write $p(t) = dG(q(t))$ we set $h(q) = H(q,0)$ so $p(t) = -t\cdot dh(\rho_t(q(t)))$ where $\rho_t$ is the inverse of $q(0)\mapsto q(0) + t\frac{\partial H}{\partial p}(q(0),0) + o(t)$ which is of the form $\mathrm{Id} + o(1)$. As a result $G(q) = -t\cdot h(q) + o(t)$. In other words if $L_t = \varphi_H^t(L)$ we have the formula $\frac{d}{dt}f_{L_t}(z) = -H(t,z)$.

Let $L\in\mathscr{L}(T^*N$ and $S$ a G.F.Q.I. for $L$. We proved in the previous section that then the numbers $c(\alpha,S)$ for $\alpha\in H^*(N)$ are independent from the choice of $S$. Similarly

PROPOSITION AND DEFINITION 7.7 (see [Tra94]). *Let $L_1, L_2$ in $\mathscr{L}_0(T^*N)$, and let $S_i$ be a G.F.Q.I. for $L_i$ equal at infinity to the quadratic forms $Q_i$ of index $d_i$. We set $d = d_1 + d_2$ and*

$$GH^*(L_1, L_2; a, b) = H^{*-d}((S_1 \ominus S_2)^b, (S_1 \ominus S_2)^a)$$

*and is called the Generating function cohomology of $(L_1, L_2)$ between $a$ and $b$. It does not depend on the choice of the $S_i$. For convenience, if $S_i$ are quadratic at infinity but do not correspond to embedded Lagrangians, we shall write $GH^*(S_1, S_2; a, b)$ for $H^{*-d}((S_1 \ominus S_2)^b, (S_1 \ominus S_2)^a)$.*

DEFINITION 7.8. *Let $L \in \mathscr{L}_0(T^*N)$. The common value of the $c(\alpha, S)$ where $S$ is any G.F.Q.I. of $L$ are denoted $c(\alpha, L)$. Let $1$ be the generator of $H^0(N)$ and $\mu_N$ be the generator of $H^n(N)$. We set $c_+(L) = c(\mu_N, L), c_-(L) = c(1_N; L)$, and*

$$\gamma(L) = c(\mu_N; L) - c(1; L)$$

*We also set $c(\alpha; L, L') = c(\alpha; S \ominus S')$, where $S$ (resp. $S'$) is a G.F.Q.I. for $L$ (resp $L'$) and*

$$\gamma(L, L') = c(\mu_N; L, L') - c(1_N; L, L')$$

*This is well defined on the set of (pairs of) Lagrangians Hamiltonianly isotopic to $0_N$.*

First of all we have that $c(\alpha,; L, L')$ is a critical value for $S \ominus S'$, that is there exists $(q, \xi, \xi')$ such that

$$\frac{\partial S}{\partial \xi}(q, \xi) = 0, \frac{\partial S'}{\partial \xi'}(q, \xi') = 0, \frac{\partial S}{\partial q}(q, \xi) - \frac{\partial S'}{\partial q}(q, \xi') = 0$$

But then $(q, \frac{\partial S}{\partial q}(q, \xi)) = (q, \frac{\partial S'}{\partial q}(q, \xi')) \in L \cap L'$ and $S(q, \xi) - S'(q, \xi') = f_L(q, \frac{\partial S}{\partial q}(q, \xi)) - f'_L(q, \frac{\partial S}{\partial q}(q, \xi))$. This proves

THEOREM 7.9 (Representation theorem). *Let $c = c(\alpha; L, L')$. Then there exists $(q, p) \in L \cap L'$ such that $f_L(q, p) - f_{L'}(q, p) = c(\alpha, L)$.*

PROOF. Indeed on $\Sigma_S$ we have $dS - pdq = dS - \lambda = 0$ since $\frac{\partial S}{\partial \xi} = 0$ on $\Sigma_S$. Thus $f_L(z_1) - f_L(z_2) = S(q_1, \xi_1) - S(q_2, \xi_2) = \int_\gamma \lambda$ where $i_S(q_j, \xi_j) = (q_j, p_j) = z_j$ (for $j = 1, 2$). □

PROPOSITION 7.10. *Let $\varphi^t$ be a Hamiltonian isotopy in $T^*N$ and $L_1, L_2 \in \mathscr{L}(T^*N)$. Then $t \mapsto c(\alpha, \varphi^t(L_1), \varphi^t(L_2))$ is constant.*

PROOF. Indeed, we have $\varphi_H^t(L_1) \cap \varphi_H^t(L_2) = \varphi_H^t(L_1 \cap L_2)$ and for $x \in L_1 \cap L_2$

$$f_{\varphi_H^t(L_j)}(\varphi_H^t(x)) = f_{L_j}(x) + \int_0^t \gamma^* \lambda - H(s, \gamma(s)) ds$$

where $\gamma(s) = \varphi_H^s(x)$. Therefore we have

$$f_{\varphi_H^t(L_1)}(\varphi_H^t(x)) - f_{\varphi_H^t(L_2)}(\varphi_H^t(x)) = f_{L_1}(x) - f_{L_2}(x)$$

does not depend on $t$. Moreover the set of $f_{L_1}(x) - f_{L_2}(x)$ for $x \in L_1 \cap L_2$ is the set of critical values of $S_1 \ominus S_2$ so, according to Sard's theorem, has empty interior. As a result, since $f_{\varphi_H^t(L_1)}(\varphi_H^t(x)) - f_{\varphi_H^t(L_2)}(\varphi_H^t(x))$ varies continuously with $t$, and has value in a set of empty interior, it must be constant. □

From Lusternik-Schnirelman's theorem (Theorem 5.39), we can deduce that $\gamma$ defines a metric on $\mathscr{L}(T^*N)$, the set of Lagrangians Hamiltonianly isotopic to the zero section.

PROPOSITION 7.11. *For any $L_1, L_2, L_3 \in \mathscr{L}(T^*N)$, the following properties hold :*

$$c_+(\overline{L}_1, \overline{L}_2) = -c_-(L_1, L_2) = c_+(L_2, L_1)$$
$$c_+(L_1, L_3) \leq c_+(L_1, L_2) + c_+(L_2, L_3)$$
$$c_-(L_1, L_3) \geq c_-(L_1, L_2) + c_-(L_2, L_3)$$
$$\gamma(L_1, L_3) \leq \gamma(L_1, L_2) + \gamma(L_2, L_3)$$
$$\gamma(L_1, L_2) = 0 \Leftrightarrow L_1 = L_2$$

*The function $\gamma$ defines a metric on $\mathscr{L}(T^*N)$ that is invariant by the action of* $\mathrm{DHam}(T^*N)$.

PROOF. Using Proposition 7.10, it is enough to deal with the case $L_2 = 0_N$. The first statement follows from Lusternik-Schnirelman (Theorem 5.39). Indeed, $\gamma(L) = 0$ implies $c(1, S) = c(\mu, S)$, hence $K_c$ the set of critical points at level $c$ is such that $\mu \in H^*(K_c)$ is nonzero. But $K_c$ can be identified to a subset of $L \cap 0_N$, hence of $N$. But on any proper subset of $N$ the class $\mu$ vanishes. As a consequence $L \cap 0_N = 0_N$, that is $0_N \subset L$ and this implies $L = 0_N$. The second statement follows from the fact that if $S$ is a G.F.Q.I. for $L$, then $-S$ is a G.F.Q.I. for $\overline{L}$, Proposition 6.2 stating that $c(1, -S) = -c(\mu, S)$ and $(-S_1) \ominus (-S_2) = -(S_1 \ominus S_2) = S_2 \ominus S_1$.

The inequalities will be a consequence of the Triangle inequality for G.F.Q.I. (see Prop 6.3). and denoting by $S_1, S_3$ G.F.Q.I. of $L_1, L_3$ we only have to prove

$$c_+(S_1 \ominus S_3) \leq c_+(S_1) + c_+(-S_3)$$

since the second inequality follows by replacing $L_i$ by $\overline{L}_i$ and the last one by subtracting the first two. Applying the triangle inequality we have

$$c(1, S_1 \ominus S_3) \geq c(1, S_1) + c(1, -S_3) = c(1, S_1) - c(\mu_N, S_3)$$
$$c(1, S_3 \ominus S_1) \geq c(1, S_3) + c(1, -S_1) = c(1, S_3) - c(\mu_N, S_1)$$

and since $c(1, S_3 \ominus S_1) = -c(\mu_N, S_1 \ominus S_3)$ this can be rewritten as

$$-c(1, S_1 \ominus S_3) \leq -c(1, S_1) + c(\mu_N, S_3)$$
$$c(\mu, S_1 \ominus S_3) \leq -c(1, S_3) + c(\mu_N, S_1)$$

that is

$$c_+(S_1 \ominus S_3) \leq c_+(S_1) + c_+(-S_3)$$

or else

$$c_+(L_1, L_3) = c_+(L_1, L_2) + c_+(L_2, L_3)$$

Finally we must prove that $\gamma(L,L')=0$ implies that $L=L'$. According to Proposition it is enough to deal with the case $L'=0_N$. Now $\gamma(L,0_N)=0$ means $c(1_N,L,0_N)=c(\mu_N,L,0_N)$. According to Theorem 5.39 the equality of the critical levels imply that $\pi^*(\mu_N)$ is non-zero in a neighborhood of $K_c$ the set of critical points at level $c=c(1_N,L,0_N)=c(\mu_N,L,0_N)$. We thus proved that $\mu_N$ in non-zero in any neighborhood of $L\cap 0_N$. But if $L\cap 0_N\neq 0_N$, since $L\cap 0_N$ is compact, it is contained in the complement of a neighborhood $V$. But $\mu_N$ vanishes on the complement of $V$, a contradiction. We thus proved that $L\cap 0_N=0_N$, that is $0_N\subset L$. But since $L$ is embedded we have $L=0_N$ (this is obvious if $N$ is connected, otherwise just argue connected component by connected component).

The symmetry $\gamma(L_1,L_2)=\gamma(L_2,L_1)$ follows immediately from the property $\gamma(-S)=\gamma(S)$ itself an obvious consequence of the equality $c(\mu,S)=-c(1,-S)$ (see Proposition 5.38).

□

EXAMPLE 7.12. Let $L_i=\mathrm{graph}(df_i)$. Then one easily checks that
$c_+(L_1,L_2)=\sup_{x\in N} f_2(x)-f_1(x)$,
$c_-(L_1,L_2)=\inf_{x\in N} f_2(x)-f_1(x)$,
$\gamma(L_1,L_2)=\mathrm{osc}(f_1-f_2)$

We may now use this to define a distance on $Ham_c(T^*N)$.

DEFINITION 7.13. *We set*

$$\widehat{c}_+(\psi)=\sup_{L\in\mathscr{L}(T^*N)} c_+(\psi(L),L)$$
$$\widehat{c}_-(\psi)=\inf_{L\in\mathscr{L}(T^*N)} c_-(\psi(L),L)$$
$$\widehat{\gamma}(\psi)=\gamma(\psi,\mathrm{Id})=\sup_{L\in\mathscr{L}(T^*N)} \gamma(\psi(L),L)$$

*In general we set* $\widehat{\gamma}(\varphi,\psi)=\widehat{\gamma}(\psi^{-1}\varphi)$

REMARK 7.14. With this definition, we do not know whether $\widehat{\gamma}(\psi)=\widehat{c}_+(\psi)-\widehat{c}_-(\psi)$.

PROPOSITION 7.15. *We have* $c_+(\psi^{-1})=-c_-(\psi)$ *and the inequalities*

$$\widehat{c}_+(\varphi\psi)\leq\widehat{c}_+(\varphi)+\widehat{c}_+(\psi)$$
$$\widehat{c}_-(\varphi\psi)\geq\widehat{c}_-(\varphi)+\widehat{c}_-(\psi)$$

*The function* $\widehat{\gamma}$ *defines a distance on* $Ham_c(T^*N)$. *In particular* $\widehat{\gamma}(\psi)=0$ *if and only if* $\psi=\mathrm{Id}$.

PROOF. The equality follows from the equality $c_+(\psi^{-1}(L),L)=c_+(L,\psi(L))=-c_-(\psi(L),L)$. The first inequality follows from the fact that

$$\widehat{c}_+(\varphi\psi)=\sup_{L\in\mathscr{L}(T^*N)} c_+(L,\varphi\psi(L))\leq$$
$$\sup_{L\in\mathscr{L}(T^*N)} c_+(L,\psi(L))+\sup_{L\in\mathscr{L}(T^*N)} c_+(\psi(L),\varphi\psi(L))$$

First if $\gamma(L,\psi(L))=0$ for all $L\in\mathscr{L}(T^*N)$ we have that for all $L$, $\psi(L)=L$. Now assume $\psi(z)\neq z$ for some $z\in T^*N$. Then let $L$ be a Lagrangian through $z$ avoiding $\varphi(z)$. Then $\psi(L)\neq L$, a contradiction. We still have to prove the existence of $L$, that is given two points in $T^*N$, there exists a Lagrangian through one of them avoiding the second one. But by the bi-transitivity of $Ham_c(T^*N)$ (see Exercice 33 from Chapter 3) this property does not depend on the choice of the pair of points. Choosing one point on the zero section and one outside proves our claim.

Now $\gamma(\psi^{-1})=\gamma(\psi)$ because $\gamma(L,\psi^{-1}(L))=\gamma(\psi(L),L)=\gamma(L,\psi(L))$.

Finally we must prove that $\widehat{\gamma}(\psi)$ is finite. This follows from Proposition 7.18. □

Note that for compact supported Hamiltonian flows in $\mathbb{R}^{2n}$ there is another possible definition of the spectral invariant, using the fact that $T^*\mathbb{R}^n\times\overline{T^*\mathbb{R}^n}\simeq T^*\Delta_{T^*\mathbb{R}^n}$ where $\Delta_{T^*\mathbb{R}^n}$ is the diagonal. The isomorphism is given by

$$\rho:(q,p,Q,P)\mapsto(q,P,p-P,Q-q)$$

Note that if $L$ is a Lagrangian in $T^*\mathbb{R}^n$ coinciding with the zero section at infinity, we can compactify $L$ to a Lagrangian in $T^*S^n$ by adding a point at infinity.

DEFINITION 7.16. *Let $\psi\in\mathrm{DHam}_c(\mathbb{R}^{2n})$ be the group of time-one maps of compact supported Hamiltonians. Set $\Gamma_\psi$ to be the compactification of the image by $\rho$ of the graph of $\psi$ in $T^*\mathbb{R}^n\times\overline{T^*\mathbb{R}^n}$. Thus $\Gamma_\psi$ is an exact Lagrangian in $T^*S^{2n}$ and we normalize $f_{\Gamma_\psi}$ by setting $f_{\Gamma_\psi}=0$ at infinity. Then we set*

$$c_\pm(\psi)=c_\pm(\Gamma_\psi)\ \textit{and}\ \gamma(\psi)=\gamma(\Gamma_\psi)$$

We shall prove in Exercise 4 that $\gamma$ is a metric. Then we can compare this metric with the Hofer norm ([Hof90]) defined on $\mathrm{Ham}(M,\omega)$ by

$$\|\psi\|_H=\inf\left\{\int_0^1\left[\sup_{x\in M}H(t,x)-\inf_{x\in M}H(t,x)\right]dt\ \middle|\ \varphi_H^1=\psi\right\}$$

PROPOSITION 7.17. *We have*

*(1)* $c_+(\varphi^{-1})=-c_-(\varphi)$
*(2)* $c_-(\varphi)\leq 0\leq c_+(\varphi)$
*(3)* $\gamma(\varphi)=0$ *if and only if* $\varphi=\mathrm{Id}$
*(4)* $c_+(\varphi\psi)\leq c_+(\varphi)+c_+(\psi)$

PROOF. We must compare the Lagrangian $\Gamma_\varphi=\{(q,P,p-P,Q-q)\mid\varphi(q,p)=(Q,P)\}$ and $\Gamma_{\varphi^{-1}}=\{(q,P,p-P,Q-q)\mid\varphi(Q,P)=(q,p)\}$. The map $(q,p,Q,P)\mapsto(P,Q,p,q)$ is symplectic, and sends $\Gamma_\varphi$ to $\overline{\Gamma_{\varphi^{-1}}}$. As a result

$$c_-(\varphi)=c(1,\Gamma_\varphi)=c(1,\overline{\Gamma_{\varphi^{-1}}})=-c(\mu,\Gamma_{\varphi^{-1}})=-c_+(\varphi^{-1})$$

For the second statement, it is enough to prove $c_-(\varphi)\leq 0$. Note that $\Gamma_\varphi$ coincides with the zero section at infinity. As a result, $S(x,\xi)=q(\xi)$ for $x$ the point at infinity. As a result if $E_x^-$ is the negative eigenspace of $q$ at $x$, we have $[E_x^-]\in H^*(S^\infty,S^{-\infty})$ represents the class $1\otimes T$. Since $S\leq 0$ on $E_x^-$, we get $c_-(\varphi)\leq 0$. This together with our first statement implies the second statement for $c_+$. As for the third statement, we know that $\gamma(\varphi)=0$ if and only if $\Gamma_\varphi=\Delta$ and this is of course equivalent to $\varphi=\mathrm{Id}$. Finally we have

$$c(1,\varphi(L))=c(1,\varphi(L),0_N)=c(1,L,\varphi^{-1}(0_N))\geq c(1,L)+c(1,\overline{\varphi^{-1}(0_N)}))$$

Applying this to $N=S^{2n}$, $L=\overline{\Gamma}_\psi\subset T^*S^{2n}$ and replacing $\varphi$ by $\mathrm{Id}\times\varphi$, using that $(\mathrm{Id}\times\varphi)(\Gamma_\psi)=\Gamma_{\varphi\psi}$ and $(\mathrm{Id}\times\varphi)(0_{S^{2n}})=\mathrm{id}\times\varphi)(\Delta)=\overline{\Gamma}_\varphi$, we get

$$c(1,\varphi\psi)\geq c(1,\psi)-c(\mu,\varphi^{-1})=c(1,\psi)+c(1,\varphi)$$

the last equality is obtained by using the first statement. Using again the first statement, we get

$$c_+(\varphi\psi)\leq c_+(\varphi)+c_+\psi)$$

□

PROPOSITION 7.18 (Spectral norm estimate I ). *Let $\varphi$ be the time-one map of the Hamiltonian $H$. We then have the inequality*

$$\gamma(\varphi)\leq\|\varphi\|_H$$

*In particular*

(1) *$\|\varphi\|_H$ defines a metric on $\mathrm{Ham}_c(T^*T^n)$ by $d_H(\varphi_1,\varphi_2)=\|\varphi_1\varphi_2^{-1}\|_H$. (This is the Hofer metric, see [Hof90])*
(2) *$H\mapsto c_+(\varphi_H)$, $H\mapsto c_-(\varphi_H)$ and $H\mapsto\gamma(\varphi_H)$ are continuous for the $C^0$-topology.*

This will follow from the more general situation dealt by the following proposition

PROPOSITION 7.19 (Spectral norm estimate I - the Lagrangian case ). *Let $\varphi_H^t$ be the flow of $H$ and $L_1,L_2$ be two Lagrangians in $\mathscr{L}(T^*N)$. Then*

$$c(\alpha,\varphi_H^t(L_1),L_2)\leq c(\alpha,L_1,L_2)+\sup\left\{-H(t,z),z\in\varphi_H^t(L_1)\cap L_2\right\}$$

*In particular setting $\|H\|_{L_2}=\sup\{-H(t,z),z\in L_2\}$, we have*

$$c(\alpha,\varphi_H^t(L_1),L_2)\leq c(\alpha,L_1,L_2)+\|H\|_{L_2}$$

Let us mention some consequences:

COROLLARY 7.20.

$$c_+(\varphi_H^t(L_1),L_2)\leq c_+(L_1,L_2)+\sup\left\{-H(t,z),z\in\varphi_t(L_1)\cap L_2\right\}$$

$$c_-(\varphi_H^t(L_1),L_2)\geq c_-(L_1,L_2)+\inf\left\{H(t,z),z\in\varphi_t(L_1)\cap L_2\right\}$$

$$c(\alpha,\varphi_H^t(L),L)\leq\|H\|_L\leq\|H\|_{C^0}$$

PROOF. The first inequality follows from applying the Proposition to $\alpha=\mu_N$. The second by taking $c_-(\varphi_H^t(L_1),L_2)=-c_+(L_2,\varphi_H^t(L_1))=-c_+(\varphi_H^{-t}(L_1),L_2)$ Applying the Proposition to $L_1=L_2$ we get the last inequality. □

We shall need the

LEMMA 7.21. *Let $f_t(x)$ be a differentiable family of smooth functions and assume for $t_0$ in some interval $I$ we have $df_{t_0}(x_0)=0$ implies $(\frac{\partial}{\partial t}f_t)_{|t=t_0}(x_0)\geq 0$. Let $c(t)=f_t(x_t)$ be a continuous path of critical values (note that $t\mapsto x_t$ does not need to be continuous). Then c(t) is increasing for $t\in I$.*

PROOF. We first assume $\frac{\partial}{\partial t}f_t(x)>0$ whenever $df_t(x)=0$. then we may perturb the family $f_t$ so that the inequality still holds $(t,x)\mapsto f_t(x)$ is smooth, and except for finitely many $t$'s, $f_t$ is Morse and has a single critical point on each critical level. As a result we can write in the complement of this finite set $c(t)=f_t(x_t)$ with $df_t(x_t)=0$ and $x_t$ is uniquely defined. Moreover $t\mapsto x_t$ is smooth, since we can apply the implicit function theorem to $df_t(x)=0$ provided $d^2f_t(x)$ is invertible. Now since

$$\frac{d}{dt}c(t)=\frac{\partial}{\partial t}(f_t(x_t))=(\frac{\partial}{\partial t}f_t)(x_t)+df_t(x_t)\frac{\partial}{\partial t}x_t=(\frac{\partial}{\partial t}f_t)(x_t)>0$$

we get that $t\mapsto c(t)$ is increasing except for finitely many values of $t$. Since it is continuous, it is increasing on the whole interval $[0,1]$.

Now for the general case, we replace $f_t(x)$ by $f_t(x)+\varepsilon t$ with $\varepsilon>0$, we get that for all $\varepsilon$ and any $t_1<t_2$ we have $c(\alpha,f_{t_1})+\varepsilon t_1\leq c(\alpha,f_{t_2})+\varepsilon t_2$. This of course implies $c(\alpha,f_{t_1})\leq c(\alpha,f_{t_2})$ i.e. $t\mapsto c(\alpha,f_t)$ is increasing.

□

PROOF. Let $S_1^t(x,\xi)$ be a G.F.Q.I. for $\psi_t(L_1)$ and $S_2(x,\eta)$ be a G.F.Q.I. for $L_2$. Then we want to estimate $c(\mu_N,S_1^t\ominus S_2)$, but for this it is enough to estimate $\frac{d}{dt}(S_1^t\ominus S_2)(x,\xi,\eta)$ for $d(S_1^t\ominus S_2)(x,\xi,\eta)=0$. But this means $\frac{\partial S_1^t}{\partial\xi}(x,\xi,\eta)=0$, $\frac{\partial S_2}{\partial\eta}(x,\xi,\eta)=0$ and $(x,\frac{\partial S_1^t}{\partial x}(x,\xi,\eta))=(x,\frac{\partial S_2}{\partial x}(x,\xi,\eta))\in\psi_t(L_1)\cap L_2$. Now $(S_1^t\ominus S_2)(x,\xi,\eta)$ is equal to $f_{L_1^t}(x,p)-f_{L_2}(x,p)$ where $(x,p)\in L_1^t\cap L_2$ corresponds to $(x,\xi,\eta)$. Then we must compute $\frac{d}{dt}(f_{L_1^t}-f_{L_2})(x,p)$. Using the formula from Remark 7.6 (2) we get

$$\frac{d}{dt}f_{L_1^t}(z)=df_{L_1}(\rho(t))\dot\rho(t)-\left[\rho^*\lambda(t)+H(t,\rho(t))\right]$$

since $\lambda=df_L$ on $L$, we have

$$\frac{d}{dt}f_{L_t}(z)=-H(t,\rho(t))$$

where $\rho(t)\in L$ and $z=\rho(0)\in L_1^t\cap L_2$. □

PROOF OF PROPOSITION 7.19. Let $S_1^t(x,\xi)$ be a G.F.Q.I. for $L_1^t=\psi_t(L_1)$ and $S_2(x,\eta)$ be a G.F.Q.I. for $L_2$. Then we want to estimate $c(\alpha,S_1^t\ominus S_2)$. On one hand we know that $c(\alpha,S_1^t\ominus S_2)=(S_1^t\ominus S_2)(x_t,\xi_t,\eta_t)$ where $(x_t,\xi_t,\eta_t)$ is a critical point of $S_1^t\ominus S_2$ thus corresponding to a point $z_t\in\psi^t(L_1)\cap L_2$. Moreover $(S_1^t\ominus S_2)(x_t,\xi_t,\eta_t)=f_{L_1^t}(z_t)-f_{L_2}(z_t)$ by the representation Theorem (Thm 7.9). By a small perturbation of $L_2$ we may assume that for $t$ outside a finite subset $E$ of $[0,1]$, $(S_1^t\ominus S_2)$ is a Morse function with distinct critical values. As a result for $t$ outside t$E$, the map $t\mapsto(x_t,\xi_t,\eta_t)$ is uniquely defined and smooth by the implicit function theorem. We will now compute $\frac{d}{dt}(S_1^t\ominus S_2)(x_t,\xi_t,\eta_t)$

and show it is bounded by $\sup\{-H(t,z), z\in\psi_t(L_1)\cap L_2\}$ on the complement of $E$. Since $t\mapsto c(\alpha, S_1^t\ominus S_2)$ is continuous this yields the bound of the Proposition.

A revoir !

The computation of $\frac{d}{dt}(S_1^t\ominus S_2)(x_t,\xi_t,\eta_t)$ is based on the fact that the action functional is generating function in a generalized sense. Let $\mathscr{P}_t(L_1,L_2)$ be the set of smooth paths $\gamma:[0,t]\longrightarrow M$ such that[1] $\gamma(0)\in L_1, \gamma(t)\in L_2$. Then

$$\mathscr{A}_t(\gamma)=f_{L_1}(z(0))+\int_0^1[\gamma^*\lambda-H(s,\gamma(s))]ds-f_{L_2}(\gamma(t))$$

is such that $D\mathscr{A}_t(\gamma)=0$ if and only if $\gamma(s)=\psi^s(\gamma(0))$ and then $\mathscr{A}_t(\gamma)=f_{L_2}(\gamma(t))-f_{L_1^t}(\gamma(t))$.

So $c(\alpha,\psi^t(L_1),L_2)=(S_1^t\ominus S_2)(x_t,\xi_t,\eta_t)=f_{L_1^t}(z_t)-f_{L_2}(z_t)=\mathscr{A}_t(\gamma_t)$ for some critical point $\gamma_t$ of $\mathscr{A}_t$, and $\gamma_t(s)=\psi^s(\gamma(0)), \gamma_t(t)=z_t$. Generically in $L_2$ and $t$, the critical values of $\mathscr{A}_t$ —which coincide with the critical values of $S_t^1\ominus S_2$— are distinct and the intersection points $z_t$ depends smoothly on $t$, so the same holds for $\gamma_t(t)$. We may thus apply the Lemma, but for convenience we want all the $\mathscr{A}_t$ to be defined on a single space, so we replace $z$ by $\zeta$ such that $\zeta(u)=\gamma(tu)$ where $\zeta\in\mathscr{P}_1(L_1,L_2)$ is a fixed space. Then substituting $s=tu$ in the formula for $\mathscr{A}_t$ we obtain

$$\mathscr{A}_t(\gamma)=f_{L_1}(\zeta(0))+t\int_0^1[\lambda(z(tu))\dot\gamma(tu)-H(tu,\gamma(tu))]du-f_{L_2}(\gamma(t))$$

and using $\zeta(u)=\gamma(tu), \dot\zeta(u)=t\dot\gamma(tu)$

$$\mathscr{A}_t(z)=f_{L_1}(\zeta(0))+\int_0^1[\lambda(\zeta(u))\dot\zeta(u)-tH(tu,\zeta(u))du-f_{L_2}(\zeta(1))$$

Now setting $\mathscr{B}_t(\zeta)$ for $\mathscr{A}_t(\gamma)$ we have

$$\frac{d}{dt}\mathscr{B}_t(\zeta)=-\frac{d}{dt}\int_0^1 tH(tu,\zeta(u))du=-\int_0^1 H(tu,\zeta(u))du-t\int_0^1 u\frac{\partial H}{\partial t}(tu,\zeta(u))du=$$
$$-\int_0^1 H(tu,\zeta(u))du-\int_0^1 u\frac{\partial}{\partial u}H(tu,\zeta(u))du=-\int_0^1\frac{d}{du}[uH(tu,\zeta(u)]du$$

the last equality follows because $d_zH(su,\zeta(u))\dot\zeta(u)=0$ if $\zeta(u)$ is a trajectory of $X_H$, and the time-one flow of $H(su,\zeta)$ is $\psi^t$. But the last term is $H(t,\zeta(1))$, and since $\zeta(1)\in L_2\cap\psi^t(L_1)$ this concludes our proof.

$\square$

REMARK 7.22. See Exercice 10 for a similar estimate for the Hofer norm.

[1]Here by $D\mathscr{A}$ we mean the variational derivative, i.e. defined for $\xi$ a vector field along $\gamma$ by setting $\exp_\gamma(\xi)(s)=\exp_{\gamma(s)}\xi(s)$, we define $D\mathscr{A}(\gamma)\xi=\frac{d}{d\varepsilon}\mathscr{A}(exp_\gamma(\varepsilon\xi))_{|\varepsilon=0}$.

PROOF OF PROPOSITION 7.18. It is enough to prove that if $\varphi_H^1 = \psi$ then

$$\gamma(\psi) \le \int_0^1 [\sup_{x\in M} H(t,x) - \inf_{x\in M} H(t,x)]\,dt$$

Now $\gamma(\psi) = \gamma(\Gamma_\psi)$ and $\Gamma_\psi$ is the image of $\overline{\Delta} = \overline{\Gamma}(\mathrm{Id})$, the compactification of the diagonal, by the extension of $\mathrm{Id}\times\varphi^t$, that is generated by $K(t,x_1,p_1,x_2,p_2) = H(t,x_1,p_1)$. But then Proposition 7.19 implies the inequality

$$c_+(\Gamma_\psi) \le \sup\left\{K(t,z) \mid t\in[0,1], z\in\overline{\Delta}\right\} = \sup\left\{H(t,x,p) \mid t\in[0,1], (x,p)\in\mathbb{R}^{2n}\right\}$$

Finally we want to prove that we can replace $\sup\left\{H(t,x,p) \mid t\in[0,1], (x,p)\in\mathbb{R}^{2n}\right\}$ by $\int_0^1 \sup_{x\in M} H(t,x)\,dt$. For this we notice that if $\tau(t)$ is an increasing diffeomorphism of $[0,1]$, then the time-one map associated to the Hamiltonian $\tau'(t)H(\tau(t),z)$ is the same as the one associated to $H(t,z)$. So let $I^+ = \int_0^1 \sup_{x\in M} H(t,x)\,dt$, and set $M(t) = \sup_{x\in M} H(t,x)$. If $M$ is strictly positive, choose $\tau$ satisfying the equation

$$\begin{cases} \tau'(t) = \frac{I^+}{M(\tau(t))} \\ \tau(0) = 0 \end{cases}$$

Since $K(t,z) = \tau'(t)H(\tau(t),z)$ has the same time-one flow as $H$ and $\max_z K(t,z) = I^+$, we are done. If $M$ vanishes, we take $\tau$ satisfying the equation

$$\left\{ \tau'(t) = (1+\delta)\frac{I^+}{M(\tau(t))+\varepsilon}\ \tau(0) = 0 \right.$$

where $\varepsilon > 0$ and $\delta$ adjusted so that $\tau(1) = 1$. This then proves that we may obtain $\varphi$ as the time-one of a Hamiltonian such that $\sup_{x\in M}\tau'(t)H(\tau(t),x) \le (1+\delta)I^+$ and since $\varepsilon$ and hence $\delta$ are arbitrarily small, this proves our claim.

□

PROOF OF PROPOSITION 7.18. □

COROLLARY 7.23. *If $H \le 0$ we have $c_+(\varphi_H) = 0$. More generally if $H_1 \le H_2$ we have*

$$c_+(\varphi_{H_1}) \le c_+(\varphi_{H_2})$$

PROOF. The first statement follows immediately from Proposition 7.19, applied to $\alpha = \mu$ and the fact that $c_+(\varphi) \ge 0$. □

REMARK 7.24. The above Corollary implies that if $\varphi$ is generated both as the time-one of a non-negative and a non-positive Hamiltonian, it must be equal to the identity : indeed, the first assumption implies $c_+(\varphi) = 0$ the second that $c_-(\varphi) = 0$, so $\gamma(\varphi) = 0$ and $\varphi = \mathrm{Id}$. This also follows by the examination of the Calabi invariant. However the property $\varphi \succeq \mathrm{Id}$ defined by $c_+(\varphi) = 0$ is not equivalent to being generated by a non-negative Hamiltonian (see Exercise 19).

When $H$ is $C^2$ small we have

PROPOSITION 7.25. *There exists $\varepsilon > 0$ such that if $H$ is autonomous and $\|H\|_{C^2} \leq \varepsilon$ we have*

$$c_+(\varphi_H^1) = \max H, c_-(\varphi_H^1) = \min H$$

PROOF. We know that for $\varepsilon$ small enough, $\varphi_H^t$ has no periodic orbit of period less than 1 other than the constants (see Proposition 3.85). By a generic perturbation, we may assume $H$ has only finitely many critical values. Now for $s$ small enough, since $\Gamma(\varphi_H^s)$ is $C^1$ close to the zero section, there is $S_s(q,P)$, a G.F.Q.I. for $\Gamma(\varphi_H^s)$ with no fibre variables. Moreover $S_s(q,P) = s\cdot H(q,P) + o(s)$ and the critical points of $S$ and $H$ coincide, since they correspond to fixed points of $\varphi_H^s$. As a result, for $s$ smaller than the difference between distinct critical values of $H$, we have $c_+(S) = \max S = s\max H$. But then by our assumption, $\frac{1}{s}c_+(\varphi_H^s)$ takes values in the set of critical values of $H$, by Sard's theorem and the continuity of this function of $s$, this must be constant, so $c_+(H) = \max H$. The case where $H$ is not generic is proved by approximation : we may write $H = C^0 - \lim H_k$ where $c_+(H_k) = \max H_k$. The same argument yields the other equality. □

COROLLARY 7.26. *The metric $\gamma$ extends to* $\mathrm{Ham}_\infty(\mathbb{R}^{2n})$*, the set of Hamiltonians such that* $\lim_{|z|\to\infty} H(t,z) = 0$*, where the limit is uniform in $t$.*

PROOF. We can write a Hamiltonian $H \in \mathrm{Ham}_\infty(\mathbb{R}^{2n})$ as $\lim_k H_k$ where the $H_k$ are compact supported : just take $\chi$ to be a function on $\mathbb{R}_+$ such that $0 \leq \chi(r) \leq 1$, equal to 1 on $[0,1]$ and vanishing on $[2,+\infty[$. Then $H_k(t,z) = \chi\left(\frac{|z|}{k}\right)H(tz)$ satisfies the inequality

$$\|H_k - H\| \leq \sup_{|z|\geq k} |H(t,z)|$$

□

Let $U$ be a bounded domain in $T^*N$. We set

DEFINITION 7.27. *We define for an open set $U$*

$$c(U) = \sup\{c_+(\varphi_H^1) \mid \mathrm{supp}(H) \subset U \textit{and bounded}\}$$
$$\gamma(U) = \sup_V \inf_\psi \{\gamma(\psi) \mid \psi(V)\cap V = \emptyset,\ V \subset U \textit{is bounded}\}$$

*where $H$ is a (time-dependent) Hamiltonian.*

*For a closed set $F$ we set $c(F) = \inf\{c(U) \mid U \supset F\}$ and $\gamma(F) = \inf\{\gamma(U) \mid U \supset F\}$.*

Its main properties are

PROPOSITION 7.28. *(Properties of the spectral capacity) We have the following properties*

1. *The functions $c,\gamma$ are symplectically invariant : if $\psi$ is a conformally symplectic map i.e. $\psi^*(\omega) = k\omega$, then $c(\psi(U)) = kc(U)$ and $\gamma(\psi(U)) = k\gamma(U)$ for any set $U$*
2. *$c,\gamma$ are monotone : if $V \subset U$ then $c(U) \leq c(V)$ and $\gamma(V) \subset \gamma(U)$.*

PROOF. For the first statement, we use that any conformally symplectic map in $\mathbb{R}^{2n}$ is isotopic to the identity among conformally symplectic maps.

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□

A consequence of the first property is the outer continuity of the capacity for certain sets, that is $\gamma(U) = \inf\{\gamma(V) \mid V \supset U\}$. The reader has already encountered these sets

DEFINITION 7.29 (see Definition 3.67). *A hypersurface $\Sigma$ in a symplectic manifold $(M,\omega)$ is of contact type if there exists a conformal vector field defined in a neighborhood of $\sigma$ and transverse to $\Sigma$. The hypersurface is of restricted contact type if the vector field is globally defined on all of $M$.*

The complement of $\Sigma$ will have two connected components. The one such that $X$ points outwards is defined as the interior. One can often relax the definition of restricted contact type by requiring that $X$ is defined in the interior of $\Sigma$. This is the only meaningful definition in the case of an ambient compact manifold, since there cannot be a global conformal vector field (the flow of such a vector field would expand the volume ).

EXAMPLES 7.30.
(1) The starshaped hypersurfaces in $T^*N$, that is those transverse to the radial vector field $p\frac{\partial}{\partial p} = \sum_{j=1}^n p_j \frac{\partial}{\partial p_j}$ are of restricted contact type, since the radial vector field is conformal.
(2) In $(\mathbb{R}^{2n},\sigma)$ the starshaped hypersurfaces that is those transverse to the radial vector field $q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p} = \sum_{j=1}^n q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j}$ are of restricted contact type.

We refer to [Lau97] for using handle attachment to obtain contact hypersurfaces with more interesting topologies. Hypersurfaces of contact type are characterized as having a neighborhood foliated by hypersurfaces having the same closed characteristics (see Proposition 3.75 in Chapter 3). We now want to prove some continuity of the capacity $\gamma$. Recall that the Hausdorff distance between $U$ and $V$ is

$$d_H(U,V) = \max\left\{\sup_{x\in V} d(x,U), \sup_{y\in U} d(y,V)\right\}$$

In other words denoting by $U_t = \{x \mid d(x,U) \leq t\}$ for $t>0$ we have $d(U,V) \leq a$ if and only if $U \subset V_a, V \subset U_a$. We may now state

PROPOSITION 7.31 (Capacity-regularity of contact boundary subsets). *If $U$ is an open set bounded by a compact hypersurface of contact type, then we have $\gamma(U) = \inf\{\gamma(V) \mid V \supset U\} = \sup\{\gamma(V) \mid \overline{V} \subset U\}$. More generally if $d$ is the Hausdorff distance, for each $\varepsilon > 0$ there exists $\delta$ such that $d_H(V,U) \leq \delta$ implies $|\gamma(V) - \gamma(U)| \leq \varepsilon$*

PROOF. Let $\rho_t$ be the flow of the conformal vector field $X$. Then for each $\tau > 0$ there exists $\delta > 0$ such that $d_H(V,U) < \delta$ implies that $\rho_{-\tau}U \subset V \subset \rho_\tau(U)$. This implies $e^{-\tau}\gamma(U)\leq\gamma(V)\leq e^{\tau}\gamma(U)$ and choosing $\tau$ such that $(e^{\tau}-1)\gamma(U)\leq\varepsilon$ (hence $(1-e^{-\tau})\gamma(U)>\varepsilon$) proves our claim.

□

A last useful tool for computations is the

PROPOSITION 7.32 (Displacement inequality). *Let $\psi$ in $Ham_0(T^*N)$ such that $\psi(U)\cap U=\emptyset$. Then*

$$c(U)\leq\gamma(\psi)$$

*As a result*

$$c(U)\leq\gamma(U)<+\infty$$

PROOF. Let us consider a Hamiltonian $H$ supported in $U$ and $\varphi_t$ be its flow. Then the fixed points of $\psi\circ\varphi_t$ are fixed points of $\psi$, since $\psi\varphi_t(x)=x$ implies $x$ does not belong to $U$ (otherwise $y=\varphi_t(x)\in U$ and then $x\in U\cap\psi(U)$). According to the representation theorem, $c_{\pm}(\psi\circ\varphi_t)=\ell_{\psi\varphi_t}(x,*)$ for $x$ a fixed point of $\psi\circ\varphi_t$, that is a fixed point of $\psi$ and as usual $*$ is a point outside the support of $\psi$. Since the set of $\ell_{\psi\varphi_t}(x,*)$ does not depend on $t$, and the set has measure 0, and depends continuously on $t$, it must be constant. So we get $c_+(\psi\varphi_1)=c_+(\psi)$ hence

$$c(\varphi_1)\leq c_+(\psi\varphi_1)+c_+(\psi^{-1})=c_+(\psi)+c_+(\psi^{-1})=\gamma(\psi)$$

□

## 3. Applications

The previous theorem allows us to prove (Conjecture 7 from Chapter 3)

PROPOSITION 7.33 (Weinstein's conjecture in $\mathbb{R}^{2n}$, see [Vit87a]). *Let $\Sigma$ be a contact type compact hypersurface bounding a domain $U$. Then there exists a closed characteristic $\gamma$ of $\Sigma$ such $c(U)$ equals the action of the characteristic, that is $\int_\gamma\lambda$ where $\lambda=i_X\omega$.*

PROOF. Let $H$ be a Hamiltonian vanishing on $U$ and, identifying a neighborhood of $\partial U$ to $(\Sigma\times[-\varepsilon,\varepsilon[,d(e^t\alpha))$, as was proved in Proposition 3.75. Set $H=h(t)$ with $h=0$ for $t\leq 0$ and $h(t)=a$ for $t\geq\varepsilon/2$. We choose $a>\gamma(U_\varepsilon)$ where $U_\varepsilon=U\cup\Sigma\times[0,\varepsilon[$. Then the periodic orbits of the flow of $H$ are

(1) The constants inside $U$ with action 0
(2) The constants outside $U$ with action $a$
(3) Closed characteristics on some $\Sigma_\varepsilon=\Sigma\times\{\varepsilon\}$. But since $\Sigma_\varepsilon$ is conformally equivalent to $\Sigma$, they have the same closed characteristics.

We must check that we cannot have only constant periodic trajectories. But if this was the case, we would have $c_+(H)\in\{0,a\}$. But the first case is impossible since the flow of $H$ is not the identity, and the second case also since $c(H)\leq c(U_\varepsilon)<a$. Therefore we must have a closed characteristic on some $\Sigma_t$ hence on $\Sigma$. □

Note that the only requirement for the proof is that $\gamma(U)$ is finite. Thus a similar proof yields

PROPOSITION 7.34. *Let $\Sigma$ be a contact type compact hypersurface bounding a compact domain $U$ in $T^*P$ where $P$ is a non-closed manifold. Then there exists a closed characteristic $\gamma$ of $\Sigma$ such $\widehat{c}(U)$ equals the action of the characteristic, that is $\int_\gamma \lambda$ where $\lambda = i_X\omega$.*

One has to be careful that $N$ not being closed, we have not defined $c_+, c_-$. But we may assume $N$ is a manifold with boundary (since $U$ is bounded) and replace $N$ by its double[2] $N$. In the sequel $c_+, c_-$ refer to the $c_+, c_-$ of the obvious extensions of the Hamiltonians to $T^*N$. Similarly $\widehat{c}_\pm$ refer to the extensions to $T^*N$.

We first need to prove the following weak form of the representation theorem (Theorem 7.9)

PROPOSITION 7.35. *Let $L_k$ be a sequence in $\mathscr{L}(T^*N)$, where $N$ is a closed manifold, such that $\lim_k c_+(\psi(L_k), L_k) = \widehat{c}_+(\psi)$. Then given any neighbourhood $U$ of the set of fixed points of $\psi$ with action $\widehat{c}_+(\psi)$, we have for $k$ large enough $L_k \cap U \neq \emptyset$.*

LEMMA 7.36. *Let $L_1, L_2$ be two exact Lagrangians in an exact Liouville manifold $(M, d\lambda)$. Let $B(z,r)$ be a symplectic ball and assume $L_1 \cap L_2 \cap B(z,r) = \{z\}$, and $T_zL_1 \cap T_zL_2 = \{0\}$. Then there is a symplectic map $\psi : B(z, r/2) \longrightarrow B(z,r)$ such that*

$$\psi(L_1 \cap B(z, r/2)) = \mathbb{R}^n \times \{0\} \cap B(z, r/2)$$

*and*

$$\psi(L_2 \cap B(z, r/2)) = \{0\} \times \mathbb{R}^n \cap B(z, r/2)$$

PROOF.

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Let $\lambda_0 = i_\rho\omega$ be the standard Liouville form on the ball, so that $\rho$ is the radial vector field. Then if $\lambda$ is the standard Liouville form, we have in $B(z,r)$ that $\lambda = \lambda_0 + df$ for some function $f$. So cutting off $f$, we may assume we have a Liouville vector field such that $X = \rho$ in $B(z,r)$. Note also that if $\psi^t$ is the flow of $X$, then $\psi^t(L)$ is an exact Lagrangian isotopy, hence induced by a Hamiltonian isotopy. Now $\psi^t(L) \cap B(z,r) = \psi^t(L \cap \psi^{-t}(B(z,r))) = \psi^t(L \cap B(z, e^{-t}r)) = e^t(L \cap B(z, e^{-t}r))$. Clearly as $t$ goes to infinity, this converges to $T_zL$. Since $Sp(2n)$ acts transitively on the set of pairs of Lagrangians, we may send $T_zL_1, T_zL_2$ on the standard Lagrangians.

□

PROOF. We argue by contradiction. Note that $c_+(\psi(L), L) = f_{\psi(L)}(z_1) - f_L(z_1)$ where $z_1 \in \psi(L) \cap L$. Write $z_1 = \psi(z_0)$, so that

$$f_{\psi(L)}(z_1) = f_L(z_0) + \int_0^1 p\dot{q} - H(t, q(t), p(t))dt$$

[2] The double of a manifold $P$ with boundary $\partial P$ is the manifold $P \cup_{\partial P} \overline{P}$ where $\overline{P}$ is $P$ with the opposite orientation.

where $(q,p):[0,1]\longrightarrow T^*N$ is the trajectory of $X_H$ form $z_0=(q(0),p(0))$ to $z_1=(q(1),p(1))$. By a small $C^\infty$ perturbation of $L$ (or of $H$), we may assume $L$ is transverse to $\psi(L)$. Then

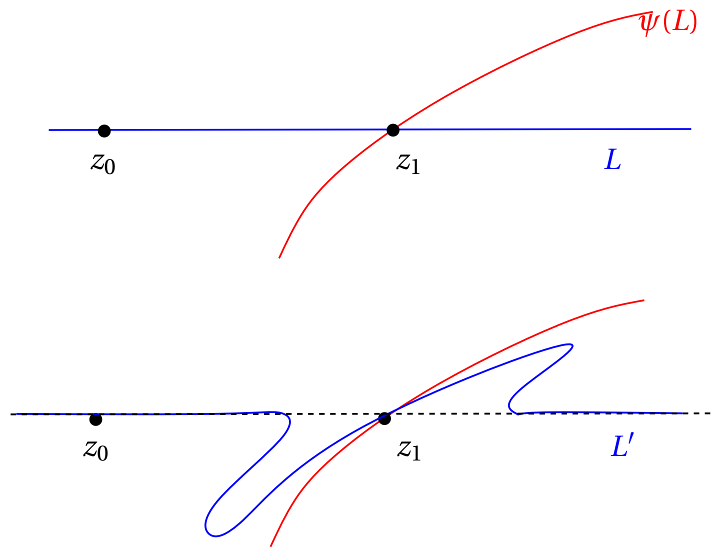


FIGURE 1. The sets $L, \psi(L)$ and $L'$

□

Finally the same proof as above works if we can prove $\widehat{\gamma}(U)<+\infty$. For this it is enough that $U$ be displaceable, i.e. there exists $H$ such that $\varphi_H(U)\cap U=\emptyset$. So we need

LEMMA 7.37. *Let $U$ be a bounded set in the cotangent bundle of a non-closed manifold. Then there exists a Hamiltonian isotopy $\psi^t$ such that $\psi^1(U)\cap U=\emptyset$*

PROOF. Now let $f$ be a function on $N$ without critical points on $N$. Then applying the Hamiltonian flow of $K(q,p)=f(q)$ we have $\varphi_K^t(q,p)=(q,p+tdf(q))$. If $C=\min\{|df(q)|\mid q\in W\}$ where $W$ is the projection of $U$ on $N$, and $TC>2\sup\{|p|\mid (q,p)\in U\}$, then $\varphi_K^T(U)\cap U=\emptyset$. We may thus take $\psi^t=\varphi_K^{tT}$

□

We may now conclude

PROOF OF PROPOSITION 7.34. The argument is the same as in Proposition 7.32. So it is enough to find a bound depending only on $U=\mathrm{supp}(\psi)$ (and not on $L$) for $c_+(\psi(L),L)$. Let $\psi$ such that $\psi(U)\cap U=\emptyset$ as provided by the above Lemma. Then $\psi(L)\cap\varphi^t(L)$ does not depend on $t$, since if $\psi(x)=\varphi^{-t}(y)$ with $x,y\in L$. But then $y\notin U$, otherwise we would have $\varphi^{-t}(y)\in U$ and this is impossible since $\psi(U)\cap U=\emptyset$. So

$y \notin U$ and $x \in L' = \psi^{-1}(L \cap (T^*N \setminus U))$ and $\psi(L) \cap \varphi^{-t}(L) = \psi(L') \cap L'$. As a result $c_+(L, \psi(\varphi^1(L))) = c_+(L, \psi(L)) \leq T\|K\|_{C^0}$.

Now let $H$ be a Hamiltonian associated to $U$ as in the proof of Proposition 7.33. Then $\widehat{c}_+(\varphi_H)$ is bounded and according to Proposition 7.35 there is a fixed point of $\varphi_H$. This orbit will be on $\Sigma_\varepsilon$ provided we can prove $\widehat{c}_+(\varphi_H) \notin \{0, a\}$ where $a = \sup_{\{} H(z) \mid z \in T^*N\}$. But $a$ can be chosen arbitrarily large, while we know that $\widehat{c}_+(\varphi_H)$ is bounded by a quantity depending only on $U$, so we only need to prove $\widehat{c}_+(\varphi_H) > 0$. But if this was not the case we would have $c_+(\varphi_H^1(L), L) \leq 0$ for all $L$. On the other hand $c_-(\varphi_H^1(L), L) \geq 0$ for all $L$ by Corollary 7.20. But this implies $\gamma(\varphi_H(L), L) = 0$ for all $L$ and $\varphi_H = \mathrm{Id}$ which is impossible. □

The Weinstein conjecture has been proved in a number of cases for other ambient symplectic manifolds (see [Vit87a; HV88; HV92] and also [Vit99] for the case of cotangent bundles of simply connected manifolds), and remarkably for all contact manifolds of dimension 3 (see [Hof93b; Tau07] and counterexamples have been found if the contact-type condition is omitted (see [Gin95; Gin97; Her99] ).

Another nice application is the following, originally proved using holomorphic curves

PROPOSITION 7.38 (Gromov's non-squeezing theorem ([Gro85]). *In* $(\mathbb{R}^{2n}, \omega)$ *we have*

$$\gamma(B^{2n}(r)) = \pi r^2 = \gamma(B^2(r) \times \mathbb{R}^{2n-2})$$

*As a result if there is a symplectic embedding from* $B^{2n}(r)$ *into* $B^2(R) \times \mathbb{R}^{2n-2}$ *then* $r \leq R$.

PROOF. It is enough to prove that $c(B^2(r)) = \gamma(B^2(r)) = \pi r^2$. Indeed $\gamma(B^2(r) \times \mathbb{R}^{2n-2}) \leq \gamma(B^2(r))$ because $\gamma(\varphi \times \mathrm{Id}) = \gamma(\varphi)$. Now for all positive $\varepsilon$, we must find $\varphi$ supported in $B^2(r)$ such that $c(\varphi) \geq \pi r^2 - \varepsilon$. But the Hamiltonian $H(q,p) = h(q^2 + p^2)$ such that $h = 0$ in $[0, r - \delta]$, $0 \leq h(t) \leq \varepsilon, h'(t) \geq 0$ on $[0, r]$ and $h'(r) > 1$ satisfies this inequality since the only non-trivial 1-periodic orbits are the circles of radius $\rho$ such that $h'(\rho) \in \pi\mathbb{Z}$, and their action is a multiple of $\pi\rho^2 - h(\rho)$ so $\geq \pi r^2 - 2\varepsilon$. □

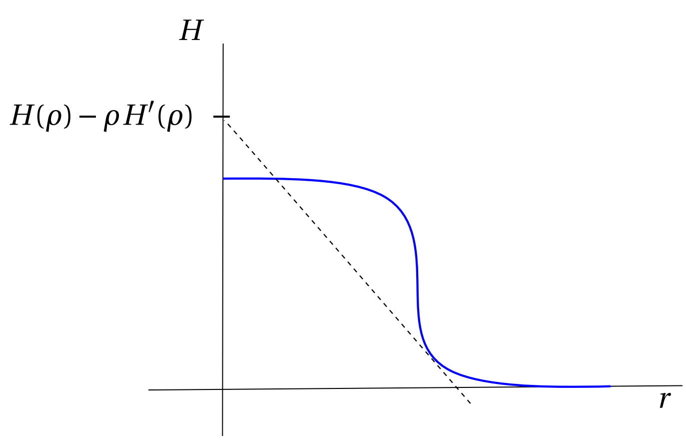


FIGURE 2. The Hamiltonian $H$

While the properties of $\gamma$ mentioned in Proposition 7.28 are shared by $U \mapsto \mathrm{vol}(U)^{2/n}$, this is not the case for this last property, since $\mathrm{vol}(B^2(r) \times \mathbb{R}^{2n-2}) = +\infty$.

COROLLARY 7.39. *Let $E(r_1, ..., r_n)$ be the ellipsoid in $\mathbb{R}^{2n}$ defined as*

$$E(r_1, ..., r_n) = \left\{ (q_1, p_1, ..., q_n, p_n) \mid \sum_{j=1}^{n} \frac{1}{r_j^2}(q_j^2 + p_j^2) \right\}$$

*with $r_1 \le r_2 \le .. \le r_n$. Then $\gamma(E(r_1, ..., r_n)) = \pi r_1^2$.*

PROOF. Obvious, since $B(r_1) \subset E(r_1, ..., r_n) \subset D^2(r_1) \times \mathbb{R}^{2n-2}$ and $\gamma(B(r_1)) = \gamma(D^2(r_1) \times \mathbb{R}^{2n-2}) = \pi r_1^2$, we then conclude by monotonicity of $\gamma$. □

Functions satisfying the conclusions of Propositions 7.28 and 7.38 are called symplectic capacities[3].

DEFINITION 7.40. *Let $c : \mathscr{P}(M, \omega) \longrightarrow \mathbb{R}_+$ be a function defined in the subsets of a symplectic manifold. We say that $c$ is a capacity if it satisfies the following properties*

*(1) The functions $c$ is symplectically invariant : if $\psi$ is a conformally symplectic map i.e. $\psi^*(\omega) = k\omega$, then $c(\psi(U)) = kc(U)$ for any set $U$.*
*(2) $c$ is monotone : if $V \subset U$ then $c(U) \le c(V)$ and $\gamma(V) \subset \gamma(U)$.*
*(3) $c$ is normalized: $c(B^{2n}(r)) = \pi r^2 = c(B^2(r) \times B^{2n-2}(R))$ for any $R \ge r$.*

We slightly changed the third property, because we do not want to assume that there is an embedding of $B^2(r) \times \mathbb{R}^{2n-2}$ into $(M, \omega)$ (even though this is usually the case for cotangent bundles at least for small $r$). Since $B^{2n}(r) \subset B^2(r) \times B^{2n-2}(r)$ we know already that $c(B^2(r) \times B^{2n-2}(r)) \ge \pi r^2$, so only the upper bound is useful. There are several symplectic capacities, defined by using different methods. Another consequence of the existence of the capacities is the following

THEOREM 7.41 (Gromov-Eliashberg's theorem). *Let $\psi_n$ be a sequence of symplectic maps of the symplectic manifold $(M, \omega)$. Assume there is a map $\psi$ such that $\psi = \lim_n \psi_n$, where the limit is for the $C^0$-topology (uniformly on compact sets). If $\psi$ is $C^1$ then $\psi$ is a symplectomorphism.*

PROOF. Assume $\psi$ is neither symplectic nor anti-symplectic, so for some $x \in M$ the same holds for $d\psi(x)$. Then taking Darboux charts at $x$ and $\psi(x)$, we can assume $\psi$ is a map from an open set of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ such that $\psi(0) = 0$ and assume $d\psi(0)$ is neither symplectic nor anti-symplectic. According to Exercise 31 of Chapter 2, we can then find an ellipsoid $E = E(r_1, \dots r_n)$ such that $\gamma(d\psi(0)E) \neq \gamma(E)$. Up to rescaling we may assume $\gamma(E) = 1$ and $\gamma(d\psi(0)E) = c \neq 1$.

Since $d\psi(0)(E)$ is convex hence of restricted contact type, the capacity-regularity of contact boundary subsets (see Proposition 7.31) implies that for each $\eta > 0$ there exists $\delta > 0$ such that $d_H(U, d\psi(0)E) < \delta$ implies

$$|\gamma(U) - \gamma(d\psi(0)E)| < \eta$$

[3] One should be careful as this word has a meaning in harmonic analysis

Take $\delta$ to correspond to $\eta = \frac{|c-1|}{2}$. Since $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi(\varepsilon x)$ $C^0$-converges to $d\psi(0)x$ as $\varepsilon$ goes to 0, we have

$$d_H(\psi_\varepsilon(E), d\psi(0)E) \leq \frac{\delta}{3}$$

for $0 < \varepsilon \leq \varepsilon_0$.

On the other hand since $\psi_n$ $C^0$ converges to $\psi$, for each $\varepsilon$ we have $d_H(\frac{1}{\varepsilon}\psi_n(\varepsilon E)), \psi_\varepsilon(E))$ converges to 0 as $n$ goes to $+\infty$ so for $n \geq N_0$ we have

$$d_H(\frac{1}{\varepsilon_0}\psi_n(\varepsilon_0 E), \psi_{\varepsilon_0}(E)) < \frac{\delta}{3}$$

As a result we have

$$d_H((\frac{1}{\varepsilon_0}\psi_n(\varepsilon_0 E), d\psi(0)E) \leq \frac{2\delta}{3}$$

hence

$$|\gamma(\frac{1}{\varepsilon_0}\psi_n(\varepsilon_0 E)) - \gamma(d\psi(0)E)| < \frac{|c-1|}{2}$$

but since $\psi_n$ is symplectic, $\gamma(\frac{1}{\varepsilon_0}\psi_n(\varepsilon_0 E)) = \gamma(E)$ and we get

$$|\gamma(E) - \gamma(d\psi(0)E)| < \frac{|c-1|}{2}$$

which is impossible since $\gamma(E) = 1$ and $\gamma(d\psi(0)E) = c$. □

REMARKS 7.42. (1) Note that the theorem is local : the proof also implies that if the limit $\psi$ exists and is smooth only in an open domain $U$, then the limit is symplectic in $U$. This is an important point, since many other symplectic rigidity statements are global.

(2) The original Gromov-Eliashberg result only claimed that if $\varphi$ is a diffeomorphism, then it is symplectic. The above proof only requires that $\varphi$ is smooth and show that $\varphi$ must be a local diffeomorphism. If the manifold is closed (or if the $\varphi_n$ preserve the boundary) it follows that $\varphi$ is a global diffeomorphism by a degree theory argument.

(3) Assuming the $\psi_n$ are Hamiltonians can we conclude $\psi$ is Hamiltonian ? This is called the $C^0$-flux conjecture One can also prove that a $C^0$ limit of maps preserving the capacities of domains with contact-type boundary must preserve the capacities of domains with contact-type boundary. We do not need to assume the maps are homeomorphisms, and in fact one we shall give examples of such maps which are not (see Exercise 13). However all known examples are invertible outside of a set of zero capacity.

These results are at the origin of $C^0$-symplectic topology.

A natural question is whether $\gamma$ could be compared with the $C^0$ distance. This was known for $\mathbb{R}^{2n}$ (see [Vit92]) the proof is easily adapted to any cotangent bundle of a non-compact manifold, for surfaces [Sey13a], and finally proved in full generality[4] by Buhovski, Seyfaddini, Humilière [BHS21].

A consequence of the above is

PROPOSITION 7.43 ($C^0$-continuity of the capacity). *Let $U \subset T^*N$ be a bounded set. For each positive $\varepsilon$ there exists a positive $\delta$ such that for any $\varphi_H^1 \in Ham_0(T^*N)$, with $H$ supported in $U$ we have*

$$d_{C^0}(\varphi, \mathrm{Id}) < \delta \Longrightarrow \gamma(\varphi) < \varepsilon$$

*In other words $\gamma$ is continuous for the $C^0$ topology.*

PROOF. Note that due to the formulation of the Proposition, there is no ned to assume $N$ is closed (since $U$ is assumed to be bounded anyway). The proof has two parts. In the first one, we prove the result assuming $\varphi$ is the identity on some open ball $B$, this is called the $\varepsilon$-trick.

LEMMA 7.44. *For any bounded set $U \subset T^*N$ there exists $C_U$ and $\varphi \in Ham_0(T^*N)$ such that*

*(1)* $\mathrm{supp}(\varphi) \subset U$
*(2)* $d(x, \varphi(x)) \geq \varepsilon$ *for $x \in V$, where $V \subset \overline{V} \subset U$*
*(3)* $|\ell_\varphi(x)| \leq C_U \varepsilon$ *for each fixed point $x$ of $\varphi$.*

PROOF. Let us assume $U$ is contained in $B_R(W)$ for $W$ a bounded set in $N$. Then let $X$ be a unit vector field[5] on $W$ and $H_\varepsilon(x,p) = 2\varepsilon\chi(x,p)\langle p, X(x)\rangle$ where $\chi = 1$ in a neighborhood of $B_R(W)$ and $\|D\chi\| \leq 1$. On $U$, the flow $\varphi_\varepsilon^t = \varphi_{H_\varepsilon}$ satisfies $d(z, \varphi_\varepsilon^1(z)) \simeq 2\varepsilon + o(\varepsilon)$ as $\varepsilon$ goes to 0. We shall assume $\varepsilon$ small enough so that $d(z, \varphi_\varepsilon(z)) \geq \varepsilon$. For $\varepsilon$ small enough, $H_\varepsilon$ satisfies the assumptions of Corollary 3.96 hence has no non-constant periodic orbits. Therefore for any periodic point of $H_\varepsilon$, we have $|\ell_{\varphi_\varepsilon}(x)| = |H_\varepsilon(x)| \leq C_{R,W}\varepsilon$ □

Let us choose $\varphi_\varepsilon$ as in the lemma. We have from (3) that $\gamma(\varphi_\varepsilon) \leq \varepsilon$. Now $\varphi_\varepsilon\psi$ has no fixed points inside $U$ since in this region $d(z, \varphi_\varepsilon(z)) \geq \varepsilon$ while $d(z, \psi(z)) < \varepsilon$. Thus the fixed points of $\varphi_\varepsilon\psi$ are the fixed points of $\varphi_\varepsilon$ and we get that $c_\pm(\varphi_\varepsilon\psi) = c_\pm(\varphi_\varepsilon)$ hence $c_+(\psi) \leq c_+(\varphi_\varepsilon\psi) + c_+(\varphi_\varepsilon^{-1}) = c_+(\varphi_\varepsilon) - c_-(\varphi_\varepsilon) = \gamma(\varphi_\varepsilon) \leq 2C\varepsilon$. Similarly $-c_-(\psi) \leq 2C\varepsilon$ and we finally conclude $\gamma(\psi) \leq 4C\varepsilon$. □

Finally we have

PROPOSITION 7.45. *Let $(H_k)_{k\geq 1}$ be a sequence of Hamiltonians on $T^*N$ and $(\varphi_k)_{k\geq 1}$ the sequence of their time-one flows.*

*(1) If $H_k \xrightarrow{C^0} H$ and $H$ is smooth, then $\varphi_k$ $\gamma$-converges to $\varphi$ the time-one flow of $H$.*

[4] Using Floer homology, on a general aspherical symplectic manifold.

[5] This always exists on a non-closed manifold. Of course we do not assume any behavior on the boundary.

(2) *If* $H_k \xrightarrow{C^0} H$ *and* $\varphi_k \xrightarrow{C^0} \varphi$ *where both* $H$ *and* $\varphi$ *are smooth, then* $\varphi$ *is the time-one flow of* $H$.

PROOF. It is enough to deal with the case $H = 0$. Indeed, let $\psi$ be the time-one flow of $H$. Then $\psi^{-1}\varphi_n$ is the time-one flow of $K_n(t,z) = H_n(t,\psi^t(z)) - H(t,\psi^t(z))$. Since $K_n$ converges to 0 in the $C^0$ topology, it is enough to prove that $\psi^{-1}\varphi_n$ $C^0$-converges to Id. But since $\gamma(\varphi_k) \leq \|H_k\|_{C^0}$ by Proposition 7.18, we have $\varphi_k \xrightarrow{\gamma} \mathrm{Id}$. Together with $\varphi_k \xrightarrow{C^0} \varphi$ and Proposition 7.43 this implies that $\varphi = \mathrm{Id}$. ☐

**3.1. Stability of special submanifolds.** The above theorem can be extended to the following

THEOREM 7.46 (see [LS94]). *Let* $(L_k)_{k\geq 1}$ *be a sequence of exact Lagrangian submanifolds. Then if* $(L_k)_{k\geq 1}$ *converges for the Hausdorff metric to the smooth manifold* $L_\infty$*, then* $L_\infty$ *is an exact Lagrangian.*

To be rewritten

LEMMA 7.47. *Let us assume the Euler characteristic of* $V$ *is zero and there is a point* $x_0$ *where* $V$ *is not Lagrangian (i.e.* $T_{x_0}L$ *is not Lagrangian). Then there exists a Hamiltonian flow* $\varphi_H^t$ *such that* $\varphi^t(V) \cap V = \emptyset$ *for* $t$ *small enough.*

PROOF. We consider the vector bundle $T_xV^\omega$ which is isomorphic to $TV$ and has dimension $n$. It will be enough to find a vector field $\xi$ near $V$ such that $\xi$ is a non-vanishing section of $TV^\omega$ and has no orbit completely contained in $V$. Let us first consider the set $W \subset U$ such that $T_xV^\omega$ is close to $T_xV$. We can assume that this is an open set with smooth boundary. Let $F$ be a smooth function on $W$ without critical points. Then we can lift $\nabla F$ to $T_xV^\omega$, to a vector field $\xi$ without zeros, and without recurrence, since $dF(x)\xi(x) < 0$. Now let us consider $U = V \setminus W$. By assumption we have a vector field $\xi$ on $\partial U$ and any extension of $\xi$ to $U$ has the sum of its indexes equal to zero (by considering the projection on $V$ of the vector field and applying Poincaré-Hopf, since $\chi(V) = 0$). We can thus extend $\xi$ to a vector field without zeros. This vector field could now have some orbit contained in $V$, but such an orbit must necessarily enter $U$. Now we cover $U$ by flow boxes $B_j$, so that any orbit completely contained in $V$ must go through one of these boxes. On each box let $\eta$ be a vector field such that $\eta(x) \in T_xV^\omega \setminus T_xV$ and perturb $\xi$ by adding a large multiple of $\eta_j$. Then the new vector field is not tangent to each trajectory through the flow box must exit $V$. But since all orbits remaining in $V$ must go through some $B_j$ we see that no trajectory remains in $V$.

We now apply Corollary 5.58 from Chapter 5 and get a Lyapounov function $H$ such that $dH(z)\xi(z) < 0$ on $V$. This implies that $X_H(z) \notin T_zV$ as this is equivalent, by duality, to $dH(z) \neq 0$ on $T_zV^\omega$. ☐

PROOF OF THEOREM 7.46. Assume first $\chi(V) = 0$ and let us argue by contradiction. Assume for $k$ large enough $L_k$ is an exact Lagrangian contained in a neighbourhood of

$V$. Let $H$ be a Hamiltonian vector field such that $\varphi_H(V)\cap V=\emptyset$. Then $\varphi_H(L_k)\cap L_k=\emptyset$ but this contradicts the proof of the Arnold conjecture. For the general case, just replace $L_k\in T^*N$ by $L_k\times 0_{S^1}\subset T^*(N\times S^1)$ which will Hausdorff converge to $V\times 0_{S^1}$. Since $\chi(V\times S^1)=0$ we conclude that $L\times 0_{S^1}$ must be Lagrangian and so must $V$. □

**3.2. Orderability of the symplectic group.** Let us consider Hamiltonian maps in $\mathrm{DHam_c}(\mathbb{R}^{2n})$, where the Hamiltonian is supported in the unit disc. Since we may normalize the Hamiltonian to vanish outside a compact set, we may define $c\pm(\varphi_H)$, which does not depend on the choice of $H$.

DEFINITION 7.48. *We say that $\varphi_H$ is positive and write* $\mathrm{Id}\prec\varphi_H$ *if* $c_-(\varphi_H)=0$.

PROPOSITION 7.49. *We have*

*(1) If $H\geq 0$ then* $\mathrm{Id}\prec\varphi_H$
*(2) If $\varphi\prec\psi$ and $\psi\prec\varphi$ then $\varphi=\psi$*
*(3) If* $\mathrm{Id}\prec\varphi$ *and* $\mathrm{Id}\prec\psi$ *then* $\mathrm{Id}\prec\varphi\circ\psi$

## 4. The Poisson bracket approach

Let us now state a Poisson bracket approach to symplectic rigidity:

THEOREM 7.50 ($C^0$-closedness of Poisson brackets. See [CV08]). *Let $(F_k)_{k\geq 1}$ and $(G_k)_{k\geq 1}$ be two sequences of functions with fixed bounded compact support. Assume*

*(1) $C^0-\lim_k F_k=F, C^0-\lim_k G_k=G$*
*(2) $C^0-\lim_k\{F_k,G_k\}=0$*
*(3) $F,G$ are smooth*

*Then* $\{F,G\}=0$

PROOF. Let $\varphi_k^t$ be the flow associated to $F_k$ and $\psi_k^t$ associated to $G_k$. Consider the isotopy

$$t\mapsto\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s}$$

It is generated for given $s$, by the Hamiltonian

$$L_s^k(t,z)=F_k(z)-F_k(\psi_k^{-s}\varphi_k^{-t}(z))$$

Indeed

$$dL_s^k(t,z)=dF_k(z)-dF_k(\psi_k^{-s}\varphi_k^{-t}(z))\,d\psi_k^{-s}(\varphi_k^{-t}(z))\circ d\varphi_k^{-t}(z)$$

and using Lemma 3.37 we get

$$X_{L_s^k}(t,z)=X_{F_k}(z)-d\varphi_k^t(\psi_k^s(z))\circ d\psi_k^s(z))X_{F_k}(\psi_k^{-s}\varphi_k^{-t}(z))$$

Now

$$\frac{d}{dt}\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s}(u) = \left(\frac{d}{dt}\varphi_k^t\right)(\psi_k^s\varphi_k^{-t}\psi_k^{-s}(u)) +$$
$$(d\varphi_k^t)(\psi_k^s\varphi_k^{-t}\psi_k^{-s}(u)) \circ (d\psi_k^s)(\varphi_k^{-t}\psi_k^{-s}(u))\left(\frac{d}{dt}\varphi_k^{-t}\right)(\psi_k^{-s}(u)) =$$
$$X_{F_k}(\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s}(u)) - d\varphi_k^t(\psi_k^s(u)) \circ d\psi_k^s(u))X_{F_k}(\psi_k^{-s}\varphi_k^{-t}(\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s}(u)))$$

and setting $z = \varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s}(u)$ we recover the above formula.

Now $L_0^k = 0$ so

$$L_s^k(z) = \int_0^s \frac{d}{ds}L_s^k(z)ds = -\int_0^s dF_k(\psi_k^{-s}\varphi_k^{-t}(z))\frac{d}{ds}\psi_k^{-s}(\varphi_k^{-t}(z))ds =$$
$$\int_0^s dF_k(\psi_k^{-s}\varphi_k^{-t}(z))X_{G_k}(\psi_k^{-s}(\varphi_k^{-t}(z))ds = \int_0^s \{F_k, G_k\}(\psi_k^{-s}(\varphi_k^{-t}(z))ds$$

Therefore for $s \geq 0$

$$\|L_s^k(z)\| \leq s\|\{F_k, G_k\}\|_{C^0}$$

and this implies that as $k$ goes to infinity $L_s^k$ converges to 0. It then follows from Proposition 7.18 that for $s, t \geq 0$

$$\gamma(\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s}) \leq st\|\{F_k, G_k\}\|_{C^0}$$

hence $\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s} \xrightarrow{\gamma} \mathrm{Id}$. On the other hand, since $F_k$ $C^0$-converges to $F$, Proposition 7.45 (1) implies that $\varphi_k^t$ $\gamma$-converges to $\varphi^t$, the flow of $X_F$. Similarly $\psi_k^s$ $\gamma$-converges to $\psi^s$, and we thus conclude $\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s}$ $\gamma$-converges to $\varphi^t\psi^s\varphi^{-t}\psi^{-s}$ and by uniqueness of the $\gamma$-limit we get

$$\varphi_k^t\psi_k^s\varphi_k^{-t}\psi_k^{-s} = \mathrm{Id}$$

Since this holds for all $s, t$, we get $\{F, G\} = 0$, using Proposition 3.54 (2)

□

REMARK 7.51. (1) The theorem is non-obvious, since $\{F, G\}$ involves the first derivatives of $F, G$, but our assumptions are only of $C^0$-convergence. However if the sequence $F_n$ is constant, then the theorem becomes obvious since we may write the condition $\{F, G_n\} = 0$ as $dG_n(X_F) = 0$ : but this is equivalent to $G_n(\varphi_F^s(z)) - G_n(z) = 0$ for all $s \in \mathbb{R}$.

(2) The theorem extends to "pseudo-representations of finite-dimensional Lie algebras" that is sequences of maps $\rho_n : \mathfrak{g} \longrightarrow C_0^\infty(M, \mathbb{R})$ such that for each $X \in \mathfrak{g}$ $C^0 - \lim \rho_n(X) = \rho(X)$. Then if the image of $\rho$ is contained in $C_0^\infty$, Humilière proved that $\rho$ is a representation of $\mathfrak{g}$ in $C_0^\infty(M, \mathbb{R})$ (see [Hum09]). The theorem actually implies Gromov-Eliashberg's theorem, at least a version for compact supported symplectic diffeomorphisms (see [Hum09], Appendix A).

(3) The analogue of the theorem becomes false if we replace the assumption $\{F_k, G_k\} = 0$ by $\{F_k, G_k\} = H$ and try to conclude that $\{F_k, G_k\} = H$ as the following exercise shows.

EXERCISE 7.52. On $T^*T^1$, take $F_n = \frac{\chi(p)}{\sqrt{n}}\cos(nq), G_n = \frac{\chi(p)}{\sqrt{n}}\sin(nq)$ and show that $F_n, G_n$ $C^0$-converge to 0 while $\{F_n, G_n\}$ does not.

We may now give a new proof of Theorem 7.41

SECOND PROOF OF THEOREM 7.41. First of all let us place ourselves in a Darboux chart and let $(r_j, \theta_j)$ be cylindrical coordinates, so that

$$\lambda = \frac{1}{2}\sum_{j=1}^n r_j^2 d\theta_j$$

and $\omega = \sum_{j=1}^n r_j dr_j \wedge d\theta_j$ with $r_j \in [0, \rho_0]$ and $\theta_j \in \mathbb{S}^1$. We set $R = (r_1, .., r_n) \in \mathbb{R}_+^n, \Theta = (\theta_1, ..., \theta_n) \in \mathbb{T}^n$.

Let now for $1 \leq j \leq n$ $\psi_j(r_1, \theta_1, ..., r_n, \theta_n) = \chi_j(r_1, ..., r_n) = \chi_j(R)$ be compact supported functions such that their differentials are linearly independent at some point $R_0$ and setting $\psi = (\psi_1, ..., \psi_n)$ we have that for $c = (\chi_1(R_0), ..., \chi_n(R_0))$ that the set $\psi^{-1}(c)$ is a Lagrangian submanifold $L$ near $(R_0, \Theta_0)$. Clearly by composing $\psi$ with a local symplectic map, we can arrange that $T_{(R_0,\Theta_0)}L$ is any Lagrange linear subspace of $T_{(R_0,\Theta_0)}M$. Let $(\Phi_n)_{n\geq 1}$ be a sequence of symplectic maps converging to $\Phi$. Then the $\psi_j$ obviously commute and so do the $\psi_j \circ \Phi_n$. If $\Phi$ is smooth, then by Theorem 7.50 the $\psi_j \circ \Phi$ also commute. As a result near $(R_0, \Theta_0)$ the set $\psi^{-1}(c)$ is a Lagrangian, $L_c$, and since $(\psi \circ \Phi)^{-1}(c) = \Phi^{-1}(\psi^{-1}(c))$ and the $\psi_j \circ \Phi$ commute, we get that $\Phi^{-1}(L_c)$ is coisotropic. Since we can choose for $T_{(R_0,\Theta_0)}L$ any Lagrangian subspace and for $(R_0, \Theta_0)$ any point in $M$ we get that the preimage by $d\Phi(z)$ of any Lagrangian subspace of $T_zM$ is coisotropic. Now if $\varphi_n$ is a sequence of symplectic maps $C^0$-converging to $\varphi$, we apply the above argument to the sequence $\Phi_n : M \times \overline{M} \longrightarrow M \times \overline{M}$ given by $\Phi_n(u, v) = (u, \varphi_n(v))$ and $L_c = \Delta$ the diagonal in $M \times \overline{M}$. Then

$$\Phi^{-1}(\Delta) = \{(u, \varphi(v)) \mid u = \varphi(v), u, v \in M\} = \{(\varphi(v), v) \mid v \in M\}$$

and this is just the graph of $\varphi$, up to a permutation of the variables. If $\varphi$ is smooth, this graph is then coisotropic hence Lagrangian, since it has dimension $\dim(M)$, and we know that this implies that $\varphi$ is symplectic. □

REMARK 7.53. A different proof was presented in [Hum09], Appendix A (see also for a variant [CV13]), which however requires the $\varphi_n$ to be defined everywhere and to coincide with the identity outside of a fixed compact set.

## 5. Spectral invariants and symplectic reduction

Let $V \subset N$ be a submanifold and consider the coisotropic submanifold $T_V^*N = \{(q, p) \in T^*N \mid x \in V\}$. Let $S$ defined on $N \times E$ be a G.F.Q.I. for a Lagrangian $L$ in $T^*N$ and assume $L$ is transverse to $T_V^*N$. We can define the reduction of $L$ to $T^*V$, $L_V = \{(q, p) \in T^*V \mid \exists (q, p') \in L \cap T_V^*N, p'_{|T_qV} = p\}$.

PROPOSITION 7.54. *The Lagrangian $L_V$ has the G.F.Q.I. $S_V = S_{V\times E}$.*

PROOF. This follows immediately from the fact that $(q, \frac{\partial S}{\partial q}(q,\xi)) \in T_V^*N$ if and only if $q \in N$ and the restriction of $\frac{\partial S}{\partial q}(q,\xi)$ to $T_qV$ is the differential of the restriction, that is $\frac{\partial S}{\partial q}(q,\xi)_{|V\times E} = \frac{\partial S_V}{\partial q}(q,\xi)$. $\square$

The following will also turn out to be useful. Let $f : M \longrightarrow N$ be a smooth map. Let $L \in T^*N$ and consider the inverse image of $L$ by the correspondence

$$\Lambda_f = \{(x, p_x, y, p_y) \mid y = f(x), p_x = p_y \circ df(x)\}$$

We consider $\Lambda_f^{-1}(L)$ in $T^*M$ (see Chapter 4 Section 2.1). Then

PROPOSITION 7.55. *We have for $\beta \in H^*(N) \setminus \{0\}, \alpha \in H^*(M) \setminus \{0\}$, $f : M \longrightarrow N$ a smooth map and $S_f(x;\xi) = S(f(x);\xi)$*

$$c(\beta, S) \le c(f^*(\beta), S_f))$$
$$c(\alpha, S_f)) \le c(f^!(\alpha), S)$$

*In particular if $S$ is a G.F.Q.I. for $L \in \mathscr{L}(T^*N)$ and $\Lambda_f^{-1} \circ L \in \mathscr{L}(T^*M)$*

$$c(\beta, L) \le c(f^*(\beta), \Lambda_f^{-1}(L))$$
$$c(\alpha, \Lambda_f^{-1}(L)) \le c(f^!(\alpha), L)$$

*where $f^!$ is the transfer map (also called the Umkehr map) from $H^k(M)$ to $H^{k+\dim(M)-\dim(N)}(N)$.*

PROOF. Indeed, if $S$ is a G.F.Q.I. for $L$, then $S_f(x;\xi) = S(f(x);\xi)$ is a G.F.Q.I. for $\Lambda_f^{-1}(L)$. Then denoting $\tilde{f}(x,\xi) = (f(x),\xi)$, we have $S_f = S \circ \tilde{f}$ and $S_f^t = \{(x,\xi) \mid S_f(x,\xi) \le t\}$ is the image by $\tilde{f}$ of $S^t$.

So we may consider the map $\tilde{f}^* : H^*(S^b, S^a) \longrightarrow H^*(S_f^b, S_f^a)$ and the commutative diagram

$$\begin{tikzcd} H^*(S^c, S^a) \arrow[r] \arrow[d, "f^*"] & H^*(S^b, S^a) \arrow[d, "f^*"] \\ H^*(S_f^c, S_f^a) \arrow[r] & H^*(S_f^b, S_f^a) \end{tikzcd}$$

If $b < c(\beta, S)$, $\beta$ has image zero by the top horizontal arrow, hence $f^*(\beta)$ has also image 0 by the lower horizontal arrow and we get that $b \le c(f^*(\beta), S_f)$. This proves the first inequality.

Setting $E_Y = Y \times \mathbb{R}^k, E_X = X \times \mathbb{R}^k$ we get by Alexander duality ([Spa66], p.296, theorem 17)

$$H_q(S^t, S^{-\infty}) \simeq H^{n-q}(E_Y \setminus S^{-\infty}, E_Y \setminus S^t) = H^{n-q}(E_Y, (-S)^{-t})$$

and similarly

$$H_q(S_f^t, S_f^{-\infty}) \simeq H^{m-q}(E_Y \setminus S_f^{-\infty}, E_Y \setminus S_f^t) = H^{m-q}(E_Y, (-S_f)^{-t})$$

where $m = \dim(X) + k, n = \dim(Y) + k$. Thus the map

$$\tilde{f}_* : H_q(E_Y, (-S)^{-t}) \longrightarrow H_q(E_Y, (-S_f)^{-t})$$

induces a map

$$\tilde{f}^! : H^{m-q}(S^t, S^{-\infty}) \longrightarrow H^{n-q}(S_f^t, S_f^{-\infty})$$

Using now the following diagram (where $d = \dim(N) - \dim(M)$)

$$\begin{tikzcd} H^{*+d}(S^c, S^a) \arrow[r] & H^{*+d}(S^b, S^a) \\ H^*(S_f^c, S_f^a) \arrow[u, "\tilde{f}^!"] \arrow[r] & H^*(S_f^b, S_f^a) \arrow[u, "\tilde{f}^!"] \end{tikzcd}$$

we see that if $b < c(\alpha, S_f)$ then $b \le c(f^!(\alpha), S)$ and this proves the second statement. ☐

REMARKS 7.56. (1) The result holds for any $S$ that is quadratic at infinity.

(2) In our case the map $f^!$ is defined as follows. First denote by $E_X = X \times \mathbb{R}^k$ (resp. $E_Y = Y \times \mathbb{R}^k$) and by $\tilde{f} : X \times \mathbb{R}^k \longrightarrow Y \times \mathbb{R}^k$ the map $(x, \xi) \mapsto (f(x), \xi)$. Then note that $S_f = S \circ \tilde{f}$ so that $S^t$ is exactly $\tilde{f}(S_f^t)$, hence $E_Y \setminus S^t$ is exactly $\tilde{f}(E_X \setminus S_f^t)$. We thus have a map

$$\tilde{f}_* : H_*(E_X \setminus S_f^t, E_X \setminus S_f^a) \longrightarrow H_*(E_Y \setminus S^t, E_Y \setminus S^a)$$

By Alexander duality ([Spa66], theorem 10 p. 342) this induces a map

$$H^{m-j}(S_f^t, S_f^a) \longrightarrow H^{n-j}(S_f^t, S_f^a)$$

where $m = \dim(X), n = \dim(Y)$.

In particular we get

COROLLARY 7.57. *Let $f : M^m \longrightarrow N^n$ be map between connected compact manifolds*

*(1)*

$$c_-(L) \le c_-(\Lambda_f^{-1}(L))$$

*hence*

$$\gamma(\Lambda_f^{-1}(L)) \le \gamma(L)$$

*(2) If $f : M \longrightarrow N$ satisfies the following condition: $f^*(\mu_N) \neq 0$ in $H^*(M)$ then*

$$\gamma(\Lambda_f^{-1}(L)) = \gamma(L)$$

PROOF. For (1) we apply the first inequality of Proposition 7.55 to $\alpha = 1$ The inequality $c_+(L) \ge c_+(\Lambda_f^{-1}(L))$ follows by replacing $L$ by $\overline{L}$ (i.e. $S$ by $-S$). Substracting the two we get the second inequality. To get equality we must only prove the reverse inequality.

Now since $f^!(\alpha \cup f^*(\mu_N)) = f^!(\alpha) \cup \mu_N$, if $f^!(\alpha) = 1$, we have $f^!(\alpha \cup f^*(\mu_N)) = \mu_N$ so that $f^*(\mu_N) \neq 0$. Conversely if $f^*(\mu_N) \neq 0$ there is a class $\alpha \in H^d(M)$ such that $\alpha \cup$

$f^*(\mu_N) = \mu_M$ and then since $f^!(\alpha) \in H^0(N)$ for degree reasons and $f^!(\alpha \cup f^*(\mu_N)) = f^!(\mu_M) = \mu_N$ on one hand, and to $f^!(\alpha)\mu_N$ on the other. As a result $f^!(\alpha) = 1$. Thus by Proposition 7.55 we have $c(\alpha, \Lambda_f^{-1}(L)) \le c(1_N, L)$, but $c(1_M, \Lambda_f^{-1}(L)) \le c(\alpha, \Lambda_f^{-1}(L))$ by Lusternik-Shnirelman' Theorem (Theorem 5.39). As a result $c(1_M, \Lambda_f^{-1}(L)) \le c(1_N, L)$ i.e. $c_-(\Lambda_f^{-1}L) \le c_-(L)$ and this implies first $c_+(\Lambda_f^{-1}L) \ge c_+(L)$ and finally

$$\gamma(\Lambda_f^{-1}(L)) \ge \gamma(L)$$

$\square$

As a consequence since if $i_M : M \longrightarrow N$ is an inclusion then $(\Lambda i_M)^{-1}$ is the symplectic reduction, we have

COROLLARY 7.58 (Spectral inequality for reduction I). *We have for $\alpha \in H^*(V), \beta \in H^*(N)$ $S$ a G.F.Q.I. over $N$, and $i_V : V \longrightarrow N$ the inclusion*

$$c(\alpha, S) \le c(i_V^*(\alpha), S_V)$$
$$c(\beta, S_V) \le c(i_V^!(\beta), S)$$

*where $i_V^!$ is the transfer map. In particular if $S$ is the G.F.Q.I. for $L \in \mathscr{L}(T^*N)$ and its reduction $L_V$ belongs to $\mathscr{L}(T^*V)$ then*

$$c(\alpha, L) \le c(i_V^*(\alpha), L_V)$$
$$c(\beta, L_V) \le c(i_V^!(\beta), L)$$

A different situation is when $N = M \times P$, then $T^*N = T^*M \times T^*P$ and we can consider the coisotropic submanifold $Z_P = \{(x, p_x, y, p_y) \in T^*N \times T^*P \mid \eta = 0\}$. Then the space $Z_P^\omega$ is the foliation $T^*N \times \{p, 0)\}$ and the manifold $Z_P/(Z_P)^\omega$ is isomorphic to $T^*N$. The reduction of $L \subset T^*N$ by $Z_P$ is $L_{Z_P} = \{(x, p_x, y, \eta) \mid (x, p_x, y, 0) \in L\}/\simeq_P$ where

$$(x, p_x, y, 0) \simeq_P (x', p_x', y', 0) \Leftrightarrow x = x', p_x = p_x'$$

An important tool to relate the spectral invariants of $L, L_V, L_{Z_P}$ is the following.

PROPOSITION 7.59 (Spectral inequality for reduction II). *Let $S(x, y, \xi)$ be a G.F.Q.I. defined on the bundle $E$ over $M \times P$. We identify $H^*(M \times P)$ to $H^*(M) \otimes H^*(P)$ by the Künneth isomorphism. Let $\alpha \in H^*(M)$. Then*

$$c(\alpha \otimes 1_P, S) \le \inf_{p \in P} c(\alpha, S_p) \le \sup_{p \in P} c(\alpha, S_p) \le c(\alpha \otimes \mu_P, S)$$

PROOF. Consider the following diagram

$$\begin{tikzcd}
H^*(M\times P) \arrow[r, "T"] \arrow[d, "i_p^*"] & H^*(E^\infty, E^{-\infty}) \arrow[r] \arrow[d] & H^*(E^t, E^{-\infty}) \arrow[d] \\
H^*(M) \arrow[r, "T_p"] & H^*(E_p^\infty, E_p^{-\infty}) \arrow[r] & H^*(E_p^t, E_p^{-\infty})
\end{tikzcd}$$

where the vertical maps are induced by the injection $i_p : M \longrightarrow M\times\{p\}$ and corresponds via the Künneth isomorphism, to the projection $H^*(M)\otimes H^*(P)\longrightarrow H^*(M)\otimes H^0(P)$. Now if $t < c(\alpha\otimes 1, S)$ then the composition of the top horizontal maps sends $\alpha\otimes 1$ to a zero. This implies that the composition of the bottom horizontal maps sends $\alpha$ to zero. But this in turn means that $t \le c(\alpha, S_p)$ and as a result $c(\alpha\otimes 1, S)\le c(\alpha, S_p)$ for all $p$ in $P$. This implies the first inequality. The second inequality is obvious. The third is obtained by using the diagram

$$\begin{tikzcd}
H^*(M\times P) \arrow[r, "T"] & H^*(E^\infty, E^{-\infty}) \arrow[r] & H^*(E^t, E^{-\infty}) \\
H^*(M) \arrow[r, "T_p"] \arrow[u, "i_p^!"] & H^*(E_p^\infty, E_p^{-\infty}) \arrow[r] \arrow[u] & H^*(E_p^t, E_p^{-\infty}) \arrow[u]
\end{tikzcd}$$

where $i_p^! : H^k(M)\longrightarrow H^{k+p}(M\times P)$ is the push-forward or shriek[6] map in cohomology. Since $i_p^!(\alpha) = \alpha\otimes\mu_P$, we get that if $t < c(\alpha, S_p)$ then the image of $\alpha$ by the lower horizontal map is zero, but this implies that the image of $\alpha\otimes\mu_P$ by the composition of the top horizontal maps is zero, that is $t\le c(\alpha\otimes\mu_P, S)$. As a result $c(\alpha, S_p)\le c(\alpha\otimes\mu_P, S)$. □

PROPOSITION 7.60 (Reduction capacity inequality). *Let $V$ be a submanifold of $N$. We have for a G.F.Q.I. $S$ the inequality $\gamma(S_V)\le\gamma(S)$. As a result if $(L_i)_V\in\mathscr{L}(T^*V)$ we have*

$$\gamma((L_1)_V, (L_2)_V)\le\gamma(L_1, L_2)$$

*If $L$ is a Lagrangian in $\mathscr{L}(T^*N\times T^*P)$ we have for each $p\in P$ $\gamma(S_p)\le\gamma(S)$ hence if $L_p\in\mathscr{L}(T^*N)$, we have*

$$\gamma((L_1)_p, (L_2)_p))\le\gamma(L_1, L_2)$$

PROOF. We have $i_V^!(\mu_V) = \mu_N$ and $i_V^*(1_N) = 1_V$. Then according to Proposition 7.58, we have $c(\mu_V, S_V)\le c(i_V^!(\mu_V), L) = c(\mu_N, S)$ and $c(1_N, S) = c(i_V^*(1_N), S)\le c(1_V, S_V)$. Substracting the inequalities we get the first part of the Proposition. Now if $S_1, S_2$ are G.F.Q.I. for $L_1$ and $L_2$ we have according to the first part of the Proposition that $\gamma((S_1\ominus S_2)_V)\le\gamma(S_1\ominus S_2)$, that is $\gamma((L_1)_V, (L_2)_V)\le\gamma(L_1, L_2)$. The second part follows from the first applied to $V = N\times\{p\}$. □

This implies

COROLLARY 7.61. *Let $L_1, L_2\in\mathscr{L}(T^*X)$ and $V\in\mathscr{L}(T^*Y)$. Then*

$$\gamma(L_1\times V, L_2\times V) = \gamma(L_1, L_2)$$

[6] also called umkehr or transfer map

PROOF. Notice that for $y \in Y$ we have $(L_i \times V)_y = L_i$ and we have

$$\gamma(L_1, L_2) \leq \gamma(L_1 \times V, L_2 \times V)$$

For the reverse inequality let $S_1(x,\xi), S_2(x,\eta)$ be G.F.Q.I. for $L_i$ and $T(y,\zeta)$ be a G.F.Q.I. for $V$. Applying the triangle inequality (Proposition 6.3) we get

$$c(1_X \boxtimes 1_Y, (S_1 \ominus S_2) \boxtimes (T \ominus T)) \geq c(1_X, (S_1 \ominus S_2)) + c(1_Y, (T \ominus T))$$

which together with

$$c(\mu_X \boxtimes \mu_Y, (S_1 \ominus S_2) \boxtimes (T \ominus T)) \leq c(\mu_X, (S_1 \ominus S_2)) + c(\mu_Y, (T \ominus T))$$

implies

$$\gamma((S_1 \ominus S_2) \boxtimes (T \ominus T)) \leq \gamma(S_1 \ominus S_2) + \gamma(T \ominus T)$$

Now $\gamma(T \ominus T) = \gamma(V, V) = 0$ hence $\gamma(L_1 \times V, L_2 \times V) \leq \gamma(L_1, L_2) + \gamma(V, V) = \gamma(L_1, L_2)$ □

In particular since a Lagrangian correspondence is the composition of a product and a symplectic reduction, we have

PROPOSITION 7.62 (Correspondence is capacity non-expansive). *Let $\Lambda$ be a Lagrangian correspondence in $T^*X \times T^*Y$ having a G.F.Q.I. $\Sigma(x, y, \eta)$. Then for any $L_1, L_2 \in \mathscr{L}(T^*X)$ we have*

$$\gamma(\Lambda \circ L_1, \Lambda \circ L_2) \leq \gamma(L_1, L_2)$$

*Similarly if $\Lambda_1, \Lambda_2$ are Lagrangian correspondences having a G.F.Q.I. and $L \in \mathscr{L}(T^*N)$, we have*

$$\gamma(\Lambda_1 \circ L, \Lambda_2 \circ L) \leq \gamma(\Lambda_1, \Lambda_2)$$

PROOF. First of all we obviously have

$$\gamma(L_1 \times \Lambda, L_2 \times \Lambda) = \gamma(L_1, L_2)$$

Then since $\Lambda \circ L$ is the reduction of $\Lambda \times L$ by the coisotropic subspace $\nu^*\Delta_X \times T^*Y$ that is by the submanifold $V = \Delta_X \times Y$, the reduction inequality implies the first inequality. The second inequality similarly follows from

$$\gamma(L \times \Lambda_1, L \times \Lambda_2) = \gamma(\Lambda_1, \Lambda_2)$$

□

COROLLARY 7.63. *Let $\varphi_H$ be the time-one flow of the compact supported Hamiltonian $H$ on $T^*N$. Then for $L \in \mathscr{L}(T^*N)$ we have*

$$\gamma(\varphi_H(L), L) \leq \gamma(\varphi_H)$$

PROOF. We take $\Lambda = \Gamma(\varphi_H)$, notice that $\Lambda \circ L = \varphi_H(L)$ and apply Proposition 7.62. □

Ajouter le cas de $c_\pm$. Conséquence : if $H \geq c$ on $L$ then $c_+(\varphi_H) \geq c$

We now apply this result to the comparison between $\gamma$ and $\widehat{\gamma}$.

COROLLARY 7.64. *We have the inequality*

$$\widehat{\gamma}(\psi) \leq \gamma(\psi)$$

PROOF. We start with a preliminary remark : if $\psi_t$ is a Hamiltonian isotopy on $\mathbb{R}^{2n}$ with compact support, then this is not the case for $\mathrm{Id} \times \psi_t$ since however large will $x$ be, we usually have $(\mathrm{id} \times \psi_t)(x, y) = (x, \psi_t(y)) \neq (x, y)$ for some $y$. However, $\mathrm{supp}(\mathrm{id} \times \psi_t) \subset \mathbb{R}^{2n} \times \mathrm{supp}(\psi_t) \subset \mathbb{R}^{2n} \times K$ for some compact $K$, hence its intersection with a neighborhood of the diagonal is compact. As a result, if $L$ is in a neighborhood of the diagonal, $(\mathrm{Id} \times \Psi_t)(L)$ is a compact supported Hamiltonian isotopy of $L$. Let $\Gamma_\psi$ be the compactification of the image by $\rho$ of the graph of $\psi$. Let $L$ be a Lagrangian in $T^*N$. Then $0_N \times \psi(L) = (\mathrm{Id} \times \psi)(0_N \times L) = \Gamma_\psi \circ (0_N \times L)$. Then applying first Corollary 7.61 and then Corollary 7.62 we get

$$\gamma(\psi(L), L) = \gamma(0_N \times \psi(L), 0_N \times L) = \gamma(\Gamma_\psi(0_N \times L), 0_N \times L) \leq \gamma(\Gamma_\psi, \Gamma_{\mathrm{Id}}) = \gamma(\psi)$$

Taking the supremum on the left hand side over all $L$ concludes the proof. □

Finally we introduce the following :

DEFINITION 7.65. *Let $H, K \in C^\infty([0,1] \times T^*M, \mathbb{R})$. We set*

$$\gamma(H, K) = \sup_{t \in [0,1]} \gamma(\varphi_H^t, \varphi_K^t)$$

*Thus $H_n \longrightarrow H$ if and only if $\varphi_{H_n}^t \longrightarrow \varphi_H^t$ uniformly in $t$ for $t \in [0,1]$.*

## 6. Notations and conventions

In our definition, an exact Lagrangian contains the information of the primitive $f_L$ of $\lambda_{|L}$. We could identify it with its lift, a Legendrian manifold, but this would lead to some confusions : for example an exact Lagrangian is embedded if $L$ is embedded, which is different than requiring that its Legendrian lift is embedded.

An exact Lagrangian brane also contains a lift $s_L$ of the tangent space to $\widetilde{\Lambda}(TM)$ the bundle over $M$ with fibre $\tilde{\Lambda}(T_xM)$ the universal cover of $\Lambda(T_xE)$. In other words $s_L$ is a section of $\widetilde{\Lambda}(TM)$ over $L$ such that $s(x)$ projects on $T_xL$. Then if we denote by $T$ the canonical action of $\pi_1(\Lambda(n)) = \mathbb{Z}$ as the group of deck transformations of $\widetilde{\Lambda}(T_xE)$ we write $L[n]$ to be $L$ with $s_L$ replaced by $T^n s_L$. Again when no confusion shall arise we only write $L$ for the brane, and $L$ will also denote the set of points of the Lagrangian brane.

The definitions for $c_+, c_-$ in the literature[7] are conflicting. We would like a positive Hamiltonian to generate a positive flow. For the standard equations, the Hamiltonian $H(q, p) = f(q)$ has flow given by $\dot{p} = -\nabla f(q)$, so the image of the zero section is the graph of a negative function.

[7]Including in the author's papers !

## 7. Comments

**7.1. First steps in symplectic rigidity.** Gromov's non-squeezing (from [Gro85]) together with the Conley-Zehnder's theorem was one of the first results in symplectic topology. It was later extended by Lalonde and McDuff ([LM95]).

Beyond non-squeezing, the question of "symplectic packing" is an exciting domain of symplectic topology : how many symplectic balls or ellipsoid or polydiscs can be disjointly embedded in a given ball or ellipsoid or polydisc. We refer to an abundant litterature on the subject and among others [Bir01; MP94; Sch05; MS12; Ush19; Ber+21].

Even though the first symplectic invariant was Gromov's "width" (see [Gro85]), the axiomatic formulation as a capacity is due to Ekeland and Hofer (see [EH89; EH90]) who defined capacities using Hamiltonian dynamics. Our version of Theorem 7.41 is a little stronger than the original, as Gromov and Eliashberg assumed that $\psi$ is a diffeomorphism (this could be weakened to assume that $\psi$ is a homeomorphism using a result by Arens in [Are46]) while here this is part of the conclusion of the theorem. This proof is from [EH89]. The original proof used Gromov's result stating that $\mathrm{Diff}(M,\omega)$ is either $C^0$ closed or $C^0$ dense in $\mathrm{Diff}(M,\omega^n)$ (see [Gro86] for the statement and proof) Our second proof using Poisson brackets was unpublished. There are other proofs, for example using symplectic shape (see [MS14]). The Laudenbach-Sikorav theorem (Theorem 7.46) in full generality also implies the Gromov-Eliashberg theorem since the graph of the $\varphi_k$ is a Lagrangian submanifold. However there is for now no local version of the Laudenbach-Sikorav theorem unless one assumes $L_k = \varphi_k(L_0)$ for a sequence of symplectic maps.

The analog of Theorem 7.41 for contact diffeomorphisms is due to Bennequin in dimension 3 using the existence of exotic contact structures on $\mathbb{R}^3$ and again that $\mathrm{Diff}(M,\xi)$ is either $C^0$ closed or $C^0$ dense in $\mathrm{Diff}(M)$ (again from [Gro86])). In higher dimension the same proof works once we know there are exotic structures on $\mathbb{R}^{2n+1}$, a result due to Niederkruger ([Nie06]). Here we cannot use the approach using capacities, since the analog of symplectic capacities do not exist on $\mathbb{R}^{2n+1}$. However a proof using contact shape is due to Müller ([Mul90]) and this again implies the stronger result that an embedding is contact if and only if it preserves the shape[8].

**7.2. Closed characteristics.** The first proof of Weinstein's conjecture was in [Vit87a], the proof we present is different from the original one, since capacities did not exist at the time. More general ambient manifolds were dealt with in [HV88] and [Vit99] for cotangent bundles (at least in the simply connected case), [FHV90] for manifolds of the type $P \times \mathbb{C}$ where $P$ is compact, in [HV92] for manifolds with certain non-vanishing Gromov-Witten invariants (see also [LT00]). All these examples are for contact hypersurfaces in a symplectic manifold (such contact manifolds are called "fillable").

[8]It is wrong to think that a smooth limit of contact maps is a contact map as the sequence of maps $(x,y,z) \mapsto (\frac{x}{n}, \frac{y}{n}, \frac{z}{n^2})$ shows.

In dimension 3 a number of cases - including non-fillable cases- and in particular for all contact forms on $S^3$ were proved in [Hof93b] and finally the conjecture was proved for any 3-dimensional contact manifold by Taubes, using the equivalence between Seiberg-Witten and Gromov-Witten invariants (see [Tau07]).

In higher dimensions, there are criteria using Symplectic homology [Vit99; She22] and Contact homology For the case of exotic structures, see [AH09], for connections with other notions, see [AFM15; CF09]. For existence of periodic orbits on hypersurfaces which are not of contact type, existence - first for a dense set of levels and then for almost all levels (for the Lebesgue measure)- was proved by Hofer-Zehnder and then Struwe (see [HZ94; Str90]). Counterexamples to existence when the contact-type condition is removed are due to V. Ginzburg and M. Herman [Gin95; Gin97; Her99]).

**7.3. Capacities and metrics.** The metric $\gamma$ was introduced in [Vit92] at the same time Hofer introduced the Hofer metric ([Hof90]). It was then extended to more general manifolds by Schwarz ([Sch00] and then Oh ([Oh05; Oh06]) in the Hamiltonian setting and by Leclercq for Lagrangian submanifolds ([Lec08]). The proof of the Gromov-Eliashberg theorem we present is essentially the one from [EH89].

**7.4. Spaces and metrics.** The $C^0$-continuity of capacities in $\mathbb{R}^{2n}$is due to the author ([Vit92]) for the capacities defined in this book and to Sikorav ([Sik90]) for Hofer's displacement energy. For the capacity, the proof relies on the so-called $\varepsilon$-shift trick, easily adapted to any open symplectic manifold. For closed manifolds this was proved for surfaces in [Sey13a], in general aspherical manifolds[9] in the amazing paper by Bukhovski, Seyfaddini, Humilière [BHS21] and extended by E. Shelukhin to $\mathbb{C}P^n$ in [She18] and Y. Kawamoto [Kaw20] to negative monotone manifolds.

The first study of the completion of the group of Hamiltonian symplectomorphism for the $\gamma$ distance[10] is due to Humilière in [Hum08]. Instead of the completion for the $\gamma$ distance, we could have used the distance between the barcodes of the functions (see [LNV13]) but it follows from Kislev-Shelukhin (see [KS22]) that this yields an equivalent distance. Theorem 7.50 is due to Cardin and the author in [CV08], but the subject was developed on one hand by V. Humiliere, in particular in [Hum09] where it was extended to “pseudo-representations” and on the other hand for bounds on the $C^0$-norm of the brackets and related questions by Entov-Polterovich and collaborators ([EP10; EP09; EPR10; EPZ07; Buh10; PR14]). See [Vit22] for recent results.

[9] i.e. such that $[\omega]\pi_2(M) = 0$, using Floer homology

[10] In fact Humilière used the $\widehat{\gamma}$ distance.

## 8. Exercises and Problems

(1) (Weinstein -Moser) (see [Wei73b]) Let $H$ be a smooth function on $\mathbb{R}^{2n}$, having a critical point at 0 and such that $H(0) = 0$ and $D^2H(0)$ is positive definite. We want to prove that $H^{-1}(c)$ has $n$ distinct closed characteristics for $c$ small enough.

This problem originates from Lyapounov's thesis in 1892 (see [Lia07] for the french translation and [Lia92]) for an english translation) and Weinstein's result was extended by [EL80a] to "pinched convex sets" and to starshaped ones in [Ber+85] and then by Y. Long and C. Zhu ([LZ02; Lon06].

(2) Prove that if $U$ is a starshaped neighborhoodof the zero section, that is $(q,p) \in U$ implies $(q,tp) \in U$ for all $t > 0$, then $\mathscr{L}(U)$ is also the set of Lagrangians $L$ such that there is a Hamiltonian isotopy supported in $U$

HINT. *Start from a Hamiltonian isotopy in $T^*N$. Show that we can assume the Hamiltonian has compact support. Conclude that $t \cdot L = \{(q,tp) \mid (q,p) \in L\}$ can be obtained from the $0_N$ by an isotopy supported in $U$. Conclude by showing that the restriction of the conformal isotopy $t \mapsto (q,tp)$ to an exact Lagrangian $L'$ is induced by a Hamiltonian isotopy with support in a neighborhood of $\bigcup_{t\in[0,1]} t \cdot L$.*

(3) Prove that if $C$ is a codimension 2 coisotropic subspace, $D(C^\omega)$ a disc in its orthogonal complement, then $c(D(C^\omega) \times C) = +\infty$.

Next exercise must be integrated to the text!

(4) We want to prove that $\gamma$ defined in Definition 7.16 is a distance. The only non-obvious fact is the triangle inequality. Let $\psi_t$ be a Hamiltonian isotopy and $L$ a Lagrangian. Let $S_t^1$ be a G.F.Q.I. for $\psi_t(L)$ and $S_t^2$ a G.F.Q.I. for $\psi_t\psi_1^{-1}(0_N)$. We first want to prove that

$$c(\alpha, \psi_1(L)) = c(\alpha, S_0^1 \ominus S_0^2) \tag{*}$$

(a) Prove that the critical values for $S_t^1 \ominus S_t^2$ do not depend on $t$.

HINT. *Prove that the critical values corresponding to intersection points of $\psi_t(L) \cap \psi_t\psi_1^{-1}(0_N) = \psi_t(L \cap \psi_1^{-1}(0_N))$. Then show that this set does not depend on t*

(b) Prove that $c(\alpha \cup \beta, \psi_1(L)) \geq c(\alpha, L) + c(\beta, \psi_1(0_N))$

HINT. *Use (*) and apply the triangle inequality to obtain $c(\alpha\cup\beta,\psi_1(L)) \geq$ $\alpha, L) + c(\beta, \overline{\psi_1^{-1}(0_N)})$. Then prove that $c(\beta, \overline{\psi_1^{-1}(0_N)}) = c(\beta, \psi_1(0_N))$ by applying once more (*) to $L = 0_N$.*

(c) Let $\varphi_t, \psi_t$ be Hamiltonian flows with compact support on $\mathbb{R}^{2n}$ and $\varphi = \varphi_1, \psi = \psi_1$. Set $\Psi = \psi \times \mathrm{Id}$ on $T^*X \times T^*X$ and $L = \Gamma_\varphi$ be the graph of $\varphi$ in

$\mathbb{R}^{2n} \times \overline{\mathbb{R}}^{2n}$ identified to $T^*\Delta_{\mathbb{R}^{2n}}$. Prove that

$$c(1,\Gamma_{\psi\varphi}) = c(1,\Psi(\Gamma_\varphi)) \geq c(1,\Gamma_\varphi) + c(1,\Psi(\Delta_{\mathbb{R}^{2N}})) = c(1,\Gamma_\varphi) + c(1,\Gamma_\psi)$$

(d) Prove that the reverse inequality holds for $c(\mu_N,\bullet)$ and conclude that

$$\gamma(\varphi\psi) \leq \gamma(\varphi) + \gamma(\psi)$$

(5) Prove that in general $\gamma(U) \neq \sup\{\gamma(\varphi_H^1) \mid \operatorname{supp}(\varphi_H^t) \subset U \text{ and bounded}\}$ in other words the assumption on the support of the Hamiltonian cannot be replaced by the assumption on the support of the isotopy.

(6) Prove that for a domain $U$, if a function $H$ is non-negative and satisfies $H \geq \gamma(U)$ on $U$ then $c_-(\varphi_H) \leq -c(U)$

(7) Prove that for a compact manifold $N$ there is no embedded Lagrangian Hamiltonianly isotopic to $0_N$- other than $0_N$- such that $\overline{L} = L$ where $\overline{L} = \{(q,-p) \mid (q,p) \in L\}$.

HINT. *Use the representation theorem (7.9) to find two points $x_\pm$ and a path $\gamma$ in $L$ from $x_-$ to $x_+$ such that $\int_\gamma \lambda > 0$. Show that if $\overline{L} = L$ there is another path $\gamma'$ from $x_-$ to $x_+$ such that $\int_{\gamma'} \lambda < 0$ and conclude. (See also Exercice 2)*

(8) Let $U$ be connected (but not simply-connected) domain in the plane. We want to prove that $c(U) = \gamma(U)$ is the area of the smallest simply-connected domain containing $U$.

(a) Prove that it is enough to prove this for an annulus that is a domain with a single hole (use the monotonicity of $c$ and $\gamma$).

(b) Prove using Moser's lemma that the case of a general annulus can be reduced to the case of a standard annulus

$$\mathbb{A}(r,R) = \left\{(q,p) \in \mathbb{R}^2 \mid r^2 \leq q^2 + p^2 \leq R^2\right\}$$

and prove that $\gamma(\mathbb{A}(r,R)) \leq \pi R^2$.

(c) Prove, using an embedding of the disc in the annulus that $c(\mathbb{A}(r,R)) \geq \pi(R^2 - r^2)$.

(d) Considering a Hamiltonian $H(q,p) = h(\frac{q^2+p^2}{2})$ where $h$ is represented on Figure 3. The intersection of the dashed lines with the vertical axis represent $h(\rho) - \rho h'(\rho)$. Since our trajectories must have $-\pi R^2 \leq h'(\rho) - h(\rho) < 0$, choosing $a > \pi R^2$ we see that $c_-(\varphi_H)$ is not represented by a constant trajectory. Similarly the trajectory with $\rho \simeq \frac{r^2}{2}$ and $h(\rho) \simeq 0$ will have $h(\rho) - \rho h'(\rho) < 0$ so cannot represent $c_-(\varphi_H)$, and the same for $\rho \simeq \frac{R^2}{2}$ and $(h(\rho) \simeq a$ (then $h(\rho) - \rho h'(\rho) > a > \pi\frac{R^2}{2}$). So the only remaining possibilities are $\rho \simeq r, h(\rho) \simeq a$ and $\rho \simeq \frac{R^2}{2}, h(\rho) \simeq 0$. In the second case we get $h(\rho) - \rho h'(\rho) \leq -\pi R^2$and in the first $c_-(\varphi_H) \simeq k\frac{r^2}{2} - a$ with $k$ such that $k\frac{r^2}{2} - a \geq -\pi R^2$. Now if we increase $R$, we see that $c_+(\varphi_H)$ will not change since it is determined by the value of $h$ near $r^2/2$. On the other hand $c_+(\varphi_H)$ must become larger than $\pi(R^2-r^2)$ large enough

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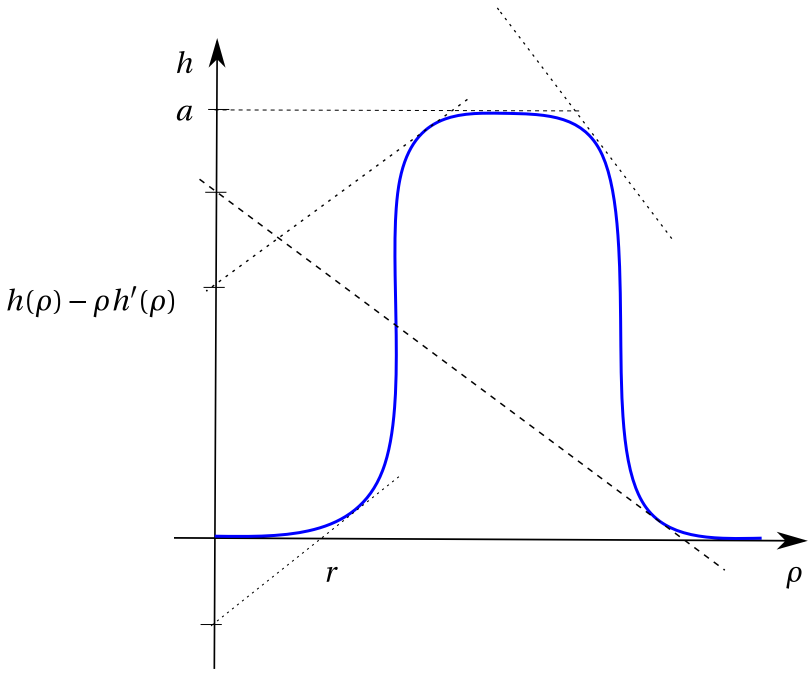


FIGURE 3. The Hamiltonian $H$ of Exercice 8.

(9) (The limit of capacity preserving maps preserves the capacity) Let $\psi_n$ be continuous maps preserving $\gamma$ (or $c$) that is $\gamma(\psi_n(U))=\gamma(U)$ for any restricted contact type domain $U$. We want to prove that if $\psi$ is the $C^0$ limit of the sequence $\psi_n$ then $\psi$ preserves the capacity of restricted contact type sets. Consider for $t\in[-\varepsilon,\varepsilon]$ the sets $U_t=\rho_t(U)$, images by the conformal vector field transverse to $\partial U$ obtained by the definition of restricted contact type. We have $\rho_{-t}(U)\subset U=\rho_0(U)\subset\rho_t(U)$ for $t>0$.
   (a) Show that $\gamma(\rho_t(U))=e^t\gamma(U)$
   (b) Prove the statement by using the fact that for $n$ large enough

$$(\psi_n(\rho_{-\varepsilon}(U)))\subset\psi(U)\subset(\psi_n(\rho_\varepsilon(U)))$$

(10) ("Usher's observation", see [Ush15], theorem 1.3) The goal of this exercise is to prove the following theorem

THEOREM. *Let $(M,\omega)$ be a symplectic manifold, $H(t,x)$ be a Hamiltonian and $A$ be a compact subset in $(M,\omega)$. Then there exists $K(t,x)$ such that* $\varphi_K^1=$

$\varphi_H^1$ *and*

$$\int_0^1 \operatorname{osc}_{x\in\varphi_K^t(A)} K_t\,dt = \int_0^1 \operatorname{osc}_{x\in A} H_t\,dt$$

COROLLARY. *Let $A$ be a compact set in $(M,\omega)$ and $H(t,x)$ a Hamiltonian. Then there exists a Hamiltonian $K(t,x)$ such that*

$$\operatorname{osc}_{[0,1]\times M} K(t,x) \le \int_0^1 \operatorname{osc}_{x\in A} H_t\,dt$$

*and $\varphi_K^1 = \varphi_H^1$ on $A$. In particular $\varphi_H^1(A) = \varphi_K^1(A)$.*

(a) Write the formula for the Hamiltonian $\overline{H}, \widehat{H}$ generating the flows $\varphi_{\overline{H}}^t = (\varphi_H^t)^{-1}$ and $\varphi_{\widehat{H}}^t = (\varphi_H^{1-t})(\varphi_H^1)^{-1}$.
(b) Write the formula for the Hamiltonian $K$ generating the flow $\varphi_K^t = (\varphi_{\widehat{H}}^t)^{-1} = \varphi_H^1 \circ ((\varphi_H)^{1-t})^{-1}$
(c) Prove that $K(t,\varphi_K^t(z)) = H(1-t,z)$
(d) Using that $\varphi_K^1 = \varphi_H^1$ prove the theorem
(e) Using the fact that if $H_1 = H$ on $\bigcup_{t\in[0,1]} {\varphi_H}^t(A)$ we have $\varphi_{H_1}^t = \varphi_H^t$ on $A$, prove the corollary.
(f) Use the above to reprove Proposition 7.19

(11) (Seyfaddini's theorem, see [Sey13b] ) Let $\mathrm{Diff}(D,\partial D)$ be the set of diffeomorphisms of $D^{2n}$ equal to Id in some (non-prescribes) neighbourhood of the boundary. We shall admit that this is a separable metric space for the $C^\infty$ metric, so that there is a countable dense subset.
(a) By plugging-in a dense sequence of suitably rescaled elements of a dense sequence, prove that there exists $\varphi \in \mathrm{Diff}(D,\partial D)$ such that the set of its conjugates $\psi\circ\varphi\circ\psi^{-1}$ where $\psi \in \mathrm{Diff}(D,\partial D)$ is $C^0$ dense in $\mathrm{Diff}(D,\partial D)$.
(b) Prove that this is not possible if we replace $\mathrm{Diff}(D,\partial D)$ by $\mathrm{Diff}_\omega^0(D,\partial D)$ the connected component of Id in $\mathrm{Diff}_\omega(D,\partial D)$.

HINT. *Prove that if $\varphi \neq \mathrm{Id}$ there exists $B(x,\varepsilon)$ such that $\varphi(B(x,\varepsilon))\cap B(x,\varepsilon) = \emptyset$ and this implies $\gamma(\varphi) \ge \pi\varepsilon^2$. Conclude that this holds for the whole orbit. Then apply Proposition 7.43.*

(12) (A capacity preserving map which is not a homeomorphism)
Let $a_n = 1+1/n, b_n = 1/n$ and $K_n$ be the domain obtained by smoothing the boundary of the rectangle $[-a_n,a_n]\times[-b_n,b_n]$ and having the same area, that is $4\frac{n+1}{n^2}$. Let $D(r_n)$ be the disc with the same area as $K_n$ ($r_n = \sqrt{4\frac{n+1}{\pi n^2}}$)
(a) Use Moser's lemma (Lemma 3.21 of Chapter 3 to find a symplectomorphism $u_n$ from $K_n\setminus K_{n+1}$ to $D(r_n)\setminus D(r_{n+1})$. Show that we may assume that $u_n$ and $u_{n+1}$ coincide in a neighborhood of $\partial K_n \cap \partial K_{n+1}$
(b) Prove that the above construction yields a symplectic map $u$ from $K_1 \setminus K_\infty = K_1\setminus[-1,1]\times\{0\}$ to $D(r_1)\setminus\{0\}$.

(c) Prove that if we extend $u$ to $v$ by sending $[-1,1]\times\{0\}$ to 0, we get a capacity preserving map from $K_1$ to $D(r_1)$ that is not a homeomorphism.
(d) Extend the above construction to higher dimensions.
(e) Is $v$ the $C^0$-limit of symplectic smooth maps ?
(f) Does $v$ have an inverse in the $\gamma$-completion $\widehat{\mathrm{DHam}}(\mathbb{R}^{2n})$ of $\mathrm{DHam}(\mathbb{R}^{2n})$.

(13) (see [MS07], p. 463, prop. 12.2.2 )

(14) (Struwe quasi-existence theorem, see [Str90]) Let $H$ be an autonomous Hamiltonian in $\mathbb{R}^{2n}$ such that $H^{-1}(c)$ is a compact regular level for $c\in[a,b]$. Then for almost all $c\in[a,b]$, $H^{-1}(c)$ has a closed characteristic.

(15) (Hofer-Zehnder capacity) (see [Hof93a]) We consider an autonomous Hamiltonian $H(z)$, its flow $\varphi_H^t$. For a point $z$, we denote by $A(z,H)$ the action $\int_0^1[p\dot{q}-H(t,z)]dt$. where $z(t)=\varphi_H^t(z)$. We shall say that $H$ is admissible if $H$ is compact supported and the only solutions of $\varphi_H^t(z)=z$ for $0<t\leq 1$ are constants (i.e. points where $dH(z)=0$). We denote by $\mathrm{Ham}_{ad}$ the set of admissible Hamiltonians. Let $U$ be a domain in $\mathbb{R}^{2n}$. We set

$$c_{HZ}(U)=\sup\left\{\|H\|_{C^0}\mid H\in\mathrm{Ham}_{ad},\mathrm{supp}(H)\subset U\right\}$$

A priori $c_{HZ}(U)\in[0,+\infty]$.
(a) Prove that for $U\subset V$ we have $c_{HZ}(U)\subset c_{HZ}(V)$
(b) Prove that $c_{HZ}(\lambda\cdot U)=\lambda^2 c_{HZ}(U)$
(c) Prove that if $\varphi$ is symplectic, $c_{HZ}(\varphi(U))=c_{HZ}(U)$
We want now to prove that if $U$ is bounded then $c_{HZ}(U)<+\infty$.
(d) Let $H$ be an admissible Hamiltonian. Prove that for all $s\in[0,1]$, $H_s(q,p)=s\cdot H(q,p)$ has no 1-periodic orbit other than the constant.
(e) Imitate the proof of Proposition 7.25 to prove that for $H$ admissible, $c_+(\varphi_H)=\max_{z\in\mathbb{R}^{2n}}H(z)$ and $c_-(\varphi_H)=\min_{z\in\mathbb{R}^{2n}}H(z)$
(f) Conclude that if $\max H>c_+(H)$, $H$ cannot be admissible
(g) Prove that $c_{HZ}(U)\leq c(U)$
(h) Prove that $c_{HZ}(B^{2n}(r))=\pi r^2$ and $C_{HZ}(Z(r))=\pi r^2$.

(16) Let $(L_k)_{k\geq 1}$ be a sequence of exact Lagrangians in $T^*N$ such that $L_k$ converges to $X$ for the Hausdorff topology, where $\dim(X)=n$ .
(a) Prove that $X$ must be Lagrangian

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(17) (Hamiltonian pseudo-representations, see [Hum09]) Let $F_n,G_n,H_n$ be sequences of compact supported Hamiltonians such that $\{F_n,G_n\}=H_n,\{F_n,H_n\}=\{G_n,H_n\}=0$.
(a) Prove that if $F=C^0-\lim F_n, G=C^0-\lim G_n, H=C^0-\lim H_n$ and $F,G,H$ are smooth, then $\{F,G\}=H,\{F,H\}=\{G,H\}=0$.

HINT. *Repeat the proof in the case of commuting Hamiltonians and apply to get the last two equalities. Now use the equality*

$$\frac{d}{ds}G_n(\varphi^s_{F_n}(x)) = H_n(\varphi^s_{F_n}(x))$$

(b) Extend to the case where we only assume $\lim_n\{F_n, G_n\} = H_n, \lim_n\{F_n, H_n\} = \lim_n\{G_n, H_n\} = 0$

See [Hum09] and [Buh10] as well as [EP10; EP09; EPR10; EPZ07] for a generalization of this result.

(18) ( See [EP10]) Let $F_k, G_k$ be smooth compact supported functions (with common compact support) such that $C^0 - \lim F_k = F, C^0 - \lim G_k = G$ and $F, G$ are smooth. We want to prove that

$$\|\{F_k, G_k\}\|_{C^0} \leq \liminf_k \|\{F_k, G_k\}\|_{C^0}$$

We remind the reader that the Hamiltonian path $\varphi^t_k\psi^s_k\varphi^{-t}\psi^{-s}_k$ is generated by the Hamiltonian

$$L^k_s(t,z) = \int_0^s \{F_k, G_k\}(\psi^{-s}_k\varphi^{-t}_k(z))ds$$

(a) Prove that $c_+(\varphi^t_k\psi^s_k\varphi^{-t}\psi^{-s}_k) \leq st\sup_z\{F_k, G_k\}$
(b) Prove that $\gamma{-}\lim(\varphi^t_k\psi^s_k\varphi^{-t}\psi^{-s}_k) = \varphi^t\psi s\varphi^{-t}\psi{-}s$ and deduce that $c_+(\varphi t\psi s\varphi^{-t}\psi{-}s) = \lim$
(c) Prove that if $F, G$ are smooth, then for $s$ small enough and $0 \leq t \leq 1$, $\varphi t\psi s\varphi^{-t}\psi{-}s$ is generated by a $C^2$ small Hamiltonian, and that as a result, using Proposition 7.25 we have $c_+(\varphi^t\psi^s\varphi^{-t}\psi^{-s}) = st\sup_z\{F, G\}(z)$
(d) Conclude that $\sup_z\{F, G\}(z) \leq \liminf\sup_z\sup_z\{F_k, G_k\}$
(e) By replacing $c_+$ by $c_-$ show that $\inf_z\{F, G\}(z) \geq \limsup\inf_z\sup_z\{F_k, G_k\}$
(f) Conclude

(19) (A partial order) We say that a Hamiltonian map is positive if $c_-(\varphi) = 0$.
(a) Prove that the set of positive Hamiltonians is a semi-group.
(b) Prove that this semi-group generates $\mathrm{DHam}_c(T^*\mathbb{R}^n)$

(20) (HLS inequality, see [Sey15], [HRS16] and [GT]) Let $F, G$ be two smooth functions supported in the bounded domains $U, V \subset \mathbb{R}^{2n}$ such that $U \cap V = \emptyset$ and $U, V$ can be "separated", i.e. moved by a symplectic isotopy : there exists a symplectic isotopy $\psi^s$ such that $\psi^s(U) \cap V = \emptyset$ for all $s$ and there is a real number $a$ such that

$$V \subset \{x_n > a\}, \psi^1(U) \subset \{x_n < a\}$$

We want to prove that

$$c(\alpha, F+G) \leq \max\{c(\alpha, F), c(\alpha, G)\}$$

We denote by $F_s = F \circ \psi^s$ and by $\varphi^t_{F_s}, \varphi^t_G$ the flows of $X_{F_s}, X_G$ and by $H_s = F_s + G$.
(a) Give an example of two disjoint sets that cannot be separated

(b) Prove that the periodic orbits of $\varphi_{F_s}^t$ are the images by $\psi^s$ of the periodic orbit of $\varphi_{F_0}^t$

(c) Prove that the set of actions of the periodic orbits of $X_{H_s}$ does not depend on $s$ and deduce that $c(\alpha, F_s + G) = c(\alpha, F + G)$

(d) Assume $U \subset B^n(0,R) \times B^n(0,R)$. Prove that the graph $\Gamma(\varphi_F^1)$ of $\varphi_F^1$, defined as

$$\Gamma(\varphi_F^1) = \{(q,P,p-P,Q-q) \mid \varphi_F^1(q,p) = (Q,P)\}$$

coincides with the zero section outside of $B^n(0,R) \times B^n(0,R)$

(e) Let $S_F(q,P;\xi)$ be the G.F.Q.I. for $\varphi_F$, so that $p - P_s = \frac{\partial S}{\partial q}(q,P_s;\xi), Q_s - q = \frac{\partial S}{\partial P}(q,P_s;\xi)$ for $\frac{\partial S}{\partial \xi}(q,P_s;\xi) = 0$. Prove that for $(q,P)$ outside $B(0,R) \times B(0,R)$, $\xi \mapsto S(q,P;\xi)$ has a unique critical point with critical value 0 (note that $S$ has been normalized to have a unique critical point with critical value 0 at infinity)

(f) Prove that we may assume $S_F(q,P;\xi) = q(\xi)$ outside $B(0,R)\times B(0,R)$ where $q$ is a non-degenerate quadratic form.

(g) Show that we may assume, by modifying $\psi$ that $\mathrm{supp}(F_1) \subset B(-2x_R,R) \times B(-2x_R,R)$ and $\mathrm{supp}(G) \subset B(2x_R,R) \times B(2x_R,R)$

(h) Assume $S_{F_1}$ is a G.F.Q.I. for $\varphi_{F_1}$ and $S_G$ a G.F.Q.I. for $\varphi_G$ Prove that we may assume that $S_{F_1}(q,P;\xi) = q(\xi)$ outside $B(-2x_R,R)\times B(-2x_R,R)$, while $S_G(q,P;\eta) = q'(\eta)$ outside $B(2x_R,R) \times B(2x_R,R)$

(i) Prove that $S_{H_1}(q,P;\xi,\eta) = S_{F_1}(q,P;\xi)+S_G(q,P;\eta)$ is a G.F.Q.I. for the graph of $\varphi_{H_1}$.

(j) Let $C$ representing a cycle in $H_*(S_G^{+\infty}, S_G^{-\infty})$ contained in $S_G^c$ Prove that after deforming $C$ inside $S_G^c$, we may assume that for $(q,P;\xi) \in C$ and $(q,P) \notin B(2x_R,R) \times B(2x_R,R)$, we have $S(q,P;\xi) \leq 0$.

HINT. *Follow the gradient of* $-\nabla_\xi S(q,P;\xi)$ *and use the fact that in the complement of* $B(2x_R,R) \times B(2x_R,R)$, *we have a single critical point with critical value* 0

(k) Let $C_1$ representing a cycle in $H_*(S_{F_1}^{+\infty}, S_{F_1}^{-\infty})$ contained in $S_{F_1}^{c_1}$ and $C_2$ representing a cycle in $H_*(S_G^{+\infty}, S_G^{-\infty})$ contained in $S_G^{c_2}$. We assume $C_1$ and $C_2$ have been deformed as in the previous question. Prove that

$$C_1 \times_{\mathbb{R}^{2n}} C_2 = \left\{(q,P;\xi,\eta)\big(q,P;\xi) \in C_1, (q,P;\eta) \in C_2\right\}$$

is contained in $S_{H_1}^c$ where $c \leq \max\{c_1,c_2\}$.

(l) Suppose $[C_1]$ is the image of $\alpha \in H_*(S^{2n})$ by the isomorphism $H_*(S^{2n}) \longrightarrow H_{*-i}(S_{F_1}^{+\infty}, S_{F_1}^{-\infty})$ and $[C_2]$ the image of $\alpha$ by the isomorphism $H_*(S^{2n}) \longrightarrow H_{*-j}(S_G^{+\infty}, S_G^{-\infty})$. We shall admit that $[C_1 \times_{\mathbb{R}^{2n}} C_2]$ is the image of $\alpha$ by the isomorphism $H_*(S^{2n}) \longrightarrow H_*(S_{H_1}^{+\infty}, S_{H_1}^{-\infty})$

(m) Prove the theorem stated

CHAPTER 8

# Applications to Hamilton-Jacobi equations

The problems we shall deal with in this chapter have a common feature : they all have some type of singularity that essentially disappears when seen from the $C^0$-symplectic point of view. For the Kepler problem, this is the collision orbits, for Hamilton-Jacobi, the caustics, and for billiards, the bouncing instant. We shall show that these have flow living in $\widehat{Ham}_0(T^*N)$.

## 1. Hamilton-Jacobi equations-the geometric approach

Let $H : T^*N \longrightarrow \mathbb{R}$ be a smooth function. We want to solve the equation with the unknown function $u \in C^1(N,\mathbb{R})$

$$\text{(HJ1)} \qquad \begin{cases} H(x, \dfrac{\partial u}{\partial x}(x)) = 0 \\ u(x) = f(x) \text{ on } \Sigma \end{cases}$$

One also considers equations involving the function $u$ itself, that is of the type

$$\text{(HJ1)} \qquad \begin{cases} H(x, \dfrac{\partial u}{\partial x}(x), u(x)) = 0 \\ \qquad u(x) = f(x) \text{ on } \Sigma \end{cases}$$

Let us first explain the construction of the local solutions by the method of characteristics.

### 1.1. The method of characteristics.

We shall illustrate this method with the equation

$$\text{(HJ1)} \qquad \begin{cases} \left\| \dfrac{\partial u}{\partial x}(x) \right\| - 1 = 0 \\ u(x) = f(x) \text{ on } \Sigma \end{cases}$$

Note first that the data of $u$ on $\Sigma$ prescribes the differential of $u$ in the direction tangent to $\Sigma$.

So if we decompose a linear form $p \in T_z^*N$ as $p = p_T + p_\nu$ where $p_\nu$ vanishes on $T_z\Sigma$, and write $du(z) = \partial_T u(z) + \partial_\nu u(z)$, we will have $= \partial_T u(z) = df(z)$, and only $\partial_\nu u(z)$ is free so that we must first solve on $\Sigma$ the equation $H(z, df(z) + p_\nu(z)) = 0$ for a function $p_\nu : \Sigma \longrightarrow T_\Sigma^* N$ such that $p_\nu(z) = 0$ on $T_z\Sigma$. In our special case, we need that $\|df(x)\| \leq 1$, and if this is the case, we have two choices for $p_\nu(z)$, one corresponding to each side of $\Sigma$. Let us assume we made such a choice, for example by choosing one side of $\Sigma$

and denoting by $\nu(x)$ the normal pointing towards this side, and then $p_\nu(z)$ such that $\langle p_\nu(z), \nu(z)\rangle > 0$. In our special situation we choose the one defined by the arrows on Figure 1. Then graph$(df)$ is a Lagrangian submanifold in $T^*\Sigma$, and it has an isotropic lift to $T^*N$ given by

$$\Lambda_f = \left\{(z, df(z) + p_\nu(z)) \mid z \in \Sigma\right\}$$

In local coordinates if $\Sigma$ is given by $x_n = 0$, $\Lambda_f$ is locally defined as the set of

$$\left(x_1, \ldots, x_{n-1}, 0, \frac{\partial f}{\partial x_1}(z), \ldots, \frac{\partial f}{\partial x_{n-1}}, p_\nu(z)\right)$$

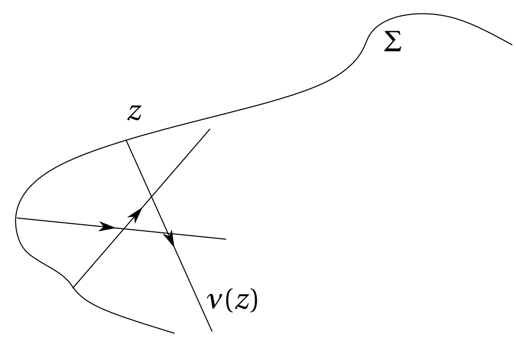


FIGURE 1. The normals to the hypersurface $\Sigma$.

Since the graph of $du$ is also a Lagrangian, it must be contained in $H^{-1}(0)$ and must contain $\Lambda_f$. If $X_H$ is transverse to $\Lambda_f$, then locally

$$L = \bigcup_{s\in\mathbb{R}} \varphi_H^s(\Lambda_f)$$

will be a Lagrangian submanifold contained in $H^{-1}(0)$. Indeed, $\varphi_H^s$ preserves $H^{-1}(0)$ and since by construction $\Lambda_f \subset H^{-1}(0)$, we will construct locally a Lagrangian. Since $\Lambda_f$ is a graph, the same will hold if consider the subset of $L$ given by

$$L_\varepsilon = \bigcup_{s\in[0,\varepsilon]} \varphi_H^s(\Lambda_f)$$

Note that in our case the flow $\varphi_H^s$ is the geodesic flow that is given the scalar product associated to the Riemanian metric defining $\| \bullet \|$: we let $v(p) \in T_zN$ to be defined by duality

$$\langle p, w\rangle = (v(p), w)_g$$

and the geodesic flow is given by

$$(x, v) \longmapsto (\exp_x(tv), d\exp_x(tv)v)$$

Not that if we set $\Sigma_t = \pi(\varphi_H^t(\Lambda_f))$ and if $\Sigma_t$ is smooth, which will be the case for small $t$, then the reduction of $L$ over $\Sigma_t$ is exactly $\varphi_H^t(\Lambda_f)$.

In our special case the flow is simply $\varphi^t(x,p) = (x + t\cdot p, p)$. By Proposition 4.13 the action of $\varphi_H$ is given by

$$f(\varphi_H^t(x_0,p_0)) = f(x_0,p_0) + \int_0^t (\varphi_H^s)^*\,[\lambda - Hds]$$

which in our special case reduces to $f(x+t\cdot p, p) = f(x,p) + t$ since $\|p\| = 1$. We see however that there are at least two issues

(1) $\Sigma_t$ is not embedded (or even immersed)
(2) at a given point $x$ $u(x)$ can be multiply defined, that is there are $(x_0,p_0)$ and $(x_0',p_0')$ in $\Lambda_f$ such that $\varphi_H^t(x_0,p_0)$ and $\varphi_H^t(x_0',p_0')$ have the same projection $x_1$ on $N$ and the values $f_L(x_1,p_1), f_L(x_1,p_1')$ are distinct. We then do not know which value to choose for $u_L(x_1)$

This is why the method of characteristics can only be used to define the solution in a neighbourhood of $\Sigma$, where we can guarantee (at least if $\Sigma$ is compact) that neither of the above problem will occur. In our special case, we see that the function $u$ is given by $u(x) = d(x,\Sigma)$ in a neighbourhood of $\Sigma$. Notice however that the function $d(x,\Sigma)$ is

(1) well defined everywhere, not just near $\Sigma$
(2) satisfies the equation $\|Du(x)\| = 1$ almost everywhere. The points where $u$ is non smooth are called focal points of the hypersurface. They form a closed set of measure zero, as they correspond (see e.g. [Mil63]) to the singularities of the map

$$\begin{aligned}\Sigma\times\mathbb{R}^+ &\longrightarrow \mathbb{R}^n\\ (x,t) &\mapsto x + tv(x)\end{aligned}$$

(or $(x,t)\mapsto \exp_x(tv(x))$ in the case of a general geodesic flow, see also Exercice 8 from Chapter 4)

To conclude, we saw that

(1) Elementary consideration of symplectic geometry allow us to solve (HJ1) locally (the method fo characteristics).
(2) There are difficulties to use this method to extend the solutions globally, because of the appearance of singularities (focal points, called shocks in the theory of Hamilton-Jacobi equations) and that solutions would be multi-valued.
(3) However we still can have globally defined solutions at the cost of admitting that solutions need not be everywhere smooth.

There are basically two methods to find global solutions of the Hamilton-Jacobi equation, the viscosity method, for which we refer to [CL83; Lio82; Bar94b] recently revived as weak-KAM solutions (see [Fat; Arn12; AS22; Ber07]), and the variational method initiated by Chaperon and Sikorav (see [Cha91; Sik89]) and developed in [OV94; Vit96].

**1.2. Variational solutions for evolution Hamilton-Jacobi equations.** In the special case where $N = M \times \mathbb{R}$ and we set $H(t,q,p)$ be a smooth Hamiltonian, we want to solve the equation

$$\text{(HJE)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}u(t,q) + H(t,q,\dfrac{\partial u}{\partial q}(t,q)) = 0 \\ \qquad\qquad\qquad u(0,q) = f(q) \end{cases}$$

The "geometric approach" consists in first replacing $u$ by the Lagrangian submanifold obtained by considering the graph of its differential, then to decide that the solution to the equation should be a Lagrangian submanifold, not necessarily a graph.

DEFINITION 8.1. *A geometric solution to the equation* (HJ1) *(resp.* (HJE)*) is a Lagrangian submanifold $L$ in $T^*N$ (resp. $T^*(N\times\mathbb{R})$), such that $H(q,p)$ vanishes in $L$ (resp. $\tau + H(t,q,p)$ vanishes on $L$). A geometric solution to the equation* (HJE) *is a Lagrangian submanifold $L$ in $T^*(N\times\mathbb{R})$, such that $\tau + H(t,q,p)$ vanishes on $L$, the projection of $L$ on $N\times\mathbb{R}$ is proper and the reduction of $L$ at $t=0$ is $G_{df}$.*

We assumed for simplicity that $N$ is compact, but the theory extends to the non-compact setting, even though one has to take care of the behaviour at infinity (see [CV08]).

REMARKS 8.2.
(1) The case (HJE) is formally a special case of (HJ1), by taking $N = M\times\mathbb{R}$ and $K(q,p,t,\tau) = \tau + H(t,q,p)$, except that of course $M\times\mathbb{R}$ is non-compact ! However it is much easier to solve (HJE) !
(2) If a Lagrangian is a geometric solution of (HJ1), it must be invariant by the flow of $H$. Indeed, the vector $X_H(z)$ is $\omega$-orthogonal to any vector in $T_z\{(q,p) \mid H(q,p)=0\}$, hence to $T_zL$. But $T_zL$ is maximal isotropic, so $X_H(z)\in T_zL$ and this implies invariance of $L$ by the flow. The same argument holds for (HJE), except that the flow is now $X_K$

Our first result is

THEOREM 8.3. *[Sik89; Cha91] There exists a unique geometric solution for equation* (HJE).

PROOF. Indeed, let $\varphi_H^s$ be the flow of $H$. Then consider the flow on $T^*(N\times\mathbb{R})$ given by $\Phi_H^s(q,p,t,\tau) = (\varphi_H^s(q,p), t+s, \tau - H(t,\varphi_H^s(q,p)))$ that is the flow of the autonomous Hamiltonian $K(q,p,t,\tau) = \tau + H(t,q,p)$ on $T^*(N\times\mathbb{R})$. Then let

$$L_0 = \{(q, df(q), 0, -H(0,q,df(q)))$$

and

$$\Lambda = \bigcup_{s\in\mathbb{R}} \varphi_H^s(L_0)\times\{s\} = \Phi_H^1(L_0)$$

We claim that $\Lambda$ is Hamiltonian isotopic to the zero section. Note that it is obvious that $\Lambda\cap\{t=t_0\}/(\tau)$ the symplectic reduction of $\Lambda$ at $t=t_0$ is Hamiltonianly isotopic to

$0_N$, since it is equal to $L_{t_0} = \varphi_H^{t_0}(L_0)$ and $L_0$ is obviously isotopic to $0_N$. In our case, set $K_u(q,p,t,\tau) = \tau + uH(t,q,p)$ and the flow of $K_u$, denoted $\Phi_{K_u}^s$ satisfies $\Lambda = \Phi_{K_1}^1(L_0)$. $\square$

As a consequence, $L_{t_0}$ has a G.F.Q.I. denoted by $S_{t_0}$ and we may assume $t \mapsto S_t$ is continuous. Moreover $S_t$ is unique.

We shall consider equation (HJE), it is called the Cauchy problem for the evolution Hamilton-Jacobi equation. This is a special case of the general Hamilton-Jacobi equation (HJ1)

Of course we shall make the assumptions on $H$ precise later on. The classical method to solve these equations is called "the method of characteristics". The idea is that if $\varphi_H^t$ is the Hamiltonian flow of $H$, we could hope that $\varphi^t(\Lambda_f) = \mathrm{graph}(du_t)$ for some function $u_t$ and then $u_t$ solves (HJE).

DEFINITION 8.4. *Let $L$ be smooth Lagrangian submanifold in $T^*N$ having $S(x,\xi)$ as Generating Function Quadratic at Infinity. Fix a coefficient field $\mathbb{K}$. We denote by $u_L(x) = c(1_x, S_x)$ and call it the* **selector** *associated to $L$.*

REMARK 8.5. For different choices of $\mathbb{K}$ the selectors may be different. See an example in [Wei13]. In the sequel we assume a choice of $\mathbb{K}$ has been made.

We then have

PROPOSITION 8.6 ([OV94]). *The function $u_L$ is continuous and there exists a closed set of zero measure, $Z_L$, such that $u_L$ is smooth on $N \setminus Z_L$ and on this set $(x, du_L(x)) \in L$.*

PROOF. Let $Z_L^1$ be the set of singular values of the projection $\pi : L \longrightarrow N$. Then $Z_L^1$ is closed of zero measure. Let $U$ be a connected component of $N \setminus Z_L^1$. Then $\pi$ restricted to $\pi^{-1}(U)$ is a covering. We set

$$Z_L^2 = \left\{ x \in N \mid \exists \eta \neq \eta',\ \frac{\partial S}{\partial \xi}(x,\eta) = \frac{\partial S}{\partial \xi}(x,\eta') = 0,\ S(x,\eta) = S(x,\eta') \right\}$$

Note that for $\eta \neq \eta'$ as above we must have $\frac{\partial S}{\partial x}(x,\eta) \neq \frac{\partial S}{\partial x}(x,\eta')$ since $i_S : \Sigma_S \longrightarrow T^*N$ is an embedding.

We now claim that $Z_L^2$ is closed in $N \setminus Z_L^1$. Indeed let $x_n \in Z_L^2$ having limit $x \in N \setminus Z_L^1$. Let $\eta_n, \eta_n'$ be a sequence corresponding to $x_n$, so that $\eta_n \neq \eta_n'$. By extracting a subsequence, we may assume $\eta_n \longrightarrow \eta, \eta_n' \longrightarrow \eta'$.

Then,

- either $\eta \neq \eta'$ and then we cannot have $\frac{\partial S}{\partial x}(x,\eta) = \frac{\partial S}{\partial x}(x,\eta')$ otherwise $i_S$ would not be an embedding. Thus we have $x \in Z_L^2$.
- or we have $\eta = \eta'$. Then the if $u_n = i_S(x_n, \eta_n), u_n' = i_S(x_n, \eta_n')$ we have $u_n \neq u_n'$ and $\pi(u_n) = \pi(u_n') = x_n$. Setting $z = \lim_n u_n = \lim_n u_n'$ we have $z \in L$ since $L$ is closed, and $\pi(z) = x$. But then $d\pi(z)$ is not onto, otherwise $\pi$ would yield a local diffeomorphism between a neighborhood $W$ of $z$ in $L$ and a neighborhood of $x$ in $N$. Then for $n$ large enough, $u_n, u_n'$ should be in $W$ but this would imply $u_n = u_n'$ a contradiction. As a result $x \in Z_L^1$.

Assume now that $Z_L^2 \cap (N \setminus Z_L^1)$ is not a set of zero measure and $x_0 \in Z_L^2$ be a point in $Z_L^2 \cap (N \setminus Z_L^1)$. Since $\pi$ is a covering near $U$, set $(x_0, \eta)$ and $(x_0, \eta')$ be the corresponding points. Then in a neighborhood of $x_0$ we may find smooth functions $\eta(x), \eta'(x)$ coinciding with $\eta, \eta'$ at $x_0$ such that $(x, \eta(x)), (x, \eta'(x)) \in \Sigma_S$.

Then the set of $x$ such that

$$S(x, \eta(x)) = S(x, \eta'(x)) \text{ and } \frac{\partial S}{\partial x}(x, \eta(x)) \neq \frac{\partial S}{\partial x}(x, \eta'(x))$$

has zero measure since this is equal, setting $f(x) = S(x, \eta(x)) - S(x, \eta'(x))$ to the set $f(x) = 0, df(x) \neq 0$. But this is a nonsingular hypersurface, so has measure zero. Since we can only have a countable number of sheets over $x_0$, we get that $Z_L^2 \cap (N \setminus Z_L^1)$ is a countable union of sets of measure zero, so has measure zero. Since $Z_L^1$ has also measure zero, this proves our first claim. We set $Z_L = Z_L^1 \cup Z_L^2$.

Finally our Proposition follows if we can prove that we can find a smooth map $\eta$ defined on $N \setminus Z_L$ such that $u_S(x) = S(x, \eta(x))$. But if $x_0 \notin Z_L$, we can find a neighborhood $U$ of $x_0$ such that $\pi$ is a trivial covering from $\pi^{-1}(U)$ to $U$. Consider the various smooth sections, $\eta_j(x)$. Since $x \notin Z_L^2$ we cannot have $S(x, \eta_j(x)) = S(x, \eta_k(x))$ for $j \neq k$ unless $\frac{\partial S}{\partial x}(x, \eta_j(x)) = \frac{\partial S}{\partial x}(x, \eta_k(x))$, but then $L$ would not be embedded.

So $u_S$ defines a unique $j$ such that $u_S(x) = S(x, \eta_j(x))$ in $U$ and then $u_S$ is smooth in $U$. Since $\eta_j$ is smooth, we have

$$du_L(x) = \frac{\partial S}{\partial x}(x, \eta_j(x)) + \frac{\partial S}{\partial \xi}(x, \eta_j(x))$$

but since the last term is zero, we get

$$(x, du_L(x)) = \left(x, \frac{\partial S}{\partial x}(x, \eta_j(x))\right) \in L$$

This concludes our proof. □

As a consequence we get

THEOREM 8.7. *Let $u_{H,f}$ be a variational solution of* (HJE). *Then $u_{H,f}$ is continuous. It is $C^\infty$ and satisfies the equation* (HJE) *outside a closed set of measure zero.*

REMARK 8.8. Let us mention here a result of Seyfaddini and the author, that is mentioned in [Vic12]. Let $L_k$ be a sequence $\gamma$-converging to a smooth Lagrangian $L$. Then $L \subset \lim_k L_k$, that is for each $z \in L$ there is a sequence $z_k \in L_k$ such that $\lim_k z_k = z$. This is a direct consequence of lemma 7 in [HLS15]. This can be proved directly as follows. Indeed, if this was not the case, we would have $B(z, r)$ such that $B(z, r) \cap L_k = \emptyset$. Then for any $\varphi_H$ generated by a Hamiltonian $H$ supported in $B(z, r)$, we have $\gamma(L_k, \varphi(L_k)) = 0$, hence $\gamma(L, \varphi(L)) = 0$. But it is easy to see by a local construction that this does not hold for all $\varphi$ supported near $z$.

## 2. Billiard's dynamics

(see [Vit00; BG89; Iri12])

Let us consider a domain bounded by a smooth closed curve in the plane, or more generally a compact domain $\Omega$, bounded by a smooth hypersurface $\Sigma = \partial\Omega$ in a Riemannian manifold $N$ (assumed to be complete). The billiard defined by $\Omega$ is the following discrete dynamical system on

$$\nu_1^+\Sigma = \{(x,p) \in T^*N \mid x \in \Sigma, \langle p, \nu(x)\rangle > 0, |p| = 1\}$$

Note that in the case of a curve, $\nu^+\Sigma$ is a circle and we get a diffeomorphism of $S^1$. For $(x,p) \in \nu^+\Sigma$ we define $(x,\overline{p})$ by $\overline{p}$ is obtained by symmetry with respect to $\nu(x)$ from $p$. The dynamical billiard is describes as follows

DEFINITION 8.9. *Let $\Omega$ be as above. We define $\Phi_\Omega(x,p) = (x',p')$ where the unique geodesic from $x$ with speed $\overline{p}$ exits $\Sigma$ at $x'$ with speed $p'$. We call $x_1, x_2, \ldots., x_n$ a bouncing trajectory if there exists $p_1, \ldots, p_n$ such that $\Phi_\Sigma(x_j,p_j) = (x_{j+1},p_{j+1})$.*

Note that if we hit $\Sigma$ tangentially, we do not record the point, called a glancing point.

PROPOSITION 8.10. *The map $\Phi_\Sigma$ is smooth provided the interior of $\Sigma$ is geodesically strictly convex. In a billiard trajectory, the $p_j$ are uniquely defined. Moreover the $x_2, \ldots, x_n$ are critical points of the function*

$$\Sigma^{n-2} \longrightarrow \mathbb{R}$$

$$(y_2, \ldots, y_{n-1}) \mapsto \sum_{j=2}^{n-2} d(y_j, y_{j+1}) + d(x_1, y_2) + d(y_{n-1}, x_n)$$

There is another way of looking at billiard trajectories by recording the geodesic connecting the two points

DEFINITION 8.11. *Let $\Omega$ be as above. A billiard trajectory is a continuous curve $\gamma : [0,l] \longrightarrow \Omega$ such that there is a finite set $B_\gamma \subset [0,l]$ such that*

*(1)* $0, l \in B_\gamma$
*(2) for $t \notin B_\gamma$ we have $\dot\gamma(t) = 1$ and $\nabla_{\dot\gamma}\dot\gamma = 0$ i.e. $\gamma$ is a geodesic*
*(3) for $t \in B_\gamma$ $\gamma$ has left and right derivatives, $\dot\gamma_-, \dot\gamma_+$ such that $\dot\gamma_- - \dot\gamma_+ \in \mathbb{R}\nu(\gamma(t))$*

## 3. Exercises and Problems

### 3.1. Applications of the selector.

(1) (Maxwell relation) Let us consider the Lagrangian $L$ in $T^*\mathbb{R}$ of Figure 2

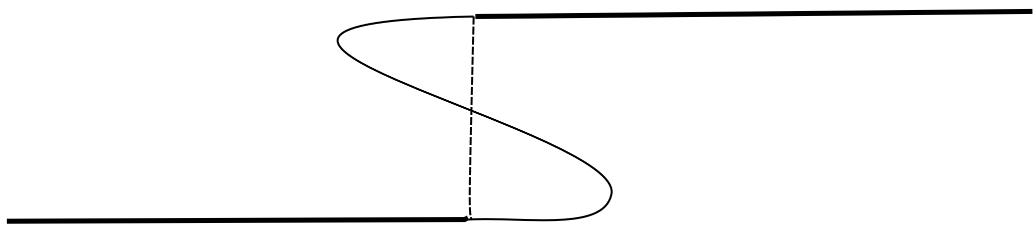

FIGURE 2. The Lagrangian $L$ and the graph of $du_L$

where the graph of the differential of the selector is the thicker line. Prove that the areas in the figure below are equal

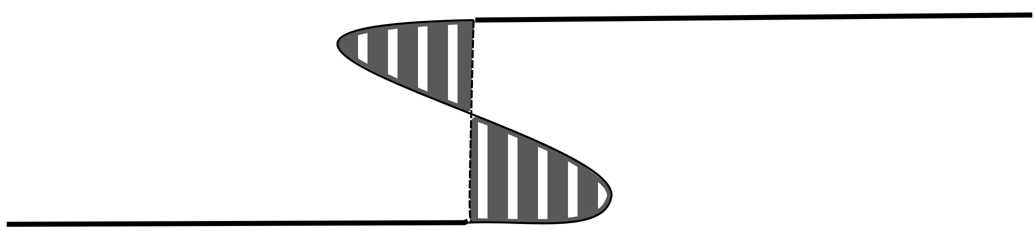

FIGURE 3. Maxwell equal area condition

(2) Let $G$ be a group acting transitively on $N$ and we consider the induced action on $T^*N$. Then the only $G$-invariant Lagrangian Hamiltonianly isotopic to the zero section is $0_N$.

HINT. *Let $u_L(x)$ be a selector associated to $L$. Then $u_{gL}(x) = u_L(gx)$ and conclude that $u_L$ is constant. Use Proposition 8.6 to show that* $0_N \subset L$.

(3) Use the method of Exercice 2 to reprove Exercice 7

(4) (Contact Hamilton-Jacobi equation) Let us consider a function $H(q,p,z)$ on the contact manifold $J^1(N,\mathbb{R})$. We defined the contact Hamiltonian $X_H$ in Definition 3.80. We also define a co-Legendrian manifold as a manifold $\Lambda$ such that at each poin t $z \in \Lambda$ we have that $T_z \cap \ker(\alpha)$ is Lagrangian in $(\ker(\alpha), d\alpha)$ and contains the Reeb vector field.

(a) Prove that the flow by a contact Hamiltonian of a co-Legendrian is co-Legendrian.
(b) Prove that If $H=0$ on a co-Legendrian submanifold $\Lambda$, then $X_H$ is tangent to $\Lambda$.
(c) Assume now that $N=M\times\mathbb{R}$

Unfinished

### 3.2. Hamilton-Jacobi PDE and flows.

(5) Let $\varphi^t$ be the Hamiltonian flow associated to $H(t,q,p)$ on $T^*N$. We set $\varphi^t(q,p)=(Q_t(q,p),P_t(q,p))$. Let

$$\Gamma(\varphi)=\left\{(t,\tau(t,q,p),q,p,Q_t(q,p),P_t(q,P))\mid t\in\mathbb{R},(q,p)\in T^*N\right\}$$

(a) Determine $\tau(t,q,p)$ so that $\Gamma(\varphi)$ is Lagrangian in $\mathbb{R}^2\times T^*N\times T^*N$
(b) We assume $N=\mathbb{R}^n$ and let $S(t,Q,p,\xi)$ be a generating function (not necessarily quadratic at infinity) for $\Gamma(\varphi)$ that is

$$\Gamma(\varphi)=\left\{(t,\frac{\partial S}{\partial t}(t,Q,p;\xi),Q,\frac{\partial S}{\partial Q}(t,Q,p;\xi))\mid \frac{\partial S}{\partial \xi}(t,Q,p;\xi)=0\right\}$$

Prove that $Q_t(q_0,p_0),P_t(q_0,p_0)$ are defined by

$$\begin{cases} 0=\dfrac{\partial S}{\partial \xi}(t,Q,p;\xi)\\ q_0=\dfrac{\partial S}{\partial p}(t,Q_t,p_0;\xi)\\ P_t=\dfrac{\partial S}{\partial Q}(t,Q_t,p;\xi)\end{cases}$$

In other words $Q_t(q_0,p_0)$ is defined by the first two equations and then $P_t$ is determined by the third one.
(c) Prove that $S$ satisfies the following Hamilton-Jacobi equation

$$\begin{cases} \dfrac{\partial S}{\partial t}(t,Q,p;\xi)+H(t,Q,\dfrac{\partial S}{\partial Q}(t,Q,p;\xi))=0\\ \dfrac{\partial S}{\partial \xi}(t,Q,p;\xi)=0\end{cases}$$

(d) What does this become if there is no $\xi$ variable ? If $H$ is of the form $\frac{1}{2}|p|^2+V(q)$ so that the equation is $p=\dot q, \dot q=\nabla V(q)$ or $\ddot q(t)+\nabla V(q(t))=0$, so that we only need to find $q(t)$ (because $p(t)=\dot q(t)$) ?

(6) (see [Jac66], Chap 30) Let $x\in\mathbb{R}^n$ and consider the equation $\ddot x=0$. We are going to transform - following Jacobi- this equation through a change of variable into a much more interesting one, proving in particular the law of addition of

abelian integrals !! Let us consider $(x_1,\dots,x_n)\in\mathbb{R}^n$ $a_1<a_2<\dots<a_n$ be fixed real numbers and consider the real roots of

$$\sum_{j=1}^{n}\frac{x_j^2}{a_j+\lambda}=1$$

We denote them by $\lambda_1>\lambda_2>\dots>\lambda_n$. The $\lambda=(\lambda_1,...,\lambda_n)$ are called the elliptic coordinates of the point $(x_1,...,x_n)$. Note that for fixed $\lambda$ the above hypersurface is a conic, and for $\lambda_k$ corresponds to a conic of signature $n-k$ (for example for $n=2$ an ellipse and a hyperbola, for $n=3$ an ellipsoid, a one-sheeted hyperboloid, and a two-sheeted hyperboloid). One can check that they intersect orthogonally, and they are confocal (i.e. have the same focal points Note that $\lambda_j$ only depends on the $x_j^2$.

We set

$$A(z)=\prod_{j=1}^{n}(z+a_j)$$

and

$$F(z;x)=A(z)-\sum_{j=1}^{n}x_j^2\prod_{k\neq j}(a_k+z)=\prod_{j=1}^{n}(z-\lambda_j)$$

We shall write $F(z)$ when the $x_j$ are implicit.

(a) Prove that indeed, $\sum_{j=1}^{n}\frac{x_j^2}{a_j+\lambda}-1=0$ has $n$ real solutions in $\lambda$ such that $-a_k<\lambda_k<-a_{k-1}$ (find the poles in $\lambda$ of the expression $\sum_{j=1}^{n}\frac{x_j^2}{a_j+\lambda}$)

(b) We want to recover $x_j^2$ from the $\lambda_j$. Prove that

$$x_k^2=-\frac{\prod_{j=1}^{n}(a_k+\lambda_j)}{\prod_{j\neq k}(a_k-a_j)}=\frac{F(-a_k)}{A'(-a_k)}$$

HINT. *$x_k^2$ is the residue of $\frac{F(z)}{A(z)}$ at $z=-a_k$*

(c) (Technical lemma 1) Prove that setting

$$M_k(\lambda)=M_k(\lambda_1,...,\lambda_n)=\sum_{j=1}^{n}\frac{x_j^2}{(a_j+\lambda_k)^2}$$

we have

$$M_k(\lambda)=\frac{F'(\lambda_k)}{A(\lambda_k)}=\frac{\prod_{j\neq k}^{n}(\lambda_k-\lambda_j)}{\prod_{j=1}^{n}(a_j+\lambda_k)}$$

HINT. *$M_k$ is the value of $\frac{d}{dz}\frac{F(z)}{A(z)}$ at $z=\lambda_k$*

(d) (Technical lemma 2) Prove that for any polynomial $R(z) = \sum_{1\le l\le n-1} p_l z^{l-1}$ of degree $n-1$ we have

$$\sum_{k=1}^{n} \frac{R(\lambda_k)}{\prod_{j\neq k}(\lambda_k - \lambda_j)} = 1$$

HINT. *Use the fact that the sum of all residues of a meromorphic function (including the one at infinity) is zero (see [Car95], chapter 3, section5). Show that for* $\frac{R(z)}{\prod_{j=1}^{n}(z-\lambda_j)}$ *the residue at infinity is* $-p_{n-1}$.

(e) Let $F(x_1,...,x_n)$ and $G(\lambda_1,...,\lambda_n) = F(x_1,...x_n)$ if $\lambda_j = \lambda_j(x_1,...,x_n)$. Prove that

$$\sum_{k=1}^{n}\left(\frac{\partial F}{\partial x_k}\right)^2 = \sum_{k=1}^{n}\frac{1}{M_k(\lambda)}\left(\frac{\partial G}{\partial \lambda_k}\right)^2$$

(f) Prove that $\sum_{k=1}^{n}\left(\frac{\partial F}{\partial x_k}\right)^2 = 1$ has the general solution

$$F(x_1,...,x_n) = p_1x_1 + ... + p_nx_n$$

with $\sum_{j=1}^{n} p_j^2 = 1$ and corresponds to the Hamiltonian $H(x,p) = \frac{1}{2}p^2$

(g) We consider the Hamilton-Jacobi equation

$$\sum_{k=1}^{n}\frac{1}{M_k(\lambda)}\left(\frac{\partial G}{\partial \lambda_k}(\lambda)\right)^2 = 1 \tag{HJ'}$$

Prove that it is equivalent to

$$\sum_{k=1}^{n}\frac{\prod_{j=1}^{n}(a_j+\lambda_k)}{\prod_{j\neq k}^{n}(\lambda_k-\lambda_j)}\left(\frac{\partial G}{\partial \lambda_k}(\lambda)\right)^2 = \sum_{k=1}^{n}\frac{R(\lambda_k)}{\prod_{j\neq k}(\lambda_k-\lambda_j)}$$

for $R$ a unitary polynomial (i.e. with leading coefficient 1) of degree $n-1$ (use question 6d)

(h) Prove that $G$ solves the above whenever $G(\lambda_1,...,\lambda_n) = \sum_{k=1}^{n} G_k(\lambda_k)$ where

$$\prod_{j=1}^{n}(a_j+\lambda_k)G_k'(\lambda_k)^2 = R(\lambda_k)$$

(i) Prove that

$$G_k(z;p) = c_k + \int_0^z \sqrt{\frac{z^{n-1}+\sum_{1\le l\le n-1} p_l z^{l-1}}{\prod_{j=1}^{n}(a_j+z)}}dz$$

solves the above for $p = (p_1,...,p_{n-1})$

(j) Prove that this yields all solutions of the Hamilton-Jacobi equation

(k) Use the previous exercice to prove that the solution $\lambda(t)$ is given by

$$\sum_{k=1}^{n}\frac{\partial G_k}{\partial p_l}(\lambda_k,p_1,...,p_{n-1}) = q_l$$

that is

$$\sum_{k=1}^{n}\int_0^{\lambda_k}\frac{z^{l-1}dz}{\sqrt{(z^{n-1}+\sum_{1\le l\le n-1}p_l z^{l-1})\prod_{j=1}^{n}(a_j+z)}}dz=\sum_{k=1}^{n}\int_0^{\lambda_k}\frac{z^{l-1}dz}{\sqrt{R(z)A(z)}}=q_l$$

(l) Fix the $(q_l, p_l)$. Prove that the $\lambda_1, ..., \lambda_n$ satisfying the equality

$$\sum_{k=1}^{n}\int_0^{\lambda_k}\frac{z^{l-1}dz}{\sqrt{R(z)A(z)}}=q_l$$

are the roots of $F(z,x)$ where $x_j=\xi_j t+\eta_j$

(m) Note that $w^2=R(z)A(z)$ defines a hyperelliptic curve of genus $2n-1$, and the integrals $\int z^{l-1}\frac{dz}{\sqrt{R(z)A(z)}}$ are called elliptic integrals of the first kind. For $n=2$ we get an ellitpic curve. Prove that translating the relation $x(t+s)+x(0)=x(t)+x(s)$ in terms of $\lambda$ implies the addition law on a cubic.

(n) Prove that the linear structure on the torus given by the Arnold-Liouville theorem coincides with the linear structure on the real part of the Jacobian curve of a hypereliiptic curve.

### 3.3. Hamilton-Jacobi equations and symplectic integrators.

(7) (Symplectic integrators I , see [HLW06] p. 182 and seq.) Let $\varphi^t(q,p)$ be the flow of the Hamiltonian $H(q,p)$, assumed to be smooth.

(a) Prove that for $t$ small enough, we have a generating function $S_t(q,P)$ for $\varphi^t$, that is

$$(Q,P)=\varphi^t(q,p)\Leftrightarrow p-P=\frac{\partial S_t}{\partial q}(q,P), Q-q=\frac{\partial S_t}{\partial P}(q,P)$$

(b) Prove that $S_t$ satisfies the Hamilton-Jacobi equation

$$\begin{cases}\dfrac{\partial S_t}{\partial t}(q,P)=H(q+\dfrac{\partial S_t}{\partial P},P)\\ S_0(q,P)=0\end{cases}$$

(c) Prove that for $t$ close to 0 we have a Taylor expansion

$$S_t(q,P)=tH(q,P)+t^2\left\langle\frac{\partial H}{\partial q}(q,P),\frac{\partial H}{\partial P}(q,P)\right\rangle+o(t^2)$$

More generally we set $S_t^{(j)}(q,P)$ to be the degree $j+1$ Taylor polynomial of the expansion of $S_t$ in $t$. Thus

$$S_t^{(0)}=tH(q,P)$$

$$S_t^{(1)}(q,P)=tH(q,P)+t^2\left\langle\frac{\partial H}{\partial q}(q,P),\frac{\partial H}{\partial P}(q,P)\right\rangle$$

etc. Compute $S_t^{(2)}$.

Denote by $\Phi_t^{(j)}(q,p)$ the map defined by the generating function $S_t^{(j)}$. That is

$$(Q,P)=\Phi_t^{(j)}(q,p)\Leftrightarrow p-P=\frac{\partial S_t^{(j)}}{\partial q}(q,P), Q-q=\frac{\partial S_t^{(j)}}{\partial P}(q,P)$$

Note that this defines an implicit numerical scheme : computing $Q,P$ from $q,p$ requires solving an implicit equation, that is $Q-q=\frac{\partial S_t^{(j)}}{\partial P}(q,P)$. However if $j=1$ and $H$ is the sum of a kinetic and potential energy (i.e. $H(q,p)=T(p)+U(q)$ the scheme becomes explicit.

(d) Prove that the scheme thus defined is of order $j$, that is

$$\|\Phi_h^{(j)}(q,p)-\varphi^h(q,p)\|\leq O(h^{j+1})$$

and that setting $nh=T$ we have

$$\|(\Phi_h^{(j)})^n(q,p)-\varphi^T(q,p)\|\leq O(nh^{j+1})=O(h^j)$$

(e) Let $H_t^{(1)}(q,p)$ be given by $H_t(q,p)=H(q,p)+tH_1(q,p)$. Prove that we can choose $H_1$ such that

$$|H_h^{(1)}(\Phi_h^{(j)}(q,p))-H_h^{(1)}(q,p)|\leq O(h^2)$$

hence

$$|H_h^{(1)}((\Phi_h^{(j)})^n(q,p))-H_h^{(1)}(q,p)|\leq O(nh^2)$$

and setting $nh=T$, the numerical scheme obtained by replacing $\varphi^T$ by $(\Phi_h^j)^n$ preserves the energy $H^{(1)}$ up to an $O(h)$.

(f) Prove that we can find $H^{(j)}(q,p)=H(q,p)+tH_1(q,p)+\ldots+t^jH_j(q,p)$ such that

$$|H_h^{(j)}((\Phi_h^{(j)})^n(q,p))-H_h^{(j)}(q,p)|\leq O(nh^{j+1})$$

so that $H^{(j)}$ is preserved by $(\Phi_h^{(j)})^n$ for $nh=T$ up to a term of order $O(h^j)$. In particular we can take $T\simeq Ch^{1-j}$, i.e. compute long time solutions, and we will quite accurately preserve the energy $H^{(j)}$.

(g) Write a python (or Matlab or your favorite programming language) code for the Hamiltonian $H(q,p)=\frac{p^2+q^2}{2}+$

(8) (Numerical schemes for Hamiltonian flows [HLW06]) Let $H(q,p)$ be a smooth Hamiltonian with flow $\varphi_H^t$ and $(Q,P)=\Phi_h(q,p)$ for $h\in\mathbb{R}_+^*$ a map. We say that $\Phi_h$ is a symplectic integrator of order $p$ if $\Phi_h$ is a symplectic map and

$$|\varphi_H^t(q,p)-(\Phi_t)(q,p)|\leq O(t^{p+1})$$

for $(q,p)$ in a bounded region. The aim is, given $H$, to construct $\Phi_h$ in a reasonable explicit way (as opposed to $\varphi^t$ which is usually not explicit) and then approximate numerically $\varphi^t$ by considering the sequence $(q_{n+1},p_{n+1})=\Phi_h(q_n,p_n)$ for $n=N$ where $Nh=t$.

(a) Consider the explicit Euler scheme

$$Q = q + h\frac{\partial H}{\partial p}(q,p), P = p - h\frac{\partial H}{\partial q}(q,p)$$

Prove that in general it is not a symplectic integrator

(b) Consider the symplectic Euler scheme defined by the implicit equation

$$Q = q + h\frac{\partial H}{\partial p}(q,P), P = p - h\frac{\partial H}{\partial q}(q,P)$$

Prove that this is a symplectic integrator of order 1. Note that $Q$ is explicit in $q, P$ but computing $P$ requires solving an implicit equation.

HINT. *$\Phi_h$ is given by a generating function ! See Exercise 7.*

(c) Prove that in general $\Phi_h$ does not preserve a first integral (i.e. there is no hypersurface preserved by $\Phi_h$.

(d) Compute the symplectic Euler scheme for $H(q,p) = \frac{1}{2}p^2 - V(q)$ (in this case the scheme is in fact explicit). Using any computing program compare the explicit and symplectic Euler scheme for $V(q) = \cos(q)$

(9) We continue Exercise 8.

(a) Prove that $z_{n+1} = z_n + hX_H(\frac{z_n+z_{n=1}}{2})$ defines a symplectic integrator of order 2

(b) Let $(a_{i,j})$ be an $s\times s$ matrix, and set $c_i = \sum_{j=1}^s a_{i,j}$ and define a level $s$ Runge-Kutta method for $\dot{z}(t) = f(t, z(t))$ to be given by solving

$$k_i = f\left(t_0 + c_i h, y_0 + h\sum_{j=1}^s a_{i,j}k_j\right)$$

and let $Z = \Phi_h(z)$ be defined by

$$Z = z + h\sum_{j=1}^s b_i k_i$$

(c) We want to prove the following theorem due to Bochev and Scovel ([BS94]):

THEOREM. *Assume the constants $a_{i,j}, b_i$ are so chosen that $\Phi_h$ preserves the levels of $Q_C(z) = \langle Cz, z\rangle$ whenever this is the case for $\dot{z}(t) = f(z(t))$. Then the Runge-Kutta method is a symplectic integrator of order $s$.*

(10) (see [Yos90] and [EPR10] for application to symplectic topology) Let $H, K$ be two non-commuting Hamiltonians. We want to approximate the flow $\varphi_{H+K}^t$ by a composition

$$\varphi_H^{a_1 t}\circ\varphi_K^{b_1 t}\circ\ldots\circ\varphi_H^{a_k t}\circ\varphi_K^{b_k t}$$

where we must determine optimal values of $a_j, b_j$ for fixed $k$ i.e. values such that the difference with $\varphi_{H+K}^t$ is as small as possible as a function of $t$.

(a) Look up the Baker-Campell-Hausdorff formula[1] to express the general expression of $\exp(X)\exp(Y)$ as an exponential of an infinite series

$$X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\frac{1}{24}[X,[Y,[Y,X]]]+\dots$$

We denote by $B(X,Y)$ the above expression. This formula holds for the exponential of any finite dimensional Lie algebra, and converges provided $X,Y$ are small enough.

HINT. *We can always write the formal series of* $\log(\exp(X)\exp(Y))$ *where* $X,Y$ *are non commuting variables. It is clear that we shall get a series with terms of the form* $X^{r_1}Y^{s_1}\dots X^{r_k}Y^{s_k}$. *What is less obvious is that the sum of the terms of degree k can be expressed in term of iterated commutators : setting* $[X,Y]=XY-YX$. *See [Eic68] for a short proof. Even less obvious is finding the minimal number of them (thus there is only one term of order* 4*, while we could expect terms of the type* $[X,[X,[Y,X]]]$*).*

(b) Give an example of element in a Lie group that can not be written as an exponential. Use this to show that the Baker-Campbell-Hausdorff formula can diverge.

HINT. *The matrix* $M=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$ *cannot be an exponential since it has negative eigenvalues and is non-diagonalizable (it would have to be the exponential of a matrix having* $\pm(2k+1)i\pi$ *as eigenvalues hence diagonalizable, a contradiction). Thus for* $J$ *the standard symplectic matrix and* $N=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}$ *we have* $\exp(\pi J)\exp(N)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}=M$ *is not an exponential.*

(c) Prove that the following version of the Baker-Campbell-Hausdorff formula holds for two smooth vector fields $X,Y$. We set $\exp(tX)$ to be the flow of $X$. We set $B_k(X,Y)$ to be the coefficient of $t^k$ in the formal power series $B(tX,tY)$. Then we have

$$\left|\exp(tX)\exp(tY)-\exp\Big(\sum_{j=0}^{k}B_j(X,Y)t^j\Big)\right|\le O(t^{k+1})$$

Note that since there are arbitrarily small diffeomorphisms which are not exponential of an autonomous vector field, it is unlikely that the Baker-Campbell-Hausdorff formula converges however small $X,Y$ are assumed to be.

HINT. *One just has to prove that both terms have a Taylor series and that these coincide up to order k. The existence of the Taylor series for* $\exp(tX)\exp(tY)$

[1] Sometimes also called Campbell-Baker-Hausdorff-Dynkin formula.

*follows from the strong form of Cauchy-Lipschitz that claims smoothness of the solutions of a smooth differential equation : this implies it has a Taylor expansion. Equality holds by a formal argument, since it holds in a formal Lie algebra.*

(d) Use the Baker-Campbell-Hausdorff formula to find a symplectic integrator of order 1. Of order 2. Of order 4.

**3.4. Thermodynamics.**

(11) (Phase diagrams, Hamilton-Jacobi and variational solutions) We consider as in Exercice 68 in Chapter 3 a closed thermodynamical system. An set of equilibria of the system is given by a Legendrian submanifold of $J^1(\mathbb{R}^n)$. In variables $(q_1, q_2, p_1, p_2, z)$ the variables $q_j$ are called "extensive" variables and the $p_j$ "intensive" variables. For example pressure and temperature are intensive variables $P, T$ while volume and entropy $V, S$ are extensive. The one form is in this case given by $dU - TdS - PdV$ and the equation $dU - TdS - PdV = 0$ translates the fact that the total variation of the energy is the sum of the heat outgoing and the work done by the system.

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## 4. Comments

The geometric approach to Hamilton-Jacobi equations coincides, for short time, with the classical method of characteristics. This method is due to Paul Charpit de Villecourt[2] who presented it at the French Academy in 1784. The manuscript was long lost, until a copy was found in some of Lacroix's papers but is still unpublished[3] (see [GE82; Sal30; Sal37]. The method of characteristics was then developed by Lagrange, Monge, Pfaff, Jacobi. Strangely enough, going in the opposite direction, from the Hamiltonian equations of dynamics to the Hamilton-Jacobi equation appeared much later in Hamilton's work and then in Jacobi's lectures on Dynamics (see [Jac66]) where it plays a crucial role (see for example Exercice 5). The need for symplectic integrators came from the very practical problem of modeling particles in accelerators over a huge number of revolutions. As such it was rediscovered several times, after pioneering (and unpublished) work by De Vogelaere (see [De 56] and [SC]) in the 1950's, the in the 1980's (see [Rut83; Cha83; Fen85; Yos90]) It was then used for other systems (see [CS90] for some examples) and in particular for long term modeling of the solar system (see [LR01])

The variational approach for the study of billiards goes back to Birkhoff ([Bir27], and we refer to recent work by [BG89]. The approach using symplectic capacities goes back to [Vit00] and was developed by many authors among them [Iri12; AO14] etc.

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[2] Sometimes known as de Ville Coer or just Paul Charpit.

[3] Jacobi insists on the importance of publishing Paul Charpit's Memoir

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