An Introduction to Symplectic Topology through Sheaf theory

Princeton, Fall 2010-New York, Spring 2011

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So do I gather strength and hope anew;
For well I know thy patient love perceives
Not what I did, but what I strove to do,–
And though the full, ripe ears be sadly few,
Thou wilt accept my sheaves.

Bringing Our Sheaves with Us (1858)
by Elizabeth Chase Allen

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CHAPTER 1

Introduction

This are the notes of graduate lectures given in the fall semester 2010 at Princeton University, and then as the Eilenberg lectures at Columbia in the spring 2011. The first part of the symplectic part of the course (chapter 2 to 4) corresponds to a course given at Beijing Unversity on 2007 and 2009, with notes by Hao Yin (Shanghai Jiaotong University). The aim of this course is to present the recent work connecting sheaf theory and symplectic topology, due to several authors, Nadler ([**Nad, Nad-Z**], [**Tam**], Guillermou-Kashiwara-Schapira [**G-K-S**]. This is completed by the approach of [**F-S-S**], and the paper [**F-S-S2**] really helped us to understand the content of these works.

Even though the goal of the paper is to present the proof of the classical Arnold conjecture on intersection of Lagrangians, and the more recent work of [**F-S-S**] and [**Nad**] on the topology of exact Lagrangians in T^*X, we tried to explore new connections between objects. We also tried to keep to the minium the requirements in category theory and sheaf theory necessary for proving our result. Even though the appendices contain some material that will be useful for those interested in pursuing the sheaf theoretical approach, much more has been omitted, or restricted to the setting we actually use[1] The experts will certainly find that our approach is "not the right one", as we take advantage of many special features of the category of sheafs, and base our approach of derived categories on the Cartan-Eilenberg resolution. We can only refer to the papers and books in the bibliography for a much more complete account of the theory.

The starting point is the idea of Kontsevich, about the homological interpretation of Mirror symmetry. This should be an equivalence between the derived category of the $D^b(\mathbf{Fuk}(\mathbf{M},\omega))$, the derived cateogory of the category having objects the (exact) Lagrangians in (M,ω) and morphisms the elements in the Floer cohomology (i.e. $\mathrm{Mor}(L_1,L_2) = FH^*(L_1,L_2)$) the derived category of coherent sheafs on the Mirror, $D^b(\mathbf{Coh}(\check{\mathbf{M}},\mathbf{J}))$. Our situation is a toy model, in which $(M,\omega) = (T^*X, d(pdq))$, and $D^b(\mathbf{Coh}(\check{\mathbf{M}},\mathbf{J}))$ is then replaced by $D^b(\mathbf{Sheaf_{cstr}}(\mathbf{X}\times\mathbb{R}))$ the category of constructible sheafs (with possibly more restrictions) on $X\times\mathbb{R}$.

There is a functor

$$SS : D^b(\mathbf{Sheaf_{cstr}}(\mathbf{X}\times\mathbb{R})) \longrightarrow D^b(\mathbf{Fuk}(\mathbf{T^*X},\omega))$$

[1]For example since the spaces on which our sheafs are defined are manifolds, we only rarely discuss assumptions of finite cohomological dimension.

determined by the singular support functor. The image does not really fall in $D^b(\mathbf{Fuk}(\mathbf{T}^*\mathbf{X},\omega))$, since we must add the singular Lagrangians, but this a more a feature than a bug. Moreover we show that there is an inverse map, called " Quantization" obtained by associating to a smooth Lagrangian L, a sheaf over X, $\mathscr{F}_L$ with fiber $(\mathscr{F}_L)_x = (CF_*(L,V_x),\partial_x)$ where V_x is the Lagrangian fiber over x and $CF_*(L,V_x),\partial_x)$ is the Floer complex of the intersection of L and V_x. This is the **Floer quantization** of L. This proves in particular that the functor SS is essentially an equivalence of categories. We are also able to explain the condition for the **Floer quantization** of L to be an actual quantization (i.e. to be well defined and provide an inverse to SS). Due to this equivalence, for complexes of sheafs $\mathscr{F}^\bullet,\mathscr{G}^\bullet$ on X, we are able to define $H^*(\mathscr{F}^\bullet,\mathscr{G}^\bullet) = H^*(\mathscr{F}^\bullet \otimes (\mathscr{G}^\bullet)^*)$ as well as $FH^*(SS(\mathscr{F}),SS(\mathscr{G}))$ and these two objects coincide. We may also define $FH^*(L,\mathscr{G})$ as $H^*(\mathscr{F}_L,\mathscr{G})$.

I thank Hao Yin for allowing me to use his lecture notes from Beijing. I am very grateful to the authors of [**Tam**], [**Nad**], [**F-S-S**] and [**F-S-S2**] and [**G-K-S**] from where theses notes drew much inspiration. It is a special pleasure to thank Stéphane Guillermou for a talk he gave at Symplect'X seminar, and many useful discussions, to Pierre Schapira for patiently explaining me many ideas of his theory and dispelling some naive preconceptions, to Paul Seidel and Mohammed Abouzaid for discussions relevant to the General quantization theorem. Finally I thank the University of Princeton, the Institute for Advanced Study and Columbia University for hospitality during the preparation of this course. A warm thanks to Helmut Hofer for many discussions and for encouraging me to turn these notes into book form.

New-York, Spring 2011

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Part 1

Elementary symplectic geometry

CHAPTER 2

Symplectic linear algebra

1. Basic facts

Let V be a finite dimensional real vector space.

DEFINITION 2.1. A symplectic form on V is a skew-symmetric bilinear nondegenerate form, i.e. a two-form satisfying:

(1)
$$\forall x, y \in V \;\; \omega(x, y) = -\omega(y, x)$$
(hence $\forall x \in V \;\; \omega(x, x) = 0$);

(2) $\forall x, \exists y$ such that $\omega(x, y) \neq 0$.

Examples:

(1) $V = \mathbb{R}^2$ with the symplectic form $\sigma_1((x, y), (x', y')) = xy' - x'y$.
(2) If $(V_1, \omega_1), (V_2, \omega_2)$ are symplectic spaces, $V = V_1 \oplus V_2$, $\omega = \omega_1 \oplus \omega_2$ defined by $\omega(v_1 + v_2, v_1' + v'2) = \omega_1(v_1, v'1) + \omega_2(v_2, v_2')$ for $v_i, v'I \in V_i$ is also symplectic.
(3) Combining the above two examples, we get a symplectic structure $\sigma_n = \sigma_1 \oplus \dots \oplus \sigma_1$ on $\mathbb{R}^{2n}$.
(4) If L is a vector space, and L^* its dual space, $L \oplus L^*$ endowed with $\sigma_L(x, x^*), (y, y^*)) = x^*(y) - y^*(x)$ is symplectic. Taking $L = \mathbb{R}$ we get the symplectic form σ_1 on $\mathbb{R}^2$ and taking $L = \mathbb{R}^n$ we get $(\mathbb{R}^{2n}, \sigma_n)$.

For a general 2-form ω on a vector space, V, we denote by $\mathrm{Ker}(\omega)$ the subspace given by
$$\mathrm{Ker}(\omega) = \{\mathrm{v} \in \mathrm{V} \mid \forall \mathrm{w} \in \mathrm{V}\, \omega(\mathrm{v}, \mathrm{w}) = 0\}$$

The second condition implies that $\mathrm{Ker}(\omega)$ reduces to zero, so when ω is symplectic, there are no "preferred directions" in V.

DEFINITION 2.2. Let (V, ω) be a symplectic vector space. We denote by $Sp(V, \omega)$ the group of automorphisms of V preserving ω. In other terms, $T \in Sp(V, \omega)$ if $\omega(Tx, Ty) = \omega(x, y)$ for all $x, y \in V$.

LEMMA 2.3. *Let W be a subspace of the symplectic space (V, ω). Setting $K = (W \cap W^\omega) = \mathrm{Ker}(\omega_{|\mathrm{W}})$, we have a decomposition $W = K \oplus S$ with S symplectic. Moreover there is a subspace $K' \subset S^\omega$ such that $K \oplus K' \oplus S$ is symplectic, where K' is identified to K^* through the map $x \mapsto \omega(x, \bullet)$. In other words, $(K \oplus K', \omega_{|K \oplus K'})$ is isomorphic to $K \oplus K^*$.*

PROOF. Indeed, if S is a complement of K, then $\omega_{|S}$ is non-degenerate, since if $v \in W \cap S^{\omega}$ we have $v \in K^{\omega} \cap S^{\omega} \subset W \cap (K \oplus S)^{\omega} = W \cap W^{\omega} \subset K$. Now the map $V^* \longrightarrow K^*$ induced by inclusion is onto and since ω induces an isomorphism from V to V^*, the map induced by ω from V to K^* is onto, and has kernel containing S. It is thus again onto on any complementary subspace of S, in particular in S^{ω}. If K' is a subspace of S^{ω} of the same dimension as K, such that the map ω restricted to K' is onto, we have the decomposition described in the lemma. ☐

PROPOSITION 2.4. *The group $SP(V,\omega)$ acts transitively on $V \setminus \{0\}$.*

PROOF. Obviously there is no symplectic vector space in dimension one. Indeed, assume first we proved the theorem for 2-dimensional symplectic spaces. So assume $\dim(V) \geq 3$. Let x, y be two vectors in V, such that $\omega(x, y) \neq 0$. We set U the vector space generated by x, y and since $\omega_{|U}$ is nondegenerate, $V = U \oplus U^{\omega}$. Let T_U such that $T_U(x) = y$. Set $T = T_U \oplus \mathrm{Id}_{U^{\omega}}$. Then T is in $Sp(V,\omega)$ and $T(x) = y$. If now we have two vectors such that $\omega(x, y) = 0$ we can find z such that $\omega(z, x), \omega(z, y) \neq 0$. Otherwise $\omega(x,z)\omega(y,z) = 0$ for all z, hence V is the union of two proper hyperplanes which is impossible[1]. When $\dim(V) = 2$, we can see that if x, y are independent, $x \mapsto y, y \mapsto -x$ is symplectic so $Sp(V,\omega)$ acts transitively on the set of pairs of linearly independent vectors. If x, y are linearly dependent, we can always find z such that the pairs x, z and z, y are pairs of independent vectors. ☐

There are special types of subspaces in symplectic manifolds. For a vector subspace F, we denote by

$$F^{\omega} = \{v \in V \mid \forall w \in F\ ,\ \omega(v, w) = 0\}$$

the symplectic orthogonal of F. From Grassmann's formula applied to the surjective map $\varphi_F : V \to F^*$ given by $\varphi_F(v) = \omega(v, \bullet)$, it follows that $\dim(F^{\omega}) = \dim(\mathrm{Ker}(\varphi_{\mathrm{F}})) = \mathrm{codim}(\mathrm{F}) = \dim(\mathrm{V}) - \dim(\mathrm{F})$. Moreover the proof of the following two formulas is left to the reader

PROPOSITION 2.5.

$$(F^{\omega})^{\omega} = F$$

$$(F_1 + F_2)^{\omega} = F_1^{\omega} \cap F_2^{\omega}$$

DEFINITION 2.6. A map $\varphi : (V_1, \omega_1) \to (V_2, \omega_2)$ is a **symplectic map** if $\varphi^*(\omega_2) = \omega_1$ that is

$$\forall x, y \in V_1, \omega_2(\varphi(x), \varphi(y)) = \omega_1(x, y)$$

It is a **symplectomorphism** if and only if it is invertible- its inverse is then necessarily symplectic. A subspace F of (V,ω) is

[1]If the field is infinite, $z \mapsto \omega(x,z)\cdot\omega(y,z)$ would be a second degree polynomial vanishing identically on the field, so it is zero, but then either $z \mapsto \omega(x,z)$ or $z \mapsto \omega(y,z)$ is identically zero, hence x or y are zero. This is also impossible for a vector space on a finite field $\mathbb{F}_q$, since a hyperplane has cardinal $q^{\dim(V)-1}$ so the union of two hyperplanes has cardinal at most $2q^{\dim(V)-1} - 1 < q^{\dim(V)}$

- isotropic if $F \subset F^{\omega}$ ($\Longleftrightarrow \omega|_F = 0$);
- coisotropic if $F^{\omega} \subset F$
- Lagrangian if $F^{\omega} = F$.

PROPOSITION 2.7. *(1) Any symplectic vector space has even dimension. If (V_1,ω_1), (V_2,ω_2) are symplectic vector spaces fo the same dimension, they are symplectomorphic.*

(2) Any isotropic subspace is contained in a Lagrangian subspace and Lagrangians have dimension equal to half the dimension of the total space.

(3) If (V_1,ω_1), (V_2,ω_2) are symplectic vector spaces with L_1, L_2 Lagrangian subspaces, and if $\dim(V_1) = \dim(V_2)$, then there symplectomorphism $\varphi : V_1 \to V_2$ such that $\varphi^*\omega_2 = \omega_1$ and $\varphi(L_1) = L_2$.

PROOF. We first prove that if I is an isotropic subspace it is contained in a Lagrangian subspace. Indeed, I is contained in a maximal isotropic subspace. We denote it again by I and we just have to prove $2\dim(I) = \dim(V)$.

Since $I \subset I^{\omega}$ we have $\dim(I) \le \dim(I^{\omega}) = \dim(V) - \dim(I)$ so that $2\dim(I) \le \dim(V)$. Now assume the inequality is strict. Then there exist a non zero vector, e, in $I^{\omega} \setminus I$, and $I \oplus \mathbb{R}e$ is isotropic and contains I. Therefore I was not maximal, a contradiction.

We thus proved that a maximal isotropic subspace I satisfies $I = I^{\omega}$ hence $2\dim(I) = \dim(V)$, and $\dim(V)$ is even.

Since $\{0\}$ is an isotropic subspace, maximal isotropic subspaces exist[2], and we conclude that we may always find a Lagrangian subspace, hence V is always even-dimensional.

This proves the first part of (1) and (2).

Let us now prove (3) and the second part of (1).

We know according to lemma 2.3 that (V_i,ω_i) is isomorphic to $L_i \oplus L_i^*$. But since L_1 is isomorphic to L_2 , we see that $L_1 \oplus L_1^*$ is isomorphic to $L_2 \oplus L_2^*$ by an isomorphism sending L_1o L_2. We shall consider a standard symplectic vector space $(\mathbb{R}^{2n},\sigma_n)$ isomorphic to $L_n \oplus L_n^*$, where $L_n = \mathbb{R}^n$

Let (V,ω) be a symplectic vector space and L a Lagrangian. We are going to prove by induction on $n = \dim(L) = \frac{1}{2}\dim(V)$ that there exists a symplectic map φ_n sending Z_n to L.

Assume this has been proved in dimension less or equal than $n-1$, and let us prove it in dimension n.

Pick any $e_1 \in L$. Since ω is nondegenerate, there exists an $f_1 \in V$ such that $\omega(e_1, f_1) = 1$. Then $f_1 \notin L$. Define

$$V' = Vect(e_1, f_1)^{\omega} = \{x \in V | \omega(x, e_1) = \omega(x, f_1) = 0\}.$$

It is easy to see that $(V', \omega_{|V'})$ is symplectic since only non-degeneracy is an issue, which follows from the fact that

$$\mathrm{Ker}(\omega_{|\mathrm{V}'}) = \mathrm{V}' \cap (\mathrm{V}')^{\omega} = \{0\}$$

[2]no need to invoke Zorn's lemma, a dimension argument is sufficient.

We now claim that $L' = L \cap V'$ is a Lagrangian in V' and $L = L' \oplus \mathbb{R}e_1$. First, since $\omega_{|L'}$ is the restriction of $\omega_{|L}$, we see that L' is isotropic. It is maximal isotropic, since otherwise, there would be an isotropic W such that $V' \supset W \supsetneq L'$, and then $W \oplus \mathbb{R}e_1$ would be a strictly larger isotropic subspace than L, which is impossible. Since $L \subset L' \oplus \mathbb{R}e_1$ our second claim follows by comparing dimensions.

Now the induction assumption implies that there is a symplectic map, φ_{n-1} from $(\mathbb{R}^{2n-2}, \sigma)$ to (V_2, ω) sending Z_{n-1} to L'. Then the map

$$\begin{aligned} \varphi_n : (\mathbb{R}^2, \sigma_2) \oplus (\mathbb{R}^{2n-2}, \sigma) &\longrightarrow (V, \omega) \\ (x_1, y_1; z) &\longrightarrow x_1 e_1 + y_1 f_1 + \varphi_{n-1}(z) \end{aligned}$$

is symplectic and sends Z_n to L.

Now the last statement of our theorem easily follows from the above: given two symplectic manifolds, $(V_1, \omega_1), (V_2, \omega_2)$ of dimension $2n$, and two lagrangians L_1, L_2, we get two symplectic maps

$$\psi_j : (\mathbb{R}^{2n}, \sigma_n) \longrightarrow (V_j, \omega_j)$$

sending Z_n to L_j. Then the map $\psi_2 \circ \psi_1^{-1}$ is a symplectic map from (V_1, ω_1) to (V_2, ω_2) sending L_1 to L_2.

□

REMARKS 2.8. (1) As we shall see, the map φ is not unique.
(2) Replacing F by F^{ω}, we see that from (2), it follows that any coisotropic vector space contains a Lagrangian one.

Since any symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \sigma)$, the group of symplectic automorphisms of (V, ω) denoted by $Sp(V, \omega) = \{\varphi \in GL(V) | \varphi^* \omega = \omega\}$ is isomorphic to $Sp(n) = Sp(\mathbb{R}^{2n}, \omega)$.

THEOREM 2.9 (Witt's theorem). *Let $(V_1, \omega_1), (V_2, \omega_2)$ be symplectic spaces such that* $\dim(V_1) \le \dim(V_2)$, *and W_1, W_2 be subspaces such that there exists $\varphi : W_1 \longrightarrow W_2$ such that $\phi^*(\omega_{2|W_2}) = \omega_{1|W_2}$. Then φ extends to a symplectic map $\tilde{\varphi} : (V_1, \omega_1), \longrightarrow (V_2, \omega_2)$.*

PROOF. If $(W_1, \omega_{1|W_1})$ is symplectic, then $V_1 = W_1 \oplus W_1^{\omega}$ and $V_2 = W_2 \oplus W_2^{\omega}$, and it is enough to prove the existence of a symplectic map $W_1^{\omega} \longrightarrow W_2^{\omega}$ whenever $\dim(W_1) \le \dim(W_2)$. This follow easily by induction. When W_1 is not symplectic, $W_1 = K_1 \oplus S_1$ with $K_1 = W_1 \cap W_1^{\omega}$, S_1 symplectic, and similarly $W_2 = K_2 \oplus \varphi(S_2)$ where necessarily $\varphi(K_1) = K_2$. Then it is enough to send $V_1' = S_1^{\omega}$ to $V_2' = \varphi(S_1)^{\omega}$ extending $\varphi : K_1 \longrightarrow K_2$. In other words it is enough to reduce to the isotropic case. But since any isotropic is contained in a Lagrangian, and any symplectic embedding of isotropic subspaces extends to embeddings of Lagrangians, we may assume W_1, W_2 are Lagrangian. But this follows from (3) above.

□

EXERCICE 2.10. Prove all results in this section for vector spaces on any field of characteristic different from 2.

We now give a better description of the set of lagrangian subspaces of (V,ω).

PROPOSITION 2.11. *(1) There is a homeomorphism between the set*

$$\Lambda_L = \{T \mid T; \textit{is Lagrangian and } T \cap L = \{0\}\}$$

and the set of quadratic forms on L^*. As a result, Λ_L is contractible, and $\Lambda(n)$ is a smooth manifold of dimension $\frac{n(n+1)}{2}$.

(2) The action of $Sp(n) = \{\varphi \in GL(V) | \varphi^*\omega = \omega\}$ on the set of pairs of transverse Lagrangians is transitive.

PROOF. For (1), we notice that $W = L \oplus L^*$ with the symplectic form

$$\sigma((e,f),(e',f')) = \langle e', f\rangle - \langle e, f'\rangle$$

is a symplectic vector space and that $L \oplus 0$ is a Lagrangian subspace.

According to the previous proposition there is a symplectic map $\psi : V \longrightarrow W$ such that $\psi(L) = L \oplus 0$, so we can work in W.

Let Λ be a Lagrangian in W with $\Lambda \cap L = \{0\}$. Then Λ is the graph of a linear map $A : L^* \to L$, more precisely

$$\Lambda = \{(Ay^*, y^*) | y^* \in L^*\}.$$

The subspace Λ is Lagrangian if and only if

$$\sigma((Ay_1^*, y_1^*),(Ay_2^*, y_2)) = 0, \text{ for all } y_1, y_2$$

i.e. if and only if

$$\langle y_1^*, Ay_2^*\rangle = \langle y_2^*, Ay_1^*\rangle$$

that is if $\langle \cdot, A\cdot\rangle$ is a bilinear symmetric form on L^*. But such bilinear form are in 1-1 correspondence with quadratic forms. The second statement immediately follows from the fact that the set of quadratic forms on an n-dimensional vector space is a vector space of dimension $\frac{n(n+1)}{2}$, and the fact that to any Lagrangian L_0 we may associate a transverse Lagrangian L_0', and L_0 is contained in the open set of Lagrangians transverse to L_0' (Well we still have to check the change of charts maps are smooth, this is left as an exercise).

To prove (2) let (L_1, L_2) and $L_1', L_2')$ be two pairs of transverse Lagrangians. By the previous proposition, we may assume $V = (L \oplus L^*, \sigma)$ and $L_1 = L_1' = L$. It is enough to find $\varphi \in Sp(V,\omega)$ such that $\varphi(L) = L, \varphi(L^*) = \Lambda$. The map $(x, y) \longrightarrow (x + Ay^*, y^*)$ is symplectic provided A is symmetric and sends $L \oplus 0$ to $L \oplus 0$ and L^* to $\Lambda = \{(Ay^*, y^*) \mid y^* \in L^*\}$. ☐

EXERCICES 1. (1) Prove that if K is a coisotropic subspace, K/K^ω is symplectic.

(2) Compute the dimension of the space of Lagrangians containing a given isotropic subspace I. Hint: show that it is the space of Lagrangians in I^ω/I.

(3) (Witt's Theorem) Let V_1 and V_2 be two symplectic vector spaces with the same dimension and $F_i \subset (V_i, \omega_i)$, $i = 1, 2$. Assume that there exists a linear isomorphism $\varphi : F_1 \cong F_2$, i.e. $\varphi^*(\omega_2)_{|F_2} = (\omega_1)_{|F_1}$. Then φ extends to a symplectic map $\widetilde{\varphi} : (V_1, \omega_1) \to (V_2, \omega_2)$. Hint: show that symplectic maps are the same thing as Lagrangians in $(V_1 \oplus V_2, \omega_1 - \omega_2)$ which are transverse to $V_1 \oplus 0$ and $0 \oplus V_2$, and the map we are looking for, correspond to Lagrangians transverse to V_1, V_2 containing $I = \{(x, \varphi(x)) \mid x \in F_1\}$. Compute the dimension of the space of non transverse ones.

(4) The action of $Sp(n)$ is not transitive on the triples of pairwise transverse Lagrangian spaces. Using the notion of index of a quadratic form prove that this has at least (in fact exactly) $n+1$ connected components. This is responsible for the existence of the Maslov index.

(5) Prove that the above results are valid over any field of any characteristic, except in characteristic 2 because quadratic forms and bilinear symmetric forms are not equivalent.

2. Complex structure

Let h be a hermitian form on a complex vector space V in the sense:
1) $h(z, z') = \overline{h(z', z)}$;
2) $h(\lambda z, z') = \lambda h(z, z')$ for $\lambda \in \mathbb{C}$;
3) $h(z, \lambda z') = \bar{\lambda} h(z, z')$ for $\lambda \in \mathbb{C}$;
4) $h(z, z) > 0$ for all $z \neq 0$.
Then

$$h(z, z') = g(z, z') + i\omega(z, z'),$$

where g is a scalar product and ω is symplectic, since $\omega(iz, z) > 0$ for $z \neq 0$.

Example: On $\mathbb{C}^n$, define

$$h((z_1, \cdots, z_n), (z'_1, \cdots, z'_n)) = \sum_{j=1}^{n} z_j \bar{z'_j} \in \mathbb{C}.$$

Then the symmetric part is the usual scalar product on $\mathbb{R}^{2n}$ while ω is the standard symplectic form.

Denote by J the multiplication by $i = \sqrt{-1}$.

PROPOSITION 2.12.

$$\begin{cases} g(Jz, z') = -\omega(z, z') \\ \omega(z, Jz') = -g(z, z') \end{cases}$$

REMARK 2.13. ω is nondegenerate because $\omega(z, Jz) = -g(z, z) < 0$ for all $z \neq 0$.

Conclusion: Any hermitian space V has a canonical symplectic form.

We will now answer the following question: can a symplectic vector space be made into a hermitian space? In how many ways?

PROPOSITION 2.14. *Let (V,ω) be a symplectic vector space. Then there is a complex structure on V such that $\omega(J\xi,\eta)$ is a scalar product. Moreover, the set $\mathscr{J}(\omega)$ of such J, called compatible almost complex structures, is contractible.*

PROOF. Let $(\cdot,\cdot)$ be any fixed scalar product on V. Then there exists A such that

$$\omega(x,y)=(Ax,y).$$

Since ω is skew-symmetric, $A^*=-A$ where A^* is the adjoint of A with respect to $(\cdot,\cdot)$. Since any other scalar product can be given by a positive definite symmetric matrix M, we look for J such that $J^2=-I$ and M such that $M^*=M$ and setting $(x,y)_M=(Mx,y)$ we have $\omega(Jx,y)=(x,y)_M$. The last equality can be rewritten as

$$(AJx,y)=(Mx,y) \text{ for all } x,y.$$

This is equivalent to finding a positive symmetric M such that $M=AJ$. It's easy to check that there is a unique solution given by $M=(AA^*)^{1/2}$ and $J=A^{-1}M$ solves $AJ=M$, $J^2=-I$ and $M^*=M$.

In summary, for any fixed scalar product $(\cdot,\cdot)$, we can find a pair (J_0,M_0) such that $\omega(J_0x,y)$ is the scalar product $(M_0\cdot,\cdot)$. If we know (J_0,M_0) is such a pair and we start from the scalar product $(M_0\cdot,\cdot)$, then we get the pair (J_0,id).

Define $\mathscr{J}(\omega)$ to be the set of all J's such that $\omega(J\cdot,\cdot)$ is a scalar product. Define $\mathscr{S}$ to be the set of all scalar products on V. By previous discussion, there is continuous map

$$\Psi:\mathscr{S}\to\mathscr{J}(\omega).$$

Moreover, if J is in $\mathscr{J}(\omega)$, Ψ maps $\omega(J\cdot,\cdot)$ to J. On the other hand, we have a continuous embedding i from $\mathscr{J}(\omega)$ to $\mathscr{S}$ which maps J to $\omega(J\cdot,\cdot)$. Clearly, $\Psi\circ i=\mathrm{id}_{\mathscr{S}}$.

Let now $M_p\in\mathscr{S}$ be in the image. Since we know $\mathscr{S}$ is contractible, there is a continuous family

$$F_t:\mathscr{S}\to\mathscr{S}$$

such that $F_0=id$ and $F_1(\mathscr{S})=M_p$. Consider

$$\tilde{F}_t:\mathscr{J}(\omega)\to\mathscr{J}(\omega)$$

given by

$$\tilde{F}_t=\Psi\circ F_t\circ i.$$

By the definition of Ψ, we know $\tilde{F}_0=id$ and $\tilde{F}_1=J_p$. This shows that $\mathscr{J}(\omega)$ is contractible. $\square$

EXERCICE 2.15. Let L be a Lagrangian subspace, show that JL is also a Lagrangian and $L\cap JL=\{0\}$.

We finally study the structure of the symplectic group,

PROPOSITION 2.16. *The group $Sp(n)$ of linear symplectic maps of (V,ω) the homotopy type of $U(n)$. It is therefore connected, and has fundamental group isomorphic to $\mathbb{Z}$ and .*

PROOF. Let $\langle Jx, y\rangle = \sigma(x, y)$ with $J^2 = -\mathrm{Id}$ and $J^* = -J$ Let $R \in Sp(n)$, then $\sigma(Rx, Ry) = \sigma(x, y)$ i.e.

$$\langle JRx, Ry\rangle = \langle x, y\rangle$$

Thus $R \in Sp(n)$ is equivalent to $R^* JR = J$.

Thus, if R is symplectic, so is R^*, since $(R^*)JRJ = J^2 = -\mathrm{Id}$ we may conclude that $(R^*)^{-1}[(R^*)JRJ]R^* = -\mathrm{Id}$, that is $JRJR^* = -\mathrm{Id}$, so that $RJR^* = J$.

Now decompose R as $R = PQ$ with P symmetric and Q orthogonal, by setting $P = (RR^*)^{1/2}$ and $Q = P^{-1}R$. Since R, R^* are symplectic so is P and hence Q. Now

$$Q^{-1}JQ = R^{-1}PJP^{-1}R = R^{-1}(PJP^{-1})R =$$

$$\hat{\mathrm{E}} - R^{-1}JP^{-2}R = R^{-1}J(RR^*)^{-1}R = R^{-1}JR^* = J$$

Thus Q is symplectic and complex, that is unitary. Then since P is also positive definite, the map $t \longrightarrow P^t$ is well defined (as $\exp(t\log(P))$ and $\log(P)$ is well defined for a positive symmetric matrix) for $s \in \mathbb{R}$ and the path $PQ \longrightarrow P^s Q$ yields a retraction form $Sp(n)$ to $U(n)$. □

EXERCICE 2.17. Prove that $Sp(n)$ acts transitively on the set of isotropic subspaces (resp. coisotropic subspaces) of given dimension (use Witt's theorem).

EXERCICE 2.18. Prove that the set $\tilde{\mathcal{J}}(\omega)$ made of complex structures J such that $\omega(J\xi, \eta) = -\omega(\xi, J\eta) > 0$ for all $\xi, \eta \in V$ is also contractible. This is equivalent to the requirement that J is an isometry for the scalar product $\omega(J\xi, \eta)$. (of course it contains $\mathcal{J}(\omega)$. Elements of $\mathcal{J}(\omega)$ are called **compatible almost complex structures** while those in $\tilde{\mathcal{J}}(\omega)$ are called **tame almost complex structures**.

CHAPTER 3

Symplectic differential geometry

1. Moser's lemma and local triviality of symplectic differential geometry

DEFINITION 3.1. A two form ω on a manifold M is symplectic if and only if
1) $\forall x \in M$, $\omega(x)$ is symplectic on T_xM;
2) $d\omega = 0$ (ω is closed).

Examples:
1) $(\mathbb{R}^{2n}, \sigma)$ is symplectic manifold.
2) If N is a manifold, then

$$T^*N = \{(q,p) | p \text{ linear form on } T_qM\}$$

is a symplectic manifold. Let $q_1, \cdots, q_n$ be local coordinates on N and let $p^1, \cdots, p^n$ be the dual coordinates. Then the symplectic form is defined by

$$\omega = \sum_{i=1}^{n} dp^i \wedge dq_i.$$

One can check that ω does not depend on the choice of coordinates and is a symplectic form. We can also define a one form, called the Liouville form

$$\lambda = p\,dq = \sum_{i=1}^{n} p^i dq_i.$$

It is well defined and $d\lambda = \omega$.

3) Projective algebraic manifolds (See also Kähler manifolds)

$\mathbb{C}P^n$ has a canonical symplectic structure σ and is also a complex manifold. The restriction to the tangent space at any point of the complex structure J and the symplectic form σ are compatible. The manifold $\mathbb{C}P^n$ has a hermitian metric h, called the Fubini-Study metric. For any $z \in \mathbb{C}P^n$, $h(z)$ is a hermitian inner product on $T_z\mathbb{C}P^n$. $h = g + i\sigma$, where g is a Riemannian metric and $\sigma(J\xi, \xi) = g(\xi, \xi)$.

Claim: A complex submanifold M of $\mathbb{C}P^n$ carries a natural symplectic structure.

Indeed, consider $\sigma|_M$. It's obviously skew-symmetric and closed. We must prove that $\sigma|_M$ is non-degenerate. This is true because if $\xi \in T_xM_{\{0\}}$ and $J\xi \in T_xM$, then $\omega(x)(\xi, J\xi) \neq 0$

DEFINITION 3.2. A submanifold in symplectic manifold (M, ω) is Lagrangian if and only if $\omega|_{T_xL} = 0$ for all $x \in M$ and $\dim L = \frac{1}{2} \dim M$. In other words T_xL is a Lagrangian subspace of $(T_xM, \omega(x))$.

We are going to prove that locally symplectic manifolds "have no geometry". A crucial lemma is

LEMMA 3.3 (Moser). *Let N be a compact submanifold in M. Let ω_t be a family of symplectic forms such that $\omega_t|_{T_N M}$ is constant. Then there is a diffeomorphism φ defined near N such that $\varphi^*\omega_1 = \omega_0$ and $\varphi|_N = id|_N$.*

PROOF. We will construct a vector field $X(t,x) = X_t(x)$ whose flow φ^t satisfies $\varphi^0 = id$ and $(\varphi^t)^*\omega_t = \omega_0$. Differentiate the last equality

$$(\frac{d}{dt}(\varphi^t)^*)\omega_t + (\varphi^t)^*(\frac{d}{dt}\omega_t) = 0.$$

Then

$$(\varphi^t)^* L_{X_t}\omega_t + (\varphi^t)^*(\frac{d}{dt}\omega_t) = 0.$$

Since φ^t is diffeomorphism, this is equivalent to

$$L_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0.$$

Using Cartan's formula

$$L_X = d \circ i_X + i_X \circ d,$$

we get

$$d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t = 0.$$

Since ω_t is nondegenerate, the map $T_xM \to (T_xM)^*$ which maps X to $\omega(X,\cdot)$ is an isomorphism. Therefore, for any one form β, the equation $i_X\omega = \beta$ has a unique solution X_β. It suffices to solve for β_t,

$$d\beta_t = -\frac{d}{dt}\omega_t.$$

with the requirement that $\beta_t = 0$ on $T_N M$ for all t, because we want $\varphi_{|N} = \mathrm{Id}_{|N}$, that is $X_t 0$ on N. On the other hand, the assumption that $\omega_t = \omega_0$ on $T_N M$ implies $(\frac{d}{dt}\omega_t) \equiv 0$ on $T_N M$. Denote the right hand side of the above equation by α, then α is defined in a neighborhood U of N. The solution of β_t is given by Poincaré's Lemma on the tubular neighborhood of N. Here by a tubular neighborhood we mean a neighborhood of N in M diffeomorphic to the unit disc bundle $D\nu_N M$ of $\nu_M N$ the normal bundle of N in M (i.e. $\nu_M N = \{(x,\xi) \in T_N M \mid \xi \perp TN\}$).

LEMMA 3.4. *(Poincaré) If α is a p-form on U, closed and vanishing on N, then there exists β such that $\alpha = d\beta$ and β vanishes on $T_N M$.*

PROOF. [1]

[1] The proof is easier if one is willing to admit that the set of exact forms is closed for the C^∞ topology, i.e. if $\alpha = d\beta_\varepsilon + \gamma_\varepsilon$ and $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$ then α is exact. This follows for example from the fact that exactness of a closed form can be checked by verifying that its integral over a finite number of cycles vanishes.

This means that for a tubular neighborhood $H^*(U,N)=0$.

Indeed, let r_t be the map on $\nu_N M$ defined by $r_t(x,\xi)=(x,t\xi)$ and V the vector field $V_t(x,\xi)=-\frac{\xi}{t}$, well defined for $t\neq 0$. This vector field satisfies $\frac{d}{dt}r_t(x,\xi)=V_t(r_t(x,\xi))$. Since r_0 sends $\nu_N M$ to its zero section, N, we have $r_0^*\alpha=0$ and $r_1=\mathrm{Id}$.

Then

$$\frac{d}{dt}(r_t)^*(\alpha)=r_t^*(L_{V_t}\alpha)=d\left(r_t^*(i_{V_t}\alpha)\right)$$

Note that $r_t^*(i_{V_t}\alpha)$ is well defined, continuous and bounded as t goes to zero, since writing (locally) (u,η) for a tangent vector to $T_{(x,\xi)}\nu_N M$

$$(r_t^*(i_{V_t}\alpha))(x,\xi)((u_2,\eta_2)....(u_p,\eta_p))=\alpha(x,t\xi)((0,\xi),(u_2,t\eta_2)...(u_p,t\eta_p))$$

remains C^1 bounded as t goes to zero. Let us denote by β_t the above form. We can write for ε positive

$$r_1^*(\alpha)-r_\varepsilon^*(\alpha)=\int_\varepsilon^1\frac{d}{dt}[(r_t)^*(\alpha)]dt=d\left(\int_\varepsilon^1(r_t)^*(i_{V_t}\alpha)dt\right)$$

Since as t goes to zero, $d(r_t^*(i_{V_t}\alpha))$ remains bounded, thus $\lim_{\varepsilon\to 0}\int_0^\varepsilon d(r_t^*(i_{V_t}\alpha))=0$ and we have that

$$\alpha=r_1^*(\alpha)-r_0^*(\alpha)=\lim_{\varepsilon\to 0}[r_1^*(\alpha)-r_\varepsilon^*(\alpha)]=$$
$$\lim_{\varepsilon\to 0}d\left(\int_\varepsilon^1(r_t)^*(i_{V_t}\alpha)dt\right)=d\left(\lim_{\varepsilon\to 0}\int_0^1(r_t)^*(i_{V_t}\alpha)dt\right)=d\beta$$

where

$$\beta=\int_0^1(r_t)^*(i_{V_t}\alpha)dt=\int_0^1\beta_t dt$$

but β_t vanishes on N, since

$$\beta_t(x,0)((u_2,\eta_2)...(u_p,\eta_p)=\alpha(x,0)((0,0),(u_2,t\eta_2)...(u_p,t\eta_p)=0$$

This proves our lemma. □

□

EXERCICE 3.5. Prove using the above lemma that if N is a submanifold of M, $H^*(M,N)$ can either be defined as the set of closed forms vanishing on TN modulo the differential forms vanishing on TN or as the set of closed form vanishing in a neighborhood of N modulo the differential of forms vanishing near N.

As an application, we have

PROPOSITION 3.6 (Darboux). *Let (M,ω) be a symplectic manifold. Then for each $z\in M$, there is a local diffeomorphism φ from a neighborhood of z to a neighborhood of o in $\mathbb{R}^{2n}$ such that $\varphi^*\sigma=\omega$.*

PROOF. According to Lecture 1, there exists a linear map $L : T_zM \to \mathbb{R}^{2n}$ such that $L^*\sigma = \omega(z)$. Hence, using a local diffeomorphism $\varphi_0 : U \to W$ such that $d\varphi_0(z) = L$, where U and W are neighborhoods of $z \in M$ and $o \in \mathbb{R}^{2n}$ respectively, we are reduced to considering the case where $\varphi_0^*\sigma$ is a symplectic form defined in U and $\omega(z) = (\varphi_0^*)\sigma$.

Define $\omega_t = (1-t)\varphi_0^*\sigma + t\omega$ in U. It's easy to check ω_t satisfies the assumptions of Moser's Lemma, therefore, there exists ψ such that $\psi^*\omega_1 = \omega_0$, i.e.

$$\psi^*\omega = \varphi_0^*\sigma.$$

Then $\varphi = \varphi_0 \circ \psi^{-1}$ is the required diffeomorphism. □

EXERCICES 1. (1) Show the analogue of Moser's Lemma for volume forms.

(2) Let ω_1, ω_2 be symplectic forms on a compact surface without boundary. Then there exists a diffeomorphism φ such that $\varphi^*\omega_1 = \omega_2$ if and only if $\int \omega_1 = \int \omega_2$.

(3) Let ω be a closed 2-form.

PROPOSITION 3.7. *(Weinstein) Let L be a closed Lagrangian submanifold in (M,ω). Then L has a neighborhood symplectomorphic to a neighborhood of $O_L \subset T^*L$. (Here, $O_L = \{(q,0) | q \in L\}$ is the zero section.)*

PROOF. The idea of the proof is the same as that of Darboux Lemma.

First, for any $x \in L$, find a subspace $V(x)$ in T_xM such that

1) $V(x) \subset T_xM$ is Lagrangian subspace;

2) $V(x) \cap T_xL = \{0\}$;

3) $x \to V(x)$ is smooth.

According to our discussion in linear symplectic space, we can find such $V(x)$ at least pointwise. To see 3), note that at each point $x \in L$ the set of all Lagrangian subspaces in T_xM transverse to T_xL may be identified with quadratic forms on $(T_xL)^*$. It's then possible to find a smooth section of such an "affine bundle".

Abusing notations a little, we write L for the zero section in T^*L. Denote by $T_L(T^*L)$ the restriction of the tangent bundle T^*L to L. Denote by T_LM the restriction of the bundle TM to L. Both bundles are over L. For $x \in L$, their fibers are

$$T_x(T^*L) = T_xL \oplus T_x(T_x^*L)$$

and

$$T_xM = T_xL \oplus V(x).$$

Construct a bundle map $L_0 : T_L(T^*L) \to T_LM$ which restricts to identity on factor T_xL and sends $T_x(T_x^*L)$ to $V(x)$. Moreover, we require

$$\omega(L_0u, L_0v) = \sigma(u,v),$$

where $u \in T_x(T_x^*L) = T_x^*L$ and $v \in T_xL$. This defines L_0 uniquely. Again, we can find φ_0 from a neighborhood of L in T^*L to a neighborhood of L in M such that $d\varphi_0|_{T_L(T^*L)} = L_0$. By the construction of L_0, one may check that

$$\varphi_0^*\omega = \sigma \text{ on } T_L(T^*L).$$

Define

$$\omega_t = (1-t)\varphi_0^*\omega + t\sigma, \quad t \in [0,1].$$

ω_t is a family of symplectic forms in a neighborhood of O_L. Moreover, $\omega_t \equiv \omega_0$ on $T_L(T^*L)$. By Moser's Lemma, there exists Ψ defined near O_L such that $\Psi^*\omega_1 = \omega_0$, i.e. $\Psi^*\sigma = \varphi_0^*\omega$. Then $\varphi_0 \circ \Psi^{-1}$ is the diffeomorphism we need. $\square$

EXERCICE 3.8. Let I_1, I_2 be two diffeomorphic isotropic submanifold in (M_1,ω_1), (M_2,ω_2). Let $E_1 = (TI_1)^{\omega_1}/(TI_1)$ and $E_2 = (TI_2)^{\omega_2}/(TI_2)$. E_1, E_2 are symplectic vector bundles over I_1 and I_2. Show that I_1 and I_2 have symplectomorphic neighborhoods if and only if $E_1 \cong E_2$ as symplectic vector bundles.

EXERCICE 3.9. Same exercise in the coisotropic situation.

2. The groups Ham and $Diff_\omega$

Since Klein's Erlangen's program, geometry has meant the study of symmetry groups. The group playing the first role here is $Diff_\omega(M)$. Let (M,ω) be a symplectic manifold. Define

$$Diff_\omega(M) = \{\varphi \in Diff(M) | \varphi^*\omega = \omega\}.$$

This is a very large group since it contains $Ham(M,\omega)$, which we will now define.

Let $H(t,x)$ be any smooth function and X_H the unique vector field such that

$$\omega(X_H(t,x),\xi) = d_x H(t,x)\xi, \quad \forall \xi \in T_x M.$$

Here d_x means exterior derivative with respect to x only.

Claim: The flow of X_H is in $Diff_\omega(M)$.

To see this,

$$\begin{aligned} \frac{d}{dt}(\varphi^t)^*\omega &= (\varphi^t)^*(L_{X_H}\omega) \\ &= (\varphi^t)^*(d \circ i_{X_H}\omega + i_{X_H} \circ d\omega) \\ &= (\varphi^t)^*(d(dH)) = 0. \end{aligned}$$

DEFINITION 3.10. The set of all diffeomorphism φ that can be obtained as the flow of some H is a subgroup $Diff(M,\omega))$ called $Ham(M,\omega)$.

To prove that $Ham(M,\omega)$ is a subgroup, we proceed as follows: first notice that the Hamiltonian isotopy can be reparametrized, and still yields a Hamiltonian isotopy $\varphi_{s(t)}$ satisfying

$$(\frac{d}{dt}\varphi_{s(t)})_{t=t_0} = s'(t)(\frac{d}{ds}\varphi_s)_{s=s(t_0)} = s'(t)X_H(s(t_0),\varphi_{s(t_0)})$$

which is the Hamiltonian flow of

$$s'(t)H(s(t),z)$$

Therefore we may use a function $s(t)$ on $[0,1]$ such that $s(0)=0, s(1/2)=1, s'(t)=0$ for t close to $1/2$ and we find a Hamiltonian flow ending at φ_1 in time $1/2$ and such that

H vanishes near $t = 1/2$. Similarly if ψ_t is the flow associated to $K(t,z)$ we may modify it in a similar way using $r(t)$ so that $K \equiv 0$ for t in a neighborhood of $[0,1/2]$. We can then consider the flow associated to $H(t,z)+K(t,z) = L(t,z)$ it will be $\varphi_{s(t)} \circ \psi_{r(t)}$ and for $t = 1$ we get $\varphi_1 \circ \psi_1$.

That φ_1^{-1} is also Hamiltonian follows from the fact that $-H(t,\varphi_t(z))$ has flow φ_t^{-1}.

EXERCICE 3.11. Show that $(\varphi^t)^{-1}\psi^t$ is the Hamiltonian flow of

$$L(t,z) = K(t,\varphi_t(z)) - H(t,\varphi_t(z))$$

This immediately proves that $Ham(M,\omega)$ is a group.

REMARK 3.12. Denote by $Diff_{\omega,0}$ the component of $Diff_\omega(M)$ in which the identity lies. It's obvious that $Ham(M,\omega) \subset (Diff_{\omega,0}(M)$.

REMARK 3.13. If $H(t,x) = H(x)$, then $H \circ \varphi^t = H$. This is what physicists call conservation of energy. Indeed H is the energy of the system, and for time-independent conservative systems, energy is preserved. This is not the case in time-dependent situations.

REMARK 3.14. If we choose local coordinates $q_1,...,q_n$ and their dual $p_1,...,p_n$ in the cotangent space, p_i, q_i, the flow is given by the ODE

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i}(t,q,p) \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}(t,q,p) \end{cases}$$

Question: How big is the quotient $Diff_{\omega_0}/Ham(M,\omega)$?

Given $\varphi \in Diff_{\omega_0}$, there is an obvious obstruction for φ to belong to $Ham(M,\omega)$. Assume $\omega = d\lambda$. Then $\varphi^*\lambda - \lambda$ is closed for all $\varphi \in Diff_\omega$, since

$$d(\varphi^*\lambda - \lambda) = \varphi^*\omega - \omega = 0.$$

If φ^t is the flow of X_H,

$$\begin{aligned} \frac{d}{dt}((\varphi^t)^*\lambda) &= (\varphi^t)^*(L_{H_X}\lambda) \\ &= (\varphi^t)^*(d(i_{X_H}\lambda) + i_{X_H}d\lambda) \\ &= (\varphi^t)^* d(i_{X_H}\lambda + H) \\ &= d((\varphi^t)^*(i_{X_H}\lambda + H)). \end{aligned}$$

This implies that $\varphi^*\lambda - \lambda$ is exact.

In summary, we can define map

$$\begin{array}{rccc} \text{Flux}: & (Diff_\omega)_0(M) & \to & H^1(M,\mathbb{R}) \\ & \varphi & \mapsto & [\varphi^*\lambda - \lambda] \end{array}$$

We know

$$Ham_\omega(M) = \ker(Flux).$$

Examples:

(1) On T^*T^1 the translation $\varphi : (x,p) \longrightarrow (x, p+p_0)$ is symplectic, but $\text{Flux}(\varphi) = p_0$.
(2) Similarly if $M = T^2$ and $\sigma = dx \wedge dy$, the map $(x,y) \longrightarrow (x, y+y_0)$ is not in $Ham(T^2,\sigma)$ for $y_0 \not\equiv 0 \mod 1$.

Indeed, since the projection $\pi : T^*T^1 \longrightarrow T^2$ is a symplectic covering, any Hamiltonian isotopy on T^2 ending in φ would lift to a Hamiltonian isotopy on T^*T^1 (if $H(t,z)$ is the Hamiltonian on T^2, $H(t,\pi(z))$ is the Hamiltonian on T^*T^1) ending to some lift of φ. But the lifts of φ are given by $(x,y) \longrightarrow (x+m, y+y_0+n)$ for $(m,n) \in \mathbb{Z}^2$, with Flux given by $y_0 + n \neq 0$.

EXERCICES 2. (1) Prove the Darboux-Weinstein-Givental theorem (also called non-linear Witt theorem. See [**A-G**] page 26): Let S_1, S_2 be two submanifolds in $(M_1,\omega_1), (M_2,\omega_2)$. Assume there is a map $\varphi : S_1 \longrightarrow S_2$ which lifts to bundle map

$$\Phi : T_{S_1}M_1 \longrightarrow T_{S_2}M_2$$

coinciding with $d\varphi$ on the subbundle TS_1, and preserving the symplectic structures, i.e. $\Phi^*(\omega_2) = \omega_1$.

Then there is a symplectic diffeomorphism between a neighborhood U_1 of S_1 and a neighborhood U_2 of S_2.

(2) Use the Darboux-Weinstein-Givental theorem to prove that all closed curves have symplectomorphic neighborhoods. Hint: Show that all symplectic vector bundle on the circle are trivial.

(3) (a) Prove that the Flux homomorphism can be defined on (M,ω) as follows. Let φ_t be a symplectic isotopy. Then $\frac{d}{dt}\varphi_t(z) = X(t,\varphi_t(z))$ and $\omega(X(t,z)) = \alpha_t$ is a closed form. Then

$$\widetilde{\text{Flux}}(\varphi) = \int_0^1 \alpha_t dt \in H^1(M,\mathbb{R})$$

depends only on the homotopy class of the path φ_t. If Γ is the image by Flux of the set of closed loops, we get a well defined map

$$\text{Flux} : Diff(M,\omega)_0 \longrightarrow H^1(M,\mathbb{R})/\Gamma$$

(b) Prove that when ω is exact, Γ vanishes and the new definition coincides with the old one.

CHAPTER 4

More Symplectic differential Geometry: Reduction and Generating functions

Philosophical Principle: Everything important is a Lagrangian submanifold.
Examples:

(1) If $(M_i, \omega_i), i = 1, 2$ are symplectic manifolds and φ a symplectomorphism between them, that is a map from M_1 to M_2 such that $\varphi^* \omega_2 = \omega_1$. Consider the graph of φ,
$$\Gamma(\varphi) = \{(x, \varphi(x))\} \subset M_1 \times M_2.$$
This is a Lagrangian submanifold of $M_1 \times \overline{M_2}$ if we define M_2 as the manifold M_2 with the symplectic form $-\omega_2$ and the symplectic form on $M_1 \times \overline{M_2}$ is given by
$$(\omega_1 \ominus \omega_2)((\xi_1, \xi_2), (\eta_1, \eta_2)) = \omega_1(\xi_1, \eta_1) - \omega_2(\xi_2, \eta_2).$$
In fact, it's easy to see $\Gamma(\varphi)$ is a Lagrangian submanifold if and only if $\varphi^* \omega_2 = \omega_1$. Note that if $M_1 = M_2$, then $\Gamma(\varphi) \cap \triangle_M = \mathrm{Fix}(\varphi)$.

(2) Let (M, J, ω) be a smooth projective manifold, i.e. a smooth manifold given by
$$M = \{P_1(z_0, \cdots, z_N) = \cdots = P_i(z_0, \cdots, z_N) = 0\}$$
where P_j are homogeneous polynomials. We shall assume the map from $\mathbb{C}^n \setminus \{0\}$ to $\mathbb{C}^r$
$$(z_0, ..., z_n) \mapsto (P_1(z_0, ..., z_n), ..., P_r(z_0, ..., z_n))$$
has zero as a regular value, so that M is a smooth manifold.

If P_j's have real coefficients, then real algebraic geometry is concerned with
$$\begin{aligned} M_{\mathbb{R}} &= \{[x_0, \cdots, x_N] \in \mathbb{R}P^N | P_j(x_0, \cdots, x_N) = 0\} \\ &= M \cap \mathbb{R}P^N. \end{aligned}$$
The problem is to “determine the relation” between M and $M_{\mathbb{R}}$". It is easy to see that $M_{\mathbb{R}}$ is a Lagrangian of (M, ω) (of course, possibly empty).

1. Symplectic Reduction

Let (M, ω) be a symplectic manifold and K a submanifold. K is said to be *coisotropic* if $\forall x \in K$, we have $T_xK \supset (T_xK)^\omega$. As x varies in K, $(T_xK)^\omega$ gives a distribution in T_xK.

LEMMA 4.1. *This distribution is integrable.*

PROOF. According to Frobenius theorem, it suffices to check that for all vector field $X, Y \in (T_xK)^\omega$, η in T_xK,

$$\omega([X,Y],\eta) = 0$$

where X and Y are vector fields in $(T_xK)^\omega$.

$d\omega(X,Y,\eta)$ vanishes, but on the other hand is a sum of terms of the form:

$X\cdot\omega(Y,\eta)$ but since $\omega(Y,\eta)$ is identically zero these terms vanishes. The same holds if we exchange X and Y.

$\eta\cdot\omega(X,Y)$ vanishes for the same reason.

$\omega(X,[Y,\eta])$ and $\omega(Y,[X,\eta])$ vanish since $[X,\eta],[Y,\eta]$ are tangent to K.

$\omega([X,Y],\eta)$ is the only remaining term. But since the sum of all terms must vanish, this must also vanish, hence $[X,Y] \in (T_xK)^\omega$ $\square$

This integrable distribution gives a foliation of K, denoted by $\mathscr{C}_K$. We can check that ω induces a symplectic form (we only need to check it is nondegenerate) on the quotient space $(T_xK)/(T_xK)^\omega$. One might expect $K/\mathscr{C}_K$ to be a a "symplectic something".

Unfortunately, due to global topological difficulties, there is no nice manifold structure on the quotient. However, in certain special cases, as will be illustrated by examples in the end of this section, $K/\mathscr{C}_K$ is a manifold, and therefore a symplectic manifold.

Let us now see the effect of the above operation on symplectic manifolds.

LEMMA 4.2. *(Automatic Transversality) If L is a Lagrangian in M and L intersects K transversally, i.e. $T_xL + T_xK = T_xM$ for $x \in K\cap L$, then L intersects the leaves of C_K transversally, $T_xL \cap T_x\mathscr{C}_K = \{0\}$, for $x \in K \cap L$.*

PROOF. Recall from symplectic linear algebra that if F_i are subspaces of a symplectic vector space, then

$$(F_1+F_2)^\omega = F_1^\omega \cap F_2^\omega.$$

We know $(T_xL)^\omega = T_xL$ and $(T_xM)^\omega = \{0\}$, then the lemma follows from $T_xL + T_xK = T_xM$. $\square$

Now, let's pretend $K/\mathscr{C}_K$ is a manifold and denote the projection by $\pi : K \to K/\mathscr{C}_K$.

1) K and L intersect transversally, so in particular $L\cap K$ is a manifold.

2) The projection $\pi : (L\cap K) \to K/\mathscr{C}_K$ is an immersion.

$$\ker d\pi(x) = T_x\mathscr{C}_K = (T_xK)^\omega.$$

$$\begin{aligned} \ker d\pi(x)|_{T_x(L\cap K)} &\subset \ker d\pi(x) \cap T_xL \\ &\subset (T_xK)^\omega \cap TxL = \{0\}. \end{aligned}$$

Therefore $d\pi(x)|_{L\cap K}$ is injective and $\pi|_{L\cap K}$ is immersion.

To summarize our findings, given a symplectic manifold (M,ω) and a coisotropic submanifold K, let L be a Lagrangian of M intersecting K transversally. Define L_K to be the image of the above immersion. Then it is a Lagrangian in $K/\mathscr{C}_K$. This operation is called *symplectic reduction*.

The only thing left to check is that L_K is Lagrangian. Let $\tilde{\omega}$ be the induced symplectic form on $K/\mathscr{C}_K$ and $\tilde{v}$ is a tangent vector of L_K. Assume the preimage of $\tilde{v}$ is v, a tangent vector of L. Since L is Lagrangian and $\tilde{\omega}$ is induced from ω, we know L_K is isotropic. It's Lagrangian by a dimension count. The same argument shows that the reduction of an isotropic submanifold (resp. coisotropic submanifold) is isotropic (resp. coisotropic).

Example 1: Let N be a symplectic manifold, and V be any smooth submanifold. Define

$$K = T^*_V N = \{(x,p) | x \in V, p \in T^*_x N\}.$$

This is a coisotropic submanifold, and its coisotropic foliation $\mathscr{C}_K$ is given by specifying the leaf through $(x,p) \in K$ to be

$$\mathscr{C}_K(x,p) = \{(x,\tilde{p}) \in K \mid \tilde{p} - p \text{ vanishes on } T_x V\}.$$

It is natural to identify $K/\mathscr{C}_K$ with T^*V.

Symplectic reduction in this case, sends Lagrangian in T^*N to Lagrangian in T^*V.

Example 2: Let N_1, N_2 are smooth manifolds and $N = N_1 \times N_2$. Suppose we choose local coordinates near a point in T^*N is written as

$$(x_1, p_1, x_2, p_2).$$

where $(x_1,p_1) \in T^*N_1, (x_2,p_2) \in T^*N_2$. Define $K = \{(x_1,p_1,x_2,p_2) | p_2 = 0\}$. The tangent space of K at a point $z = (x_1,p_1,x_2,p_2)$ is given by

$$(v_1, w_1, v_2, 0),$$

$$(T_z K)^{\omega} = \{(0,0,0,w_2)\}.$$

Then we can identify $K/\mathscr{C}_K$ with T^*N_1.

Symplectic reduction sends a Lagrangian in T^*N to a Lagrangian in T^*N_1.

1.1. Lagrangian correspondences. Let Λ be a Lagrangian submanifold in $\overline{T^*X} \times T^*Y$. Then it induces a correspondence from T^*X to T^*Y as follows: consider a set $C \subset T^*X$, and $C \times \Lambda \subset T^*X \times \overline{T^*X} \times T^*Y$. Now, denote by Δ_{T^*X} the diagonal in $T^*X \times \overline{T^*X}$. The submanifold $K = \Delta_{T^*X} \times T^*Y$ is coisotropic, and we define $\Lambda \circ C$ as $C \times \Lambda \cap K/\mathscr{K} \subset K/\mathscr{K} = T^*Y$. When C is a submanifold, then $\Lambda \circ C$ is a submanifold provided $C \times T^*Y$ is transverse to Λ.

If C is isotropic or coisotropic, it is easy to check that the same will hold for $\Lambda \circ C$. In particular if L is a Lagrangian submanifold, the correspondence maps $\mathscr{L}(T^*X)$ to $\mathscr{L}(T^*Y)$ (well, not everywhere defined) can alternatively be defined as follows : take the symplectic reduction of Λ by $L \times \overline{T^*Y}$. This is well defined at least when L is generic. We denote it by $\Lambda \circ L$. In particular if Λ_1 is a correspondance from T^*X to T^*Y and Λ_2 a correspondence from T^*Y to T^*Z then

$$\Lambda_2 \circ \Lambda_1 = \{(x,\xi,z,\zeta) \mid \exists (y,\eta), (x,\xi,y,\eta) \in \Lambda_1, (y,\eta,z,\zeta) \in \Lambda_2\}$$

Note that Λ^a (sometimes denoted as Λ^{-1}) is defined as $\Lambda^a = \{(x,\xi,y,\eta) \mid (y,\eta,x,\xi) \in \Lambda\}$. This is a Lagrangian correspondence from T^*Y to T^*X. The composition $\Lambda \circ \Lambda^a \subset T^*X \times \overline{T^*X}$ is, in general, not equal to the identity (i.e. Δ_{T^*X}, the diagonal in T^*X), even though this is the case if Λ is the graph of a symplectomorphism. A fundamental example is the correspondance associated to a smooth map $f : X \longrightarrow Y$. Then

$$\Lambda_f = \{(x,\xi,y,\eta) \in T^*X \times T^*Y \mid f(x) = y, \eta \circ df(x) = \eta\}$$

Then if $g : Y \longrightarrow Z$, we have $\Lambda_{g\circ f} = \Lambda_g \circ \Lambda_f$.

EXERCICE 4.3. Compute $\Lambda \circ \Lambda^a$ for $\Lambda = V_x \times V_y$, where V_x is the cotangent fiber over x.

2. Generating functions

Our goal is to describe Lagrangian submanifolds in T^*N. Let $\lambda = pdx$ be the Liouville form of T^*N. Given any 1-form α on N, we can define a smooth manifold

$$L_\alpha = \{(x,\alpha(x)) | x \in N, \alpha(x) \in T^*_x N\} \subset T^*N.$$

LEMMA 4.4. *L_α is Lagrangian if and only if α is closed.*

PROOF. Let $i : N \to TN$ be the embedding map $i(x) = (x,\alpha(x))$. Notice that

$$\lambda|_{L_\alpha} = \alpha$$

i.e.

$$i^*(\lambda) = \alpha.$$

Lagrangian condition is $(d\lambda)|_{L_\alpha} = 0$, i.e. $d\alpha = 0$. □

DEFINITION 4.5. If $\lambda|_L$ is exact, we say L is exact Lagrangian.

In particular, L_α is exact if and only if $\alpha = df$ for some function f on N. In this case,

$$L_\alpha \cap O_N = \{x | \alpha(x) = df(x) = 0\} = Crit(f),$$

where O_N is the zero section of TN.

REMARK 4.6. 1) If L is C^1 close to O_N, then $L = L_\alpha$ for some α. To see this, L_α is 'graph' of α in TN and a C^1 perturbation of a graph is a graph.

2) If L is exact, C^1 close to O_N, then $L = L_{df}$. Therefore, $\#(L \cap O_N) \geq 2$, if we assume N is compact. (f has at least two critical points, corresponding to maximum and minimum, and we may find more with more sophisticated tools.)

Arnold Conjecture: If $\varphi \in Ham_\omega(T^*N)$ and $L = \varphi(O_N)$, then $\#(L \cap O_N) \geq \mathrm{cat}_{\mathrm{LS}}(N)$, where $\mathrm{cat}_{\mathrm{LS}}(N)$ is the minimal number of critical points for a function on N.

DEFINITION 4.7. A generating function for L is a smooth function $S : N \times \mathbb{R}^k \to \mathbb{R}$ such that

1) The map

$$(x,\xi) \quad \mapsto \quad \frac{\partial S}{\partial \xi}(x,\xi)$$

has zero as a regular value. As a result $\Sigma_S = \{(x,\xi) | \frac{\partial S}{\partial \xi}(x,\xi) = 0\}$ is a submanifold. (Note that $\partial S/\partial \xi$ is a vector of dimension k, so Σ_S is a manifold with the same dimension as N, but may have a different topology.)

2)

$$\begin{array}{rccc} i_S: & \Sigma_S & \to & T^*N \\ & (x,\xi) & \mapsto & (x, \frac{\partial S}{\partial x}(x,\xi)) \end{array}$$

has image $L = L_S$.

LEMMA 4.8. *If for some given S satisfying 1) of the definition and L_S is given by 2), then L_S is an immersed Lagrangian in T^*N.*

PROOF. Since S is a function from $N \times \mathbb{R}^k$ to $\mathbb{R}$, the graph of dS in $T^*(N \times \mathbb{R}^k)$ is a Lagrangian in $T^*(N \times \mathbb{R}^k)$. We will use the symplectic reduction as in the Example 2 in the last section. Define K as a submanifold in $T^*(N \times \mathbb{R}^k)$,

$$K = T^*N \times \mathbb{R}^k \times \{0\}.$$

K is coisotropic as shown in Example 2. Locally, the graph of dS is given by

$$gr(dS) = \{(x, \xi, \frac{\partial S}{\partial x}(x,\xi), \frac{\partial S}{\partial \xi}(x,\xi))\}.$$

Then

$$\Sigma_S = gr(dS) \cap K.$$

The regular value condition in 1) ensures that $gr(dS)$ intersects K transversally. By symplectic reduction, we know i_S is an immersion and L_S is a Lagrangian in T^*N because $gr(dS)$ is Lagrangian in $T^*(N \times \mathbb{R}^k)$. □

REMARK 4.9. If L_S is embedded, we have

$$L_S \cap O_N \simeq Crit(S).$$

Question: Which L have a generating function?
Answer: (Giroux) It is given by conditions on the tangent bundle TL.

DEFINITION 4.10. Let S be a generating function on $N \times \mathbb{R}^k$. We say that S is quadratic at infinity if there exists a nondegenerate quadratic form Q on $\mathbb{R}^k$ such that

$$S(x,\xi) = Q(\xi) \quad \text{for } |\xi| >> 0.$$

For simplicity, we will use GFQI to mean generating function quadratic at infinity.

PROPOSITION 4.11. *Let S be a generating function of L_S such that*
(1) $\|\nabla(S-Q)\|_{C^0} \le C$,
(2) $\|S-Q\|_{C^0(B(0,r))} \le Cr$,
then there exists $\tilde{S}$ GFQI for L_S.

PROOF. (sketch) Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that $\rho \equiv 1$ on $[0, A]$, $\rho \equiv 0$ on $[B, +\infty)$ and $-\varepsilon \le \rho' \le 0$. Define

$$S_1(x,\xi) = \rho(|\xi|)S(x,\xi) + (1-\rho(|\xi|))Q(\xi)$$

We are going to prove that

$$\frac{\partial}{\partial \xi} S_1(x,\xi) = 0 \Longleftrightarrow \frac{\partial}{\partial \xi} S(x,\xi) = 0$$

Indeed,

$$\frac{\partial}{\partial \xi} S_1(x,\xi) = \frac{\partial}{\partial \xi}\left(\rho(|\xi|)\left(S(x,\xi) - Q(\xi)\right) + Q(\xi)\right)$$

$$= \rho'(|\xi|)\frac{\xi}{|\xi|}\left(S(x,\xi) - Q(\xi)\right) + \rho(|\xi|)\frac{\partial}{\partial \xi}(S-Q)(x,\xi) + A_Q\xi = 0$$

For this one must have, if $|A\xi| \ge k|\xi|$

$$c|\xi| \le \varepsilon \|S - Q\|_{C^0} + \|\nabla(S-Q)\|_{C^0} \le \varepsilon C|\xi| + C$$

therefore for ε small enough, this implies

$$|\xi| \le \frac{C}{c - \varepsilon C}$$

and this remains bounded for ε small enough. If we choose A large enough so that it is larger than $\frac{C}{c-\varepsilon C}$, then $S_1 = S_0$ and therefore Σ_{S_1} and Σ_S coincide, and also i_{S_1} and i_{S_0}. $\square$

THEOREM 4.12. *(Sikorav, [**Sik1**]) N is compact. Let $L = \varphi(O_N)$ and $\varphi \in Ham(T^*N)$. Then L has a GFQI.*

PROOF. (Brunella, [**Bru**]) Consider a "special" case $N = \mathbb{R}^N$ and $\varphi \in Ham^0(\mathbb{R}^N)$. By superscript 0, we mean compactly supported.

There is a ' 'correspondence" between function $h : N \times N \to \mathbb{R}$ and maps $\varphi_h : T^*N \to T^*N$ given by

$$\varphi_h(x_1, p_1) = (x_2, p_2) \quad \Longleftrightarrow \quad \begin{cases} p_1 = \frac{\partial}{\partial x_1} h(x_1, x_2) \\ p_2 = -\frac{\partial}{\partial x_2} h(x_1, x_2) \end{cases}$$

The graph of φ_h is a submanifold in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ with symplectic form given by $\omega = dp_1 \wedge dx_1 - dp_2 \wedge dx_2$. It's a Lagrangian if and only if φ_h is a symplectic diffeomorphism.

The graph of dh is a submanifold in $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ with the natural symplectic structure and it's Lagrangian.

Note that the first is a graph of a map T^*N to T^*N while the second is the graph of a map $N \times N$ to $\mathbb{R}^l \times \mathbb{R}^l$ (in particular the first is transverse to $\{0\}(T^*N)$, while the second is transverse to $\{0\} \times \mathbb{R}^l$).

There is a symplectic isomorphism between $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ and $T^*(\mathbb{R}^n \times \mathbb{R}^n)$, given by

$$(x_1, p_1, x_2, p_2) \mapsto (x_1, x_2, p_1, -p_2).$$

and this maps the graph of dh to the graph of φ_h.

Set $h_0(x_1, x_2) = \frac{1}{2}|x_1 - x_2|^2$, then

$$\varphi_{h_0}(x_1, p_1) = (x_1 - p_1, p_1).$$

If h is C^2 close to h_0, then $gr(dh)$ is C^1 close to $gr(dh_0)$, under isomorphism, $\Gamma(\varphi_{h_0})$, since C^1 perturbation of a graph is a graph, we know (up to isomorphism) $gr(dh) = \Gamma(\varphi_h)$. Since $gr(dh)$ is always Lagrangian, φ_h is symplectic isomorphism.

REMARK 4.13. We can do the same with $-h_0$.

$$\varphi_{-h_0} = (\varphi_{h_0})^{-1}.$$

REMARK 4.14. We can do the inverse. Any symplectic map φ C^1 close to φ_{h_0} is of the form φ_h.

PROPOSITION 4.15 (Chekanov's composition formula). *Let L be a Lagrangian in $T^*\mathbb{R}^n$. L coincides with O_N outside a compact set and has a GFQI $S(x,\xi)$. If $h = h_0$ near infinity, then $\varphi_h(L)$ has GFQI*

$$\tilde{S}(x, \xi, y) = h(x, y) + S(y, \xi).$$

REMARK 4.16. $\tilde{S}$ is only approximately quadratic at infinity. We use the last proposition to make it real GFQI.

For the proof of the claim, check that $L_{\tilde{S}}$ is $\varphi_h(L)$.

$$\frac{\partial \tilde{S}}{\partial \xi}(x, \xi, y) = 0 \Longleftrightarrow \frac{\partial S}{\partial \xi}(y, \xi) = 0.$$

$$\frac{\partial \tilde{S}}{\partial y}(x, \xi, y) = 0 \Longleftrightarrow \frac{\partial h}{\partial y}(x, y) + \frac{\partial S}{\partial y}(y, \xi) = 0.$$

A point in $L_{\tilde{S}}$ is

$$\begin{aligned}(x, \frac{\partial \tilde{S}}{\partial x}(x, \xi, y)) &= (x, \frac{\partial h}{\partial x}(x, y)) \\ &= \varphi_h(y, -\frac{\partial h}{\partial y}(x, y)) \\ &= \varphi_h(y, \frac{\partial S}{\partial y}(y, \xi)).\end{aligned}$$

$(y, \frac{\partial S}{\partial y}(y, \xi))$ is a point in L_S.

If k is close to $-h_0$, $\varphi_k \circ \varphi_h(L)$ has GFQI. If $k = -h_0$, then $(\varphi_{h_0}^{-1} \circ \varphi_h)(L)$ has GFQI.

Any C^1 small symplectic map ψ can be given as

$$\varphi_h = \varphi_{h_0} \circ \psi.$$

So the conclusion is for any ψ C^1 close to the identity, if L has GFQI, then $\psi(L)$ has GFQI.

Now take $\varphi^t \in Ham(T^*N)$.

$$\varphi^1 = \varphi^1_{\frac{N-1}{N}} \circ \varphi^{\frac{N-1}{N}}_{\frac{N-2}{N}} \cdots \varphi_0^{\frac{1}{N}}.$$

Each factor is C^1 small. Then If L has GFQI, then $\varphi^1(L)$ has GFQI.

□

3. The Maslov class

The Maslov or Arnold-Maslov class is a topological invariant of a Lagrangian submanifold, measuring how much its tangent space "turns" with respect to a given Lagrangian distribution.

Let $E = V \oplus V^*$, with the standard symplectic form $\sigma(x,p)(x',p')) = \langle p', x\rangle - \langle p, x'\rangle$. We shall identify V with $V \oplus 0$ and V^* with $0 \oplus V^*$. Then any linear space L close to $V \oplus 0$ is the graph of a linear map $A_L : V \longrightarrow V^*$. Then L is Lagrangian if and only if A_L is self-adjoint. Thus if we have a smooth path $[-1,1] \longrightarrow \Lambda(E)$ of Lagrangians close to V, such that $L(t) \cap V \oplus 0 \neq 0$ if and only if $t = 0$, we get a one-parameter family of self-adjoint $A(t) \in S(V)$, such that $A(t)$ is invertible if and only if $t \neq 0$. Then the index[1] of the quadratic form associated to $A(t)$ is constant except when t goes through zero. We then define $\Sigma_V = \{L \in \Lambda(E) \mid L \cap V \neq 0\}$, and say that the path $L(t)$ crosses Σ_V positively if the index of $A(t)$ increases as t goes from -1 to $+1$. Unfortunately a general path cannot, even near a point where it is crossing Σ_V be assumed to be a graph of A. However, we may use generating quadratic forms, in the same spirit as generating functions.

DEFINITION 4.17. Let $Q : V \oplus W$ be a quadratic form with associated self-adjoint map $A_Q : V \oplus W \longrightarrow (V \oplus W)^*$. We see that Q is a generating form for $L \subset E$ if

$$i_W^* \circ A_Q : V \oplus W \longrightarrow W^*$$

is onto, and L is the reduction of the graph of A_Q in $(V \oplus W) \oplus (V \oplus W)^*$ by $V \oplus 0 \oplus (V \oplus W)^*$, that is

$$L = L_Q = \{(x, i_V^* A_Q(x)) \mid i_W^* \circ A_Q(x) = 0\}$$

Note that the assumption that $i_W^* \circ A_Q$ is onto is equivalent to the transverailty of $gr(A_Q) = \{(x, A_Q(x)) \mid x \in V \oplus W\}$ and $C_V = V \oplus 0 \oplus (V \oplus W)^*$. Note also that setting $x = v + \xi$ with $v \in V, \xi \in W$, the function $S(v;\xi) = Q(x)$ is a GFQI for L_Q. Now we can just imitate the discussion on te previous section to construct a generating function for $T_0(L_Q)$ where $T_0(v,p) = (v+p,p)$. Indeed, the graph of T is in $E \oplus \overline{E}$ which can be identified to $(V \oplus W) \oplus (V \oplus W)^*$ and is generated by $\Phi_{T_0}(v,w) = \frac{1}{2}|v-w|^2$. Note that T has a generating quadratic form (with no extra fiber variable) is just equaivalent to the transversality of the graph of T is transverse to $(v \oplus W)^*$. So this is an open property, and will hold for any T close enough to T_0.

[1] that is the number of negative eigenvalues

Using again Chekanov's formula, we get

PROPOSITION 4.18. *Let L_Q have generating quadratic form Q, and the graph of T have generating quadratic form (with no fiber variable) $\Phi_T(v,w)$. Then $T(L)$ is generated by the quadratic generating form*

$$Q'(w;v,\xi) = Q(v,\xi) + \Phi(v,w)$$

Note that the same applies for T_0^{-1}, generated by $\Phi_{T_0^{-1}}(v,w) = -\frac{1}{2}|v-w|^2$. And applying this to the composition $T \circ T_0^{-1}$, this will apply to any linear symplectic map close to identity.

COROLLARY 4.19. *Let $L = L_Q$ have generating quadratic form Q, and T be close to the identity. The $T(L)$ has a generating quadratic form. Moreover there is a continuous map from a neighbourhood U of the identity in $Sp(E)$ to the set of generating quadraitc forms on $V \oplus W$ (for W large enough), such that $T \mapsto Q_T$ is continuous and Q_T is a generating quadratic form for $T(L)$. Moreover if Q is defined on $V \oplus \mathbb{R}^r$, then Q_T is defined over $V \oplus \mathbb{R}^{r+4n}$ and $Q_{\mathrm{Id}}(z,v,w) = Q \oplus |v|^2 \ominus |w|^2$, where $v,w \in \mathbb{R}^{2n}$.*

DEFINITION 4.20. We denote by $\mathfrak{G}$ the space of all generating quadratic forms. It identified with the union of $\mathfrak{G}_r$, the set of generating quadratic forms defined on $V \oplus \mathbb{R}^r$. We embed $\mathfrak{G}_r$ into $\mathfrak{G}_{r+2}$ by identifying $Q(w,w_1,...,w_r)$ to $Q(v,w_1,...,w_r) + w_{r+1}^2 - w_{r+2}^2$.
It is endowed with the topology of the limit $\mathfrak{G} = \lim_r \mathfrak{G}_r$. The map $Q \longrightarrow L_Q$ defines a map from $\mathfrak{G} \longrightarrow \Lambda(E)$.

THEOREM 4.21 (cf [**Theret**]). *The map $Q \mapsto L_Q$ is a Serre fibration.*

PROOF. Let K be any compact topological space, and consider a map $F : I^n \times [0,1] \longrightarrow \Lambda(E)$, denoted by $(k,t) \mapsto L_{(k,t)}$. We assume there is a map $f : I^n \times \{0\} \longrightarrow \mathfrak{G}$ lifting F, that is $L_{f(k)} = L_{F(k,0)}$. Now we use the fact that the projection $Sp(E) \longrightarrow \Lambda(E)$ given by $T \mapsto T(V)$ is a Serre fibration, so F lifts to $G : I^n \times [0,1] \longrightarrow Sp(E)$ such that $L_{F(k,t)} = T_{G(k,t)} L_0$. Then let us write for simplicity $T_{G(k,t)} = T_{k,t}$. We have for $t_2 - t_1$ small enough, that $T_{k,t_2} \circ T_{k,t_1}^{-1}$ is close to the identity. So if $\tilde{F}$ is defined over $K \times [0,t_1]$, with values in $\mathfrak{G}_r$, then according to Corollary 4.19, we may find a map $\tilde{F}$ defined over $K \times [0,t_2]$ and values in $\mathfrak{G}_{r+2n}$. By compactness of the interval, we can define $\tilde{F}$ over $K \times [0,1]$. $\square$

In the sequel we shall somtimes use a decomposition of E as a sum of V and W where W is a Lagrangian subspace transverse to V. Now let $L : [0,1] \longrightarrow \Lambda(E)$ be a continous path, and Q_t a generating function of $L(t)$ depending continuously on t. Then

DEFINITION 4.22. Let $L : [0,1] \longrightarrow \Lambda(E)$ be a continuous path, and $t \longrightarrow Q_t$ generating quadratic form. Then we set $i(L;V,V^*) = \mathrm{ind}(Q_1) - \mathrm{ind}(Q_0)$. This is called the Conley-Zehnder index of the path $(L(t))_{t\in[0,1]}$.

The Conley-Zehnder index has the following properties

PROPOSITION 4.23. *We have*

(1) *If* $t \mapsto \dim(L(t) \cap V)$ *is constant on* $[0,1]$, *then* $i(L,V,W) = 0$.
(2) *the value of* $i(L;V,W)$ *does not depend on* W, *we denote it by* $i(L,V)$
(3) *If* $L_1 \star L_2$ *is the concatenation of the paths* L_1, L_2 *such that* $L_1(1) = L_2(0)$, *then* $i(L_1 \star L_2, V) = i(L_1, V) + i(L_2, V)$
(4) *If* $L^{-1}(t) = L(1-t)$ *is the opposite path, then* $i(L^{-1}, V) = -i(L,V)$
(5) *If* T *is a symplectomorphism such that* $TV = V, TV^* = V^*$ *then* $i(TL,V) = i(L,V)$
(6) *If* E' *is another symplectic vector space,* V' *a Lagrangian subspace of* E', *and* $L'(t)$ *a path in* $\Lambda(E')$ *then* $i(L \times L', V \times V') = i(L,V) + i(L',V')$.

PROOF. □

Note that if L_1, L_2 are Lagrangian subspaces transverse to V, there is a unique

PROPOSITION 4.24. *Let* L_1, L_2 *be two paths with same endpoints. Then* $i(L_1,V) = i(L_2,V)$ *if and only if* L_1 *and* L_2 *are homotopic with fixed endpoints.*

DEFINITION 4.25. Let $R : [0,1] \longrightarrow Sp(n)$ be path in the symplectic group such that $R(0) = \mathrm{id}$. We set $i(R) = i(gr(R(t)), \Delta)$ where $gr(R) = \{(x, Rx) \mid x \in \mathbb{R}^{2n}\}$ in $\mathbb{R}^{2n} \oplus \overline{\mathbb{R}}^{2n}$, where $\Delta = gr(\mathrm{id}) = \{(x,x) \mid x \in \mathbb{R}^{2n}\}$. this is called the Conley-Zehnder index of R.

PROPOSITION 4.26. *We have* $i(R_1 \star R_2) = i(R_1 \cdot R_2)$. *Moroever*

$$|i(R_1 \cdot R_2) - i(R_1) - i(R_2)| \leq i(R_1(1), R_2(1))$$

(1)

4. Contact and homogeneous symplectic geometry

4.1. Contact geometry, symplectization and contactization. Let (N, ξ) be a pair constituted of a manifold N, and a hyperplane field ξ on N. This means that locally, there is a non-vanishing 1-form α such that $\xi = \mathrm{Ker}(\alpha)$.

DEFINITION 4.27. The pair (N, ξ) is a **contact manifold** if integral submanifolds of ξ (i.e. submanifolds everywhere tangent to ξ) have the smallest possible dimension, i.e. $\frac{\dim(N)-1}{2}$. An integral submanifold of dimension $\frac{\dim(N)-1}{2}$ in a contact manifold is called a **Legendrian submanifold**.

It is easy to check that if locally $\xi = \mathrm{Ker}(\alpha)$, the contact type condition is equivalent to requiring that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing. Note also that the global existence if α is equivalent to the co-orientability of ξ. Sometimes we assume the existence of α. This is always possible, at the cost of going to a double cover.

Examples:

(1) the standard example is $\mathbb{R}^{2n+1}$, with coordinates $q_1, ..., q_n, p^1, ..., p^n, z$ and $\xi = \ker(\alpha)$ with $\alpha = dz - p^1 dq_1 - ... - p^n dq_n$.

(2) A slightly more general case is $J^1(N)$ for any manifold N. This is the set of (q,p,z) where $z \in N$, $p \in T_q^*N$ and $z \in \mathbb{R}$, the contact form being $dz - pdq$. Note that for any smooth function f on N, the set $j^1 f = \{(q, df(q), f(q)0 \mid q \in N\}$ is Legendrian. Moreover any Legendrian graph is of this form.

(3) The manifold $ST^*N = \{(q,p) \in T^*N \mid |p| = 1\}$, where $|\bullet|$ is induced by any riemannian metric on N, endowed with the restriction of the Liouville form. The same holds for $PT^*N = ST^*N/\simeq$ where $(q,p_1) \simeq (q,p_2)$ if and only if $p_1 = \pm p_2$.

EXERCICE 4.28. Prove that $PT^*\mathbb{R}^n$ is contactomorphic to J^1S^{n-1}. There is a natural contactomorphism called Euler coordinates: a point $(q,p) \in PT^*(\mathbb{R}^n)$ corresponds in a unique way to a to a point in $\mathbb{R}^n$ and a linear hyperplane (i.e. the pair $(q, \ker(p))$), that may be replaced by the parallel linear hyperplane through this point. In other words we identify $PT^*\mathbb{R}^n$ to the set of pairs constituted of an affine hyperplanes and a point on the hyperplane. The hyperplane may be associated to its normal vector, q, in S^{n-1}, the distance from the origin to the hyperplane, a real number z, and a vector in the hyperplane, connecting the orthogonal projection of the origin on the hyperplane and the point, p. Now (q,p,z) are in $J^1(S^{n-1})$ because p is orthogonal to q , provided we use the canonical metric in $\mathbb{R}^n$ to identify vectors and covectors.

There are two constructions relating symplectic and contact manifolds.

DEFINITION 4.29 (Symplectization of a contact manifold). Let (N,ξ) be a contact manifold, with contact form α. Then $(N \times \mathbb{R}_+^*, d(t\alpha))$ is a symplectic manifold called the symplectization of (N,ξ).

PROPOSITION 4.30 (Uniqueness of the Symplectization). *If* $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta) = \xi$ *we have a symplectomorphism between* $(N \times \mathbb{R}_+^*, d(t\alpha))$ *and* $(N \times \mathbb{R}_+^*, d(t\beta))$. *Indeed, we have* $\beta = f\alpha$ *where* f *is a non-vanishing function on* N. *Then the map* $F : (z,t) \mapsto (z, f(z)\cdot t)$ *satisfies* $F^*(t\alpha) = tf(z)\alpha = t\beta$, *so realizes a symplectomorphism* $F : (N \times \mathbb{R}_+^*, d(t\beta)) \to (N \times \mathbb{R}_+^*, d(t\alpha))$.

PROPOSITION 4.31. *Let* (M,ω) *be a symplectic manifold. Assume* $\omega = d\lambda$. *Then* $(M \times \mathbb{R}, dz - \lambda)$ *is a contact manifold. If we only know that* ω *is an integral class, and* P *is the circle bundle over* M *with first Chern class* ω, *then the canonical* $U(1)$*-connection,* θ *on* P *with curvature* ω *makes* (P,θ) *into a contact manifold*[2].

EXERCICE 4.32. State and prove the analogue of Darboux and Weinstein's theorem in the contact setting.

PROPOSITION 4.33 (Symplectization of a Legendrian submanifold). *Let* L *be a Legendrian submanifold in* (N,ξ). *Then* $L \times \mathbb{R}_+^*$ *is a Lagrangian in the symplectization of* (N,ξ). *Let* L *be a Lagrangian in* (M,ω) *with* ω *exact. Assume* L *is exact, that is* λ_L *is an*

[2]The 1-form θ is the unique S^1 invariant form such that $d\theta = \pi^*(\omega)$. In both cases, we call the manifold the contactization or the prequantization of (M,ω).

exact form (it is automatically closed, since ω vanishes on L). Then L has a lift to a Legendrian Λ in $(M\times\mathbb{R}, dz-\lambda)$, unique up to a translation in z. Similarly if ω is integral, and the holonomy of θ along L is integral, we have a Legendrian lift Λ of L, unique up to a rotation in $U(1)$.

The proof is left as an exercise.

4.2. Homogeneous symplectic geometry. We now show that contact structures are equivalent to homogeneous symplectic structures. Indeed,

DEFINITION 4.34. A homogeneous symplectic manifold is a symplectic manifold (M,ω) endowed with a smooth proper and free action of $\mathbb{R}_+^*$, such that denoting by $\frac{\partial}{\partial\lambda}$ the vector field associated to the action, we have $L_{\lambda\frac{\partial}{\partial\lambda}}\omega=\omega$.

Clearly the symplectization of a contact manifold is a homogeneous symplectic manifold. We now prove the converse.

Example: Let M be a smooth manifold. We denote by $\mathring{T}^*M$ the manifold $T^*M\setminus 0_M$ endowed with the obvious action $\lambda\cdot(q,p)=(q,\lambda,p)$. This is the symplectization of ST^*M.

PROPOSITION 4.35 (Homogeneous symplectic geometry is contact geometry). *Let (M,ω) be a homogeneous symplectic manifold. Then (M,ω) is symplectomorphic (by a homogeneous map) to the symplectization of $(M/\mathbb{R}_+^*, i_X\omega)$*

PROOF. Let $X=\frac{1}{\lambda}\frac{\partial}{\partial\lambda}$, and consider the form $\alpha(\xi)=\omega(X,\xi)$ which is well defined on the quotient $C=M/\mathbb{R}_+^*$. this is a contact form on C, since $i_X\omega\wedge(d(i_X\omega)^{n-1}=i_X\omega\wedge(L_X\omega)^n=i_X\omega\wedge\omega^{n-1}=\frac{1}{n}i_X(\omega^n)$, and since tangent vectors to C are identified to tangent vectors to M transverse to C, this does not vanish. Let t be a coordinate on M such that $dt(X)=1$, and $\widetilde{\omega}=d(t\pi^*(\alpha))$, then (M,ω) is equal to $(M,d(t\alpha))$. Indeed, let us consider two vectors, first of all in the case where one is X and the other is in $dt(Y)=0$. Then $\widetilde{\omega}(X,Y)=(dt\wedge\alpha+td\alpha)(X,Y)=dt(X)\alpha(Y)=(i_X\omega)(Y)=\omega(X,Y)$. Now assume Y,Z are bot in $\ker(dt)$. Then $\widetilde{\omega}(Y,Z)=dt\wedge t\alpha(Y,Z)+td\alpha(Y,Z)$ but $d\alpha=di_X\omega=\omega$ so that $\widetilde{\omega}(Y,Z)=\omega(Y,Z)$. □

EXERCICE 4.36. Prove that $\mathring{T}^*(M\times\mathbb{R})$ is symplectomorphic to $T^*M\times\mathbb{R}\times\mathbb{R}_+^*$, the symplectization of $J^1(M)$. Hint: prove that the contact manifold $J^1(M)$ is contactomorphic to an open set of $ST^*(M\times\mathbb{R})$.

PROPOSITION 4.37 (Symplectization of a contact map). *Let $\Phi:(N,\xi)\to(P,\eta)$ be a contact transformation, that is a diffeomorphism such that $d\Phi$ sends ξ to η. Then there exists a homogeneous lift of Φ*

$$\widetilde{\Phi}:(\widetilde{N},\omega_\xi)\to(\widetilde{P},\omega_\eta).$$

Conversely any homogeneous symplectomorphism from $(\widetilde{N},\omega_\xi)\to(\widetilde{P},\omega_\eta)$ is obtained in this way.

PROOF. Assume that $\Phi^*(\beta) = \alpha$ where $\mathrm{Ker}(\alpha) = \xi, \mathrm{Ker}(\beta) = \eta$. Then this induces a symplectic map $\widetilde{\Phi}$ between $(N \times \mathbb{R}_+^*, d(t\alpha))$ and $(P \times \mathbb{R}_+^*, d(t\beta))$ and by uniqueness of the symplectization (or rather the fact that it does not depend on the choice of the contact form) we are done. Conversely if $\Psi^*\omega_\eta = \omega_\xi$ that is $\Psi^* d(t\beta) = dt\alpha$, in other words, $d\left(\Psi^*(t\beta) - t\alpha\right) = 0$. If the map is exact, this means, $\Psi^*\beta = \alpha + df$ □

EXERCICES 1. (1) Prove that the above lift is functorial, that is the lift of $\Phi \circ \Psi$ is $\widetilde{\Phi} \circ \widetilde{\Psi}$

(2) Let $\varphi : T^*M \to T^*M$ be an exact symplectic map, that is a map such that $\varphi^*(\lambda) - \lambda$ is exact. Prove that there is a lift of φ to a contact map $\widetilde{\varphi} : J^1M \to J^1M$. Prove that if (N, α) is a contact manifold and ψ a diffeomorphism of N such that $\psi^*(\alpha) = \alpha$ (note that this is stronger than requiring that ψ is a contact diffeomorphism, that is $\psi^*(\alpha) = f \cdot \alpha$ for some nonzero function f) then ψ lifts in turn to a homogeneous symplectic map $(N \times \mathbb{R}_+^*, d(t\alpha))$ to itself.

(3) Prove that the symplectization of $J^1(M)$ is $T^*(M) \times \mathbb{R} \times \mathbb{R}_+^*$ and explicit the symplectomorphism obtained from the above $\widetilde{\varphi}$ by symplectization. Thus to any symplectomorphism $\varphi : T^*M \to T^*M$ we may associate a homogeneous symplectomorphism

$$\Phi : T^*(M) \times \mathbb{R} \times \mathbb{R}_+^* \to T^*(M) \times \mathbb{R} \times \mathbb{R}_+^*$$

Prove that the lift is functorial. That is the lift of $\varphi \circ \psi$ is $\Phi \circ \Psi$.

As a result of Proposition 4.35 we have

COROLLARY 4.38. *An exact Lagrangian submanifold L in $(M, \omega = d\lambda)$ has a unique lift $\widehat{L}$ to the (homogeneous) symplectization of its contactization,* $(\widehat{M}, \Omega) = (M \times \mathbb{R}_*^+ \times \mathbb{R}, dt \wedge d\tau - dt \wedge \lambda)$.

PROOF. Indeed, let $f(z)$ be a primitive of λ on L. Set $\widehat{L} = \{(z, t, \tau) \mid z \in L, \tau = f(z)\}$. Then, $d(td\tau - t\lambda)$ restricted to $\widehat{L}$ equals zero. □

PROPOSITION 4.39. *Let L be an exact Lagrangian. Then L is a conical (or homogeneous) Lagrangian in T^*X if and only if* $\lambda_L = 0$.

PROOF. Let X be the homogeneous vector field, that is the vector fleld such that $i_X\omega = \lambda$. Then since for every vector $Y \in TL$ we have $\lambda(Y) = \omega(X, Y) = 0$ since both X and Y are tangent to L, we have $\lambda_L = 0$. □

Locally, L is given by a homogeneous generating function, that is a generating function $S(q, \xi)$ such that $S(q, \lambda \cdot \xi) = \lambda \cdot S(q, \xi)$.

PROPOSITION 4.40 (See [**Duis**], page 83.). *Let L be a germ of homogeneous Lagrangian. Then L is locally defined by a homogeneous generating function.*

PROOF. Indeed, let $S(x, \xi)$ be a generating function for L. Then $S_\lambda(x, \xi) = S(x, \lambda \cdot \xi)$ is also a generating function for L, since □

EXERCICE 4.41. Let $S(q,\xi)$ be a (local) generating function for L. What is the generating function for $\widehat{L}$?

EXERCICE 4.42. Let S be a smooth hypersurface in M, and $\pi : T^*M \to M$ be the projection.

(1) Prove that if $\nu^*S = \{(x,p) \mid x \in S, p_{|T_xS} = 0\}$ is the conormal to S, then $S = \pi(\nu^*S)$.
(2) Prove that for any homogeneous Lagrangian, L, in T^*M, $\pi_{|L}$ is a map of rank at most $n-1$ (find a trivial kernel).
(3) Prove that if L' is homogeneous Lagrangian and C^1 close to L (i.e. $L' \cap DT^*M$ is C^1 close to $L \cap DT^*M$), then L' is the conormal of some hypersurface S'. Hint: prove that $\pi(L')$ is a (non-empty) smooth hypersurface.

PROPOSITION 4.43. *Let Σ be a germ of hypersurface near z in a homogeneous symplectic manifold. Then after a homogenous symplectic diffeomorphism we may assume Σ is either in $\{q_1 = 0\}$ or $\{p_1 = 0\}$.*

PROOF. Let us consider a transverse germ, V, to X.Then V is transverse to Σ, and denote $\Sigma_0 = V \cap \Sigma$. By a linear change of variable, we may assume the tangent space $T_z\Sigma$ □

CHAPTER 5

Generating functions for Hamiltonians on cotangent bundles of compact manifolds.

In the previous lecture, we proved that if $L_0 = O_{\mathbb{R}^n}$ outside a compact set and has GFQI, and φ is compactly supported Hamiltonian map of $T^*\mathbb{R}^n$, then $\varphi(L)$ has a GFQI.

Let us return to the general case: let N be a compact manifold. For l large enough, there exists an embedding $i : N \hookrightarrow \mathbb{R}^l$. It gives rise to an embedding $\tilde{i}$ of T^*N into $T^*\mathbb{R}^l$, obtained by choosing a metric on $\mathbb{R}^l$. This can be defined as

$$\begin{array}{ccc} T^*N & \hookrightarrow & T^*\mathbb{R}^l \\ (x,p) & \mapsto & (\tilde{x}(x,p), \tilde{p}(x,p)) \end{array}$$

where $\tilde{(}x)(x,p) = i(x)$ and $\tilde{p}(x,p) = p \circ \pi(x)$. $\pi(x)$ is the orthogonal projection $T\mathbb{R}^l \to T_xN$.

It's easy to check that $\tilde{i}^* \tilde{p} d\tilde{x} = p dx$, i.e. $\tilde{i}$ is a symplectic map(embedding). Moreover, if we denote by $N \times (\mathbb{R}^l)^*$ the restriction of $T^*\mathbb{R}^l$ to N, then it's coisotropic as in Example 1 of symplectic reduction. To any Lagrangian in $T^*\mathbb{R}^l$(transversal to $N \times (\mathbb{R}^l)^*$), we may associate the reduction, that is a Lagrangian of T^*N.

Let $\tilde{L} \subset T^*\mathbb{R}^l$ be a Lagrangian. Assume $\tilde{L}$ coincides with $O_{\mathbb{R}^l}$ outside a compact set and $\tilde{L}$ is transverse to $N \times (\mathbb{R}^l)^*$. Denote its symplectic reduction by $\tilde{L}_N = \tilde{L}_{N\times(\mathbb{R}^l)^*} = \tilde{L} \cap (N \times (\mathbb{R}^l)^*)/\sim$.

Claim: For $\varphi \in Ham(T^*N)$, if $\tilde{L}$ has GFQI, then $\varphi(\tilde{L}_N)$ has GFQI.

REMARK 5.1. If $\tilde{L}$ has $\tilde{S} : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}$ as GFQI, then $\tilde{L}_N$ has $\tilde{S}|_{N\times\mathbb{R}^k}$ as GFQI.

For the proof of the claim, we will construct $\tilde{\varphi}$ with compact support such that

$$(\tilde{\varphi}(\tilde{L}))_N = \varphi(\tilde{L}_N).$$

Then, the claim follows from the last remark and first part of the proof. Assume φ is the time one map of φ^t associated to $H(t,x,p)$, where (x,p) is coordinates for T^*N. Locally, we can write (x,u,p,v) for points in $\mathbb{R}^l$ so that $N = \{u = 0\}$. We define

$$\tilde{H}(t,x,u,p,v) = \chi(u)H(t,x,p),$$

where χ is some bump function which is 1 on N and 0 outside a neighborhood of N. By the construction, $X_{\tilde{H}} = X_H$ on $N \times (\mathbb{R}^l)^*$. $\tilde{\varphi} = \tilde{\varphi}^1$, the time one flow of $\tilde{H}$, is the map we need.

The theorem follows by noticing that if we take $\tilde{L} = O_{\mathbb{R}^l}$, which is the same as zero section outside compact set and has GFQI, then $\tilde{L}_N = O_N$. $\square$

Exercice: Show that if L has a GFQI, then $\varphi(L)$ has GFQI for $\varphi \in Ham(T^*N)$.
Hint. If $S: N \times \mathbb{R}^k \to \mathbb{R}$ is a GFQI for L, then L is the reduction of $gr(dS)$.

REMARK 5.2. 1) O_N is generated by

$$\begin{array}{rccc} S: & N \times \mathbb{R} & \to & \mathbb{R} \\ & (x,\xi) & \mapsto & \xi^2 \end{array}$$

2) There is no general upper bound on k (the minimal number of parameter of a generating functions needed to produce all Lagrangian.)
Reason: Consider a curve in T^*S^1

1. Applications

We first need to show that GFQI has critical points. Let us consider a smooth function f on noncompact manifold M satisfying (PS) condition.
(PS): If a sequence (x_n) satisfying $df(x_n) \to 0$ and $f(x_n) \to c$, then (x_n) has a converging subsequence.

REMARK 5.3. Clearly, the limit of the subsequence is a critical point at level c.

REMARK 5.4. A GFQI satisfies (PS). It suffices to check this for a nondegenerate quadratic form Q. Let $Q(x) = \frac{1}{2}(A_Q x, x)$, then $dQ(x) = A_Q(x)$. Since Q is nondegenerate, we know A_Q is invertible and

$$dQ(x_n) \to 0 \Longrightarrow A_Q x_n \to 0 \Longrightarrow x_n \to 0.$$

PROPOSITION 5.5. *If f satisfies (PS) and $H^*(f^b, f^a) \neq 0$, then f has a critical point in $f^{-1}([a,b])$, where $f^\lambda = \{x \in M | f(x) \leq \lambda\}$.*

PROPOSITION 5.6. *For $b >> 0$ and $a << 0$ we have*

$$H^*(S^b, S^a) \cong H^{*-i}(N).$$

PROOF. One can replace S by Q since $S = Q$ at infinity. Define

$$Q^\lambda = \{\xi | Q(\xi) \leq \lambda\}.$$

$$\begin{aligned} H^*(S^b, S^a) &= H^*(N \times Q^b, N \times Q^a) \\ &= H^*(N) \times H^*(Q^b, Q^a). \end{aligned}$$

Since Q is a quadratic form, it's easy to see $H^*(Q^b, Q^a)$ is the same as $H^*(D^-, \partial D^-)$ where D^- is the disk in the negative eigenspace of Q (hence has dimension $index(Q)$, the number of negative eigenvalues). □

Conjecture:(Arnold) Let $L \subset T^*N$ be an exact Lagrangian. Is there $\varphi \in Ham(T^*N)$ such that $L = \varphi(O_N)$?

REMARK 5.7. L_S is always exact since $\lambda|_{L_S} = dS|_{\Sigma_S}$.

$$L_S = \{(x, \frac{\partial S}{\partial x}(x,\xi)) | \frac{\partial S}{\partial \xi}(x,\xi) = 0\}.$$

$$\lambda|_{L_S} = pdx = \frac{\partial S}{\partial x}(x,\xi)dx = dS,$$

since for points on L_S, $\frac{\partial S}{\partial \xi} = 0$.

A recent result by Fukaya, Seidel and Smith ([**F-S-S**]) grants that under quite general assumptions, the degree of the projection $\deg(\pi : L \to N) = \pm 1$ and $H^*(L) = H^*(N)$.

EXERCICE 5.8. Prove that if L has GFQI S, then $\deg(\pi : L \to N) = \pm 1$.

Indication: Choose a generic point $x_0 \in N$. The degree is the multiplicity with sign of the intersection of L and the fiber over x_0. That is counting the number of ξ with $\frac{\partial S}{\partial \xi}(x_0,\xi) = 0$, i.e. the number of critical points of function $\xi \mapsto S(x_0,\xi)$ with sign

$$(-1)^{index(\frac{d^2S}{d\xi^2}(x_0,\xi))}.$$

Therefore

$$\deg(\pi : L \to N) = \sum_{\xi_j} (-1)^{index(\frac{d^2S}{d\xi^2}(x_0,x_j))}$$

where the summation is over all ξ_j with $\frac{\partial S}{\partial \xi}(x_0,\xi_j) = 0$. The summation is finite since S has quadratic infinity and the sum is the euler number of the pair (S^b, S^a) for large b and small a. Finally, check that for all quadratic form Q, the euler number of (Q^b, Q^a) is ± 1.

By the previous claim, for large b and small a

$$H^*(S^b, S^a) \cong H^{*-i}(N).$$

Since N is compact, we know $H^*(N) \neq 0$. This implies that S has at least one critical point and $(L_S \cap O_N) \neq \emptyset$.

THEOREM 5.9 (Hofer ([**Hofer**]). *Let N be a compact manifold and $L = \varphi(O_N)$ for some $\varphi \in Ham(T^*N)$, then*

$$\#(L \cap O_N) \geq cl(N).$$

If all intersection points are transverse, then

$$\#(L \cap O_N) \geq \sum b_j(N).$$

Here

$$cl(N) = \max\{k | \exists \alpha_1, \cdots, \alpha_{k-1} \in H^*(N) \setminus H^0(N) \text{ such that } \alpha_1 \cup \cdots \cup \alpha_{k-1} \neq 0\}$$

and

$$b_j(N) = \dim H^j(N).$$

COROLLARY 5.10.

$$\#(L \cap O_N) \geq 1.$$

We shall postpone the proof of the theorem. However we may prove the corollary: since by the Theorem of Sikorav, L has GFQI, and by proposition 1.4 and 1.3 it must have a critical point. Some calculus of critical levels as in the next lectures will allow us to recover the full strength of Hofer's theorem.

THEOREM 5.11. *(Conley-Zehnder*[**Co-Z**]*) Let $\varphi \in Ham(T^{2n})$, then*

$$\#Fix(\varphi) \geq 2n+1.$$

If all fixed points are nondegenerate, then

$$\#Fix(\varphi) \geq 2^{2n}.$$

REMARK 5.12. $2n+1$ is the cup product length of T^{2n} and 2^{2n} is the sum of Betti numbers of T^{2n}.

PROOF. Let (x_i, y_i) be coordinates of T^{2n}. We will write (x, y) for simplicity. The symplectic form is given by $\omega = dy \wedge dx$. Consider $T^{2n} \times \overline{T^{2n}}$ with coordinates (x, y, X, Y), whose symplectic form is given by

$$\omega = dy \wedge dx - dY \wedge dX.$$

With this ω, the graph of φ, $\Gamma(\varphi)$ is a Lagrangian. Consider another symplectic manifold T^*T^{2n}, denote the coordinates by (a, b, A, B). Note that x, y, X, Y, a, b take value in $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and A, B takes value in $\mathbb{R}^n$.

It has the natural symplectic form as a cotangent bundle

$$\omega = dA \wedge da + dB \wedge db.$$

Define a map $F : T^*T^{2n} \to T^{2n} \times \overline{T^{2n}}$

$$F(a, b, A, B) = \left(\frac{2a-B}{2}, \frac{2b+A}{2}, \frac{2a+B}{2}, \frac{2b-A}{2}\right) \bmod \mathbb{Z}^n.$$

It's straightforward to check that F is a symplectic covering.

Let $\triangle_{T^{2n}}$ be the diagonal in $T^{2n} \times \overline{T^{2n}}$. It lifts to $O_{T^{2n}} \subset T^*T^{2n}$ and the projection π induces a bijection between $O_{T^{2n}}$ and $\triangle_{T^{2n}}$. Of course $O_{T^{2n}}$ is only one component in the preimage of $\triangle_{T^{2n}}$ corresponding to $A = B = 0$ (other components are given by $A = A_0, B = B_0$ where $A_0, B_0 \in \mathbb{Z}^n$. Now assume φ is the time one map of $\varphi^t \in Ham(T^{2n})$.

$$\Gamma(\varphi^t) = (id \times \varphi^t)(\triangle_{T^{2n}}).$$

This Hamiltonian isotopy lifts to a Hamiltonian isotopy Φ^t of T^*T^{2n} such that

$$\pi \circ \Phi^t = \phi^t \circ \pi.$$

Then the restriction of the projection to $\Phi^t(O_{T^{2n}})$ remains injective, since

$$\pi(\Phi^t(u)) = \pi(\Phi^t(v))$$

implies

$$\phi^t(\pi(u)) = \phi^t(\pi(v))$$

but since π is injective on $O_{T^{2n}}$ and ϕ^t is injective, this implies $u = v$.

Therefore to distinct points in $\Phi^t(O_{T^{2n}}) \cap O_{T^{2n}}$ correspond distinct points in $\Gamma(\varphi) \cap \triangle_{T^{2n}} = Fix(\varphi)$.

According to Hofer's theorem, the first set has at least $2n+1$ points, so the same holds for the latter. $\square$

REMARK 5.13. The theorem doesn't include all fixed point φ. Indeed, we could have done the same with any other component of $\pi^{-1}(\triangle_{T^{2n}})$ (remember, they are parametrized by pairs of vectors $(A_0, B_0) \in \mathbb{Z}^n \times \mathbb{Z}^n$), and possibly obtained other fixed points. What is so special about those we obtained ? It is not hard to check that they correspond to periodic contractible trajectories on the torus. Indeed, a closed curve on the torus is contractible if and only if it lifts to a closed curve on $\mathbb{R}^{2n}$. Now, our curve is $\Phi^t(a,b,0,0)$ and projects on $(id \times \varphi^t)(x,y,x,y) = (x,y,\phi^t(x,y))$. Since $\Phi^1(a,b,0,0) \in O_{T^{2n}}$, we may denote $\Phi^1(a,b,0,0) = (a',b',0,0)$, and since $\phi^1(x,y)) = (x,y)$, we have $a' = x = a, b' = y = b$. Thus $\Phi^t(a,b,0,0)$ is a closed loop projecting on $(id\times\varphi^t)(x,y,x,y)$, this last loop is therefore contractible, hence the loop $\varphi^t(x,y)$ is also contractible.

Historical comment: Conley-Zehnder proof of the Arnold conjecture for the torus came before Hofer's theorem. It was the first result in higher dimensional symplectic topology, followed shortly after by Gromov's non-squeezing.

THEOREM 5.14. *(Poincaré and Birkhoff) Let φ be an area preserving map of the annulus, shifting each circle (boundary) in opposite direction, then $\#Fix(\varphi) \geq 2$.*

PROOF. Assume φ is the time one map of a Hamiltonian flow φ^t associated to $H = H(t,r,\theta)$, where (r,θ) is the polar coordinates of the annulus$(1 \leq r \leq 2)$. Assume without loss of generality

$$\frac{\partial H}{\partial r} > 0 \text{ for } r = 2$$

and

$$\frac{\partial H}{\partial r} < 0 \text{ for } r = 1.$$

One can extend H to $[\frac{1}{2}, \frac{5}{2}] \times S^1$ such that $\frac{\partial H}{\partial r}(r,\theta) < 0$ for $r < 1$, $\frac{\partial H}{\partial r}(r,\theta) > 0$ for $r > 2$ and

$$H(t,r,\theta) = -r \text{ on } [\frac{1}{2}, \frac{2}{3}]$$

and

$$H(t,r,\theta) = r \text{ on } [\frac{7}{3}, \frac{5}{2}].$$

Take two copies of this enlarged annulus and glue them together to make a torus. Then $\#Fix(\varphi) \geq 3$. At least one copy has two fixed points. $\square$

2. The calculus of critical values and first proof of the Arnold Conjecture

Let N be a compact manifold and $\varphi \in Ham(T^* N)$, then $L = \varphi(O_N)$ is a Lagrangian. We have proved the following

THEOREM 5.15. *L has a GFQI.*

There are several consequences

- Hofer's theorem: $\#(\varphi(O_N) \cap O_N) \geq 2$; (In fact Hofer's theorem says more.)
- Conley-Zehnder theorem: $\#Fix(\varphi) \geq 2n+1$ for $\varphi \in Ham(T^{2n})$;
- Poincaré-Birkhoff Theorem.

Today, we are going to talk about 1) Uniqueness of GFQI of L and 2) Calculus of critical levels.

REMARK 5.16. Theorem 5.15 extends to continuous family, i.e. if φ_λ is a continuous family of Hamiltonian diffeomorphisms and $L_\lambda = \varphi_\lambda(O_N)$, then there exists a continuous family of GFQI S_λ.

REMARK 5.17. The Theorem 1.1 (you mean 5.15? Yes (Claude) holds also for Legendrian isotopies(Chekanov). Let $J^1(N,\mathbb{R}) \equiv T^* N \times \mathbb{R}$ and define

$$\alpha = dz - pdq.$$

DEFINITION 5.18. Λ is called a Legendrian if and only if $\alpha|_\Lambda = 0$.

Example: Given a smooth function $f \in C^\infty(N,\mathbb{R})$, the submanifold defined by

$$z = f(x), p = df, q = x$$

is a Legendrian. One similarly associates to a generating function, $S : N \times \mathbb{R}^k \longrightarrow \mathbb{R}$ a legendrian submanifold (under the same transversality assumptions as for the Legendrian case)

$$\Lambda_S = \{(x, \frac{\partial S}{\partial x}(x,\xi), S(x,\xi)) \mid \frac{\partial S}{\partial \xi} = 0\}$$

Denote the projection from $T^* N \times \mathbb{R}$ to $T^* N$ by π. Then any Legendrian submanifold projects down to an (exact) Lagrangian. Moreover, any exact Lagrangian can be lifted to a Legendrian. Note however that there are legendrian isotopies that do not project to Lagrangian ones. So Chekanov's theorem is in fact stronger than Sikorav's theorem, even though the proof is the same.

2.1. Uniqueness of GFQI.

Let $\varphi \in Ham(T^* N)$ and $L = \varphi(O_N)$. Denote a GFQI for L by S. We will show that we can obtain different GFQI by the following three operations.

Operation 1:(Conjugation) If smooth map $\xi : N \times \mathbb{R}^k \to \mathbb{R}^k$ satisfies that for each $x \in N$, $\xi(x,\cdot) : \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism, then we claim:

$$\tilde{S}(x,\eta) = S(x,\xi(x,\eta))$$

is again GFQI for L.

Recall from the definition of generating function

$$L_{\tilde{S}} = \{(x, \frac{\partial \tilde{S}}{\partial x}(x,\eta)) | \frac{\partial \tilde{S}}{\partial \eta}(x,\eta) = 0\}$$

and

$$L_S = \{(x, \frac{\partial S}{\partial x}(x,\xi)) | \frac{\partial S}{\partial \xi}(x,\xi) = 0\}.$$

Since $\frac{\partial \xi}{\partial \eta}$ is invertible, the chain rule says $\frac{\partial \tilde{S}}{\partial \eta}(x,\eta)$ and $\frac{\partial S}{\partial \xi}(x,\xi(x,\eta))$ simultaneously. On such points,

$$\frac{\partial \tilde{S}}{\partial x}(x,\eta) = \frac{\partial S}{\partial x}(x,\xi(x,\eta)) + \frac{\partial S}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \frac{\partial S}{\partial x}(x,\xi(x,\eta)).$$

Operation 2: (Stabilization) If q is a nondegenerate quadratic form, then

$$\tilde{S}(x,\xi,\eta) = S(x,\xi) + q(\eta)$$

is a GFQI for L.

The reason is

$$\frac{\partial \tilde{S}}{\partial x}(x,\xi,\eta) = \frac{\partial S}{\partial x}(x,\xi)$$

and

$$\frac{\partial \tilde{S}}{\partial \xi} = \frac{\partial \tilde{S}}{\partial \eta} = 0 \iff \begin{cases} A_q \eta = 0 \Longrightarrow \eta = 0 \\ \frac{\partial S}{\partial \xi}(x,\xi) = 0 \end{cases}$$

where A_q is given by $(A_q\eta,\eta) = q(\eta)$ for all η and is invertible since q is nondegenerate.

Operation 3: (Shift) By adding constant,

$$\tilde{S}(x,\xi) = S(x,\xi) + c.$$

The GFQI is unique up to the above operations in the sense that

THEOREM 5.19 (Uniqueness theorem for GFQI). *If S_1, S_2 are GFQI for $L = \varphi(O_N)$, then there exists $\tilde{S}_1$, $\tilde{S}_2$ obtained from S_1 and S_2 by a sequence of operations 1,2,3 such that $\tilde{S}_1 = \tilde{S}_2$.*

PROOF. For the proof, see [**Theret**]. □

The main consequence of this theorem is that given $L = \varphi(O_N)$, for different choices of GFQI, we know the relation between $H^*(S^b, S^a)$. It suffices to trace how $H^*(S^b, S^a)$ changes by operation 1,2,3.

It's easy to see that $H^*(S^b, S^a)$ is left invariant by operation 1, because the pair (S^b, S^a) is diffeomorphic to $(\tilde{S}^b, \tilde{S}^a)$.

For operation 3,

$$H^*(\tilde{S}^b, \tilde{S}^a) = H^*(S^{b-c}, S^{a-c}).$$

For operation 2, we claim without proof for $b > a$

$$H^*(\tilde{S}^b, \tilde{S}^a) = H^{*-i}(S^b, S^a)$$

where i is the index of q.

REMARK 5.20. The theorem holds for $L = \varphi(O_N)$ only, no result is known for general L). Moreover, the theorem holds for families.

2.2. Calculus of critical levels. In this section, we assume M is a manifold and $f \in C^\infty(M,\mathbb{R})$ is a smooth function satisfying (PS) condition. Given $a < b < c$, there is natural embedding map

$$(f^b, f^a) \hookrightarrow (f^c, f^a).$$

It induces

$$H^*(f^c, f^a) \to H^*(f^b, f^a).$$

DEFINITION 5.21. Let $\alpha \in H^*(f^c, f^a)$. Define

$$c(\alpha, f) = \inf\{b|\text{ image of } \alpha \text{ in } H^*(f^b, f^a) \text{ is not zero}\}.$$

Since the embedding also induces

$$H_*(f^b, f^a) \hookrightarrow H_*(f^c, f^a),$$

the same can be done for $\omega \in H_*(f^c, f^a) \setminus \{0\}$.

DEFINITION 5.22. For $\omega \in H_*(f^c, f^a) \setminus \{0\}$, define

$$c(\omega, f) = \inf\{b|\omega \text{ is in the image of } H_*(f^b, f^a)\}.$$

PROPOSITION 5.23. *$c(\alpha, f)$ and $c(\omega, f)$ are critical values of f.*

PROOF. Prove the first one only. Proof for the other is similar. Let $\gamma = c(\alpha, f)$, assume γ is not a critical value. Since f satisfies (PS) condition, we have

$$H^*(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}) = 0.$$

Study the long exact sequence for the triple $(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}, f^a)$,

$$H^*(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}) \to H^*(f^{\gamma-\varepsilon}, f^a) \to H^{(}f^{\gamma+\varepsilon}, f^a) \to H^{*+1}(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon})$$

Since the first and the last space are $\{0\}$, we know

$$H^*(f^{\gamma-\varepsilon}, f^a) \cong H^*(f^{\gamma+\varepsilon}, f^a).$$

By the definition of γ, the image of α in $H^*(f^{\gamma-\varepsilon}, f^a)$ is zero, but the image of α in $H^*(f^{\gamma+\varepsilon}, f^a)$ is not zero. This is a contradiction. $\square$

Recall Alexander duality:

$$AD: \quad H^*(f^c, f^a) \quad \to \quad H_{n-*}(X - f^a, X - f^c) = H_{n-*}((-f)^{-a}, (-f)^{-c}).$$

PROPOSITION 5.24. *Assume that M is a compact, connected and oriented manifold, then for $\alpha \in H^*(f^c, f^a) \setminus \{0\}$,*

1) $c(\alpha, f) = -c(AD(\alpha), -f)$;

2) $c(1, f) = -c(\mu, -f)$ where $1 \in H^0(M)$ and $\mu \in H^n(M)$ are generators. (In fact, any nonzero element will do since they are all proportional. Here we assumed $a = -\infty$ and $c = +\infty$.)

PROOF. 1) Diagram chasing on the following diagram, using the fact that $X \setminus f^a = (-f)^{-a}$.

$$\begin{array}{ccc} H_*(f^c, f^a) & \xrightarrow{AD} & H^{n-*}(X \setminus f^a, X \setminus f^c) \\ \downarrow & & \downarrow \\ H_*(f^b, f^a) & \xrightarrow{AD} & H^{n-*}(X \setminus f^a, X \setminus f^b) \\ \downarrow & & \downarrow \\ H_*(f^b, f^c) & \xrightarrow{AD} & H^{n-*}(X \setminus f^c, X \setminus f^b) \end{array}$$

2) It suffices to show

$$c(1, f) = \min(f) \text{ and } c(\mu, f) = \max(f).$$

□

THEOREM 5.25. *(Lusternik-Schnirelmann) Assume $\alpha \in H^*(f^c, f^a) \setminus \{0\}$ and $\beta \in H^*(M) \setminus H^0(M)$, then*

(5.1) {star} $$c(\alpha \cap \beta, f) \geq c(\alpha, f)$$

If equality holds in equation 9.47 with common value γ, then for any neighborhood U of $K_\gamma = \{x | f(x) = \gamma, df(x) = 0\}$, we have $\beta \neq 0$ in $H^*(U)$.

REMARK 5.26. If $\beta \notin H^0(M)$ and equality in (9.47) holds, then $H^p(U) \neq 0$ for all U and some $p \neq 0$. This implies K_γ is infinite. Otherwise, take U to be disjoint union of balls then $H^p(U) = 0$ for all $p \neq 0$, which is a contradiction. One can even show that K_γ is uncountable by the same argument.

COROLLARY 5.27. *Let $f \in C^\infty(M, \mathbb{R})$ with compact M, then*

$$\#Crit(f) \geq cl(M).$$

PROOF. Inequality 9.47 is obvious because $\alpha = 0$ in $H^*(f^b, f^a)$ implies $\alpha \cap \beta = 0$ in $H^*(f^b, f^a)$.

If equality in (9.47) holds, for any given U, take ε sufficiently small so that

1) There exists a saturated neighborhood $V \subset U$ of K_γ for the negative gradient flow of f between $\gamma + \varepsilon$ and $\gamma - \varepsilon$, in the sense that any flow line coming into V will either go to K_γ for all later time or go into $f^{\gamma-\varepsilon}$.(Never come out of V between $\gamma + \varepsilon$ and $\gamma - \varepsilon$). Moreover by (PS) we may assume V contains all critical points in $f^{\gamma+\varepsilon} \setminus f^{\gamma-\varepsilon}$.

2) (PS) condition ensures a lower bound for $|\nabla f|$ for all $x \in f^{\gamma+\varepsilon} \setminus (V \cup f^{\gamma-\varepsilon})$.

Let $X = -\nabla f$ and consider its flow φ^t.

$$\frac{d}{dt} f(\varphi^t(x)) = -\left|\nabla f\right|^2 (\varphi^t(x)).$$

Therefore, we have

- If $x \in f^{\gamma-\varepsilon}$, then $\varphi^t(x) \in f^{\gamma-\varepsilon}$.
- (V is saturated) $x \in V$ implies $\varphi^t(x) \in V \cup f^{\gamma-\varepsilon}$.

- For $x \notin V$ and $x \in f^{\gamma+\varepsilon}$. By 1), $\varphi^t(x) \notin V$ and as long as $f(\varphi^t(x)) \geq \gamma - \varepsilon$, we have (due to 2))
$$\left|\nabla f\right|(\varphi^t(x)) \geq \delta_0.$$
This implies that there exists $T > 0$ such that for $x \in f^{\gamma+\varepsilon} \setminus V$,
$$f(\varphi^T(x)) < \gamma - \varepsilon.$$

In conclusion, we get an isotopy $\varphi^T : f^{\gamma+\varepsilon} \to f^{\gamma-\varepsilon} \cup V \subset f^{\gamma-\varepsilon} \cup U$.

Assume $\beta = 0$ in $H^*(U)$. By definition $\alpha = 0$ in $H^*(f^{\gamma-\varepsilon}, f^a)$, then $\alpha \cup \beta = 0$ in $H^*(f^{\gamma-\varepsilon} \cup U, f^a)$. But φ^T is an isotopy, we know
$$\alpha \cup \beta = 0 \text{ in } H^*(f^{\gamma+\varepsilon}, f^a).$$
This is a contradiction to $c(\alpha \cup \beta, f) = \gamma$. □

2.3. The case of GFQI. If S is a GFQI for L, we know
$$H^*(S^\infty, S^{-\infty}) \cong H^{*-i}(N)$$
where i is the index of the nondegenerate quadratic form associated with S.

Due to this isomorphism, to each $\alpha \in H^*(N)$, we associate $\tilde{\alpha} \in H^*(S^\infty, S^{-\infty})$. Define
$$c(\alpha, S) = c(\tilde{\alpha}, S).$$

We claim the next result but omit the proof.

PROPOSITION 5.28. *For $\alpha_1, \alpha_2 \in H^*(N)$,*
$$c(\alpha_1 \cup \alpha_2, S_1 \oplus S_2) \geq c(\alpha_1, S_1) + c(\alpha_2, S_2),$$
where
$$(S_1 \oplus S_2)(x, \xi_1, \xi_2) = S_1(x, \xi_1) + S_2(x, \xi_2).$$

REMARK 5.29. The isomorphism mentioned above is precisely
$$\begin{array}{ccc} H^*(N) \otimes H^*(D^-, \partial D^-) & = & H^*(S^\infty, S^{-\infty}) \\ \alpha \otimes T & \mapsto & \tilde{T} \cup p^*\alpha \end{array}$$
where $p : N \times \mathbb{R}^k \to N$ is the projection.

$$\begin{array}{ccccccc} H^*((S_1 \oplus S_2)^\infty, (S_1 \oplus S_2)^{-\infty}) & \cong & H^*(N) & \otimes & H^*(D_1^-, \partial D_1^-) & \otimes & H^*(D_2^-, \partial D_2^-) \\ \tilde{T} \cup p^*\alpha & & \alpha & & T_1 & & T_2 \end{array}$$

$$\begin{aligned} \tilde{T} \cup p^*\alpha &= \tilde{T}_1 \cup \tilde{T}_2 \cup p^*(\alpha_1 \cup \alpha_2) \\ &= (\tilde{T}_1 \cup p^*\alpha_1) \cup (\tilde{T}_2 \cup p^*\alpha_2) \end{aligned}$$

Part 2

Sheaf theory and derived categories

CHAPTER 6

Categories and Sheaves

1. The language of categories

DEFINITION 6.1. A category $\mathscr{C}$ is a pair $(\mathrm{Ob}(\mathscr{C}), \mathrm{Mor}_{\mathscr{C}})$ where

- $\mathrm{Ob}(\mathscr{C})$ is a class of Objects [1]
- Mor is a map from $\mathscr{C} \times \mathscr{C}$ to a class, together with a composition map

$$\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \longrightarrow \mathrm{Mor}(A,C)$$
$$(f,g) \mapsto g \circ f$$

The composition is :

(1) associative
(2) has an identity element, $\mathrm{id}_A \in \mathrm{Mor}(A,A)$ such that $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ for all $f \in \mathrm{Mor}(A,B)$.

The category is said to be small if $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Mor}_{\mathscr{C}}$ are actually sets. It is locally small if $\mathrm{Mor}_{\mathscr{C}}(A,B)$ is a set for any A,B in $\mathrm{Ob}(\mathscr{C})$.

DEFINITION 6.2. A **functor** between the categories $\mathscr{C}$ and $\mathscr{D}$ is a "pair of maps", one from $\mathrm{Ob}(\mathscr{C})$ to $\mathrm{Ob}(\mathscr{D})$ the second one sending $\mathrm{Mor}_{\mathscr{C}}(A,B)$ to $\mathrm{Mor}_{\mathscr{D}}(F(A),F(B))$ such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Examples:

(1) The category **Sets** of sets, where objects are sets and morphisms are maps. The subcategory **Top** where objects are topological spaces and morphisms are continuous maps.
(2) The category **Group** of groups, where objects are groups and morphisms are group morphisms. It has a subcategory, **Ab** with objects the abelian groups and morphisms the group morphisms. This is a **full subcategory**, which means that $\mathrm{Mor}_{\mathbf{Group}}(A,B) = \mathrm{Mor}_{\mathbf{Ab}}(A,B)$ for any pair A,B of abelian groups; the set of morphisms between two abelian groups does not depend on whether you consider them as abelian groups or just groups. An example of a subcategory which is **not** a full subcategory is given by the subcategory **Top** of **Sets**, since not all maps are continuous.

[1] The class of Objects can be and often is a "set of sets". There is clean set-theoretic approach to this, using "Grothendieck Universe", but we will not worry about these questions here (nor elsewhere...).

(3) The category **R-mod** of R-modules, where objects are left R-modules and morphisms are R-modules morphisms.
(4) The category **k-vect** of k-vector spaces, where objects are k-vector spaces and morphisms are k-linear maps.
(5) The category **Man** of smooth manifolds, where objects are smooth manifolds and morphisms are smooth maps.
(6) Given a manifold M, the category **K-Vect(M)** of smooth K-vector bundles over M and morphisms are smooth linear fiber maps. The category **K-Vect** of K-vector bundles over any manifold.
(7) If P is a partially ordered set (a poset), **Ord(P)** is a category with objects the element of P, and morphisms $\mathrm{Mor}(x,y)=\emptyset$ unless $x\leq y$ in which case $\mathrm{Mor}(x,y)=\{*\}$.
(8) The category **Pos** of partially ordered sets (i.e. posets), where morphisms are monotone maps, i.e. maps $f:X\to Y$ such that $x_1\leq x_2 \Longrightarrow f(x_1)\leq f(x_2)$.
(9) if X is a topological space, **Open(X)** is the category where objects are open sets, and morphisms are such that $\mathrm{Mor}(U,V)=\{*\}$ if $U\subset V$, and $\mathrm{Mor}(U,V)=\emptyset$ otherwise. (since the set of open sets in X is a set partially ordered by inclusion, this is a special case of **Ord(P)**).
(10) If G is a group, the category **Group (G)** has objets the one element set $\{*\}$ and $\mathrm{Mor}_{\mathscr{C}}(*,*)=G$, where composition corresponds to multiplication.
(11) For any category **C**, we can define the category **Rep (C)** of functors from **C** to **Vect**, **Fun(C,Vect)**. Objects are functors from **C** to **Vect**. For example if G is a group, the category **Rep(Group (G))** has objects $R(G)$ the set of group representations of G and morphisms the set of morphisms between group representations.
(12) The category **FreeQuiver** with two objects, X_0 called vertex, X_1 called edge, and beside the identity morphism, two morphisms $h,t: X_0\longrightarrow X_1$ called "head" and "tail".
(13) The simplex category **Simplex** also denoted Δ whose objects are sets $[n]=\{0,1,...,n\}$ for $i\geq -1$ ($[-1]=\emptyset$) and morphisms are the monotone maps. The following maps $\delta_i^n:[n-1]\to[n]$ obtained by letting i out, and $\sigma_i^n:[n+1]\to[n]$ such that $\sigma_i(i)=\sigma_i(i+1)=i$ and otherwise injective generate the set of morphisms. They satisfy the relations
    (a) $\delta_i^{n+1}\delta_j^n=\delta_{j-1}^{n+1}\delta_i^n$ for $i<j$
    (b) $\sigma_i^n\sigma_j^{n+1}n=\sigma_j^n\sigma_{i-1}^{n+1}$ for $i>j$
    (c) $\delta_i^n\sigma_j^n=\sigma_{j-1}^n\delta_i^n$ if $i<j$
    (d) $\delta_i^n\sigma_j^n=\mathrm{id}$ for $i=j$ or $i=j+1$
    (e) $\delta_i^n\sigma_j^n=\sigma_j\delta_{i-1}$ for $i>j+1$

    These relations generate all the relations between the δ_i^n,σ_j^n. One should think of i as the i-th vertex of a simplex, and then δ_i^n,σ_i^n are i-th face map and degeneracy map. Note that all the morphisms are generated by composition of the δ_i^n, σ_i^n. The category **Simplex** is a full subcategory of **Pos**.

(14) There is functor **Simplex** to **Top** sends $[n]$ to the n-simplex $\{(x_0,...,x_n) \mid 0 \leq x_0 \leq x_1 \leq ... \leq x_n \leq 1\}$. It sends δ_i^n to the map that sends the $n-1$-simplex to the i-th face of the n-simplex. This is called the geometric realization of **Simplex**.

(15) The category **Cell** of cell complexes. A cell complex is a topological space X, endowed with a partition X_σ, $\sigma \in P_X$, such that
   (a) Each point has a neighborhood containing finitely many X_σ
   (b) each X_σ is homeomorphic to B^K for some k,
   (c) $\bar{X}_\sigma \cap X_\tau \neq \emptyset$ implies $X_\sigma \subset \bar{X}_\tau$ and we say X_σ is a face of X_τ.
   (d) moreover $(\bar{X}_\sigma, X_\sigma)$ is homeomorphic to $(\bar{B}^k, B^k)$.

   We say (X, P_X) is an **open cell complex**, if it only satisfies the first three axioms, and $X \cup \{\infty\}$, its Alexandrov compactification satisfies the fourth. This gives P_X a structure of Poset, and we associate to (X, P_X) the category **Ord(**P_X**)**.

(16) Given a category $\mathscr{C}$, the opposite category is the category denoted $\mathscr{C}^{op}$ having the same objects as $\mathscr{C}$, but such that $\mathrm{Mor}_{\mathscr{C}^{\mathbf{op}}}(A,B) = \mathrm{Mor}_{\mathscr{C}}(B,A)$ with the obvious composition map: if we denote by $f^* \in \mathrm{Mor}_{\mathscr{C}^{\mathbf{op}}}(A,B)$ the image of $f \in \mathrm{Mor}_{\mathscr{C}}(B,A)$, we have $f^* \circ g^* = (g \circ f)^*$. In some cases there is a simple identification of $\mathscr{C}^{op}$ with a natural category (example: the opposite category of **k-vect** is the category with objects the space of linear forms on a vector space).

(17) Given a category $\mathscr{C}$, we can consider the quotient category by isomorphism. The standard construction, at least if the category is not too large, is to choose for each isomorphism class of objects a given object (using the axiom of choice), and consider the subcategory $\mathscr{C}'$ of $\mathscr{C}$ generated by these objects.

Examples: A functor from **Group(G)** to **Group(H)** is a morphism from G to H. There are lots of **forgetful functors**, like **Group** to **Sets**. There is also a functor from **Top** to **Pos** sending X to the set of its open subsets ordered by inclusion.

DEFINITION 6.3. A functor is **fully faithful** if for any pair X, Y the map $F_{X,Y} : Mor(X,Y) \to \mathrm{Mor}(F(X), F(Y))$ is bijective. We say that F is an **equivalence of categories** if it is fully faithful, and moreover for any $X' \in \mathscr{D}$ there is X such that $F(X)$ is isomorphic to X'.

Note that for an equivalence of categories, we only require that F is a bijection between equivalence classes of isomorphic objects.

Examples:

(1) Let us consider the category **StVect** with objects $\mathbb{N}$ and $\mathrm{Mor}(m,n) = L(\mathbf{K}^m, \mathbf{K}^n)$. There is an obvious functor from **StVect** to **K-Vect** associating to n the vector space $\mathbb{R}^n$. The functor is fully faithful since the morphisms in **StVect** coincide with those in **K-Vect**. It is not an equivalence of categories, since there are

vector spaces non-isomorphic to $\mathbf{K}^n$. However if we replace **K-Vect** by **K-Vect-finite** the category of finite dimensional vector spaces, the functor is now an equivalence.

(2) Given a category, $\mathscr{C}$ we may, using the axiom of choice, associate to each object A in $\mathscr{C}$, a representative $[A]$ of its equivalence class and an isomorphism $i_A : A \longrightarrow [A]$. We denote the class of such representatives by $\mathscr{C}'$, as defined in example (17), page 55. For each pair A, B in $\mathscr{C}'$, we defined $\mathrm{Mor}_{\mathscr{C}'}(A,B) = \mathrm{Mor}_{\mathscr{C}}(A,B)$. Then there is a functor $C : \mathscr{C} \longrightarrow \mathscr{C}'$ sending each object to its representative, and sending $f \in \mathrm{Mor}_{\mathscr{C}}(A,B)$ to $i_B f \circ i_A^{-1} \in \mathrm{Mor}_{\mathscr{C}'}([A],[B])$. This yields an equivalence of categories $S : \mathscr{C} \longrightarrow \mathscr{C}'$.

There is also a notion of transformation of functors.

DEFINITION 6.4. If $F : \mathbf{A} \to \mathbf{B}, G : \mathbf{A} \to \mathbf{B}$ are functors, a transformation of functors is a family of maps parametrized by X, $T_X \in \mathrm{Mor}(F(X), G(X))$ making the following diagram commutative for every f in $\mathrm{Mor}(X,Y)$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

Notice that some categories are categories of categories, the morphisms being the functors. This is the case for **Group** with objects the set of categories of the type **Group(G)**, or of **Pos** whose objects are the **Ord(P)**. We may also, given two categories, **A**, **B** define the category with objects the functors from **A** to **B**, and morphisms the transformations of these functors. We shall see for example that **presheaves over** X (see the next section) are nothing but functors defined on the category **Open(X)**. And so on, and so forth....

Example:

(1) For any category **C** a new category **Quiv(C)** of functors from **Quiver** to **C**.

1.1. Special objects and morphisms.

DEFINITION 6.5. An initial object in a category is an element I such that for any object A, $\mathrm{Mor}(I,A)$ has exactly one element. A terminal object T is an object such that $\mathrm{Mor}(A,T)$ is a singleton for each A. Equivalently T is a terminal object if and only if it is an initial object in the opposite category.

Examples: $\emptyset$ in **Sets**, $\{e\}$ in **Group**, $\{0\}$ in **R-mod** or **K-Vect**, the smallest object in **P** if it exists, $[-1]$ in **Simplex**. Finally as in maps, we have the notion of monomorphism and epimorphisms

DEFINITION 6.6. An element $f \in \mathrm{Mor}(B,C)$ is a **monomorphism** if for any $g_1, g_2 \in \mathrm{Mor}(A,B)$ the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. An element $f \in \mathrm{Mor}(A,B)$ is an

epimorphism if for any $g_1, g_2 \in \mathrm{Mor}(B, C)$ the property $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. An **isomorphism** is a morphism $f \in \mathrm{Mor}(A, B)$ such that there exists g such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{id}_A$. It is easy to check that g is then unique, and is denoted by f^{-1}.

EXERCICES 1.
(1) Which of the categories defined on page 53 have an initial object ? A terminal object ?
(2) Prove that an initial object is unique, up to a unique isomorphism.
(3) Is being an isomorphism equivalent to being both a monomorphism and an epimorphism ? Such a category is said to be "balanced"). . Prove that if $\mathrm{Mor}(A, B)$ is finite for all A, B then the category is balanced.
(4) Prove that in the category **Sets** monomorphisms and epimorphisms are just injective and surjective maps. Is **Sets** balanced ?
(5) In the category **Groups**, prove that epimorphisms are surjective morphisms, Prove that the category **Groups** is balanced.

    Hint to prove that an epimorphism is onto: prove that for any proper subgroup H of G (not necessarily normal), there is a group K and two different morphisms g_1, g_2 in $\mathrm{Mor}(G, K)$ such that $g_1 = g_2$ on H. For this use the action of G on the classes of H/G to reduce the problem to $\mathfrak{S}_{q-1} \subset \mathfrak{S}_q$ and prove that there are two different morphisms $\mathfrak{S}_q \to \mathfrak{S}_{q+1}$ equal to the inclusion on $\mathfrak{S}_{q-1}$.
(6) Prove that the injection $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category **Rng** of commutative rings with unit. Hint: use the fact that a morphism $g : \mathbb{Q} \to R$ must be injective[2].
(7) Prove that in the category **Top** an epimorphism is surjective. Is the category balanced ? Give an example of a balanced subcategory. Find also a subcategory of **Top** such that any continuous map with dense image is an epimorphism.
(8) Which of the categories in the list starting on page 53 are balanced ?
(9) Prove that the composition of two monomorphisms (resp. epimorphism) is a monomorphism (resp. epimorphism)

2. Additive and Abelian categories

DEFINITION 6.7. An additive category is a category such that

(1) It has a 0 object which is both initial and terminal. The zero map in $\mathrm{Mor}(A, B)$ is defined as the unique composition $A \to 0 \to B$.
(2) $\mathrm{Mor}(A, B)$ is an abelian group, 0 is the zero map, composition is bilinear.
(3) It has finite biproducts (see below for the definition).

[2]There are in fact two possible definitions for a ring morphism: either it is just a map such that $f(x+y) = f(x)+f(y)$, $f(xy) = f(x)f(y)$ or we add the hypothesis $f(1) = 1$. In the latter case $\mathrm{Mor}(\mathbb{Q}, R) = \emptyset$ unless R has zero characteristic.

A category has finite products if for any A_1, A_2 there exists an object denoted $A_1 \times A_2$ and maps $p_k : A_1 \times A_2 \longrightarrow A_k$ such that

$$\mathrm{Mor}(Y, A_1) \times \mathrm{Mor}(Y, A_2) \simeq \mathrm{Mor}(Y, A)$$

the bijection being given by the map $f \to (p_1 \circ f, p_2 \circ f)$ and which are universal in the following sense[3]. For any maps $f_1 : Y \to A_1$ and $f_2 : Y \to A_2$ there is a **unique** map $f : Y \to A_1 \times A_2$ making the following diagram commutative

$$\begin{tikzcd}
& Y \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
A_1 & A_1 \times A_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & A_2
\end{tikzcd}$$

It has finite coproducts if given any A_1, A_2 there exists an object denoted $A_1 + A_2$ and maps $i_k : A_k \longrightarrow A_1 + A_2$ such that

$$\mathrm{Mor}(A_1, Y) \times \mathrm{Mor}(A_2, Y) = Mor(A_1 + A_2, Y)$$

and this is given by $g \to (g \circ i_1, g \circ i_2)$. In other words for any $g_1 : A_1 \to Y$ and $g_2 : A_2 \to Y$ there exists a **unique** map $g : A_1 + A_2 \to Y$ making the following diagram commutative

$$\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[dr, "g_1"'] & A_1 + A_2 \arrow[d, dashed, "f"] & A_2 \arrow[l, "i_2"] \arrow[dl, "g_2"] \\
& Y &
\end{tikzcd}$$

the category has finite **biproducts** if it has both products and coproducts, these are equal and moreover

(1) $p_j \circ i_k$ is id_{A_j} if $j = k$ and 0 for $j \neq k$,
(2) $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{A_1 \oplus A_2}$.

We then denote the biproduct of A_1 and A_2 by $A_1 \oplus A_2$. According to exercise 4 on the facing page, if biproducts exist, they are **unique** up to a **unique isomorphism**. Note also that the definition of product and coproducts makes sense in any category, while the definition of biproducts requires a law group on $\mathrm{Mor}(A, B)$ and the existence of a 0 map.

[3]The maps (p_1, p_2) correspond to Id_A under the identification of $\mathrm{Mor}(A, A)$ and $\mathrm{Mor}(A, A_1) \times \mathrm{Mor}(A, A_2)$. The maps i_1, i_2 mentioned later are obtained similarly.

EXERCICES 2. (1) What are products and coproducts in the category **Sets**, **Groups** **Top**, K-**Vect(M)** ?
(2) Which of the categories defined on 53 are additive ?

DEFINITION 6.8. A **kernel** for a morphism $f \in \mathrm{Mor}(A,B)$ is a pair (K,k) where $K \xrightarrow{k} A \xrightarrow{f} B$ such that $f \circ k = 0$ and if $g \in \mathrm{Mor}(P,A)$ and $f \circ g = 0$ there is a **unique** map $h \in \mathrm{Mor}(P,K)$ such that $g = k \circ h$. In other words whenever $g \circ f = 0$ whe have existence of the dotted map below

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, "f"] & B \\ & P \arrow[u, "g"'] \arrow[ul, dashed, "h"] & \end{tikzcd}$$

A **cokernel** is a pair (C,c) such that $c \circ f = 0$ and if $g \in \mathrm{Mor}(B,Q)$ is such that $g \circ f = 0$ there is a **unique** $d \in \mathrm{Mor}(C,Q)$ such that $d \circ c = g$. In other words whenever $f \circ g = 0$ whe have existence of the dotted map below

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "c"] \arrow[dr, "g"] & C \arrow[d, dashed, "d"] \\ & & Q \end{tikzcd}$$

A **Coimage** is the kernel of the cokernel. An **Image** is the cokernel of the kernel.

EXERCICES 3. (1) Identify Kernel and Cokernel in the category of R-modules.
(2) In the category **Groups**, prove that the cokernel of f is $G/N(\mathrm{Im}(f))$, where $N(H)$ is the normalizer[4] of H in G, but epimorphisms are surjective morphisms. In particular, to have cokernel 0 is not equivalent to being an epimorphism. Prove that the category **Groups** is balanced.
(3) Prove that a kernel is a monomorphism, that is if (K,k) is the kernel of $A \xrightarrow{f} B$, then $k : K \to A$ is a monomorphism. Prove that a cokernel is an epimorphism (use the uniqueness of the maps).
(4) It is a general fact that solutions to **universal problems**, if they exists, are unique up to unique isomorphism. Prove this for products, coproducts, Kernels and Cokernels.
(5) Assuming an additive category has both kernels and cokernels, prove that there is a unique map from $\mathrm{Coim}(f)$ to $\mathrm{Im}(f)$. Use the following diagram, justifying the existence of the dotted arrows

[4] i.e. the largest subgroup such that H is a normal subgroup of $N(H)$.

(6.1)
$$\begin{tikzcd} \mathrm{Ker(f)} \arrow[r, "i"] & A \arrow[rr, "f"] \arrow[d, "u"] & & B \arrow[r, "p"] & \mathrm{Coker}(f) \\ & \mathrm{Coim}(f) = \mathrm{Coker}(i) \arrow[urr, dashed, "\psi"] & & \mathrm{Ker(p)} = \mathrm{Im(f)} \arrow[u, "v"] & \end{tikzcd}$$

Then $p \circ \psi = 0$: since u is an epimorphism according to Exercise 4 (3), and $p \circ \psi \circ u = p \circ f = 0$, and this implies $p \circ \psi = 0$ hence ψ factors through Ker(p). We now have the following diagram with the unique map σ

(6.2)
$$\begin{tikzcd} \mathrm{Ker(f)} \arrow[r, "i"] & A \arrow[rr, "f"] \arrow[d, "u"] & & B \arrow[r, "p"] & \mathrm{Coker}(f) \\ & \mathrm{Coim}(f) = \mathrm{Coker}(i) \arrow[urr, "\psi"] \arrow[rr, dashed, "\sigma"] & & \mathrm{Ker(p)} = \mathrm{Im(f)} \arrow[u, "v"] & \end{tikzcd}$$

DEFINITION 6.9 (Abelian category). An **abelian category** is an additive category such that

(1) It has both kernels and cokernels
(2) The natural map from the coimage to the image of a morphism (see the map σ in Exercise 3, (5)) is an isomorphism.

The second statement can be replaced by the more intuitive one: every morphism $f : A \to B$ has a factorization

(2')
$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, "u"] \arrow[rr, bend left, "f"] & \mathrm{Im}(f) \arrow[r, "v"] & B \arrow[r] & \mathrm{Coker}(f) \end{tikzcd}$$

where u and v are the natural maps (see Exercise 3, (5)).

EXERCICES 4. (1) Prove that the factorization of morphisms (2') is equivalent to property (2).
(2) Prove that in (2'), u is an epimorphism and v a monomorphism.
(3) Prove that in an abelian category, the kernel of f is zero if and only if f is a monomorphism. Prove that $\mathrm{Coker}(f) = 0$ if and only if v is an isomorphism from $\mathrm{Im}(f)$ to B and this in turn means f is an epimorphism. If a map (in a non-abelian category) is both mono and epi, is it an isomorphism (f is an isomorphism if and only if there exists g such that $f \circ g = \mathrm{id}, = g \circ f = \mathrm{id}$) ? Consider the case of a group morphism for example.
(4) Which one of the categories from the list of examples starting on page 53 are abelian ?
(5) The opposite of an abelian category is an abelian category

PROPOSITION 6.10. *Let $\mathscr{C}$ be an abelian category. Then a morphism which is both a monomorphism and an epimorphism is an isomorphism. Therefore an abelian category is balanced.*

PROOF. Notice first that $0 \to A$ has cokernel equal to (A, Id). Similarly the kernel of $B \to 0$ is (B, Id). Assuming f is both an epimorphism and a monomorphism, we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "\mathrm{Id}"] & B \arrow[r, "p"] & 0 \\
\mathrm{Coim}(f) = \mathrm{Coker}(i) = A \arrow[rr, dashed, "\sigma"] & & \mathrm{Ker(p)} = \mathrm{Im(f)} = \mathrm{B} \arrow[u, "\mathrm{Id}"'] &
\end{tikzcd} \tag{6.3}$$

and the result follows from the fact that σ is an isomorphism. ☐

DEFINITION 6.11. In an abelian category, the notion of **exact sequence** is defined as follows. A sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $g \circ f = 0$ and the map from $\mathrm{Im}(f)$ to Ker(g) is an isomorphism. The exact sequence is said to be **split** if there is a map $h : C \to B$ such that $g \circ h = \mathrm{Id}_C$.

Whenever $g \circ f = 0$ we get a map w from $\mathrm{Im}(f)$ to Ker(g). It is obtained from the following diagram

(6.4) {6-4}

$$\begin{tikzcd}
\mathrm{Im}(f) = \mathrm{Ker(p)} \arrow[rr, dashed, "w"] \arrow[dr, "v"] & & \mathrm{Ker(g)} \arrow[dl, "i"'] \\
A \arrow[u, "u"] \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "p"] & C \\
& \mathrm{Coker}(f) &
\end{tikzcd}$$

Here u, v come from the canonical factorization of f. We claim that $g \circ v = 0$ since $g \circ v \circ u = g \circ f = 0$ and u is an epimorphism according to Exercise 4, (2). As a result v factors through a map $w : \mathrm{Im}(f) \to \mathrm{Ker(g)}$.

Note that $0 \to A \xrightarrow{f} B$ is exact if and only if f is a monomorphism, and $A \xrightarrow{f} B \to 0$ is exact if and only if f is an epimorphism.

EXERCICE 6.12. (1) Prove that w is a monomorphism. Hint: use that $v = i \circ w$ and that v is a monomorphism (see Exercice 4, (2))

(2) Prove that if an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split, that is there is a map $h : C \to B$ such that $g \circ h = \mathrm{Id}_C$, then $B \simeq A \oplus C$. Prove that the same conclusion holds if there exists k such that $k \circ f = \mathrm{Id}_A$.

Hint: prove that there exists a map $k : B \to A$ such that $\mathrm{Id}_B = f \circ k + h \circ g$. Indeed, $g \circ h \circ g = g$ and since g is an epimorphism, and $g \circ (\mathrm{Id}_B - h \circ g) = 0$, we get that since $f : A \to B$ is the kernel of g, that $(\mathrm{Id}_B - h \circ g) = f \circ k$ for some map $k : B \to A$.

Now $f \oplus h : A \oplus C \to B$ is an isomorphism, with inverse $k \oplus g : B \to A \oplus C$.

Note that Property (2') can be replaced by either of the following conditions:

(2") any monomorphism is a kernel, and any epimorphism is a cokernel. In other words, monomorphism $0 \to A \xrightarrow{f} B$ can be completed to an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, and any $B \xrightarrow{g} C \to 0$ can be completed to an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$

(2"') If $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence, then we have a factorization

$$A \xrightarrow{f} B \xrightarrow{g_1} \mathrm{Coker}(g) \xrightarrow{g_2} C$$

where the last map is monomorphism.

Moreover

PROPOSITION 6.13. *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence, then $(A, f) = \mathrm{Ker(g)}$ *and* $(C, g) = \mathrm{Coker}(f)$.

PROOF. Consider $0 \to A \xrightarrow{f} B$. We claim the map $A \xrightarrow{u} \mathrm{Im}(f)$ is an isomorphism. It is a monomorphism, because the factorization (2') of f is written $0 \to A \xrightarrow{u} \mathrm{Im}(f) \xrightarrow{v} B$. Moreover it is an epimorphism according to Exercise 4, (3).

Since the map w from (*) is an isomorphism (due to the exactness of the sequence), we have the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Im}(f) = \mathrm{Ker(p)} \arrow[rr, dashed, "\simeq"', "w"] \arrow[dr, "v"] & & \mathrm{Ker(g)} \arrow[dl, "i"'] & \\
0 \arrow[r] & A \arrow[u, "u"', "\simeq"] \arrow[r, "f"] & B \arrow[r, "g"] & C &
\end{tikzcd}$$

and thus the isomorphism $(w \circ u)$ identifies (A, f) is isomorphic to $(\mathrm{Ker(g)}, \mathrm{i})$. We leave the proof of the dual statement to the reader. ☐

A sequence as above is called a **short exact sequence**.

DEFINITION 6.14. Let F be a functor between additive categories. We say that F is additive if the associated map from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(F(A), F(B))$ is a morphism of abelian groups. Let F be an additive functor between abelian categories. We say that the functor F is **exact** if it transforms an exact sequence in an exact sequence. It is **left-exact** if it transforms an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ to an exact sequence $0 \to F(A) \xrightarrow{F(f)} B \xrightarrow{F(g)} F(C)$. It is **right-exact**, if it transforms an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ to an exact sequence $A \xrightarrow{F(f)} B \xrightarrow{F(g)} C \to 0$.

Example: In an abelian category, $\mathscr{C}$

(1) The functor $X \to \mathrm{Mor}(X, A)$ (from $\mathscr{C}$ to **Ab**) is left-exact. Indeed, consider an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, and the corresponding sequence $0 \to \mathrm{Mor}(X, A) \xrightarrow{f_*} \mathrm{Mor}(X, B) \xrightarrow{g_*} \mathrm{Mor}(X, C)$ is exact, since the fact that f is a monomorphism is equivalent to the fact that f_* is injective, while the fact that $\mathrm{Im}(f_*) = \mathrm{Ker(g_*)}$ follows from the fact that $A \xrightarrow{f} B$ is the kernel of g (according to Prop. 6.13), so that for any X and $u \in \mathrm{Mor}(X, B)$ such that $g \circ u = 0$, there exists a unique v making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ & X \arrow[u, "u"'] \arrow[ul, dashed, "v"] & \end{tikzcd}$$

(2) The contravariant functor $M \to \mathrm{Mor}(M, X)$ is right-exact. This means that it transforms $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ to $0 \to \mathrm{Mor}(C, X) \xrightarrow{g^*} \mathrm{Mor}(B, X) \xrightarrow{f^*} \mathrm{Mor}(A, X)$.
(3) In the category R-**mod**, the functor $M \to M \otimes_R N$ is right-exact. It is not left exact, since in the category of $\mathbb{Z}$-modules, the exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ tensored by $\mathbb{Z}/2\mathbb{Z}$ becomes $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}$. The failure of exactness is due to torsion and gives rise to the **Tor** functor.
(4) If a functor has a **right-adjoint** it is right-exact, if it has a **left-adjoint**, it is left-exact (see Lemma 7.24, for the meaning and proof).

EXERCICE 6.15. (1) Let $\mathscr{C}$ be a small category and $\mathscr{A}$ an abelian category. Prove that the category $\mathscr{C}^{\mathscr{A}}$ of functors from $\mathscr{C}$ to $\mathscr{A}$ is an abelian category.
(2) Prove that for a functor to be exact, it is enough to transform short exact sequences to short exact sequences. Hint: if

$$A_1 \xrightarrow{f_1} A_2 \to .... \xrightarrow{f_n} A_n$$

is exact if and only if $f_{j+1} \circ f_j = 0$ and all the sequences $0 \to \mathrm{Ker(f_j)} \to \mathrm{A_j} \to \mathrm{Ker(f_{j+1})} \to 0$ are (short) exact sequences. This can be summarized in the following diagram where $C_j = \mathrm{Ker(f_j)}$

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
 & & & C_2 \arrow[ur] \arrow[dr] & & & & C_4 \arrow[ur] \arrow[dr] & & & \\
 & & A_1 \arrow[ur] \arrow[rr] & & A_2 \arrow[rr] \arrow[dr] & & A_3 \arrow[rr] \arrow[ur] & & A_4 \arrow[rr] \arrow[dr] & & A_5 \\
 & C_1 \arrow[ur] & & & & C_3 \arrow[ur] \arrow[dr] & & & & C_5 \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}
$$

Conversely prove that if in the above commutative diagram if all the diagonal short sequences are exact, then the horizontal sequence is exact

3. The category of Chain complexes

To any abelian category $\mathscr{C}$ we may associate the category **Chain($\mathscr{C}$)** of **chain complexes**. Its objects are sequences

$$\ldots \overset{d_{m-1}}{\to} I_m \overset{d_m}{\to} I_{m+1} \overset{d_{m+1}}{\to} I_{m+2} \overset{d_{m+2}}{\to} I_{m+3} \ldots .$$

where the **boundary maps** d_m satisfy the condition $d_m \circ d_{m-1} = 0$. Its morphisms, called chain maps, are commutative diagrams

$$
\begin{tikzcd}
\ldots \arrow[r, "d_{m-1}"] & I_m \arrow[r, "d_m"] \arrow[d, "u_m"] & I_{m+1} \arrow[r, "d_{m+1}"] \arrow[d, "u_{m+1}"] & I_{m+2} \arrow[r, "d_{m+2}"] \arrow[d, "u_{m+2}"] & I_{m+3} \arrow[r, "d_{m+3}"] \arrow[d, "u_{m+3}"] & \ldots . \\
\ldots \arrow[r] & J_m \arrow[r, "\partial_m"] & J_{m+1} \arrow[r, "\partial_{m+1}"] & J_{m+2} \arrow[r, "\partial_{m+2}"] & I_{m+3} \arrow[r, "\partial_{m+3}"] & \ldots
\end{tikzcd}
$$

It has several natural subcategories, in particular the subcategory of bounded complexes **Chain**$^b(\mathscr{C})$, complexes bounded from below **Chain**$^+(\mathscr{C})$, complexes bounded from above **Chain**$^-(\mathscr{C})$. The cohomology $\mathscr{H}^m(A^\bullet)$ of the chain complex $A^\bullet$ is given by $\ker(d_m)/\mathrm{Im}(d_{m-1})$. We may consider $\mathscr{H}^m(A^\bullet)$ as a chain complex with boundary maps equal to zero.

EXERCICES 5. (1) Show that the definition of $\mathscr{H}(A^\bullet)$ indeed makes sense in an abstract category: one must prove that there is a natural mapping $\mathrm{Im}(d_{m-1}) \to \mathrm{Ker(d_m)}$ (see the map w from diagram (6.4) page 61) and $\mathscr{H}^m(C^\bullet)$ is defined as the cokernel of this map.

(2) Determine the kernel and cokernel in the category **Chain($\mathscr{C}$)**.

PROPOSITION 6.16. *Let $\mathscr{C}$ be an abelian category. Then* $\mathbf{Chain}^b(\mathscr{C})$, $\mathbf{Chain}^+(\mathscr{C})$, $\mathbf{Chain}^-(\mathscr{C})$ *are abelian categories.*

The map from **Chain** $(\mathscr{C})$ *to* **Chain** $(\mathscr{C})$ *induced by taking homology is a functor. In particular any morphism $u = (u_m)_{m\in\mathbb{N}}$ from the complex $A^\bullet$ to the complex $B^\bullet$ induces a map $u_* : \mathscr{H}(A^\bullet) \to \mathscr{H}(B^\bullet)$. If moreover u, v are* **chain homotopic**, *that is there exists a map $s = (s_m)_{m\in\mathbb{N}}$ such that $s_m : I_m \to J_{m-1}$ and $u - v = \partial_{m-1} \circ s_m + s_{m+1} \circ d_m$ then $\mathscr{H}(u) = \mathscr{H}(v)$.*

PROOF. The proof is left to the reader or referred for example to [**Weib**]. □

The abelian category $\mathscr{C}$ is a subcategory of **Chain**$(\mathscr{C})$ by identifying A to $0 \to A \to 0$ and it is then a full subcategory.

DEFINITION 6.17. A chain map $u : A^\bullet \to B^\bullet$ is a **chain homotopy equivalence** if and only if there exists a chain map $v : B^\bullet \to A^\bullet$ such that $u\circ v$ and $v\circ u$ are chain homotopic to the Identity.

DEFINITION 6.18. A map $u : A^\bullet \to B^\bullet$ is a quasi-isomorphism if the induced map $\mathscr{H}(u)$ is an isomorphism from $\mathscr{H}(A^\bullet)$ to $\mathscr{H}(B^\bullet)$. Two chain complexes $A^\bullet, B^\bullet$ are quasi-isomorphic if and only if there exists a chain complex, $C^\bullet$, and chain maps $u^\bullet : C^\bullet \to A^\bullet$ and $v^\bullet : C^\bullet \to B^\bullet$ such that $u^\bullet, v^\bullet$ are quasi-isomorphisms (i.e. induce an isomorphism in cohomology).

A fundamental result in homological algebra is the existence of long exact sequences associated to a short exact sequence.

PROPOSITION 6.19. *Given a short exact sequence of chain complexes,*

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$$

there exists a map $\delta_m : \mathscr{H}^m(C^\bullet) \xrightarrow{\delta} \mathscr{H}^{m+1}(A^\bullet)$ such that we have a long exact sequence

$$.. \to \mathscr{H}^m(A^\bullet) \xrightarrow{f^*} \mathscr{H}^m(B^\bullet) \xrightarrow{g^*} \mathscr{H}^m(C^\bullet) \xrightarrow{\delta} \mathscr{H}^{m+1}(A^\bullet) \to ...$$

PROOF. See any book on Algebraic topology or [**Weib**] page 10. □

REMARK 6.20. If the exact sequence is split (i.e. there exists $h : C^\bullet \to B^\bullet$ such that $g \circ h = \mathrm{Id}_C$), then we can construct a **sequence of chain maps**,

$$... \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{\delta} A^\bullet[1] \xrightarrow{f[1]} B^\bullet \xrightarrow{g[1]} ...$$

where we set $(A^\bullet[k])^n = A^{n+k}$ and $\partial_{A^\bullet[k]} = (-1)^k\partial$, and such that the long exact sequence is obtained by taking the cohomology of the above sequence.

This does not hold in general, but these **distinguished triangles** play an important role in triangulated categories (of which the Derived category is the main example), where exact sequences do not make much sense.

Finally, the Freyd-Mitchell theorem tells us that if $\mathscr{C}$ is a small abelian category[5], then there exists a ring R and a **fully faithful and exact**[6] functor $F : \mathscr{C} \to \mathbf{R-mod}$ for some R. The functor F identifies A with a subcategory of R-Mod : F yields an equivalence between $\mathscr{C}$ and a subcategory of R-Mod in such a way that kernels and cokernels computed in $\mathscr{C}$ correspond to the ordinary kernels and cokernels computed in R-Mod. We can thus, whenever this simplifies the proofs, assume that an abelian category is a subcategory of the category of R-modules. As a result, all diagram theorems in an abelian categories, can be proved by assuming the objects are R-modules, and the maps are R-modules morphisms, and in particular maps between sets[7].

We refer to [**Weib**] for the sketch of a proof, but let us mention a crucial ingredient in the proof of Freyd-Mitchell theorem, Yoneda's lemma.

LEMMA 6.21 (Yoneda's lemma). *Given two objects A, A' in $\mathscr{C}$, and assume for all C there is a bijection $i_C : \mathrm{Mor}(A,C) \to \mathrm{Mor}(A',C)$, commuting with all the maps $f^* : \mathrm{Mor}(C,A) \to \mathrm{Mor}(B,A)$ induced by $f : B \to C$. Then A and A' are isomorphic.*

The functor $h^A : C \to \mathrm{Mor}(C,A)$ is a functor from $\mathscr{C}$ to sets. The natural transformations from h^A to a functor F from $\mathscr{C}$ to **Sets** are in 1-1 correspondence with elements of $F(A)$, and natural transformations from h^A to h^B are in 1-1 correspondence with $Mor(B,A)$. Thus if denote by $\mathbf{Set}^{C^{op}}$ the category of functors from $\mathscr{C}^{op}$ to **Sets**, we get a fully faithful functor (also called en embedding) from $\mathscr{C}$ to $\mathbf{Set}^{C^{op}}$, given by $C \longrightarrow h^C$. This is called Yoneda's embedding. When $\mathscr{C}$ has some extra properties, we may replace **Sets** by a more appropriate category. For example if $\mathscr{C}$ is abelian we may replace **Sets** by **Ab**, the category of Abelian groups. Then the category $\mathbf{Ab}^{C^{op}}$, also denoted by $\mathrm{Mor}(\mathscr{C},\mathbf{Ab})$, of **additive functors** between $\mathscr{C}^{op}$ and **Ab** is an abelian category. It is called, for reasons we let the reader guess, the category of Modules over $\mathscr{C}$.

As a consequence of the Freyd-Mitchell theorem, we see that all results of homological algebra obtained by diagram chasing are valid in any abelian category. For example we have :

LEMMA 6.22 (Snake Lemma). *In an abelian category, consider a commutative diagram:*

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

---

[5]Remember that this means that objects and morphism are in fact sets.

[6]A functor F is fully faithful if $F_{X,Y} : \mathrm{Mor}(X,Y) \to \mathrm{Mor}(F(X),F(Y))$ is bijective.

[7]see http://unapologetic.wordpress.com/2007/09/28/diagram-chases-done-right/ and the appendix to this chapter for an alternative approach to this specific problem.

where the rows are exact sequences and 0 is the zero object. Then there is an exact sequence relating the kernels and cokernels of a, b, and c:

$$\mathrm{Ker(a)} \longrightarrow \mathrm{Ker(b)} \longrightarrow \mathrm{Ker(c)} \xrightarrow{d} \mathrm{Coker}(a) \longrightarrow \mathrm{Coker}(b) \longrightarrow \mathrm{Coker}(c)$$

Furthermore, if the morphism f is a monomorphism, then so is the morphism $\mathrm{Ker(a)} \longrightarrow \mathrm{Ker(b)}$*, and if g' is an epimorphism, then so is* $\mathrm{Coker}(b) \longrightarrow \mathrm{Coker}(c)$.

PROOF. First we may work in the abelian category generated by the objects and maps of the diagram. This will be a small abelian category. According to the Freyd-Mitchell theorem, we may assume the objects are R-modules and the morphisms are R-modules morphisms. Note that apart from the map d, whose existence we need to prove, the other maps are induced by f, g, f', g'. Note also that the existence of d in the general abelian category follows from the R-module case and the Freyd-Mitchell theorem, since the functor provided by the theorem is fully-faithful. Let us construct d. Let $z \in \mathrm{Ker(c)}$, then $z = g(y)$ because g is onto, and $g'b(y) = 0$, hence $b(y) = f'(x')$ and we set $x' = d(z)$. We must prove that x' is well defined in $\mathrm{Coker}(f') = A'/a(A)$. For this it is enough to see that if $z = 0$, $y \in \mathrm{Ker(g)} = \mathrm{Im(f)}$ that is $y = f(x)$, and so if $bf(x) = b(y) = f'(x')$, we have $f'(x') = f'(a(x))$ and since f' is monomorphism, we get $x' = a(x)$.

Let us now prove the maps are exact at Ker(b). Let $v \in \mathrm{Ker(b)}$ (i.e. $b(v) = 0$) such that $g(v) = 0$. Then by exactness of the top sequence, $v = f(u)$ with $u \in A$. We have $f'a(u) = b(f(u)) = b(v) = 0$, and since f' is injective, $a(u) = 0$ that is $u \in \mathrm{Ker(a)}$. □

EXERCICE 6.23 (The five lemma). Consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[r, "k"] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "k'"] & E' \end{tikzcd}$$

where the lines are exact. Then if a, b, d, e are isomorphisms, then so is c.

4. Presheaves and sheaves

Let X be a topological space, $\mathscr{C}$ a category.

DEFINITION 6.24. A $\mathscr{C}$-**presheaf** on X is a functor from the category $\mathbf{Open(X)}^{op}$ to the category $\mathscr{C}$.

DEFINITION 6.25. A presheaf $\mathscr{F}$ of R-modules on X is defined by associating to each open set U in X an R-module, $\mathscr{F}(U)$, such that If $V \subset U$ there is a unique module morphism $r_{V,U} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ such that $r_{W,V} \circ r_{V,U} = r_{W,U}$ and $r_{U,U} = \mathrm{id}$. Equivalently, a presheaf is a functor from the category $\mathbf{Open(X)}^{op}$ to the category R-**mod**.

Notation: if $s \in \mathscr{F}(U)$ we often denote by $s_{|V}$ the element $r_{V,U}(s) \in \mathscr{F}(V)$. From now on we shall, unless otherwise mentioned, mostly deal with presheaves in the category

R-**mod**. Our results extend to sheaves in any abelian category. The reader can either check this for himself (most proofs translate verbatim to a general abelian category), or use the Freyd-Mitchell theorem (see page 66).

DEFINITION 6.26. A presheaf $\mathscr{F}$ on X is a **sheaf** if whenever $(U_j)_{j\in I}$ are open sets in X covering U (i.e. $\bigcup_{j\in I} U_j = U$), the map

$$\mathscr{F}(U) \longrightarrow \{(s_j)_{j\in I} \in \prod_{j\in I} \mathscr{F}(U_j),\ \ r_{U_j,U_j\cap U_k}(s_j) = r_{U_k,U_j\cap U_k}(s_k)\}$$

is bijective.

This means that elements of $\mathscr{F}(U)$ are defined by **local properties**, and that we may check whether they are equal to zero by local considerations. We denote by R-**Presheaf(X)** and R-**Sheaf(X)** the category of R-modules presheaves or sheaves.

EXERCICE 6.27. Does the above definition imply that for a sheaf, $\mathscr{F}(\emptyset)$ is the terminal object in the category ? Use the fact that in category with products, the empty product is the terminal object. What happens in the category[8] **Rings** ? Can one replace the original category by a smaller one such that this condition holds. To simplify matters, one usually adds this condition to the definition of a sheaf, and we stick to this tradition.

Examples:

(1) The skyscraper R-sheaf over x, denoted R_x is given by $R_x(U) = 0$ if $x \notin U$ and $R_x(U) = R$ for $x \in U$, the map $R_x(V) \to R_x(U)$ being the obvious map (identity if both are equal to R, and the 0-map otherwise).
(2) Let $f : E \to X$ be a continuous map, and $\mathscr{F}(U)$ be the sheaf of continuous sections of f defined over U, that is the set of maps $s : U \to E$ such that $f \circ s = \mathrm{id}_U$.
(3) Let $E \to X$ be a map between manifolds, and Π be a subbundle of T_zE. Consider $\mathscr{F}(U)$ to be the set of sections $s : X \to E$ such that $ds(x) \subset \Pi(s(x))$.
(4) If $f : Y \to X$ is a map, then we define a sheaf as $\mathscr{F}_f(U) = f^{-1}(U)$. This is a sheaf of **Open(Y)** on X but can also be considered as a sheaf of sets, or a sheaf of topological spaces.
(5) Set $\mathscr{F}(U)$ to be the set of constant functions on U. This is a presheaf. It is not a sheaf, because local considerations can only tell whether a function is **locally constant**. On the other hand the sheaf of locally constant functions is indeed a sheaf. It is called **the constant sheaf**, and denoted R_X. It can also be defined by setting $\mathscr{F}(U)$ to be the set of locally constant functions from U to the discrete set R.
(6) Let $\mathscr{F}$ be a sheaf. We say that $\mathscr{F}$ is **locally constant** if and only if $\mathscr{F}$ every point is contained in an open set U such that the sheaf $\mathscr{F}_U$ defined on U by

[8] most authors don't accept the single element ring where $0 = 1$ as a ring, which causes difficulties here.

$\mathscr{F}_U(V) = \mathscr{F}(V)$ for $V \subset U$ is a constant sheaf. There are non-constant locally constant sheaves, for example the set of locally constant sections of the $\mathbb{Z}/2$ Möbius band, defined by $M = [0,1] \times \{\pm 1\}/\{(0,1) = (1,-1)\}$.

(7) If A is a closed subset of X, then k_A, **the constant sheaf over** A is the sheaf such that $k_A(U)$ is the set of locally constant functions from $A \cap U$ to k.

(8) If U is an open set in X, then k_U, **the constant sheaf over** U is defined by $k_U(V)$ is the subset of $k(U \cap V)$ made of sections of the constant sheaf k_X with support a closed subset of V. This means that $k(U \cap V) = k^{\pi_0(U \cap V)}$, where $\pi_0(U \cap V)$ is the number of connected component of $U \cap V$ such that $\overline{U \cap V} \subset U$. See exercise 6.32 for this apparently strange definition.

(9) The sheaf $C^0(U)$ of continuous functions on U is a sheaf. The same holds for $C^p(U)$ on a C^p manifold, or $\Omega^p(U)$ the space of smooth p-forms on a smooth manifold, or $\mathscr{D}(U)$ the space of distributions on U, or $\mathscr{T}^p(U)$ the set of p-currents on U.

(10) If X is a complex manifold, the sheaf of holomorphic functions $\mathscr{O}_X$ is a sheaf. Similarly if E is a holomorphic vector bundle over X, then $\mathscr{O}_X(E)$ the set of holomorphic sections of the bundle E.

(11) The functor **Top** $\to$ **Chains** associating to a topological space M its singular cochain complex $(C^*(M,R),\partial)$ yields a presheaf of R-modules by associating to U, the R-module of singular cochains on U, $C^*(U,R)$. Its associated sheaf is denoted by $\mathbf{C}^*(U,R)$. It can be defined directly as the set of cochains invariant by barycentric subdivision. If s sends a simplex to the sum of its parts obtained by barycentric subdivision, $s : C_*(U,\mathbb{R}) \longrightarrow C_*(U,\mathbb{R})$ and s^* its adjoint $s^* : C^*(U,\mathbb{R}) \longrightarrow C^*(U,\mathbb{R})$. Then $\mathbf{C}^*(U,\mathbb{R})$ is defined as the set of cochains invariant by s^* (i.e. $s^*\alpha = \alpha$). Since ti is a sheaf, we have the exact sequence

$$0 \to \mathbf{C}^*(U \cup V) \to \mathbf{C}^*(U) \oplus \mathbf{C}^*(V) \to \mathbf{C}^*(U \cap V) \to 0$$

On the other hand using the functor **Top** $\longrightarrow$ **R** $-$ **mod** given by $U \to H^*(U)$, we get a presheaf of R-modules by $\mathscr{H}(U) = H^*(U)$. This is not a sheaf, because Mayer-Vietoris is a long exact sequence

$$\dots \to H^{*-1}(U \cup V) \to H^*(U \cup V) \to H^*(U) \oplus H^*(V) \to H^*(U \cap V) \to H^{*+1}(U \cup V) \to \dots$$

not a short exact sequence, so two elements in $H^*(U)$ and $H^*(V)$ with same image in $H^*(U \cap V)$ do come from an element in $H^*(U \cup V)$, but this element is not unique: the indeterminacy is given by the image of the coboundary map $\delta : H^{*-1}(U \cap V) \to H^*(U \cup V)$. The stalk of this presheaf is $\lim_{U \ni x} H^*(U)$, the local cohomology of X at x. If X is a manifold, the Poincaré lemma tells us that this is R in degree zero and 0 otherwise.

(12) Let **Simplex** be the category of the simplex. A presheaf over this category is called a simplicial set. This is a functor $\mathbf{Simplex}^{op}$ to $\mathscr{C}$, or the determination of objects $\mathscr{C}_n$ and maps δ_j^n, σ_j^n satisfying the simplicial identities

(a) $\delta_i^{n+1}\delta_j^n = \delta_{j-1}^{n+1}\delta_i^n$ for $i < j$
(b) $\sigma_i^n\sigma_j^{n+1}n = \sigma_j^n\sigma_{i-1}^{n+1}$ for $i > j$
(c) $\delta_i^n\sigma_j^n = \sigma_{j-1}^n\delta_i^n$ if $i < j$
(d) $\delta_i^n\sigma_j^n = \mathrm{id}$ for $i = j$ or $i = j+1$
(e) $\delta_i^n\sigma_j^n = \sigma_j\delta_{i-1}$ for $i > j+1$

This definition anticipates the fact that the notion of sheaves does not actually need a topological space, but a **Site** (see the Appendix).

(13) Similarly a **cellular sheaf** on the cell complex (X, P_X) is a functor from **Ord**(P_X) to $\mathscr{C}$. Note that the natural functor **Cell(X)** to **Open(X)**^{op} sends a sheaf over X to a cellular sheaf.

EXERCICE 6.28. Prove that a locally constant sheaf is the same as local coefficients. In particular prove that on a simply connected manifold, all locally constant sheaf are of the form $k_X \otimes V$ for some vector space V.

Because a sheaf is defined by local considerations, it makes sense to define the germ of $\mathscr{F}$ at x. The following definition makes sense if the category has direct limits.

DEFINITION 6.29. Given a family $(A_\alpha)_{\alpha\in J}$ of objects indexed by a totally ordered set, J, and morphisms $f_{\alpha,\beta} : A_\alpha \to A_\beta$ defined for $\alpha \leq \beta$, the direct limit of the sequence is an object A toghether with maps $f_\alpha : A_\alpha \to A$ satisfying the universal property: for each family of maps $g_\alpha : A_\alpha \to B$ such that $f_{\alpha,\beta} \circ g_\beta = g_\alpha$, we have a map $\varphi : B \to A$ making the following diagram commutative :

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & B \\
A_\alpha \arrow[u, "f_\alpha"] \arrow[ur, "g_\alpha"] \arrow[r, "f_{\alpha,\beta}"'] & A_\beta \arrow[u, "g_\beta"'] \arrow[ul, "f_\beta"']
\end{tikzcd}$$

Note that if we restrict ourselves to metric spaces, for example manifolds, we only need this concept for $J = \mathbb{N}$.

Note that direct limits do not necessarily exist. However they do exist in the category of abelian groups, or on the category of sheaves of R-modules.

DEFINITION 6.30. Let $\mathscr{F}$ be a presheaf on X and assume that direct limits exists in the category where the sheaf takes its values[9]. The **stalk** (or **germ**) of $\mathscr{F}$ at x, denoted $\mathscr{F}_x$ is defined as the **direct limit**

$$\varinjlim_{U \ni x} \mathscr{F}(U)$$

[9]Such a category is said to be cocomplete

An element in $\mathscr{F}_x$ is just an element $s \in \mathscr{F}_U$ for some $U \ni x$, but two such objects are identified if they coincide in a neighborhood of x: they are "germs of sections". For example if $\mathbb{C}_X$ is the constant sheaf, $(\mathbb{C}_X)_x = \mathbb{C}$.

REMARK 6.31. (1) Be careful, the data of an element s_x in $\mathscr{F}_x$ for each x, does not in general, define an element in $\mathscr{F}(X)$. On the other hand if it does, the element is then unique.
(2) For any closed F, we denote by $\mathscr{F}(F) = \lim_{U \supset F} \mathscr{F}(U)$. Be careful, for V open, it is not true that $\mathscr{F}(V) = \lim_{U \supsetneq V} \mathscr{F}(U)$, since there can be sections on V that do not extend to a neighborhood (e.g. continuous functions on V going to infinity near ∂V do not extend). For example, on a Hausdorff space, a skyscraper sheaf at x_0 is the unique sheaf such that $\mathscr{F}_{x_0} = k$ and $\mathscr{F}_x = 0$ for $x \neq x_0$.
(3) Using the stalk, we see that any sheaf can be identified with the sheaf of continuous sections of the map $\mathscr{E} = \bigcup_{x\in X} \mathscr{F}_x \to X$ sending $\mathscr{F}_x$ to x. The main point is to endow $\bigcup_{x\in X} \mathscr{F}_x$ with a suitable topology, and this topology is rather strange, for example the fibers are always totally disconnected. Indeed, the topology is given as follows: open sets in $\bigcup_{x\in X} \mathscr{F}_x$ are generated by $U_s = \{s(x) \mid x \in U, s \in \mathscr{F}(U)\}$. Note that the topology of $\mathscr{E}$ is rather wild, and if $\mathscr{F}$ is the set of sections of a continuous map $f : E \to B$, then the set $\mathscr{E}$ **does not** in general coincide with E.
(4) For a section $s \in \mathscr{F}(X)$ define the support $\mathrm{supp}(s)$ of s as the set of x such that $s(x) \in \mathscr{F}_x$ is nonzero. Note that this set is closed, or equivalently the set of x such that $s(x) = 0$ is open, contrary to what one would expect, before a moment's reflection shows that the stalk is a set of germs, and if a germ of a function is zero, the germ at nearby points are also zero.

EXERCICE 6.32. Prove that for A, closed, $(k_A)_x = k$ for $x \in A$ and 0 otherwise and that for U open, $(k_U)_x$ is k for $x \in U$ and 0 otherwise. Prove that if U is an open set, and we set $\mathscr{F}_U(V)$ to be the set of locally constant functions on $U \cap V$, then $(\mathscr{F}_U)_x = k$ for $x \in \overline{U}$ and 0 otherwise. Thus we have $\mathrm{supp}(k_A) = A, \mathrm{supp}(k_U) = U, \mathrm{supp}(\mathscr{F}_U) = \overline{U}$.

First we set

DEFINITION 6.33. Let $\mathscr{F}, \mathscr{G}$ be presheaves. A morphism f from $\mathscr{F}$ to $\mathscr{G}$ is a family of maps $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that $r_{V,U} \circ f_U = f_V \circ r_{V,U}$. Such a morphism induces a map $f_x : \mathscr{F}_x \to \mathscr{G}_x$.

4.1. Sheafification. The notion of stalk will allow us to associate to each presheaf a sheaf. Let $\mathscr{F}$ be a presheaf.

DEFINITION 6.34. The sheaf $\widetilde{\mathscr{F}}$ is defined as follows. Define $\widetilde{\mathscr{F}}(U)$to be the subset of $\prod_{x\in U} \mathscr{F}_x$ made of families $(s_x)_{x\in U}$ such that for each $x \in U$, there is $W \ni x$ and $t \in \mathscr{F}(W)$ such that for all y in W $s_y = t_y$ in $\mathscr{F}_y$.

Clearly we made the property of belonging to $\widetilde{\mathscr{F}}$ local, so this is a sheaf (Check !). Contrary to what one may think, even if we are only interested in sheaves, we cannot avoid presheaves or sheafification.

PROPOSITION 6.35. *Let $\mathscr{F}$ be a presheaf, $\widetilde{\mathscr{F}}$ the associated sheaf. Then $\widetilde{\mathscr{F}}$ is characterized by the following universal property: there is a natural morphism $i : \mathscr{F} \to \widetilde{\mathscr{F}}$ inducing an isomorphism $i_x : \mathscr{F}_x \to \widetilde{\mathscr{F}}_x$, and such that for any $f : \mathscr{F} \to \mathscr{G}$ morphisms of presheaves such that $\mathscr{G}$ is a sheaf, there is a unique $\tilde{f} : \widetilde{\mathscr{F}} \to \mathscr{G}$ making the following diagram commutative*

$$\begin{tikzcd} \mathscr{F} \arrow[r, "i"] \arrow[dr, "f"'] & \widetilde{\mathscr{F}} \arrow[d, "\tilde{f}"] \\ & \mathscr{G} \end{tikzcd}$$

PROPOSITION 6.36. *Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then*

(1) *If for all x we have $f_x = 0$, then $f = 0$*
(2) *If for all x we have f_x is injective, then f_U is injective*
(3) *If for all x, the map f_x is an isomorphism, then so is f_U*

PROOF. Let $s \in \mathscr{F}(U)$. Then $f_x = 0$ implies that for all $x \in U$ there is a neighborhood U_x such that $f_U(s)_x = 0$. This implies that $f_U(s) = 0$, hence $f = 0$ Let us now assume that $f_U(s) = 0$, and let us prove $s = 0$. Indeed, since $f_x s_x = 0$ we have $s_x = 0$ for all $x \in U$. But this implies $s = 0$ in $\mathscr{F}(U)$ by the locality property of sheaves. Finally, if f_x is bijective, it is injective and so is f_U. We have to prove that if moreover f_x is surjective, so is f_U. Indeed, let $t \in \mathscr{G}(U)$. By assumption, for each x, there exists s_x defined on a neighborhood V_x of x, such that $f_{V_x}(s_x) = t_x$ on $W_x \subset V_x$ containing x. We may of course replace V_x by W_x. By injectivity, such a s_x is unique. If s_x is defined over W_x, and s_y over W_y then on $W_x \cap W_y$ we have $f_{V_x}(s_x) = f_{W_x}(s_y) = t_{W_x \cap W_y}$, hence $s_x = s_y$ on $W_x \cap W_y$. As a result, according to the definition of a sheaf, there exists s equal to s_x on each W_x and $f(s) = t$. As a result the map f_U has a unique inverse, g_U for each open et U and we may check that g_U is a sheaf morphism, and $g \circ f = \mathrm{Id}_{\mathscr{F}}, f \circ g = \mathrm{Id}_{\mathscr{G}}$. □

Of course we do not have a surjectivity analogue of the above, because it does not hold in general.

In terms of categories, R-**Presheaf(X)** being the category of presheaves, and R-**Sheaf(X)** the category of sheaves of R-modules, these are abelian categories. The 0 object is the sheaf associating the R-module 0 to any open set. This is equivalent to $\mathscr{F}_x = 0$ for all x. The biproduct of $\mathscr{F}_1, \mathscr{F}_2$ is the sheaf associating to U the R-module $\mathscr{F}_1(U) \oplus \mathscr{F}_2(U)$. Clearly $\mathrm{Mor}(\mathscr{F}, \mathscr{G})$ is abelian and makes R-**Sheaf(X)** into an additive category. We also have that $\mathrm{Ker(f)(U) = Ker(f_U)}$. Indeed, this defines a sheaf on X, since if s_j satisfies $f_{U_j}(s_j) = 0$ and $s_{U_j} = s_{U_k}$ on $U_j \cap U_k$, then$f_U(s) = 0$. On the other hand $\mathrm{Im}(f)(U)$ is not defined as $\mathrm{Im}(f_U)$, since this is not a sheaf. Indeed, $t_{U_j} = f_{U_j}(s_j)$ and

$t_j = t_k$ on $U_j \cap U_k$ does not imply that $t_j = t_k$ on $U_j \cap U_k$, so here is no way to guarantee that there exists s such that $t = f(s)$. However $\mathrm{Im}(f_U)$ defines a presheaf. Then the Image in the category of Sheaves, denoted by $\mathrm{Im}(f)$ is the sheafification of $\mathrm{Im}(f_U)$. The same holds for $\mathrm{Coker}(f)$. Indeed, the universal property of sheafification means that if $f : \mathscr{F} \to \mathscr{G}$ is a morphism, and $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{p} \mathscr{H}$ is the cokernel in the category of presheaves, so that for any sheaf $\mathscr{L}$ such that

$$\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{L}$$

satisfies $g \circ f = 0$, we have a pair $(\mathscr{C}, q)$ such that there exits a unique h making this diagram commutative

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "f"] & \mathscr{G} \arrow[r, "g"] \arrow[dr, "q"] & \mathscr{L} & \\
 & & \mathscr{H} \arrow[u, dashed, "h"] \arrow[r, "i_{\mathscr{H}}"] & \widetilde{\mathscr{H}}
\end{tikzcd}$$

But if $\mathscr{L}$ is a sheaf, the map $\mathscr{H} \xrightarrow{h} \mathscr{L}$ lifts to a map $\widetilde{\mathscr{H}} \xrightarrow{\tilde{h}} \mathscr{L}$. Now set $\tilde{q} = i_{\mathscr{H}} \circ q$, it is easy to check that $(\widetilde{\mathscr{H}}, \tilde{q})$ has the universal property we are looking for, hence this is the cokernel of f in the category R-**Sheaf (X)**. Because $(i_{\mathscr{H}})_x$ is an isomorphism, we see that $\mathrm{Coker}(f)_x = \mathrm{Coker}(f_x)$.

To conclude, we have an inclusion functor from R-**Presheaf(X)** to R-**Sheaf(X)**, and the sheafification functor: $Sh : R$-**Presheaf(X)** $\to R$-**Sheaf(X)**.

⚠ CAUTION: It follows from the above that the Image in the category of presehaves does not coincide with the Image in the category of sheaves. Since we mostly work with sheaves, $\mathrm{Im}(f)$ will designate the Image in the category of sheaves, unless otherwise mentioned.

Now the definition of an exact sequence in the abelian category of sheaves translates as follows.

DEFINITION 6.37. A sequence of sheaves over X, $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$ is exact, if and only if for all $x \in X$, $\mathscr{F}_x \xrightarrow{f_x} \mathscr{G}_x \xrightarrow{g_x} \mathscr{H}_x$ is exact.

Example:

(1) Let $U = X \setminus A$ where A is a closed subset of X. Then we have an exact sequence

$$0 \to k_{X \setminus A} \to k_X \to k_A \to 0$$

obtained from the obvious maps.

(2) Given a sheaf $\mathscr{F}$ and a closed subset A of X, we have as above an exact sequence

False ? At l
wrong not

$$0 \to \mathscr{F}_{X\setminus A} \to \mathscr{F} \to \mathscr{F}_A \to 0$$

were $\mathscr{F}_A(U) = \mathscr{F}(U \cap A)$ while $\mathscr{F}_{X\setminus A}(U)$ is the set of sections of $\mathscr{F}(U \cap (X \setminus A))$ with closed support contained in $X \setminus A$.

Now consider the functor Γ_U from $\mathbf{R-Sheaf(X)} \longrightarrow \mathbf{R-mod}$ given by $\Gamma_U(\mathscr{F}) = \mathscr{F}(U)$. We have that a short exact sequence, i.e. a sequence $0 \to \mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C} \to 0$ such that for each x the sequence $0 \to \mathscr{A}_x \xrightarrow{f_x} \mathscr{B}_x \xrightarrow{g_x} \mathscr{C}_x \to 0$ is exact, then

$$0 \to \mathscr{A}(U) \xrightarrow{f_U} \mathscr{B}(U) \xrightarrow{g_U} \mathscr{C}(U)$$

is exact, and f_U is injective by proposition 6.35, but the map g_U is not necessarily surjective. However Γ_U is left-exact. Indeed, we wish to prove that $\mathrm{Im}(f_U) = \mathrm{Ker(g_U)}$. Because $g_x \circ f_x = 0$ we have $g_U \circ f_U = 0$, so that $\mathrm{Im}(f_U) \subset \mathrm{Ker(g_U)}$. Let us prove the reverse inclusion. Let $t \in \mathrm{Ker(g_U)}$. Then for each $x \in U$, there exists s_x such that on some neighborhood U_x we have $t_x = f_x(s_x)$, and by injectivity of f_x, s_x is unique. This implies that on $U_x \cap U_y$, $s_x = s_y$. But this implies that the s_x are restrictions of an element in $\mathscr{A}(U)$.

We just proved

PROPOSITION 6.38. *For any open set, U, the functor $\Gamma_U : \mathbf{Sheaf(X)} \to R-mod$ is left exact.*

EXERCICE 6.39. Let $(U_j)_{j\in I}$ be a basis of neighbourhoods of X. In other words any open set of X is the union of a subfamily of U_j. Let us assume that for any $i, j \in I$ there is $k \in I$ such that $U_i \cap U_j = U_k$. For example this is the case for the family of open sets of the type $U \times V$ in $X \times Y$. Then a sheaf is determined by the $\mathscr{F}(U_j)$. More precisely if we have a $\mathscr{F}(U_j)$ and the maps $\mathscr{F}(U_j) \longrightarrow \mathscr{F}(U_k)$ for $U_k \subset U_j$, and we have that whenever $U_j \cup U_k = U_l$,

$$\mathscr{F}(U_l) = \{(s_j, s_k) \in \mathscr{F}(U_j) \times \mathscr{F}(U_k) \mid s_j = s_k \text{ on } U_j \cap U_k\}$$

this defines a unique sheaf on X, by setting $\mathscr{F}(U) = \lim_{U_i \subset U} \mathscr{F}(U_i)$. Moreover if two sheaves $\mathscr{F}, \mathscr{G}$ on X coincide on the U_j, i.e. for all $j \in I$ there are isomorphisms $\varphi : \mathscr{F}(U_j) \longrightarrow \mathscr{G}(U_j)$ compatible with the restrictions, then $\mathscr{F} = \mathscr{G}$.

DEFINITION 6.40. Let $s \in \Gamma(U, \mathscr{F})$. Then $\mathrm{supp}(s)$ is the complement of the largest open set $V \subset U$ such that $s_{|V} = 0$. or else, because we are dealing with sheaves, $\bigcup_{V \subset U, s_{|V}=0} V$. By definition this is a closed set.

Note that if $s \in \Gamma(V, \mathscr{F})$ is such that $\mathrm{supp}(s) \subset U \subset \bar{U} \subset V$ then s extends by 0 to an element in $\Gamma(X, \mathscr{F})$, by setting $\tilde{s}_{|V} = s$, $\tilde{s}_{|X\setminus V} = 0$.

5. Appendix: Freyd-Mitchell without Freyd-Mitchell

If the only application of the Freyd-Mitchell theorem was to allow us to prove theorems on abelian categories as if the objects were modules and the maps module morphisms, there would be the following simpler approach. Let us first prove that **pull-back** exist in any abelian category.

Consider the diagram:

(6.5)
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

DEFINITION 6.41. The above diagram has a pull-back (P, i, j) where $i \in \mathrm{Mor}(P, X)$, $j \in \mathrm{Mor}(P, Y)$ if for any Q and maps $u \in \mathrm{Mor}(Q, X)$, $v \in \mathrm{Mor}(Q, Y)$ such that $f \circ u = g \circ v$ there is a unique map $\rho \in \mathrm{Mor}(Q, P)$ such that $i \circ \rho = u$, $j \circ \rho = v$.

(6.6)
$$\begin{tikzcd} Q \arrow[dr, dashed, "\rho"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & P \arrow[r, "i"] \arrow[d, "j"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

We can construct a pull-back in any abelian category by taking for (P, i, j) the kernel of the map $f - g : X \oplus Y \to Z$. Then $(f - g) \circ (u, v) = 0$ and the existence and uniqueness of ρ follows form existence and uniqueness of the dotted map in the definition of the kernel.

Let us define the relation $x \in_m A$ to mean $x \in \mathrm{Mor}(B, A)$ for some B, and identify x and y if and only if there are epimorphisms u, v such that $x \circ u = y \circ v$. This is obviously a reflexive and symmetric relation. We need to prove it is transitive through the following diagram

(6.7)
$$\begin{tikzcd} \bullet \arrow[r, "u'"] \arrow[d, "v'"] & \bullet \arrow[r, "t"] \arrow[d, "u"] & \bullet \arrow[d, "x"] \\ \bullet \arrow[r, "v"] \arrow[d, "w"] & \bullet \arrow[r, "y"] \arrow[d, "y"] & A \\ \bullet \arrow[r, "z"] & A & \end{tikzcd}$$

The existence of u', v' follows from pull-back from the other diagrams. Moreover u', v' are epimorphisms, so $x \equiv z$ since $x \circ (t \circ u') = z \circ (w \circ v')$. Let us denote by $\overline{A}$ the set of $x \in_m A$ modulo the equivalence relation. $\overline{A}$ is an abelian group:

(1) 0 is represented by the zero map, and any zero map in $\mathrm{Mor}(B,A)$ is equivalent to it.
(2) if $x \in_m A$, then $-x \in_m A$.
(3) If $f \in \mathrm{Mor}(A,A')$ and $x \in_m A$ then $f \circ x \in_m A'$. We denote this by $f(x)$.

Now

(1) if f is a monomorphism, if and only if $f \circ x = 0$ implies $x = 0$. This is also equivalent to $f(x) = f(x')$ implies $x = x'$.
(2) the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $g \circ f = 0$ and for any y such that $g(y) = 0$ we have $y = f(x)$

Indeed, $f(x) = 0$ means there is an epimorphism u such that $f \circ x \circ u = 0$. Since f is a monomorphism this implies $x \circ u = 0$ that is $x \equiv 0$. The second statement follows from the fact that $f(x) = f(x')$ is equivalent to $f(x - x') = 0$.

We thus constructed a functor from $\mathscr{C}$ to **Sets**. Its image is an abelian subcategory of the category of sets, and Freyd-Mitchell tells us that this is a category of R-modules, for some R, but the first embedding is enough for “diagram chasing with elements”.

6. Appendix: Presheaves and sheaves on sites. Simplicial sets

Presheaves are functors from $\mathbf{Open(X)}^{op}$ to a category $\mathbf{C}$. It is tempting to extend this notion to more general categories than $\mathbf{Open(X)}^{op}$. The categories for which this has nice properties are called **sites**, which are small categories endowed with a Grothendieck topology . This is the replacement for the notion of open cover of a topological space. Let $\mathbf{S}$ be a small category with pull-backs. We consider for an object X in $\mathbf{S}$, families $\{f_i : U_i \longrightarrow X \mid i \in I\}$

A Grothendieck topology on $\mathbf{S}$ is a set $G(X)$ of such families for each object X, called **coverings** of X, having the following properties

(1) If $f : Y \longrightarrow X$ is an isomorphism then $\{f : Y \longrightarrow X\} \in G(X)$
(2) If $\{f_i : U_i \longrightarrow X \mid i \in I\} \in G(X)$ and $g : Y \longrightarrow X$ is morphism, then the family of pull-backs $\{h_i : U_i \times_X Y \longrightarrow Y \mid i \in I\}$ is in $G(Y)$
(3) If $\{f_i : U_i \longrightarrow X \mid i \in I\} \in G(X)$ and for each i we have $\{g_i : V_{i,j} \longrightarrow U_i \mid j \in I\} \in G(U_i)$ then $\{f_i \circ g_{i,j} : V_{i,j} \longrightarrow X \mid i \in I\} \in G(X)$

DEFINITION 6.42. A Site is a pair $(\mathbf{S}, G)$ of a small category with pull-backs and a Grothedieck topology.

Note that what we defined is actually called in the litterature a basis for a Grothendieck topology, but defining a basis defines a topology.

DEFINITION 6.43. A presheaf on a site is a functor P from $\mathbf{S}^{op}$ to $\mathbf{C}$. A matching family associated to $x_i \in P(U_i)$ where $\{f_i : U_i \longrightarrow X\}$ is in $G(X)$ is a family such that $x_i \circ \pi^1_{i,j} = x_j \circ \pi^2_{i,j}$ where $\pi^1_{i,j}$ (resp $\pi^2_{i,j}$) is the first (resp. second) projection of $U_i \times_X U_j$ and an amalgamation of the $(x_i)_{i \in I}$ is an element $x \in P(X)$ such that $x \circ f_i = x_i$. Finally P is a sheaf if and only if every matching family has a unique amalgamation.

Example:

(1) If X is a topological space, the category **Open(X)** such that $G(X)$ is the set of open covers of X is a site. Then the definition of presheaf or sheaves coincides with the usual one on the topological space X.
(2) Let Δ be the simplex category. Its objets are the sets $[n] = \{1,2,....,n\}$ and the morphisms are the monotone maps, generated by δ_i^n, σ_i^n. A presheaf on Δ with values in **Sets** is called a **simplicial set** . That is Σ is a simplicial set if for each n we have a set $\Sigma(n) = \{\sigma_j^n \mid j \in J_n\}$ and to each map $\delta_i^n : [n-1] \to [n]$ we associate a map $\partial_i^n : \Sigma(n) \longrightarrow \Sigma(n-1)$, and to $\sigma_i^n : [n+1] \to [n]$ we associate a map $s_i^n : \Sigma(n) \longrightarrow \Sigma(n+1)$. The geometrix realization of Σ is the topological space
$$X(\Sigma) = \coprod_n \Sigma(n) \times \Delta^n / \simeq$$
where $(u, \delta_i^n(t)) = (\partial_i^n(u), t)$ and $(u, \sigma_i^n(t)) = (s_i^n(u), t)$. We endow $X_q(\Sigma) = \coprod_{1 \leq n \leq q} \Sigma(n) \times \Delta^n / \simeq$ with the quotient topology, and $X(\Sigma)$ with the union topology. The map $GR : \Sigma \longrightarrow X(\Sigma)$ is a functor from **SSet** to **Top**. The functor GR has a right adjoint, calld the Singular complex, $ST(X)_n = C^0(\Delta_n, X)$, with obvious morphisms.

7. Appendix: Fibrant replacement

Let N be a manifold, and assume it is triangulated, so it is the geometric realization of a simplicial set. Assume for each simplex σ we consider its barycenter x_σ and that we have a complex $(\Gamma_\sigma, \delta_\sigma)$ that we "see" over x_σ. Also for each path $\gamma_{\sigma,\tau}$ from x_σ to x_τ, we have a chain map $\varphi_{\sigma,\tau}$ from $(\Gamma_\sigma, d_\sigma)$ and (Γ_τ, d_τ). However we don't have $\varphi_{\tau,\rho}\varphi_{\sigma,\tau} - \varphi_{\tau',\rho}\varphi_{\sigma,\tau} = h_{\sigma,\rho}$ where τ, τ' are the two facets of σ containing ρ.

From now on we denote δ for δ_σ. We want to construct a sheaf over N such that its cohomology over each point is $H^*(\Gamma_\sigma, d)$, or rather each fiber is chain homotopy equivalent to $(\Gamma_\sigma, d_\sigma)$. For this we proceed as follows using the so-called cobar construction of Adams. remember that the chain complex of $X(\sigma)$ $C_d(X(\sigma))$ can be described as the set of linear combinations of $\sigma_d^j \in \Sigma(d)$, with boundary defined by $\partial_d = \sum_{i=0}^d (-1)^i \delta_i^d$. It can be discretized as follows.

For $\dim(\sigma_j) = 1$, and assuming the σ_j are adjacent edges, take the union of the edges defined by the $x_{\sigma_0}, x_{\sigma_1}, ..., x_{\sigma_q}$. This yields an element in $C_0(\Omega_{\sigma_0,\sigma_q})$. Now if for example σ_0, σ_2 are edges, and σ_1 is a facet, having the two edges in its boundary (as consecutive edges having common vertex $\sigma_{0,1}$), we consider the family of paths starting from $[\sigma_0, \sigma_{0,2}]] \cup [\sigma_{0,2}, \sigma_2]$ and deform it to the path $[\sigma_0, \sigma_1] \cup [\sigma_1, \sigma_2]$ in the obvious way as on Figure 1. This show how to associate an element in $C_1(\Omega_{\sigma_0,\Sigma_q})$ whenever we have a sequence $x_{\sigma_0}, x_{\sigma_1}, ..., x_{\sigma_q}$ such that $d = \sum_j (\dim(\sigma_j) - 1)$ equals 1. If for example σ_1, σ_2 are contiguous facets, we get a

Let $C_*(\Omega_{\sigma,\tau})$ be the chain complex associate to the space of paths from x_σ to x_τ. Let

$$\mathscr{C}^* = \bigoplus_\tau C_*(\Omega_{q,x_\tau}) \otimes \Gamma_\tau^*$$

and D be the differential defined by

$$D(\alpha \otimes x) = \sum_{\beta \in C_*((\Omega_{\sigma,\tau})} (\alpha \star \beta) \otimes h_\alpha(x)$$

Here $\star$ represents the concatenation of paths.

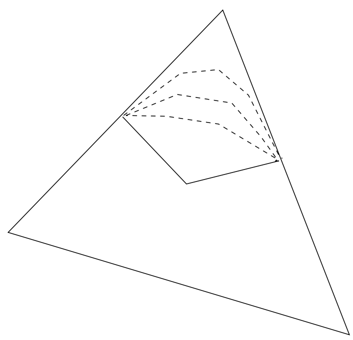

FIGURE 1. The cobar construction

Claim: $D^2 = 0$

CHAPTER 7

More on categories and sheaves.

1. Injective objects and resolutions

Let I be an object in a category.

DEFINITION 7.1. The object I is said to be injective, if for any maps h, f such that f is a monomorphism, there exists g making the following diagram commutative

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, dashed, "g"] \\ & & I \end{tikzcd}$$

This is equivalent to saying that $A \to \mathrm{Mor}(A, I)$ sends monomorphisms to epimorphisms. Note that g is by no means unique ! An injective sheaf is an injective object in R-**Sheaf (X)**.

PROPOSITION 7.2. *If I is injective in an abelian category $\mathscr{C}$, the functor $A \to \mathrm{Mor}(A, I)$ from $\mathscr{C}$ to* **Ab** *is exact.*

PROOF. According to Example 1 on page 63, Mor is left exact. So sends an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

to

$$0 \longrightarrow \mathrm{Mor}(C, I) \xrightarrow{g^*} \mathrm{Mor}(B, I) \xrightarrow{f^*} \mathrm{Mor}(A, I)$$

but we just translated the injectivity property into the statement that the functor $A \to \mathrm{Mor}(A, I)$ sends monomorphisms to epimorphisms, so $f^* : \mathrm{Mor}(B, I) \longrightarrow \mathrm{Mor}(A, I)$ is an epimorphism, so the sequence

$$0 \longrightarrow \mathrm{Mor}(C, I) \xrightarrow{g^*} \mathrm{Mor}(B, I) \xrightarrow{f^*} \mathrm{Mor}(A, I) \longrightarrow 0$$

is exact. $\square$

DEFINITION 7.3. A category has enough injectives, if any object A has a monomorphism into an injective object.

EXERCICE 7.4. Prove that in the category **Ab** of abelian groups, the group $\mathbb{Q}/\mathbb{Z}$ is injective. Prove that **Ab** has enough injectives (prove that a sum of injectives is injective).

In a category with enough injectives, we have the notion of **injective resolution**.

PROPOSITION 7.5 ([**Iv**], p.15). *Assume $\mathscr{C}$ has enough injectives, and let B be an object in $\mathscr{C}$. Then there is an exact sequence*

$$0 \to B \overset{i_B}{\to} J_0 \overset{d_0}{\to} J_1 \overset{d_1}{\to} J_2 \to ....$$

where the J_k are injectives.This is called an **injective resolution** *of B. Moreover given an object A in $\mathscr{C}$ and a map $f : A \to B$ and a resolution of A (not necessarily injective), that is an exact sequence*

$$0 \to A \overset{i_A}{\to} L_0 \overset{d_0}{\to} L_1 \overset{d_1}{\to} L_2 ....$$

and an injective resolution of B as above, then there is a morphism (i.e. a family of maps $u_k : L_k \to J_k$) such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "f"] & L_0 \arrow[r, "d_0"] \arrow[d, "u_0"] & L_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & L_2 \arrow[r, "d_2"] \arrow[d, "u_2"] & \dots . \\
0 \arrow[r] & B \arrow[r, "i_B"] & J_0 \arrow[r, "\partial_0"] & J_1 \arrow[r, "\partial_1"] & I_2 \arrow[r, "\partial_2"] & \dots
\end{tikzcd}$$

Moreover any two such maps are homotopic (i.e. $u_k - v_k = \partial_{k-1} s_k + s_{k+1}\delta_k$, where $s^k : I_k \to J_{k-1}$).

PROOF. The existence of a resolution is proved as follows: existence of J_0 is by definition of having enough injectives. Then let $M_1 = \mathrm{Coker}(i_B)$ so that $0 \to B \overset{d_0}{\to} J_0 \overset{f_0}{\to} M_1 \to 0$ is exact. A map $0 \to M_1 \to J_1$ induces a map $0 \to B \overset{i_B}{\to} J_0 \overset{d_0}{\to} J_1$, exact at J_0. Continuing this procedure we get the injective resolution of B. Now let $f : A \to B$ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r, "i_A"] \arrow[d, "f"] & L_0 \\
0 \arrow[r] & B \arrow[r, "i_B"] & J_0
\end{tikzcd}$$

Since J_0 is injective, i_A is a monomorphism, then $i_B \circ f$ lifts to a map $u_0 : L_0 \to J_0$. Let us now assume inductively that the map u_k is defined, and let us define u_{k+1}. We decompose using property (2) of Definition 6.9:

$$\begin{tikzcd}
L_{k-1} \arrow[r, "d_{k-1}"] \arrow[d, "u_{k-1}"] & L_k \arrow[r, "d_k"] \arrow[d, "u_k"] & L_{k+1} \\
J_{k-1} \arrow[r, "\partial_{k-1}"] & J_k \arrow[r, "\partial_k"] & J_{k+1}
\end{tikzcd}$$

as

$$\begin{tikzcd} L_{k-1} \arrow[r, "d_{k-1}"] \arrow[d, "u_{k-1}"] & L_k \arrow[r, "d_k"] \arrow[d, "u_k"] & \mathrm{Coker}(d_{k-1}) \arrow[r, "i_k"] \arrow[d, dashed, "v_{k+1}"] & L_{k+1} \\ J_{k-1} \arrow[r, "\partial_{k-1}"] & J_k \arrow[r, "\partial_k"] & \mathrm{Coker}(\partial_{k-1}) \arrow[r, "j_k"] & J_{k+1} \end{tikzcd}$$

Since $(\partial_k \circ u_k) \circ d_{k-1} = 0$, there exists by definition of the cokernel a map $v_{k+1} : \mathrm{Coker}(d_{k-1}) \to \mathrm{Coker}(\partial_{k-1})$, making the above diagram commutative. Then since i_k is monomorphism (due to exactness at L_k) and J_{k+1} is injective, the map $j_k \circ v_{k+1}$ factors through i_k so that there exists $u_{k+1} : L_{k+1} \to J_{k+1}$ making the above diagram commutative. The construction of the homotopy is left to the reader. □

PROPOSITION 7.6. *The category* **R – Sheaf(X)** *has enough injectives.*

PROOF. The proposition is proved as follows.

Step 1: One proves that for each x there is an injective $\mathscr{D}(x)$ such that $\mathscr{F}_x$ injects into $\mathscr{D}(x)$. In other words we need to show that R-mod has enough injectives. We omit this step since it is trivial for $\mathbb{C}$-sheaves (any vector space is injective).

Step 2: Construction of $\mathscr{D}$. The category **R-mod** has enough injectives, so choose for each x a map $\tilde{\jmath}_x : \mathscr{F}_x \to \mathscr{D}(x)$ where $\mathscr{D}(x)$ is injective, and consider the sheaf $\mathscr{D}(U) = \prod_{x \in U} \mathscr{D}(x)$. Thus a section is the choice for each x of an element $\mathscr{D}(x)$ (without any "continuity condition"). One should be careful. The sheaf $\mathscr{D}$ does not have $\mathscr{D}(x)$ as its stalk: the stalk of $\mathscr{D}$ is the set of germs of functions (without continuity condition) $x \mapsto \mathscr{D}(x)$ for x in a neighborhood of x_0. Obviously, $\mathscr{D}_{x_0}$ surjects on $\mathscr{D}(x_0)$. However, for each $\mathscr{F}$ we have $\mathrm{Mor}(\mathscr{F}, \mathscr{D}) = \prod_{x \in X} Hom(\mathscr{F}_x, \mathscr{D}(x))$: indeed, an element $(f_x)_{x \in X}$ in the right hand side will define a morphism f by $s \to f_x(s_x)$, and vice-versa, an element f in the left hand side, defines a family $(f_x)_{x \in X}$ by taking the value $f_x(s_x) = f(s)_x$. So $\tilde{j}_x$ defines an element j in $\mathrm{Mor}(\mathscr{F}, \mathscr{D})$. Clearly $\mathscr{D}$ is injective since for each x, there exists a lifting g_x

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}_x \arrow[r, "f"] \arrow[dr, "h"'] & \mathscr{G}_x \arrow[d, dashed, "g_x"] \\ & & \mathscr{D}(x) \end{tikzcd} \quad .$$

and the family (g_x) defines a morphism $g : \mathscr{G} \to \mathscr{D}$ (one may need the axiom of choice to choose g_x for each x).

Step 3: Let $\mathscr{F}$ an object in R-**Sheaf(X)** and $\mathscr{D}$ be the above associated sheaf. Then the obvious map $i : \mathscr{F} \to \mathscr{D}$ induces an injection $i_x : \mathscr{F}_x \to \mathscr{D}(x)$ hence is a monomorphism.

□

When R is a field, there is a unique injective sheaf with $\mathscr{D}(x) = R^q$. It is called the canonical injective R^q-sheaf. Let us now define

DEFINITION 7.7. Let $\mathscr{F}$ be a sheaf, and consider an injective resolution of $\mathscr{F}$

$$0 \to \mathscr{F} \xrightarrow{d_0} \mathscr{I}_0 \xrightarrow{d_1} \mathscr{I}_1 \xrightarrow{d_2} \mathscr{I}_2 \ldots.$$

Then the cohomology $\mathscr{H}^*(X,\mathscr{F})$ (also later denoted $R^*\Gamma(X,\mathscr{F})$) is the (co)homology of the sequence

$$0 \to \mathscr{I}_0(X) \xrightarrow{d_{0,X}} \mathscr{I}_2(X) \xrightarrow{d_{1,X}} \mathscr{I}_2(X) \ldots.$$

In other words $\mathscr{H}^m(X,\mathscr{F}) = \mathrm{Ker}(\mathrm{d_{m,X}})/\mathrm{Im}(\mathrm{d_{m-1,X}})$. According to Proposition 7.5 this definiton is independent from the choice of the resolution, since any two of them are homotopic.

Check that $\mathscr{H}^0(X,\mathscr{F}) = \mathscr{F}(X)$. Note that the second sequence is not an exact sequence of R-modules, because exactness of a sequence of sheaves means exactness of the sequence of R-modules obtained by taking the stalk at x (for each x). In other words, the functor from **Sheaf(X)** to R-**mod** defined by $\Gamma_x : \mathscr{F} \to \mathscr{F}_x$ is exact, but the functor $\Gamma_U : \mathscr{F} \to \mathscr{F}(U)$ is not.

This is a general construction that can be applied to any left-exact functor defined on an abelian category with enough injectives: take an injective resolution of an object, apply the functor to the resolution after having removed the object, and compute the cohomology. According to Proposition 7.5, this does not depend on the choice of the resolution, since two resolutions are chain homotopy equivalent, and F sends chain homotopic maps to chain homotopic maps, hence preserves chain homotopy equivalences . This is the idea of derived functors, that we are going to explain in full generality (i.e. applied to chain complexes). It is here applied to the functor Γ_X. It is a way of measuring how this left exact functor fails to be exact: if the functor is exact, then $\mathscr{H}^0(X,\mathscr{F}) = \mathscr{F}(X)$ and $\mathscr{H}^m(X,\mathscr{F}) = 0$ for $m \geq 1$.

For the moment we set

DEFINITION 7.8 (The derived functors R^jF). Let $\mathscr{C}$ be a category with enough injectives, and F be a left-exact functor. Then $R^jF(A)$ is obtained as follows: take an injective resolution of A,

$$0 \to A \xrightarrow{i_A} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \to \ldots.$$

then $R^jF(A)$ is the j-th cohomology of the complex

$$0 \to F(I_0) \xrightarrow{F(d_0)} F(I_1) \xrightarrow{F(d_1)} F(I_2) \to \ldots.$$

We say that A is F-acyclic, if $R^jF(A) = 0$ for $j \geq 1$.

Note that the left-exactness of F implies that we always have $R^0F(A) = A$. Since according to Proposition 7.5, the $R^jF(A)$ do not depend on the choice of the resolution, an injective object is acyclic: take $0 \to I \to I \to 0$ as an injective resolution, and notice that the cohomology of $0 \to I \to 0$ vanishes in degree greater than 0.

However, as we saw in the case of sheaves, injective objects do not appear naturally. So to be able to do computations, we would like to be able to use resolutions with a wider class of objects

DEFINITION 7.9. A **flabby** sheaf is a sheaf such that the map $\mathscr{F}(U) \to \mathscr{F}(V)$ is onto for any $V \subset U$.

Notice that by composing the restriction maps, $\mathscr{F}$ is flabby if and only if $\mathscr{F}(X) \to \mathscr{F}(V)$ is onto for any $V \subset X$. This clearly implies that the restriction of a flabby sheaf is flabby.

PROPOSITION 7.10. *An injective sheaf is flabby. A flabby sheaf is* Γ_X*-acyclic.*

PROOF. First note that the sheaf we constructed to prove that **Sheaf(X)** has enough injectives is clearly flabby. Therefore any injective sheaf $\mathscr{I}$ injects into a flabby sheaf, $\mathscr{D}$. Moreover there is a map $p : \mathscr{D} \to \mathscr{I}$ such that $p \circ i = \mathrm{id}$, since the following diagram yields the arrow p

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D} \arrow[d, dashed, "p"] \\ & & \mathscr{I} \end{tikzcd}$$

As a result, we have diagrams

$$\begin{tikzcd} \mathscr{D}(U) \arrow[r, "p_U"] \arrow[d, "s_{V,U}"] & \mathscr{I}(U) \arrow[d, "r_{V,U}"] \\ \mathscr{D}(V) \arrow[r, "p_V"] & \mathscr{I}(V) \end{tikzcd}$$

Since $p_U \circ i_U = \mathrm{id}$, we have that p_U is onto, hence $r_{V,U}$ is onto.

We now want to prove the following: let $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{G} \to 0$ be an exact sequence, where $\mathscr{E}, \mathscr{F}$ are flabby. Then $\mathscr{G}$ is flabby.

Let us first consider an exact sequence $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{G} \to 0$ with $\mathscr{E}$ flabby. We want to prove that the map $v(X) : \mathscr{F}(X) \to \mathscr{G}(X)$ is onto. Indeed, let $s \in \Gamma(X, \mathscr{G})$, and a maximal set for inclusion, U, such that there exists a section $t \in \Gamma(U, \mathscr{F})$ such that $v(t) = s$ on U. We claim $U = X$ otherwise there exists $x \in X \setminus U$, a section t_x defined in a neighborhood V of x such that $v(t_x) = s$ on V. Then $t - t_x$ is defined in $\Gamma(U \cap V, \mathscr{F})$, but since $v(t - t_x) = 0$, we have, by left-exactness of $\Gamma(U \cap V, -)$, $t - t_x = u(z)$ for $z \in \Gamma(U \cap V, \mathscr{E})$. Since $\mathscr{E}$ is flabby, we may extend z to X, and then $t = t_x + u(z)$ on $U \cap V$. We may the find a section $\tilde{t} \in \Gamma(U \cup V, \mathscr{F})$ such that $\tilde{t} = t$ on U and $\tilde{t} = t_x + u(z)$ on V. Clearly $v(\tilde{t})_U = s_{|U}$ and $v(\tilde{t})_V = v(t_x) + vu(z) = v(t_x) = s_{|V}$, hence $v(\tilde{t}) = s$ on $U \cup V$. This contradicts the maximality of U.

As a result, we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}(X) \arrow[r, "u(X)"] \arrow[d, "\rho_{X,U}"] & \mathscr{F}(X) \arrow[r, "v(X)"] \arrow[d, "\sigma_{X,U}"] & \mathscr{G}(X) \arrow[r] \arrow[d, "\tau_{X,U}"] & 0 \\ 0 \arrow[r] & \mathscr{E}(U) \arrow[r, "u(U)"] & \mathscr{F}(U) \arrow[r, "v(U)"] & \mathscr{G}(U) \arrow[r] & 0 \end{tikzcd}$$

and $\rho_{U,X}, \sigma_{U,X}$ are onto. This immediately implies that $\tau_{X,U}$ is onto. Finally, let us prove that a flabby sheaf $\mathscr{F}$ is acyclic. We consider the exact map $0 \to \mathscr{F} \to \mathscr{I}$ where $\mathscr{I}$ is injective. Using the existence of the cokernel, this yields an exact sequence $0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{K} \to 0$. By the above remark, $\mathscr{K}$ is flabby. Consider then the long exact sequence associated to the short exact sequence of sheaves:

$$0 \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{I}) \to H^0(X,\mathscr{K}) \to H^1(X,\mathscr{F}) \to H^1(X,\mathscr{I}) \to H^1(X,\mathscr{K}) \to ...$$

We prove by induction on n that for any $n \geq 1$ and any flabby sheaf, $H^n(X,\mathscr{F}) = 0$. Indeed, we just proved that $H^0(X,\mathscr{I}) \to H^0(X,\mathscr{K})$ is onto, and we know that $H^1(X,\mathscr{I}) = 0$. this implies $H^1(X,\mathscr{F}) = 0$. Assume now, that for any flabby sheaf and $j \leq n$, H^j vanishes. Then the long exact sequence yields

$$.. \to H^n(X,\mathscr{K}) \to H^{n+1}(X,\mathscr{F}) \to H^{n+1}(X,\mathscr{I}) \to ...$$

Since $\mathscr{I}$ is injective, $H^{n+1}(X,\mathscr{I}) = 0$ and since $\mathscr{K}$ is flabby $H^n(X,\mathscr{K}) = 0$ hence $H^{n+1}(X,\mathscr{F})$ vanishes. □

Examples: Flabby sheaves are much more natural than injective ones, and we shall see they are just as useful. The sheaf of distributions, that is $\mathscr{D}_X(U)$, the dual of $C_0^\infty(U)$, the sheaf of differential forms with distribution coefficients, the set of singular cochains defined on X (see Exercise 1)... are all flabby.

A related notion is the notion of soft sheaves. A soft sheaf is a sheaf such that the map $\mathscr{F}(X) \to \mathscr{F}(K)$ is surjective for any closed set K. Of course, we define $\mathscr{F}(K) = \lim_{K \subset U} \mathscr{F}(U)$. In other words, an element defined in a neighborhood of K has an extension (maybe after reducing the neighborhood) to all of X. The sheaves of smooth functions, smooth forms, continuous functions... are all soft. A fine sheaf is a sheaf admitting partitions of unity.

DEFINITION 7.11. A fine sheaf is a sheaf such that for any open cover $(U_\alpha)_{\alpha\in A}$ of X, the section Id in $\mathrm{Hom}(\mathscr{F},\mathscr{F})$ can be written as $\sum_\alpha \rho_\alpha$ where $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$.

The sheaf $C^\infty(U)$ of smooth functions is fine, since there are partitions of unity, but not flabby since smooth functions on open sets do note necessarily extend to X. There are flabby sheaves that are not fine.

PROPOSITION 7.12. *Every fine sheaf is soft*

PROOF. Indeed if K is closed and s is a section defined in a neighbourhood V of K, then $X = V \cup (X \setminus K)$ and there $\rho_V, \rho_{X\setminus K}$, and $\rho_V(s)$

□

We refer to subsection 3.1 for applications of these notions.

EXERCICES 1. (1) Prove that for a locally contractible space, the sheaf of singular cochains is flabby. Prove that the singular cohomology of a locally contractible space X is isomorphic to the sheaf cohomology $H^*(X,k_X)$.
(2) Prove that soft sheaves are acyclic.

Application: The Mayer-Vietoris sequence

PROPOSITION 7.13. *Let U, V be open sets in X, and $\mathscr{F}$ a sheaf on X. Then there is a long exact sequence*

$$\dots \longrightarrow H^p(U \cup V, \mathscr{F}) \longrightarrow H^p(U, \mathscr{F}) \oplus H^p(V, \mathscr{F}) \longrightarrow H^p(U \cap V) \longrightarrow H^{p+1}(U \cup V, \mathscr{F}) \longrightarrow \dots$$

PROOF. Let $\mathscr{I}^\bullet$ be a complex of injective sheaves quasi-isomorphic to $\mathscr{F}^\bullet$. By the definition of a sheaf, we have a short exact sequence

$$0 \longrightarrow \mathscr{I}^\bullet_{|U\cup V} \longrightarrow \mathscr{I}^\bullet_{|U} \oplus \mathscr{I}^\bullet_{|V} \longrightarrow \mathscr{I}^\bullet_{|U\cap V} \longrightarrow 0$$

where the first arrow is $s \mapsto (s_{|U}, s_{|V})$ and the second one is $(s, t) \mapsto (s-t)_{U\cap V}$ and the above is the corresponding long exact sequence (see Proposition 6.19, page 65) associated to this short exact sequence. $\square$

2. Operations on sheaves. Sheaves in mathematics.

2.1. General constructions. First of all, if $\mathscr{F}$ is sheaf over X, and U an open subset of X, we denote by $\mathscr{F}_{|U}$ the sheaf on U defined by $\mathscr{F}_{|U}(V) = \mathscr{F}(V)$ for all $V \subset U$. For clarity, we define $\Gamma(U, \bullet)$ as the functor $\mathscr{F} \to \Gamma(U, \mathscr{F}) = \mathscr{F}(U)$.

DEFINITION 7.14. Let $\mathscr{F}, \mathscr{G}$ be sheaves over X. We define $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ as the sheaf associated to the presheaf $U \mapsto \mathrm{Mor}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$. We define $\mathscr{F} \otimes \mathscr{G}$ to be the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes \mathscr{G}(U)$. The same constructions hold for sheaves of modules over a sheaf of rings $\mathscr{R}$, and we then write $\mathscr{H}om_{\mathscr{R}}(\mathscr{F}, \mathscr{G})$ and $\mathscr{F} \otimes_{\mathscr{R}} \mathscr{G}$.

REMARK 7.15.
(1) Note that sometimes $\mathrm{Mor}(\mathscr{F}, \mathscr{G})$ is denoted by $Hom(\mathscr{F}, \mathscr{G})$. This is not a sheaf, while $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf. There is however a connection between the two definitions: $\mathrm{Mor}(\mathscr{F}, \mathscr{G}) = \Gamma(X, \mathscr{H}om(\mathscr{F}, \mathscr{G}))$.
(2) Note that $\mathrm{Mor}(\mathscr{F}_{|U}, \mathscr{G}_{|U}) \neq Hom(\mathscr{F}(U), \mathscr{G}(U))$ in general, since an element f in the left hand side defines compatible $f_V \in \mathscr{H}om(\mathscr{F}(V), \mathscr{G}(V))$ for all open sets V in U, while the right-hand side does not.
(3) Note that tensor products commute with direct limits, so $(\mathscr{F} \otimes \mathscr{G})_x = \mathscr{F}_x \otimes \mathscr{G}_x$. On the other hand Mor does not commute with direct limits, so $\mathscr{H}om(\mathscr{F}, \mathscr{G})_x$ is generally different from $\mathscr{H}om(\mathscr{F}_x, \mathscr{G}_x)$.

Let $f : X \to Y$ be a continuous map. We define a number of functors associated to f as follows.

DEFINITION 7.16. Let $f : X \to Y$ be a continuous map, $\mathscr{F} \in$ **Sheaf(X)**, $\mathscr{G} \in$ **Sheaf(Y)** The sheaf $f_*\mathscr{F}$ is defined by

$$f_*(\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$$

The sheaf $f^{-1}(\mathscr{G})(U)$ is the sheaf associated to the presheaf $Pf^{-1}(\mathscr{F}) : U \mapsto \lim_{V \supset f(U)} \mathscr{G}(V)$. We also define $\mathscr{F} \boxtimes \mathscr{G}$ as follows. If p_X, p_Y are the projections of $X \times Y$ on the respective factors, we have $\mathscr{F} \boxtimes \mathscr{G} = p_X^{-1}\mathscr{F} \otimes p_Y^{-1}(\mathscr{G})$. When $X = Y$ and d is the diagonal map, we define $d^{-1}(\mathscr{F} \boxtimes G) = \mathscr{F} \otimes \mathscr{G}$. This is the sheaf associated to the presheaf $U \to \mathscr{F}(U) \otimes \mathscr{G}(U)$.

REMARK 7.17. ([**Iv**] page 149, th. 2.2 and page 104, (5.8)) Note that if $i : U \to X$ is the inclusion of an open set, we have $i^{-1}(\mathscr{F})(V) = \mathscr{F}(V)$ for an open set V, such that $V \subset U$.

Note also that if $j : X \to Y$ is an inclusion of a paracompact space, and Z a compact subset of X, we have

$$(j^{-1}\mathscr{G})(Z) = \lim_{j(Z)\subset V} \mathscr{G}(V)$$

Indeed, a section s of $(j^{-1}\mathscr{G})(Z)$ yields for each x a germ $\sigma_x \in \mathscr{G}(V_x)$ where V_x is open in Y and $s_x = s_y$ on $V_x \cap V_y \cap Z$, that is $s_x = s_y$ on a neighborhood W_{xy} of $V_x \cap V_y \cap Z$ (W_{xy} is a priori smaller than $V_x \cap V_y$). Now for each x we have V_x, and since $\bigcup_{x\in Z} V_x$ covers Z, we may consider a finite subcover. We are thus reduced to the following problem: we have sections s_1, s_2 defined on U_1, U_2 containing V_1, V_2 in Z. Let $W \subset Y$ be an open set containing $U_1 \cap U_2 \cap Z$. Then there are sets $U_1' \subset U_1, U_2' \subset U_2$ such that $U_1 \cap Z \subset U_1', U_2 \cap Z \subset U_2'$ and $U_1' \cap U_2' \subset W$. Indeed, take U_1'', U_2'' to be disjoint neighborhoods of $(Z \setminus W) \cap U_1$ and $(Z \setminus W) \cap U_2$ contained in U_1, U_2, which is always possible since

$$(Z \setminus W) \cap U_1 \cap (Z \setminus W) \cap U_2 = (Z \cap U_1 \cap U_2) \setminus W = \emptyset$$

and we are in a locally compact space, since $Z \setminus W$ is compact. Then $U_1' = U_1'' \cup W, U_2' = U_2'' \cup W$ satisfies our assumptions.

In particular

LEMMA 7.18. *We have* $(f^{-1}\mathscr{F})_x = \mathscr{F}_{f(x)}$.

PROOF. indeed the stalk of the sheaf associated to a presheaf is the same as the stalk of the presheaf. Now we must compute $\lim_{x\in U} \lim_{V\supset f(U)} \mathscr{F}(V)$, but continuity of f implies that this is the same as $\lim_{f(x)\in V} \mathscr{F}(V)$. □

Examples:

(1) Let $i : \{x\} \longrightarrow X$ be the inclusion of the point x. Then $i^{-1}(\mathscr{F})$ is a sheaf over a singleton, and defined by $i^{-1}(\mathscr{F})(x) = \mathscr{F}_x$.
(2) Let $\mathscr{F}, \mathscr{G}$ be complexes. Then we have $R\Gamma(U \times V, \mathscr{F} \boxtimes \mathscr{G}) = R\Gamma(U, \mathscr{F}) \boxtimes R\Gamma(V, \mathscr{G})$ where if $(\mathscr{I}^p, d^p)$ is an injective resolution for $\mathscr{F}$ and $(\mathscr{J}^q, \delta^q)$ an injective resolution of $\mathscr{G}$, we define $R\Gamma(U, \mathscr{F}) \boxtimes R\Gamma(V, \mathscr{G})$ is the quasi-isomorphism type of $\mathscr{K}^n(U \times V) = \sum_{p+q=n} \mathscr{I}^p(U) \otimes \mathscr{J}^q(V), \partial = d \otimes 1 + 1 \otimes \delta$.
(3) Let f be a diffeomorphism from X to Y. Then for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y we have $f^{-1} \circ f_*(\mathscr{F}^\bullet) = \mathscr{F}^\bullet$ and $f_* \circ f^{-1}(\mathscr{G}^\bullet) = \mathscr{G}^\bullet$.

Now let $U \subset X$. We claim $(j^{-1}\mathscr{G})(U) = \lim_{j(U)\subset V} \mathscr{G}(V)$, so that in this case the presheaf $Pj^{-1}\mathscr{F}$ is actually a sheaf. Indeed, let K_n be a sequence such that $\bigcup_n K_n = U$. Then a section $s \in (j^{-1}\mathscr{G})(U)$ restricts to a section on K_n, and this section according to the above argument extends to a section t_n defined on an open set V_n in Y containing K_n. Now possibly reducing V_n, we may assume $t_n = t_m$ on $V_n \cap V_m$ for all $m < n$. But then this defines a section of $\mathscr{F}$ on $\bigcup_n V_n$, that is a neighborhood of U.

2.2. Sheaves associated to open or closed subsets.

DEFINITION 7.19. Let $\mathscr{F}$ be a sheaf on X. Then for A a closed set, $j : A \to X$ the inclusion, the sheaf $\mathscr{F}_{|A}$ is the sheaf on A, given by $j^{-1}(\mathscr{F})$. $\mathscr{F}_A$ is the sheaf on X given by $\mathscr{F}_A = j_* j^{-1}(\mathscr{F})$. Thus $\mathscr{F}_A(U) = \lim_{V \supset U \cap A} \mathscr{F}(V)$. For U open we set $\mathscr{F}_U = \ker(\mathscr{F} \to \mathscr{F}_{X-U})$. For Z locally closed, we write $Z = U \cap A$ with U open, A closed and set $\mathscr{F}_Z = (\mathscr{F}_U)_A$. This is the unique sheaf such that $(\mathscr{F}_Z)_{|Z} = \mathscr{F}_{|Z}$ and $(\mathscr{F}_Z)_{|X \setminus Z} = 0$

PROPOSITION 7.20. *For A a closed set we have $(\mathscr{F}_A)_x = \mathscr{F}_x$ for $x \in A$ and $(\mathscr{F}_A)_x = 0$ for $x \in X \setminus A$.*

2.3. Functors on sheaves.

PROPOSITION 7.21. *The functors f_*, f^{-1} are respectively left-exact and exact. Moreover , let f, g be continuous maps, then $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.*

PROOF. For the first statement, let us prove that f^{-1} is exact. We use the fact that $f^{-1}(\mathscr{G})_x = \mathscr{G}_{f(x)}$. Thus an exact sequence $0 \to \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \to 0$ is transformed into the sequence $0 \to f^{-1}(\mathscr{F}) \xrightarrow{u \circ f} f^{-1}(\mathscr{G}) \xrightarrow{v \circ f} f^{-1}(\mathscr{H}) \to 0$ which has germs

$$0 \to (f^{-1}(\mathscr{F}))_x \xrightarrow{u(f(x))} (f^{-1}(\mathscr{G}))_x \xrightarrow{v(f(x))} (f^{-1}(\mathscr{H}))_x \to 0$$

equal to

$$0 \to \mathscr{F}_{f(x))} \xrightarrow{u(f(x))} \mathscr{G}_{f(x)} \xrightarrow{v(f(x))} \mathscr{H}_{f(x)} \to 0$$

which is exact. Now we prove that f_* is left-exact. Indeed, consider an exact sequence $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} G$. By left-exactness of Γ_U, the sequence

$$0 \to \mathscr{E}(U) \xrightarrow{u(U)} \mathscr{F}(U) \xrightarrow{v(U)} G(U)$$

is exact, hence for any $V \subset Y$, the sequence

$$0 \to \mathscr{E}(f^{-1}(V)) \xrightarrow{v(f^{-1}(V))} \mathscr{F}(f^{-1}(V)) \xrightarrow{v(f^{-1}(V))} G(f^{-1}(V))$$

is exact, which by taking limits on $V \ni x$ implies the exactness of

$$0 \to (f_*\mathscr{E})_x \xrightarrow{(f_* u)_x} (f_*\mathscr{F})_x \xrightarrow{(f_* v)_x} (f_*\mathscr{G})_x.$$

$\square$

PROPOSITION 7.22. *We have* $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F}) = \mathrm{Mor}(f^{-1}(\mathscr{G}), \mathscr{F})$. *We say that f_* is* **right-adjoint** *to f^{-1} or that f^{-1} is left adjoint to f_*.*

PROOF. We claim that an element in either space, is defined by the following data, called a f-homomorphism: consider for each x a morphism $k_x : \mathscr{G}_{f(x)} \to \mathscr{F}_x$ such that for any section s of $\mathscr{G}(U)$, $k_x \circ s(f(x))$ is a (continuous) section of $\mathscr{F}(U)$. Notice that there are in general many x such that $f(x) = y$ is given, and also that a f-homomorphism is the way one defines morphisms in the category **Sheaves** of sheaves over all manifold (so that we must be able to define a morphism between a sheaf over

X and a sheaf over Y). Now, we claim that an element in $\mathrm{Mor}(f^{-1}(\mathscr{G}),\mathscr{F})$ defines k_x, since $(f^{-1}(\mathscr{G}))_x = \mathscr{G}_{f(x)}$, so a map sending elements of $f^{-1}(\mathscr{G})(U)$ to elements of $\mathscr{F}(U)$ localizes to a map k_x having the above property. Conversely, given a map k_x as above, let $s \in f^{-1}(\mathscr{G})(U)$. By definition, for each point $x \in U$ there exists a section $t_{f(x)}$ defined near $f(x)$ such that $s = t_{f(x)}$ near x. Now define $s'_x = k_x t_{f(x)}$. We have that $s'_x \in F_x$, and by varying x in U, this defines a section of $\mathscr{F}(U)$. So k_x defines a morphism from $f^{-1}(\mathscr{G})$ to $\mathscr{F}$.

Now an element in $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F})$ sends for each U, $\mathscr{G}(U)$ to $\mathscr{F}(f^{-1}(U))$, hence an element in $\mathscr{G}_{f(y)}$ to an element in some $\mathscr{F}(f^{-1}(V_{f(y)}))$, where $V_{f(y)}$ is a neighborhood of $f(y)$, which induces by restriction an element in $\mathscr{F}_y$, hence defines k_x. Vice-versa, let $s \in \mathscr{G}(V)$ then for $y \in V$ and $x \in f^{-1}(y)$, we define $s'_x = k_x s_y$. The section s'_x is defined on V_x a neighborhood of x, and by assumption $k_x s_{f(x)}$ is continuous, so s' is continuous.

We thus identified the set of f-homomorphism both with $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F})$ and with $\mathrm{Mor}(f^{-1}(\mathscr{G}),\mathscr{F})$, which are thus isomorphic.

EXERCICE 7.23. Prove that $f_*\mathscr{H}om(f^{-1}(\mathscr{G}),\mathscr{F}) = \mathscr{H}om(\mathscr{G}, f_*\mathscr{F})$.

□

The notion of adjointness is important in view of the following.

PROPOSITION 7.24. *Any right-adjoint functor is left exact. Any left-adjoint functor is right-exact.*

PROOF. Let F be right-adjoint to G, that is $\mathrm{Mor}(A, F(B)) = \mathrm{Mor}(G(A), B)$. We wish to prove that F is left-exact. The exactness of the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is equivalent to showing that for all X,

$$0 \to \mathrm{Mor}(X, A) \xrightarrow{f^*} \mathrm{Mor}(X, B) \xrightarrow{g^*} \mathrm{Mor}(X, A) \tag{7.1}$$

Indeed, exactness of the sequence is equivalent to the fact that $A \xrightarrow{f} B$ is the kernel of g, or else that for any X, and $u : X \to A$ such that $g \circ u = 0$, there exists a unique $v : X \to B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ & X \arrow[ul, dashed, "v"] \arrow[u, "u"'] & \end{tikzcd}$$

The existence of v implies exactness of (7.1) at $\mathrm{Mor}(X, B)$, while uniqueness yields exactness at $\mathrm{Mor}(X, A)$.

As a result, left-exactness of F is equivalent to the fact that for each X, and each exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence

$$0 \to \mathrm{Mor}(X, F(A)) \xrightarrow{F(f)^*} \mathrm{Mor}(X, F(B)) \xrightarrow{F(g)^*} \mathrm{Mor}(X, F(A))$$

is exact. But this sequence coincides with

$$0 \to \mathrm{Mor}(G(X), A) \xrightarrow{f^*} \mathrm{Mor}(G(X), B) \xrightarrow{g^*} \mathrm{Mor}(G(X), C)$$

its exactness follows from the left-exactness of $M \to \mathrm{Mor}(X, M)$.

□

Note that in the literature, f^{-1} is sometimes denoted f^*. Note also that if f is the constant map, then $f_*\mathscr{F} = \Gamma(X, \mathscr{F})$, so that $R^j f_* = R^j\Gamma(X, \bullet)$ is the functor $\mathscr{F} \mapsto H^j(X, \mathscr{F})$.

EXERCICE 7.25. Show that Sheafification is the right adjoint functor to the inclusion of sheaves into presheaves. Conclude that Sheafification is a left-exact functor. Prove it also right exact.

COROLLARY 7.26. *The functor f_* maps injective sheaves to injective sheaves. The same holds for Γ_X.*

PROOF. Indeed, we have to check that $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, f_*(\mathscr{I}))$ is an exact functor. But this is the same as checking that $\mathscr{F} \to \mathscr{H}om(f^{-1}\mathscr{F}, \mathscr{I})$ is exact. Now $F \to f^{-1}(\mathscr{F})$ is exact, and since $\mathscr{I}$ is injective, $\mathscr{G} \to \mathscr{H}om(\mathscr{G}, \mathscr{I})$ is exact. Thus $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, f_*(\mathscr{I}))$ is the composition of two exact functors, hence is exact. The second statement is a special case of the first by taking f to be the constant map.

□

There is at least another simple functor: $f_!$ given by

DEFINITION 7.27. $f_!(\mathscr{F})(U) = \{s \in \mathscr{F}(f^{-1}(U)) \mid f : \mathrm{supp}(s) \to U \text{ is a proper map}\}$.

If f is proper, then $f_!$ and f_* coincide. Even though $f_!$ has a right-adjoint $f^!$, we shall not construct this as it requires a slightly complicated construction, extending Poincaré duality, the so-called Poincaré-Verdier duality (see [**Iv**] chapter V).

Example:

(1) Let A be a closed subset of X, and k_A be the constant sheaf on A, and $i : A \to X$ be the inclusion of A in X. Then $i_! = i_*$ and $i_*(k_A) = k_A$ and $i^{-1}(k_A) = k_X$. Thus if $i : A \to X$ is the inclusion of the closed set A in X, and $\mathscr{F}$ a sheaf on X, then $\mathscr{F}_A = i_* i^{-1}(\mathscr{F})$. This does not hold for A open, as we shall see in a moment.
(2) Let U be an open set in X and j the inclusion. Then $\mathscr{F}_U = j_! j^{-1}(\mathscr{F})$. This formula in fact holds for U locally closed (i.e. the intersection of a closed set and an open set).
(3) We have, with the above notations,

$$j^{-1} \circ j_* = j^{-1} \circ j_! = i^! \circ i_* = i^{-1} \circ i_* = \mathrm{id}$$

Note that the above operations extend to complexes of sheaves:

DEFINITION 7.28. Let $A^\bullet, B^\bullet$ be two bounded complexes. Then we define $(A^\bullet \otimes B^\bullet)^m = \sum_j A^j \otimes B^{m-j}$ with boundary map $d_m(u_j \otimes v_{m-j}) = \partial_j u_j \otimes v_{m-j} + u_j \otimes \partial_{m-j} v_{m-j}$. and $\mathcal{H}om(A^\bullet, B^\bullet)^m = \sum_j Hom(A^j \otimes B^{m+j})$, with boundary map $d_m f = \sum_p \partial_{m+p} f^p + (-1)^{m+1} f^{p+1} \partial_p$.

Finally we define the functor $\Gamma_Z : \mathbf{Sheaf(X)} \to \mathbf{Sheaf(X)}$ by

DEFINITION 7.29. Let Z be a closed set. Let $\mathscr{F} \in \mathbf{Sheaf(X)}$. Then the sheaf $\Gamma_Z \mathscr{F}$ is defined by $\Gamma_Z \mathscr{F}(U) = \ker(\mathscr{F}(U) \to \mathscr{F}(U \setminus Z))$. If Z is only locally closed, of the form $V \cap A$ with V open and A closed, we set $\Gamma_Z(U) = \Gamma_A(U \cap V)$.

EXERCICE 7.30.
(1) Check that in the definition of Γ_Z for Z locally closed, the definition is indeed independent on the way we write Z as $A \cap V$.
(2) Here Z is a closed subset of X. Check the following statements:
   (a) Show that the support of Γ_Z is contained in Z.
   (b) Show that Γ_Z is a left exact functor from **Sheaf(X)** to **Sheaf(X)**.
   (c) Show that Γ_Z maps injectives to injectives.
   (d) Show that $\mathscr{F}_Z = k_Z \otimes \mathscr{F}$ and $\Gamma_Z(\mathscr{F}) = \mathscr{H}om(k_Z, \mathscr{F})$.

PROPOSITION 7.31. *The functor Γ_Z is left-exact. It sends flabby sheaves to flabby sheaves (and in particular injective sheaves to acyclic sheaves).*

PROOF. One checks that Γ_Z is left-exact from the left-exactness of the functor $\mathscr{F} \to \mathscr{F}_{|X \setminus Z}$. Applying the Snake lemma (Lemma 6.22) to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r, "f"] \arrow[d, "a"] & \mathscr{G} \arrow[r, "g"] \arrow[d, "b"] & \mathscr{H} \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathscr{F}_{X \setminus Z} \arrow[r, "f'"] & \mathscr{G}_{X \setminus Z} \arrow[r, "g'"] & \mathscr{H}_{X \setminus Z} &
\end{tikzcd}$$

yields exactness of the sequence $0 \to \mathrm{Ker(a)} \to \mathrm{Ker(b)} \to \mathrm{Ker(c)}$ that is exactness of $0 \to \Gamma_Z(\mathscr{F}) \to \Gamma_Z(\mathscr{G}) \to \Gamma_Z(\mathscr{H})$.

We must now prove that if $\mathscr{F}$ is flabby, $\Gamma_Z(X, \mathscr{F}) \to \Gamma_Z(U, \mathscr{F})$ is onto. Let $s \in \Gamma_Z(U, \mathscr{F})$, that is an element in $\mathscr{F}(U)$ vanishing on $U \setminus Z$. We may thus first extend s by 0 on $X \setminus Z$ to the open set $(X \setminus Z) \cup U$. By flabbiness of $\mathscr{F}$ we then extend s to X. $\square$

We could denote $\Gamma_Z(U, \mathscr{F})$ by $\Gamma(U, U \setminus Z, \mathscr{F})$ or in general for two open sets $V \subset U$, $\Gamma(U, V, \mathscr{F}) = \Gamma_{X \setminus V}(U, \mathscr{F})$. We denote the j-th derived functor of $\mathscr{F} \mapsto \Gamma_Z(U, \mathscr{F})$ by $H^j_Z(U, \mathscr{F})$, or also by $H^j(U, U \setminus Z, \mathscr{F})$.

2.4. Sheaves and D-modules. Note that the rings we shall consider in this subsection are non-commutative, a situation we had not explicitly considered above. A D-module is a module over the ring D_X of algebraic differentials operators over an algebraic manifold X. Let O_X be the sheaf of rings of holomorphic functions, Θ_X the sheaf of rings of first order linear differential operators (i.e. derivations) on O_X (i.e. holomorphic vector fields), and D_X the sheaf of noncommutative rings generated by

O_X and Θ_X, that is the sheaf of holomorphic differential operators on X. A D-module is a module over the ring D_X. More generally, given a sheaf of rings $\mathscr{R}$, we can consider $\mathscr{R}$-modules, that is for each open U, $\mathscr{F}(U)$ is an $\mathscr{R}(U)$-module and the restriction morphism is compatible with the $\mathscr{R}$-module structure. What we did for R-modules also holds for $\mathscr{R}$-modules.

Let us show how D-modules appear naturally. Let P be a general differential operator, that is, locally, $Pu = (\sum_{j=1}^m P_{1,j}u_j,...,\sum_{j=1}^m P_{q,j}u_j)$ and we want to solve $\sum_{j=1}^m P_{i,j}u_i = v_j$, and let us start with $v = 0$. The operator P yields a linear map $D_X^p \to D_X^q$ and we may consider the map

$$\begin{array}{rcl} \Phi(u) : D_X^p & \longrightarrow & O_X \\ (Q_j)_{1\le j\le p} & \longrightarrow & \sum_{j=1}^p Q_j u_j \end{array}$$

so that if $(u_1,...,u_p)$ is a solution of our equation, then $\Phi(u)$ vanishes on $D_X \cdot P_1 + ... + D_X P_q$ where

$$P_j = \begin{pmatrix} P_{1,j} \\ \vdots \\ P_{q,j} \end{pmatrix}$$

Conversely, a map $\Phi : D_X^p \longrightarrow O_X$ vanishing on $D_X \cdot P_1 + ... + D_X P_q$ yields a solution of our equation, setting $u_j = \Phi(0,..,1,0...0)$.

Then, let $\mathscr{M}$ be the D-module $D_X/(D_X \cdot P)$, the set of solutions of the equation corresponds to $\mathrm{Mor}(\mathscr{M}, O_X)$. The Riemann-Hilbert correspondence RH sends $\mathscr{M}$ to $\mathscr{M} \otimes_{D_X} \Omega_X$ and is a functor from D-modules to **Sheaf**(**X**). We have $SS(RH(\mathscr{M})) = \mathrm{Char}(\mathscr{M})$ where Char is the characteristic variety. One could take this as the definition of the Characteristic variety ([**Hotta**]).

3. Injective and acyclic resolutions

One of the goals of this section, is to show why the injective complexes can be used to define the derived category. One of the main reasons, is that on those complexes, quasi-isomorphism coincides with chain homotopy equivalence. We also explain why acyclic resolutions are enough to compute the derived functors, and finally work out the examples of the deRham and Čech complexes, proving that they both compute the cohomology of X with coefficients in the constant sheaf.

We start with the following

PROPOSITION 7.32. *Let $f : A^\bullet \to I^\bullet$ be a quasi-isomorphism where the I^p are injective. Then there exists $g : I^\bullet \to A^\bullet$ such that $g \circ f$ is homotopic to* id.

We first construct the mapping cone of a map.

DEFINITION 7.33 (Mapping cone construction). Let $f^\bullet : A^\bullet \to B^\bullet$ be morphism of chain complexes, and $C(f)^\bullet = A^\bullet[1] \oplus B^\bullet$ with boundary map

$$d = \begin{pmatrix} -\partial_A & 0 \\ -f & \partial_B \end{pmatrix}$$

Then there is a short exact sequence of chain complexes

$$0 \longrightarrow B^\bullet \xrightarrow{u=\begin{pmatrix}0\\1\end{pmatrix}} C(f)^\bullet \xrightarrow{v=\begin{pmatrix}1\\0\end{pmatrix}} A^\bullet[1] \longrightarrow 0$$

The above exact sequence (or distinguished triangle) yields a long exact sequence in homology:

$$\longrightarrow H^n(A^\bullet, \partial_A) \xrightarrow{H^n(f)} H^n(B^\bullet, \partial_B) \xrightarrow{H^n(u)} H^n(C(f)^\bullet, d) \xrightarrow{\delta_f^n} H^{n+1}(A^\bullet, \partial_A) \longrightarrow ...$$

where the connecting map can be identified with $H^\bullet(f)$ and $\delta_f^* = H^*(f)$ coincides with the connecting map defined in the long exact sequence of Proposition 6.19. Note that $H^n(A^\bullet[1], \partial_A) = H^{n+1}(A^\bullet, \partial_A)$. Now we see that $H^n(f)$ is an isomorphism if and only if $H^n(C(f)^\bullet, d) = 0$ for all n. Thus under the assumptions of the proposition (with $B^\bullet = I^\bullet$), we have an acyclic complex $(C(f)^\bullet, d)$, and a map $v : C(f)^\bullet \to A^\bullet[1]$.

We claim that it is sufficient to prove that this map is homotopic to zero. Indeed, let s be such a homotopy. It induces a map $s^\bullet : C(f)^\bullet \to A^\bullet$ such that $-\partial_A s(a,b) + sd(a,b) = a$ or else

$$-\partial_A s(a,b) + s(-\partial_A(a), -f(a) + \partial_B(b)) = a$$

so setting $g(b) = s(0,b)$ and $t(a) = s(-a,0)$ we get (apply successively to $(0,-b)$ and $(a,0)$),

$$\partial_A g(b) - g(\partial_B b) = 0$$

so g is a chain map, and

$$\partial_A t(a) + gf(a) + t\partial_A(a) = a$$

so gf is homotopic to Id_A.

The proposition thus follows from the following lemma.

LEMMA 7.34. *Any morphism from an acyclic complex $C^\bullet$ to an injective complex $I^\bullet$ is homotopic to* 0.

Let f be the morphism. We will construct the map s such that $f = \partial s + sd$ by induction using the injectivity. Assume we have constructed the solid maps and we wish to construct the dotted one in the following (non commutative !) diagram, such that $f_{m-1} = \partial_{m-2} s_{m-1} + s_m d_{m-1}$.

$$\begin{tikzcd}
\cdots\cdot \arrow[r] & C^{m-2} \arrow[r, "d_{m-2}"] \arrow[d, "f_{m-2}"] \arrow[dl, "s_{m-2}"'] & C^{m-1} \arrow[r, "d_{m-1}"] \arrow[d, "f_{m-1}"] \arrow[dl, "s_{m-1}"] & C^{m} \arrow[r, "d_m"] \arrow[d, "f_m"] \arrow[dl, dashed] & \cdots \\
\cdots\cdot \arrow[r, "\partial_{m-3}"] & I^{m-2} \arrow[r, "\partial_{m-2}"] & I^{m-1} \arrow[r, "\partial_{m-1}"] & I^{m} \arrow[r, "\partial_m"] & \cdots
\end{tikzcd}$$

The horizontal maps are not injective, but we may replace them by the following commutative diagram

$$\begin{tikzcd}
& {} \arrow[d, "d_{m-2}"] & \\
& C^{m-1} \arrow[d] \arrow[dr, "d_{m-1}"] \arrow[dd, bend right=60, "f_{m-1}-\partial_{m-2}s_{m-1}"'] & \\
0 \arrow[r] & \mathrm{Im}(d_{m-1}) = \mathrm{Ker}(\mathrm{d_m}) \arrow[r, "d'_m"] \arrow[d, dashed, "w"'] & C^{m} \arrow[dl, dashed, "s_m"] \\
& I^{m-1} &
\end{tikzcd}$$

.

where we first prove the existence of w and then the existence of s_m. The existence of w follows from the fact that there is a morphism $\mathrm{Ker(d_{m-1})} \to \mathrm{ker(f_{m-1} - \partial_{m-2}s_{m-1})}$ since $\mathrm{Ker(d_{m-1}) = Im(d_{m-2})}$ and we just have to check that $(f_{m-1}-\partial_{m-2}s_{m-1})\circ d_{m-2} = 0$ which is obvious from the diagram and the induction assumption, because

$$f_{m-1} \circ d_{m-2} = \partial_{m-2} \circ f_{m-2} =$$
$$\partial_{m-2} \circ (\partial_{m-3}s_{m-2} + s_{m-1}d_{m-2}) = \partial_{m-2}s_{m-1}d_{m-2}$$

The injectivity of d'_m follows from the exactness of the sequence, and the existence of s_m follows from the injectivity of I^{m-1}.

Notice that proposition 7.32 implies

COROLLARY 7.35. *Let $I^\bullet$ be an acyclic chain complex of injective elements, and F is any left-exact functor, then $F(I^\bullet)$ is also acyclic.*

PROOF OF THE COROLLARY. Indeed, since the 0 map from $I^\bullet$ to itself is a quasi-isomorphism, we get a homotopy between $\mathrm{id}_{I^\bullet}$ and 0. In other words $\mathrm{id}_{I^\bullet} = ds + sd$. As a result $F(\mathrm{id}_{I^\bullet}) = F(d)F(s) + F(s)F(d) = dF(s) + F(s)d$ and this implies that $F(\mathrm{id}_{I^\bullet}) : F(I^\bullet) \to F(I^\bullet)$ is homotopic to zero, which is equivalent to the acyclicity of $F(I^\bullet)$. □

Note that this implies that to compute the right-derived functor, we may replace the injective resolution by any F-acyclic resolution, that is resolution by objects L_m such that $H^j(L_m) = 0$ for all $j \neq 0$:

COROLLARY 7.36. *Let $0 \to A \to L_0 \to L_1 \to ...$ be a resolution of A such that the L_j are F-acyclic, that is $R^m F(L_j) = 0$ for any $m \geq 1$. Then $RF(A)$ is quasi-isomorphic to the chain complex $0 \to F(L_0) \to F(L_1) \to ....$. In particular $R^m F(A)$ can be computed as the cohomology of this last chain complex.*

PROOF. Let $I^\bullet$ be an injective resolution of A. There is according to 7.5 a morphism $f : L^\bullet \to I^\bullet$ extending the identity map. Because the map f is a quasi-isomorphism (there is no homology except in degree zero, and then by assumption f_* induces the identity), according to the previous result there exists $g : I^\bullet \to L^\bullet$ such that $g \circ f$ is homotopic to the identity. But then $F(g) \circ F(f)$ is homotopic to the identity, and $F(f)$ is an isomorphism between the cohomology of $F(I^\bullet)$, that is $RF^*(A)$ and that of $F(L^\bullet)$.

Manque de montrer que $F(f) \circ F(g)$ est homotope Ĺ l'identitŐ □

Note that the above corollary will be proved again using spectral sequences in Proposition 8.13 on page 107.

Note that if $\mathscr{I}$ is injective, $0 \to \mathscr{I} \to \mathscr{I} \to 0$ is an injective resolution, and then clearly $H^0(X, \mathscr{I}) = \Gamma(X, \mathscr{I})$ and $H^j(X, \mathscr{I}) = 0$ for $j \geq 1$. A sheaf such that $H^j(X, \mathscr{F}) = 0$ for $j \geq 1$ is said to be Γ_X**-acyclic** (or **acyclic** for short).

EXERCICE 7.37. Let $A^\bullet, B^\bullet$ be complexes of injective objects quasi-isomorphic to $0 \to A \to 0$ and $0 \to B \to 0$. Then any map $A^\bullet \to B^\bullet[-1]$ is homotopic to 0. Show that a map $A^\bullet \to B^\bullet[1]$ does not need to be homotopic to zero.

3.1. Complements: DeRham, singular and Čech cohomology. We shall prove here that DeRham or Čech cohomology compute the usual cohomology.

Let $\mathbb{R}_X$ be the constant sheaf on X. Let Ω^j be the sheaf of differential forms on X, that is $\Omega^j(U)$ is the set of differential forms defined on U. This is clearly a soft sheaf, and we claim that we have a resolution

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} .... \xrightarrow{d} \Omega^n \to 0$$

where d is the exterior differential. This is a resolution as follows from the exactness of

$$0 \to (\mathbb{R}_X)_x = \mathbb{R} \xrightarrow{i} \Omega^0_x \xrightarrow{d} \Omega^1_x \xrightarrow{d} \Omega^2_x \xrightarrow{d} \Omega^3_x \xrightarrow{d} .... \xrightarrow{d} \Omega^n_x \to 0$$

which in turn follows from the Poincaré lemma, since for U contractible, we already have the exactness of

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \xrightarrow{d} .... \xrightarrow{d} \Omega^n(U) \to 0$$

and x has a fundamental basis of contractible neighborhoods. Since soft sheaves are acyclic, we may compute $H^*(X, \mathbb{R}_X)$ by applying $\Gamma(X, \bullet)$ to the above resolution. That is the cohomology of

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \Omega^3(X) \xrightarrow{d} .... \xrightarrow{d} \Omega^n(X) \to 0$$

or else the DeRham cohomology.

Similarly, if Z is a closed subset, then $k_Z = i_* i^{-1} k_X$, where $i : Z \to X$ is the inclusion. Because i^{-1} is exact and i_* sends injective to injectives, $i_* i^{-1}(\Omega^j_X)$ are acyclic, and k_Z has a resolution by the Ω^j_Z, where $\Omega^\bullet_Z$ is the sheaf of germs of forms near Z. In other words, $\Omega^j_Z(U) = \{\omega \in \Omega^j(U)\}/\simeq$ where $\omega_1 \simeq \omega_2$ if and only if $\omega_1 = \omega_2$ in a neighborhood of Z. Then $H^j(X, k_Z) = H^j(\Omega^\bullet_Z)$.

3.2. Singular cohomology. Let $f : X \to Y$ be a continuous map between topological spaces, and $\mathscr{C}^*_f$ be the complex of singular cochains over f, that is $\mathscr{C}^q_f(U)$ is the set of singular q-cochains over $f^{-1}(U)$. There is of course a boundary map $\delta : \mathscr{C}^q_f(U) \to \mathscr{C}^{q+1}_f(U)$. For $X = Y$ and $f = \mathrm{Id}$ this is just the sheaf of singular cochains on X. If moreover the space X is locally contractible, the sequence

$$0 \to k_X \to \mathscr{C}^0 \xrightarrow{\delta} \mathscr{C}^1 \xrightarrow{\delta} \mathscr{C}^2 \to ....$$

yields a resolution of the constant sheaf, the exactness of the sequence at the stalk level follows from its exactness on any contractible open set U. Thus, since the $\mathscr{C}^q$ are flabby, the cohomology $H^*(X, k_X)$ is computed as the cohomology of the complex

$$0 \to \mathscr{C}^0(X) \xrightarrow{\delta} \mathscr{C}^1(X) \xrightarrow{\delta} \mathscr{C}^2(X) \to$$

3.3. Čech cohomology. Let $\mathscr{F}$ be a sheaf of R-modules on X.

DEFINITION 7.38. Given a covering $\mathfrak{U}$ of X by open sets U_j, an element of $C^q(\mathfrak{U}, \mathscr{F})$ consists in defining for each $(q+1)$-uple $(U_{i_0}, ..., U_{i_q})$ an element $s(i_0, ..., i_q) \in \mathscr{F}(U_{i_0} \cap ... \cap U_{i_q})$ such that $s(i_{\sigma(0)}, i_{\sigma(1)}, ..., i_{\sigma(q)}) = \varepsilon(\sigma) s(i_0, ..., i_q)$.

If $s \in \check{C}^q(\mathfrak{U}, \mathscr{F})$ we define $(\delta s)(i_0, i_1, ..., i_{q+1}) = \sum_j (-1)^j s(i_0, i_1, ., \widehat{i_j} .., i_{q+1})$. This construction defines a sheaf on X as follows: to an open set V we associate the covering of V by the $U_j \cap V$, and there is a natural map induced by restriction of the sections of $\mathscr{F}$, $\check{C}^q(\mathfrak{U}, \mathscr{F}) \to \check{C}^q(\mathfrak{U} \cap V, \mathscr{F})$ obtained by replacing U_j by $U_j \cap V$. Thus the Čech complex associated to a covering is a sheaf over X. We may consider the complex of sheaves

$$0 \to \mathscr{F} \xrightarrow{i} \check{C}^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} ... \xrightarrow{\delta} \check{C}^q(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} \check{C}^{q+1}(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} ...$$

PROPOSITION 7.39. *The Čech complex associated to $\mathscr{F}$ is a resolution of $\mathscr{F}$*

⚠ CAUTION: This is usually NOT an injective resolution. Except in special cases, it cannot be used directly to compute the cohomology of $\mathscr{F}^\bullet$.

PROOF. We must prove that the complex of sheaves

$$0 \to \mathscr{F} \xrightarrow{i} \check{C}^0(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} \ldots \xrightarrow{\delta} \check{C}^q(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} \check{C}^{q+1}(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} \ldots$$

is exact. Let $x \in X$ and $U \in \mathfrak{U}$ such that $x \in U$. To $s \in \check{C}^q(\mathfrak{U},\mathscr{F})_x$ we associate $Ks \in \check{C}^{q-1}(\mathfrak{U},\mathscr{F})_x$ as follows: $Ks(U_{i_0},\ldots,U_{i_{q-1}}) = s(U,U_{i_0},\ldots,U_{i_{q-1}})$ which makes sense since for $V \subset U$ containing x, , $V \cap U \cap U_{i_0} \cap \ldots \cap U_{i_{q-1}} = V \cap U_{i_0} \cap \ldots \cap U_{i_{q-1}}$

Now we have

$$K(\delta s)(U_{i_0},\ldots,U_{i_q}) = K\sum_j (-1)^j s(U_{i_0},U_{i_1},.,\widehat{U}_{i_j},..,U_{i_{q+1}})$$

$$\delta K(s))(U_{i_0},\ldots,U_{i_q}) = \delta s(U,U_{i_0},\ldots,U_{i_q}) = s(U_{i_0},\ldots,U_{i_q})) - \sum_j (-1)^j s(U,U_{i_0},..,\widehat{U}_{i_j},\ldots,U_{i_q})$$

so we see that $\delta K + K\delta = Id$ and this implies that the sequence is exact. □

However if for all $(i_0,\ldots,i_q)$, the $H^j(U_{i_0} \cap \ldots \cap U_{i_q},\mathscr{F})$ are zero for $j \geq 1$, we say we have an acyclic cover, and the cohomology of $\check{C}^q(\mathfrak{U},\mathscr{F})$ computes the cohomology of the sheaf $\mathscr{F}$. This will follow from a spectral sequence argument (see exercise 8.30).

3.4. Direct and inverse limits. We start with

EXERCICES 2. (1) Let $\mathscr{A}$ be a sheaf over $\mathbb{N}$, $\mathbb{N}$ being endowed with the topology for which the open sets are $\{1,2,\ldots,n\}$, $\mathbb{N}$ and $\emptyset$. Prove that a sheaf over $\mathbb{N}$ is equivalent to a sequence of R-modules, A_n and maps

$$\ldots \to A_n \to A_{n-1} \to \ldots. \to A_0$$

and that $H^0(\mathbb{N},\mathscr{A}) = \varprojlim A_n$ is the inverse limit of the sequence. Describe $\lim^1(A_n)_{n\geq 1} \overset{def}{=} H^1(\mathbb{N},\mathscr{A})$

(2) Show that the above sheaf is flabby if and only if the maps $A_n \to A_{n-1}$ are onto, and that the sheaf is acyclic if and only if the sequence satisfies the Mittag-Leffler condition: the image of A_k in A_j is stationary as k goes to infinity.

(3) Prove that the inverse limit functor $\varprojlim$ is left-exact, while the direct limit functor $\varinjlim$ is exact. Prove that

$$H^*(\varinjlim_\alpha C_\alpha) = \varinjlim_\alpha H^*(C_\alpha)$$

Let $\mathscr{C}$ be an abelian category, I be a directed poset, that is a poset where every pair of elements has an upper bound. We define $F(I,\mathscr{C})$ to be the set of directed systems, that is $i \mapsto X_i$ and for $i \leq j$ there is a map $f_{ij} : X_i \longrightarrow X_j$ such that $f_{ii} = \mathrm{id}$ and $f_{ij} \circ f_{jk} = f_{ik}$. This is called a directed system associated to I. It is the same as the category of covariant functors from $Ord(I)$ to $\mathscr{C}$. The usual case is as above, $I = \mathbb{N}$ with the standard order. In some situations we have an inverse limit functor

$$\varprojlim : F(I,\mathscr{C}) \longrightarrow \mathscr{C}$$

where the limit is an object L toghether with maps $f_i : L \longrightarrow X_i$ such that $f_i \circ f_{ij} = f_j$, and this is universal, i.e. for any other object N and morphisms $g_i \in \mathrm{Mor}(N, X_i)$ such that $g_i \circ f_{ij} = g_j$, there is a unique $u \in \mathrm{Mor}(N, L)$ such that $g_i = f_i \circ u$.

Inverse limits do not necessarily exist. They however do exist in the category of Abelian groups, or in the category of sheaves. Moreover the functor $\varprojlim$ is left-exact, so we may define $R^j \lim = \lim^j$. In particular if $\lim^1$ vanishes, the functor $\varprojlim$ is exact.

DEFINITION 7.40 (Mittag-Leffler condition). A directed system (indexed by $\mathbb{N}$) satisfies the Mittag-Leffler condition if for all k, the map $A_n \longrightarrow A_k$ is an epimorphism for n large enough.

From exercise 1 we deduce

THEOREM 7.41. *If A_k is an inverse system satisfying the Mittag-Leffler condition. Then $\lim^1(A_k)$ vanishes. As a result if we have an exact sequence $0 \longrightarrow A_k \longrightarrow B_k \longrightarrow C_k \longrightarrow 0$ and $(A_k)_k$ satisfies the Mittag-Leffler condition, then we have an exact sequence*

$$0 \longrightarrow \varprojlim(A_k) \longrightarrow \varprojlim(B_k) \longrightarrow \varprojlim(C_k) \longrightarrow 0$$

THEOREM 7.42. *Limits exist in the category $D^b(X)$. Moreover if for $t \in \mathbb{R}$ we define open sets U_t such that $\overline{U}_t \subset U_s$ for $t < s$, then we have*

$$R\Gamma(U_t, \mathscr{F}^\bullet) = \varprojlim_{s<t} R\Gamma(U_s, \mathscr{F}^\bullet)$$

PROOF. Let $\mathscr{I}^\bullet$ be a complex of injective sheaves quasi isomorphic to $\mathscr{F}^\bullet$. Indeed the system $\Gamma(U_s, \mathscr{I}^\bullet) \longrightarrow \Gamma(U_{s'}, \mathscr{I}^\bullet)$ is onto for any $s > s'$.

$\square$

REMARK 7.43. If we have a sequence $\mathscr{F}_k$ of sheaves, with maps $\mathscr{F}_k \longrightarrow \mathscr{F}_{k-1}$, then we cannot make sense of $\lim R\Gamma(X, \mathscr{F}_k)$. Indeed, let J_k^p, I_k^p be injective resolution of $\mathscr{F}_k$. Then for each k the complexes $I_k^\bullet$ and $J_k^\bullet$ are quasi-isomorphic. But we cannot conclude that $\lim_k I_k^\bullet$ and $\lim_k J_k^\bullet$ are quasi-isomorphic.

add sth on

PROPOSITION 7.44. *Let $\mathscr{F}^\bullet$ be in $D^b(X)$. Assume for an increasing sequence U_k of open sets we have that $\mathscr{F}^\bullet(U_k)$ is acyclic. Then setting $U = \bigcup_k U_k$, we have that $\mathscr{F}^\bullet(U)$ is acyclic.*

PROOF. First, we may replace $\mathscr{F}^\bullet$ by a quasi-isomorphic complex of injective sheaves, $\mathscr{I}^\bullet$. Let $(s_k)_{k\geq 1}$ is a sequence in $\mathscr{I}^p(U_k)$ defining an element $s \in \mathscr{I}^p(U)$ such that $\partial s_k = 0$,. Then, assume we constructed $v_1, ..., v_{k-1}$ such that $\rho_{j,j-1}(v_j) = v_{j-1}$ and $s_j = \partial v_j$. Now $s_k = \partial v_k'$. However there is no guarantee that $\rho_{k,k-1}(v_k') = v_{k-1}$. However since $\rho_{k,k-1}\partial v_k = \partial v_{k-1}$, we have $\partial(\rho_{k,k-1} v_k - v_{k-1}) = 0$ so by assumption we may write $\rho_{k,k-1} v_k' - v_{k-1} = \partial w_{k-1}$. By surjectivity of $\mathscr{I}^p(U_k) \longrightarrow \mathscr{I}^p(U_{k-1})$ resulting form the injectivity of $\mathscr{I}^p$, we may find w_k such that $\rho_{k,k-1}(w-k) = w_{k-1}$. Then set $v_k = v_k' + \partial w_k$. We have $\partial v_k' = \partial v_k = s_k$ and $\rho_{k,k-1}(v_k) = \rho_{k,k-1}(v_k') + \rho_{k,k-1}(\partial w_k) =$

$\rho_{k,k-1}(v'_k) + \partial w_{k-1} = v_{k-1}$. So we may by induction construct a sequence (v_k) which defines an element v in $\mathscr{I}^{p-1}(U)$ and $\partial v = s$. This proves that $\mathscr{I}^\bullet(U)$ is acyclic. □

REMARK 7.45. This is a special case of the following theorem.

THEOREM 7.46. *Let $X_k^\bullet$ be a sequence of complexes, with chain maps $\rho_{k,l} : X_k^\bullet \to X_l^\bullet$. Let $X^\bullet = \varprojlim_k X_k^\bullet$. Assume the sequence $H^{j-1}(X_k^\bullet)$ satisfies the Mittag-Leffler condition. Then $H^j(X^\bullet) = \varprojlim_k H^j(X_k^\bullet)$.*

In our situation since the $H^j(X_k^\bullet)$ are assumed to vanish, they automatically satisfy the Mittag-Leffler condition.

4. Appendix: More on injective objects

Let us first show that the functor $A \to \mathrm{Mor}(A, L)$ is left exact, regardless of whether L is injective or not. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. Since f is a monomorphism $\mathrm{Mor}(f) : \mathrm{Mor}(B, L) \to \mathrm{Mor}(A, L)$ given by the map $u \to u \circ f$, is injective, by definition of monomorphisms. We thus only have to prove $\mathrm{Im}(\mathrm{Mor}(g)) = \mathrm{Ker}(\mathrm{Mor(f)})$. Assume $u \in \mathrm{Ker}(\mathrm{Mor(f)})$ so that $u \circ f = 0$. According to proposition 6.13, $(C, g) = \mathrm{Coker}(f)$, so by definition of the cokernel we get the factorization $u = v \circ g$.

LEMMA 7.47. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence such that A is injective. Then there exists $w : B \to A$ such that $w \circ f = \mathrm{id}_A$. As a result there exists of $u : C \to B$ and $v : B \to A$ such that $\mathrm{id}_B = f \circ v + u \circ g$, and the sequence splits.*

PROOF. The existence of w follows from the definition of injectivity applied to $h = \mathrm{id}_A$. The map w is then given as the dotted map. Now since $f = f \circ w \circ f$ we get $(\mathrm{id}_A - f \circ w) \circ f = 0$, hence by definition of the Cokernel, and the fact that $C = \mathrm{Coker}(g)$, there is a map $u : C \to B$ such that $(\mathrm{id}_A - f \circ w) = u \circ g$. This proves the formula $\mathrm{id}_B = f \circ v + u \circ g$ with $v = w$. As a result, $g = g \circ \mathrm{Id}_B = g \circ f \circ v + g \circ u \circ g$, and $g \circ f = 0$, and since g is an epimorphism and $g = g \circ u \circ g$ we have $\mathrm{Id}_C = g \circ u$ and the sequence is split according to Definition 6.11 and Exercise 6.12. □

LEMMA 7.48. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence with A, B injective. Then C is injective.*

PROOF. Indeed, the above lemma implies that the sequence splits, $B \simeq A \oplus C$, but the sum of two objects is injective if and only if they are both injectives: injectivity is a lifting property, and to lift a map to a direct sum, we must be able to lift to each factor. □

As a consequence any additive functor F will send a short exact sequences of injectives to a short exact sequences of injectives, since the image by F will be split, and a split sequence is exact. The same holds for a general exact sequence since it decomposes as $0 \to I_0 \to I_1 \to \mathrm{Ker}(\mathrm{d}_2) = \mathrm{Im}(\mathrm{d}_1) \to 0$. Since I_0, I_1 are injectives, so is $\mathrm{Ker}(\mathrm{d}_2) =$

Im(d$_1$). Now we use the exact sequence $0 \to \mathrm{Im}(d_1) \to I_2 \to \mathrm{Ker}(d_3) = \mathrm{Im}(d_2) \to 0$ to show that $\mathrm{Ker}(d_3) = \mathrm{Im}(d_2)$ is injective. Finally all the $\mathrm{Ker}(d_j)$ and $\mathrm{Im}(d_j)$ are injective. But this implies that the sequences $0 \to \mathrm{Im}(d_{m-1}) \to I_m \to \mathrm{Ker}(d_{m+1}) = \mathrm{Im}(d_m) \to 0$ are split, hence $0 \to F(\mathrm{Im}(d_{m-1})) \to F(I_m) \to F(\mathrm{Ker}(d_{m+1})) = F(\mathrm{Im}(d_m)) \to 0$ is split hence exact. This implies (Check !) that the sequence $0 \to F(I_0) \to F(I_1) \to F(I_2) \to F(I_3) \to$ is exact.

LEMMA 7.49 (Horseshoe lemma). *Let $0 \to A \to B \to C \to 0$ be an exact sequence, and, $I_A^\bullet$, $I_C^\bullet$ be injective resolutions of A and C. Then there exists an injective resolution of B, $I_B^\bullet$, such that $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ is an exact sequence of complexes. Moreover, we can take $I_B^\bullet = I_A^\bullet \oplus I_C^\bullet$.*

PROOF. See [**Weib**] page 37. One can also use the Freyd-Mitchell theorem. ☐

PROPOSITION 7.50. *Let $\mathscr{C}$ be an abelian category with enough injectives. Let $f : A \to B$ be a morphism. Assume for any injective object I, the induced map $f^* : \mathrm{Mor}(B, I) \to \mathrm{Mor}(A, I)$ is an isomorphism, then f is an isomorphism.*

PROOF. Assume f is not a monomorphism. Then there exists a non-zero $u : K \to A$ such that $f \circ u = 0$. We first assume u is a monomorphism. Let $\pi : K \to I$ be a monomorphism into an injective I. Then there exists $v : A \to I$ such that $v \circ u = \pi$. Let $h : B \to I$ be such that $v = h \circ f$. We have $h \circ f \circ u = v \circ u = \pi$ but also $f \circ u = 0$ hence $h \circ f \circ u = 0$ which implies $\pi = 0$ a contradiction. Now we still have to prove that u may be supposed to be injective. But the map u can be factored as $t \circ s$ where $s : K \to \mathrm{Im}(u)$ and $t : \mathrm{Im}(u) \to A$ and t is mono and s is epi. Thus since $f \circ u = 0$, we have $f \circ t \circ s = 0$, but since s is epimorphisms, we have $f \circ t = 0$ with t mono. Assume now f is not an epimorphism; Then there exists a nonzero map $v : B \to C$ such that $v \circ f = 0$. We now send C to an injective I by a monomorphism π. Then $(\pi \circ v) \circ f = 0$, and $\pi \circ v$ is nonzero, since π is a monomorphism. We thus get a non zero map $\pi \circ v \in \mathrm{Mor}(B, I)$ such that its image by f^* in $\mathrm{Mor}(A, I)$ is zero. ☐

As an example we consider the case of sheaves. Let $\mathscr{F}, \mathscr{G}$ be sheaves over X, and $f : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves. We consider an injective sheaf, $\mathscr{I}$, then $\mathrm{Mor}(\mathscr{F}, \mathscr{I}) = \bigcup_x \mathrm{Mor}(\mathscr{F}_x, \mathscr{I}(x))$, so that the map f^* on each component will give $f_x : \mathscr{F}_x \to \mathscr{G}_x$. If this map is an isomorphism, then f is an isomorphism (this was proved in Prop. 6.36).

One should be careful: the map f must be given, and the fact that $\mathscr{F}_x$ and $\mathscr{G}_x$ are isomorphic for all x does not imply the isomorphism of $\mathscr{F}$ and $\mathscr{G}$.

EXERCICE 7.51. Prove that if $\mathscr{C}$ is an abelian category, the category $\mathscr{I}$ of injective objects in $\mathscr{C}$ is a full abelian sub-category.

TO BE COMPLETED

4.1. Appendix: Poincaré-Verdier Duality. Let $f : X \to Y$ be a continuous map between manifolds. We want to define the map $f^!$, and then of course $Rf^!$, adjoint of $f_!$ and $Rf_!$. This is the sheaf theoretic version of Poincaré duality.

EXERCICES 3. (1) Use the above to prove that for $\mathscr{F}^\bullet$ a complex of sheaves over X, we have $\mathscr{H}(\mathscr{F}^\bullet_x) = \lim_U H^*(U, \mathscr{F}^\bullet)$. In other words the presheaf $U \mapsto H^*(U, \mathscr{F}^\bullet)$ has stalk $\mathscr{H}(\mathscr{F}^\bullet_x)$, and of course the same holds for the associated sheaf. So the stalk of the sheaf associated to the presheaf $U \mapsto H^*(U, \mathscr{F}^\bullet)$ is the homology of the stalk complex $\mathscr{F}^\bullet_x$.

CHAPTER 8

Derived categories, spectral sequences, application to sheaves

One of the main reasons to introduce derived categories is to do without spectral sequences. It may then seem ironic to base our presentation of derived categories on spectral sequences, via Cartan-Eilenberg resolutions. We could then rephrase our point of view: the goal of spectral sequences is to actually do computations. The derived category allows us to make this computation simpler hence more efficient by applying the spectral sequence only once at the end of our categorical reasoning. This is a common method in mathematics: we keep all information in an algebraic object, and only make explicit computations after performing all the algebraic operations.

1. The categories of chain complexes

As we mentioned in the prevous chapter, given $\mathscr{C}$ an abelian category with enough injectives, one can consider the different categories of chain complexes, $\mathbf{Ch^b}(\mathscr{C})$, $\mathbf{Ch}^+(\mathscr{C})$, $\mathbf{Ch}^-(\mathscr{C})$ respectively of chain complexes bounded, bounded from below, and bounded from above. We denote by $A^\bullet$ an object in $\mathbf{Ch}^+(\mathscr{C})$, we write it as

$$\dots \xrightarrow{d_{m-1}} A^m \xrightarrow{d_m} A^{m+1} \xrightarrow{d_{m+1}} A^{m+2} \xrightarrow{d_{m+2}} \dots$$

The functor $\mathscr{H}(A^\bullet)$ denotes the cohomology of this chain complex, that is $\mathscr{H}^m(A^\bullet) = \mathrm{Ker(d_m)/Im(d_{m-1})}$. We can see this is a complex with zero differential, so that $\mathscr{H}$ is a functor from $\mathbf{Ch}(\mathscr{C})$ to itself. When $\mathscr{F}^\bullet$ is a complex of sheaves, one should be careful not to confuse this with $H^*(X,\mathscr{F}^m)$ obtained by looking at the sheaf cohomology of each term, nor is it equal to something we have not defined yet, $H^*(X,\mathscr{F}^\bullet)$ that is computed from a spectral sequence involving both $\mathscr{H}$ and H^* as we shall se later.

Because we are interested in cohomologies, we will identify two chain homotopic chain complexes, but replacing chain complexes by their cohomology loses too much information. There are two notions which are relevant. The first is chain homotopy. The second is quasi-isomorphism.

DEFINITION 8.1. A chain map $f^\bullet$ is a **quasi-isomorphism**, if the induced map $\mathscr{H}(f^\bullet):\mathscr{H}(A^\bullet)\to\mathscr{H}(B^\bullet)$ is an isomorphism.

It is easy to construct two quasi-isomorphic chain complexes which are not homotopy equivalent. For example the following $\mathbb{Z}$-modules sequences.

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to 0$$

and

$$0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

are quasi-isomorphic, by the quasi-isomorphism induced by the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. But they are not homotopy equivalent : there is no non trivial chain map from the second complex to the first, since there is no nonzero map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$. The same example shows that the relation "there is a quasi-isomorphism from $A^\bullet$ to $B^\bullet$ is transitive and reflexive but not symmetric. So we must in general use a slightly sophisticated relation on chain complexes.

DEFINITION 8.2. Two chain complexes $A^\bullet, B^\bullet$ are quasi-isomorphic if and only if there exists $C^\bullet$ and chain maps $f^\bullet : C^\bullet \to A^\bullet$ and $g^\bullet : C^\bullet \to B^\bullet$ such that $f^\bullet, g^\bullet$ are quasi-isomorphisms (i.e. induce an isomorphism in cohomology).

We shall restrict ourselves to derived categories of bounded complexes. The derived category is philosophically the category of chain complexes quotiented by the relation of quasi-isomorphisms. This is usually acheived in two steps. We first quotient out by chain-homotopies, because it is easy to prove that homotopy between maps is a transitive relation, and only afterwords by quasi-isomorphism, for which transitivity is more complicated.

Note that if

$$0 \to A \to B^1 \to B^2 \to B^3 \to \ldots$$

is a resolution of A, then $0 \to A \to 0$ is quasi-isomorphic to $0 \to B^1 \to B^2 \to B^3 \to \ldots.$ Indeed the map $i : A \to B^1$ induces obviously a chain map and a quasi-isomorphism

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "i_0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots. \\
0 \arrow[r] & B^1 \arrow[r, "\partial_1"] & B^2 \arrow[r, "\partial_2"] & B^3 \arrow[r, "\partial_3"] & \ldots
\end{tikzcd}$$

The idea of the derived category, is that it is a universal category such that any functor sending quasi-isomorphisms to isomorphisms, factors through the derived category. Because we do not use this property, we shall give here a particular construction, in a case sufficiently general for our purposes: the case when the category $\mathscr{C}$ is a **category having enough injectives**. We refer to the bibliography for the general construction.

DEFINITION 8.3. Let $\mathscr{C}$ be an abelian category. The homotopy category, $\mathbf{K^b}(\mathscr{C})$ is the category having the same objects as $\mathbf{Chain^b}(\mathscr{C})$ and morphisms are equivalence classes of chain maps for the chain homotopy relation : $\mathrm{Mor}_{\mathbf{K^b}(\mathscr{C})}(A,B) = \mathrm{Mor}_{\mathscr{C}}(A,B)/\simeq$ i.e. $f \simeq g$ means that f is chain homotopic to g.

Note that $\mathbf{K^b}(\mathscr{C})$ is additive but is **not** an abelian category: by moding out by the chain homotopies, we lost the notion of kernels and cokernels. As a result there is no good notion of exact sequence. However $\mathbf{K^b}(\mathscr{C})$ is a **triangulated category**. We shall not go into the details of this notion here, but to remark that this is related to the property that short exact sequences of complexes only yield long exact sequences in homology. Before taking homology, a long exact sequence is a sequence of complexes

$$..\to A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1] \to ..$$

usually only homotopy exact. Let **Inj(**$\mathscr{C}$**)** be the category of injective objects. This is a full subcategory of $\mathscr{C}$. Let $\mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$ be the same category constructed on injective objects. To each chain complex, we can associate a chain complex of injective objects as follows:

Let

$$... \xrightarrow{d_{m-1}} A^m \xrightarrow{d_m} A^{m+1} \xrightarrow{d_{m+1}} A^{m+2} \xrightarrow{d_{m+2}} ...$$

be the chain complex, and for each A^m an injective resolution

$$0 \longrightarrow A^m \xrightarrow{i_m} I_0^m \xrightarrow{d_0^m} I_1^m \xrightarrow{d_1^m} I_2^m \xrightarrow{d_2^m} ...$$

By slightly refining this construction, we get the notion of Cartan-Eilenberg resolution:

DEFINITION 8.4. A Cartan-Eilenberg resolution of $A^\bullet$ is a commutative diagram, where the lines are injective resolutions:

$$\begin{tikzcd}
 & {} \arrow[d, "\partial^{m-2}"] & {} \arrow[d, "\partial_0^{m-2}"] & {} \arrow[d, "\partial_1^{m-2}"] & {} \arrow[d, "\partial_2^{m-2}"] & \\
0 \arrow[r] & A^{m-1} \arrow[r, "i_{m-1}"] \arrow[d, "\partial^{m-1}"] & I_0^{m-1} \arrow[r, "\delta_0^{m-1}"] \arrow[d, "\partial_0^{m-1}"] & I_1^{m-1} \arrow[r, "\delta_1^{m-1}"] \arrow[d, "\partial_1^{m-1}"] & I_2^{m-1} \arrow[r, "\delta_2^{m-1}"] \arrow[d, "\partial_2^{m-1}"] & .. \\
0 \arrow[r] & A^{m} \arrow[r, "i_{m}"] \arrow[d, "\partial^{m}"] & I_0^{m} \arrow[r, "\delta_0^{m}"] \arrow[d, "\partial_0^{m}"] & I_1^{m} \arrow[r, "\delta_1^{m}"] \arrow[d, "\partial_1^{m}"] & I_2^{m} \arrow[r, "\delta_2^{m}"] \arrow[d, "\partial_2^{m}"] & ... \\
0 \arrow[r] & A^{m+1} \arrow[r, "i_{m+1}"] \arrow[d, "\partial^{m+1}"] & I_0^{m+1} \arrow[r, "\delta_0^{m+1}"] \arrow[d, "\partial_0^{m+1}"] & I_1^{m+1} \arrow[r, "\delta_1^{m+1}"] \arrow[d, "\partial_1^{m+1}"] & I_2^{m+1} \arrow[r, "\delta_2^{m+1}"] \arrow[d, "\partial_2^{m+1}"] & ... \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

Moreover they must satisfy the following conditions

(1) If $A^m = 0$, then for all j, the I_j^m are zero.
(2) The lines yield injective resolutions of $\mathrm{Ker}(\partial^{\mathrm{m}}), \mathrm{Im}(\partial^{\mathrm{m}})$ and $\mathscr{H}^m(A^*)$. In other words, the $\mathrm{Im}(\partial_j^m)$ are an injective resolution of $\mathrm{Im}(\partial^m)$, the $\mathrm{Ker}(\partial_{\mathrm{j}}^{\mathrm{m}})/\mathrm{Im}(\partial_{\mathrm{j}}^{\mathrm{m-1}})$ are an injective resolution of $\mathrm{Ker}(\partial^{\mathrm{m}})/\mathrm{Im}(\partial^{\mathrm{m-1}}) = \mathscr{H}^{\mathrm{m}}(\mathrm{A}^\bullet, \partial)$. This implies that the $\mathrm{Ker}(\partial_{\mathrm{j}}^{\mathrm{m}})$ are an injective resolution of $\mathrm{Ker}(\partial^{\mathrm{m}})$.

REMARK 8.5. We decided to work in categories of finite complexes. This raises a question: are Cartan-Eilenberg resolutions of such complexes themselves finite. Clearly this is equivalent to asking whether an object has a finite resolution. The answer is positive over manifolds: they have cohomological dimension n, so we can always find resolutions of length at most n (see [**Bre**] chap 2, thm 16.4 and 16.28). If we want to work with bounded from below complexes, we do not need this result, but then we shall need to be slightly more careful about convergence results for spectral sequences, even though there is no real difficulty. The case of complexes unbounded from above and below is more complicated- because of spectral sequence convergence issues- and we shall not deal with it.

Now we claim

PROPOSITION 8.6. *(1) Every chain complex has a Cartan-Eilenberg resolution.*
(2) Let $A^\bullet, B^\bullet$ be two complexes, $I^{\bullet,\bullet}$ and $J^{\bullet,\bullet}$ be Cartan-Eilenberg resolutions of $A^\bullet, B^\bullet$, and $f : A^\bullet \to B^\bullet$ be a chain map. Then f lifts to a chain map $\tilde{f} : I^{\bullet,\bullet} \to J^{\bullet,\bullet}$. Moreover two such lifts are chain homotopic.

PROOF. (see [**Weib**]) Set $B^m(A^\bullet) = \mathrm{Im}(\partial^m)$, $Z^m(A^\bullet) = \mathrm{Ker}(\partial^m)$ and $H^m(A^\bullet)$, and consider the exact sequence $0 \to B^m(A^\bullet) \to Z^m(A^\bullet) \to H^m(A^\bullet) \to 0$. Starting from injective resolutions $I^\bullet_{B^m}$ of $B^m(A)$ and $I^\bullet_{H^m}$ of $H^m(A^\bullet)$, the Horseshoe lemma (lemma 7.49 on page 99) yields an exact sequence of injective resolutions $0 \to I^\bullet_{B^m} \to I^\bullet_{Z^m} \to I^\bullet_{H^m} \to 0$. Applying the Horseshoe lemma again to $0 \to Z^m(A^\bullet) \to A^m \to B^{m+1}(A^\bullet) \to 0$ we get an injective resolution $I^\bullet_{A^m}$ of A^m and exact sequence $0 \to I^\bullet_{Z^m} \to I^\bullet_{A^m} \to I^\bullet_{B^{m+1}} \to 0$. Then $I^\bullet_{A^m} \xrightarrow{\partial^\bullet_m} I^\bullet_{A^{m+1}}$ is the composition of $I^\bullet_{A^m} \to I^\bullet_{B^{m+1}} \to I^\bullet_{Z^{m+1}} \to I^\bullet_{A^{m+1}}$. This proves (1). Property (2) is left to the reader. □

Note: a chain homotopy between $f, g : I^{\bullet,\bullet} \to J^{\bullet,\bullet}$ is a pair of maps $s^h_{p,q} : I^{p,q} \to J^{p+1,q}$ and $s^v_{p,q} : I^{p,q} \to J^{p,q+1}$ such that $g - f = (\delta s^h + s^h\delta) + (\partial s^v + s^v\partial)$. This is equivalent to requiring that $s^h + s^v$ is a chain homotopy between $Tot(I^{\bullet,\bullet})$ and $Tot(J^{\bullet,\bullet})$.

PROPOSITION 8.7. *Let I^m_j be the double complex as above, and $Tot(I^{\bullet,\bullet})$ be the chain complex given by $T^q = \oplus_{j+m=q} I^m_j$ and $d = \partial + (-1)^m\delta$, in other words $d_{|I^m_j} = d^m_j + (-1)^m\delta^m_j$. Then $A^\bullet$ is quasi-isomorphic to $T^\bullet$.*

LEMMA 8.8 (Tic-Tac-Toe). *Consider the following bi-complex*

$$
\begin{tikzcd}
 & {} \arrow[d, "\partial^{m-2}"] & {} \arrow[d, "\partial_0^{m-2}"] & {} \arrow[d, "\partial_1^{m-2}"] & {} \arrow[d, "\partial_2^{m-2}"] & \\
0 \arrow[r] & A^{m-1} \arrow[r, "i_{m-1}"] \arrow[d, "\partial^{m-1}"] & I_0^{m-1} \arrow[r, "\delta_0^{m-1}"] \arrow[d, "\partial_0^{m-1}"] & I_1^{m-1} \arrow[r, "\delta_1^{m-1}"] \arrow[d, "\partial_1^{m-1}"] & I_2^{m-1} \arrow[r, "\delta_2^{m-1}"] \arrow[d, "\partial_2^{m-1}"] & .. \\
0 \arrow[r] & A^{m} \arrow[r, "i_{m}"] \arrow[d, "\partial^{m}"] & I_0^{m} \arrow[r, "\delta_0^{m}"] \arrow[d, "\partial_0^{m}"] & I_1^{m} \arrow[r, "\delta_1^{m}"] \arrow[d, "\partial_1^{m}"] & I_2^{m} \arrow[r, "\delta_2^{m}"] \arrow[d, "\partial_2^{m}"] & \dots \\
0 \arrow[r] & A^{m+1} \arrow[r, "i_{m+1}"] \arrow[d, "\partial^{m+1}"] & I_0^{m+1} \arrow[r, "\delta_0^{m+1}"] \arrow[d, "\partial_0^{m+1}"] & I_1^{m+1} \arrow[r, "\delta_1^{m+1}"] \arrow[d, "\partial_1^{m+1}"] & I_2^{m+1} \arrow[r, "\delta_2^{m+1}"] \arrow[d, "\partial_2^{m+1}"] & \dots \\
 & {} & {} & {} & {} &
\end{tikzcd}
$$

Assume the lines are exact (i.e. i_m *is injective and* $\mathrm{Im}(i_m) = \ker(\delta_0^m)$ *and* $\mathrm{Im}(\delta_j^m) = \mathrm{Ker}(\delta_{\mathrm{j+1}}^{\mathrm{m}})$*). Then the maps* i_m *induce a quasi-isomorphism between the total complex* $T^q = \oplus_{j+m=q} I_j^m$ *endowed with* $d = \partial + (-1)^m \delta$ *and the chain complex* $A^\bullet$*.*

PROOF. The proof is the same as the proof of the spectral sequence computing the cohomology of a bicomplex, except that here we get an exact result. Let us write for convenience $\overline{\delta} = (-1)^m \delta$. Then notice that the maps i_m yield a chain map between $A^\bullet$ and $T^\bullet$. Indeed, if $u_m \in A^m$, $(\partial + \overline{\delta})(i_m(u_m)) = \partial_0^m i_m(u_m)$ since $\overline{\delta}_0^m \circ i_m = 0$. But $\partial_0^m i_m(u_m) = i_{m+1}\partial_m(u_m) = 0$ since u_m is ∂_m-closed. Similarly if u_m is exact, $i_m(u_m)$ is exact, so that i_m induces a map in cohomology. We must now prove that this induces an isomorphism in cohomology. Injectivity is easy: suppose $i_m(u_m) = (\partial + \overline{\delta})(y)$. Because there is no element left of I_0^m, we must have $y = y_0^{m-1}$ hence $i_m(u_m) = \partial_0^{m-1}(y_0^{m-1})$ and $\overline{\delta}_0^{m-1}(y_0^{m-1}) = 0$. This implies by exactness of the lines that $y_0^{m-1} = i_{m-1}(u_{m-1})$, and

$$i_m(u_m) = \partial_0^{m-1}(y_0^{m-1}) = \partial_0^{m-1}(i_{m-1}(u_{m-1}) = i_m \partial_{m-1}(u_{m-1})$$

injectivity of i_m implies that $u_m = \partial_m u_{m-1}$, so u_m was zero in the cohomology of $A^\bullet$. We finally prove surjectivity of the map induced by i_m in cohomology.

Indeed, let $x = \sum_{j+m=q} x_j^m$ such that $(\partial + \overline{\delta})(x) = 0$. Looking at the component of $(\partial + \overline{\delta})(x)$ in I_j^m we see that this is equivalent to $\partial x_{j-1}^{m-1} + \overline{\delta} x_j^{m-1} = 0$. Since the complexes are bounded, there is a smallest $j = j_0$ such that $x_j^m \neq 0$. Then we have $\overline{\delta} x_{j_0}^{m_0-1} = 0$ (since $x_{j_0-1}^{m_0} = 0$), and by exactness of $\overline{\delta}$, we have $x_{j_0}^{m_0-1} = \overline{\delta} y_{j_0}^{m_0}$. Then $x - (\partial + \overline{\delta})(y_{j_0}^{m_0})$ has for all components in I_j^m vanishing for $j \geq j_0 - 1$. By induction, we see that we can replace x by a $(\partial + \overline{\delta})$ cohomologous element with a single component w_0^m in I_0^m and since $(\partial + \overline{\delta})(w_0^m) = 0$, we have $w_0^m = i_m(u_m)$ and we easily check $\partial(u_m) = 0$. □

If we are talking about an element in $\mathscr{C}$ identified with the chain complex $0 \to A \to 0$ the total complex above is quasi-isomorphic to an injective resolution of A. Then if F is a left-exact functor, we denote $RF(A)$ to be the element

$$0 \to F(I_0) \to F(I_1) \to ....$$

in $\mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$. And $R^jF(A)$ is the j-th homology of the above. sequence[1]. But if we want to work in the category of chain complexes, we must give a a meaning to $RF(A^\bullet)$ for a complex $A^\bullet$.

REMARK 8.9. The idea of the total complex of a double complex has an important consequence: we will never have to consider triple, quadruple or more complicated complexes, since these can all eventually be reduced to usual complexes.

DEFINITION 8.10. Assume $\mathscr{C}$ is a category with enough injectives. The derived category of $\mathscr{C}$, denoted $D^b(\mathscr{C})$ is defined as $\mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$. The functor $D : \mathbf{Chain^b}(\mathscr{C}) \to D^b(\mathscr{C})$ is the map associating to $\mathscr{F}^\bullet$ the total complex of a Cartan-Eilenberg resolution of $\mathscr{F}^\bullet$.

REMARKS 8.11.
(1) The category $D^b(\mathscr{C})$ has the following fundamental property. Let F be a functor from $\mathbf{Chain^b}(\mathscr{C})$ to a category $\mathscr{D}$, which sends quasi-isomorphisms to isomorphisms, then F can be factored through $D^b(\mathscr{C})$: there is a functor $G : D^b(\mathscr{C}) \to \mathscr{D}$ such that $F = G \circ D$.
(2) We need to choose for each complex, a Cartan-Eilenberg resolution of it, and the functor $D : \mathbf{Chain^b}(\mathscr{C}) \to D^b(\mathscr{C})$ depends on this choice. However, chosing for each complex a resolution yields a functor, and any two functors obtained in such a way are isomorphic (I would hope...).
(3) Note that if $A^\bullet, B^\bullet$ are objects in $\mathbf{Chain^b}(\mathscr{C})$, then the morphisms from $A^\bullet$ to $B^\bullet$ do not coincide (but are contained in) the set of morphisms from $A^\bullet$ to $B^\bullet$ considered as objects in the derived category. In other words the functor D is not a full functor. For example, if A is an object, identified with $0 \to A \to 0$, there is no nonzero morphism from A to $A[1]$ in the category of chain complexes, but there can be a morphism between their image in the derived category, because their injective representatives do not need to be concentrated in a single degree. However there is no morphism in the derived category from A to $A[-1]$ (see exercise 7.37 of Chapter 8).
(4) We shall indifferently say, for two objects in the derived category, that they are quasi-isomorphic or isomorphic. The former is mostly used when we pay attention to a specific complex representing an object in the derived category.

DEFINITION 8.12. Assume $\mathscr{C}$ is a category with enough injectives, and $D^b(\mathscr{C}) = \mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$ its derived category. Let F be a left-exact functor. Then the **right-derived**

[1]The notation does not convey the idea that information is lost from $RF(A)$ to $R^jF(A)$, as always when taking homology.

functor of F, $RF : D^b(\mathscr{C}) \to D^b(\mathscr{D})$ is obtained by associating to $A^\bullet$ the image by F of the total complex of a Cartan-Eilenberg resolution of $A^\bullet$.

Note that Proposition 8.6 (2) shows that $RF(A^\bullet)$ does not depend on the choice of the Cartan-Eilenberg resolution. Most of the time, we only compute $RF(A)$ for an element A in $\mathscr{C}$. For this take an injective resolution of A.

Examples:

(1) Let $\mathscr{F}^\bullet$ be a complex of sheaves. Then, $H^m(X,\mathscr{F}^\bullet)$ is defined as follows: apply Γ_X to a Cartan-Eilenberg resolution of $F^\bullet$, and take the cohomology. In other words, $H^m(X,\mathscr{F}^\bullet) = (R^m\Gamma_X)(\mathscr{F}^\bullet)$. As we pointed out before, this is different from $H^m(X,\mathscr{F}^q)$. But we shall see that there is a spectral sequence with $E_2 = H^p(X,\mathscr{F}^q)$ (resp. $E_2^{p,q} = H^p(X,\mathscr{H}^q(\mathscr{F}^\bullet))$) converging to $H^{p+q}(X,\mathscr{F}^\bullet)$.

(2) Computing Tor. Let M be an R-module, and $0 \to M \to I_1 \to I_2 \to ....$ be an injective resolution. Let F be the $\otimes_R N$ functor, then $R^jF(M) = Tor^j(M,N)$ is the j-th cohomology of $RF(M)$ given by $0 \to F(I_1) \to F(I_2) \to F(I_3) \to ....$ For example the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ has the resolution

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z} \xrightarrow{g} \mathbb{Q}/\mathbb{Z} \to 0$$

where the map f sends 1 to $\frac{1}{2}$ and $g(x) = 2x$. Then $Tor(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})$ is the complex $0 \to \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \xrightarrow{\bar{g}} \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \to 0$. This is isomorphic to $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \to 0$, so that $Tor^0(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $Tor^1(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, while $Tor^k(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = 0$ for $k \geq 2$. However this is usually done using projective resolutions, which cannot be done for sheaves, since they do not have enough projectives:

we start from

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

which yields

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Finally the notion of spectral sequence allows us to replace injective resolutions by acyclic ones, as we already proved in corollary 7.35:

PROPOSITION 8.13. *Let* $0 \to A \to B^1 \to B^2 \to B^3 \to ...$ *be a resolution by* F*-acyclic objects, that is* $R^jF(B^m) = 0$ *for all* $j \geq 1$ *and all* m*. Then* $RF(A)$ *is quasi-isomorphic to* $0 \to F(B^1) \to F(B^2) \to F(B^3) \to ...$

PROOF. The proposition tells us that injective resolutions are not necessary to compute derived functors: F-acyclic ones are sufficient. Indeed we saw that $0 \to A \to 0$ is quasi-isomorphic to

$$0 \longrightarrow B^1 \xrightarrow{\partial_1} B^2 \xrightarrow{\partial_2} B^3 \xrightarrow{\partial_3} ...$$

To compute the image by RF of this last complex, we use again the Cartan-Eilenberg resolution of the above exact sequence.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^1 \arrow[r, "i_1"] \arrow[d, "\partial_1"] & I_1^0 \arrow[r, "\delta_0^1"] \arrow[d, "\partial_1^0"] & I_1^1 \arrow[r, "\delta_1^1"] \arrow[d, "\partial_1^1"] & I_2^1 \arrow[r, "\delta_2^1"] \arrow[d, "\partial_1^2"] & \ldots \\
0 \arrow[r] & B^2 \arrow[r, "i_2"] \arrow[d, "\partial_2"] & I_0^2 \arrow[r, "\delta_0^2"] \arrow[d, "\partial_2^0"] & I_1^2 \arrow[r, "\delta_1^2"] \arrow[d, "\partial_2^1"] & I_2^2 \arrow[r, "\delta_2^2"] \arrow[d, "\partial_2^2"] & \ldots \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

We must then apply F to the above diagram, and we must compute the cohomology of the total complex obtained by removing the column containing the B^j. But by assumptions the horizontal lines remain exact, since the B^j are F-acyclic, while

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F(B^1) \arrow[r, "i_1"] \arrow[d, "\partial_1"] & F(I_1^0) \arrow[r, "\delta_0^1"] \arrow[d, "\partial_1^0"] & F(I_1^1) \arrow[r, "\delta_1^1"] \arrow[d, "\partial_1^1"] & F(I_2^1) \arrow[r, "\delta_2^1"] \arrow[d, "\partial_1^2"] & \ldots \\
0 \arrow[r] & F(B^2) \arrow[r, "i_2"] \arrow[d, "\partial_2"] & F(I_0^2) \arrow[r, "\delta_0^2"] \arrow[d, "\partial_2^0"] & F(I_1^2) \arrow[r, "\delta_1^2"] \arrow[d, "\partial_2^1"] & F(I_2^2) \arrow[r, "\delta_2^2"] \arrow[d, "\partial_2^2"] & \ldots \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

Since the horizontal lines remain exact by assumption, using Tic-Tac-Toe, we can represent any cohomology class of the total complex $F(Tot(I^{p,q}))$ by a closed element in $F(B^{p+q})$. $\square$

2. Spectral sequences of a bicomplex. Grothendieck and Leray-Serre spectral sequences

Apart from simple situations, we cannot apply the Tic-Tac-Toe lemma to a general bicomplex. However one should hope to recover at least **some** information on total cohomology, from the homology of lines and columns.

Let us start with the algebraic study. Let $(K^{p,q},\partial,\delta)$ be a double (or bigraded)complex. In other words, $\partial_{p,q} : K^{p,q} \to K^{p,q+1}$ and $\delta_{p,q} : K^{p,q} \to K^{p+1,q}$ each define a complex. We moreover assume that ∂ and δ commute. This yields a third chain complex, called the total complex, given by $Tot(K^{\bullet,\bullet})^m = \oplus_{p+q=m} K^{q,p}$ and $D_m = \sum_{p+q=m} \partial^{p,q} + (-1)^p \delta^{p,q}$.

DEFINITION 8.14. A spectral sequence is a sequence of bigraded complexes $(E_r^{p,q}, d_r^{p,q})$, such that $d_r^2 = 0$, $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$, such that $E_{r+1}^{p,q} = H(E_r^{p,q}, d_r^{p,q})$. The spectral sequence is said to converge to a graded complex F^p endowed with a homogeneous increasing filtration F_m, if for r large enough, $F^p_m/F^p_{m-1} = E_r^{m,p-m}$.

Remember that the filtration F_m is homogeneous if $F_m = \bigoplus_p (F_m \cap F^p)$. Note that our definition of a converging spectral sequence is not the most general definition, since convergence could be reached in infinite time. This will not happen in our situation, as long as we stick with bounded complexes (and bounded resolutions). Note that the map ∂ obviously induces a boundary map on $H_\delta^{p,q}(K^{\bullet,\bullet}) = H^{p,q}(K^{\bullet,\bullet},\delta) \to H_\delta^{p+1,q}(K^{\bullet,\bullet})$.

THEOREM 8.15 (Spectral sequence of a total complex). *There is a spectral sequence from $H_\partial H_\delta(K^{\bullet,\bullet})$ converging to $H^{p+q}(Tot(K^{\bullet,\bullet}))$.*

REMARK 8.16. Of couse the differentials ∂ and δ play symmetric roles, so there is also a spectral sequence from $H_\delta H_\partial(K^{\bullet,\bullet})$ converging to the same limit, $H^{p+q}(Tot(K^{\bullet,\bullet}))$. In general the two different spectral sequences give different informations.

PROOF. Our proof spells out in detail some of the ideas found in [**B-T**] and is also inspired by Vakil's notes ([**Vak**], p.62 ff.). We added the obvious idea, that a spectral sequence is an approximation scheme to the total cohomology, in pretty much the same way as numerical schemes (Newton method, Runge-Kutta, finite element method, etc...) approximate solutions of ordinary equations.

For simplicity we assume $K^{p,q} = 0$ for p or q nonpositive. This is called a first-quadrant spectral sequence. We set $K^n_p = \sum_{l\geq p} K^{p,n-p}$. For us an element will be "small", if it belongs to $K^{p,q}$ for p large. This is in fact more obvious on the filtration, small elements are those in K^n_p for p large.

Then a cohomology class in $H^m(Tot(K^{\bullet,\bullet}), d = \partial+\overline{\delta})$ is just a sequence $\mathbf{x} = (x_0,\dots,x_m)$ of elements in $K^{p,m-p}$ such that $\partial x_0 = 0$ and $\overline{\delta} x_j + \partial x_{j+1} = 0$ for $1 \leq j \leq m-1$ and finally $\overline{\delta} x_m = 0$. This is represented by the **zig-zag**, where the zeros indicate that the sum of the images of the arrows abutting there is zero.

$$\begin{tikzcd}
0 & & & & \\
x_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & x_m \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0
\end{tikzcd}$$

FIGURE 1. $\mathbf{x} = x_0 + ... + x_m$ a cocycle in $Tot(K^{\bullet,\bullet})^m$

This is well defined modulo addition of coboundaries, that correspond to sequences $(y_0, ..., y_{m-1})$, such that $x_0 = \partial y_0, x_j = \overline{\delta} y_j + \partial y_{j+1}, \overline{\delta} y_{m-1} = x_m$, that is represented as follows

$$\begin{tikzcd}
x_0 & & & & \\
y_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_1 & & & \\
& y_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_2 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_m & \\
& & & y_{m-1} \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_m
\end{tikzcd}$$

FIGURE 2. $\mathbf{y} = y_0 + ... + y_{m-1}$ and $\mathbf{x} = x_0 + ... + x_m$ is the D-coboundary of $\mathbf{y}$

Note that a number of x_j could be 0, but unwritten x_j are always zero.
The main ideas of the spectral sequence, are

(1) That $H^n(K)$ can be approximated by the images $F^p(H^n(K))$ of the $H^n(K_p)$ under the inclusions $K^n_p \subset K^n$.
(2) that a zig-zag as in Figure 1 can be approximated by "truncated zig-zags of length r", so we have an approximation $S^{p,n-p}_r$ of $H^n(K_p)$. We set $S^{p,n-p}_r / (\mathrm{Im}(d) + S^{p+1,n-p-1}_{r-1})$

(3) More precisely the filtration $F^p(H^n(K))$ is such that

$$F^p(H^n(K))\big/F^{p+1}(H^n(K)) = E_\infty^{p,n-p}$$

(4) that we can compute $E_r^{p,q}$ as the cohomology of $(E_{r-1}^{p,q}, d_r)$. This is not obvious when we make the approximation scheme, but a crucial tool in applications.

Replace the $K^{p,q}$ by $E_r^{p,q}$ as follows:
the space $H_r^{p,q}$ is a quotient of the set $C_r^{p,q}$ of sequences $\mathbf{x} = x_0 + ... + x_{r-2}$ such that

(1) $x_j \in K^{p-j,q+j}$
(2) $\partial x_0 = 0$ and $\overline{\delta} x_j + \partial x_{j+1} = 0$ for $j \geq 1$
(3) there exists x_{r-1} satisfying $-\overline{\delta} x_{r-2} = \partial x_{r-1}$

It will be convenient to use the notation $\mathbf{x} = x_0 + ... + x_{r-2} + (x_{r-1})$, where the parenthesis mean that only the existence of x_{r-1} matters and not its value[2]. Another possible notation would be to replace x_{r-1} by $x_{r-1} + \ker(\partial) \cap K^{p-r+1,q+r-1}$ (so that (x_{r-1}) designates an element in $K^{p-r+1,q+r-1}/\mathrm{Ker}(\partial))$. An element of $S_r^{p,q}$ is thus represented by the **zig-zag**

$$\begin{tikzcd}
0 & & & & \\
x_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & x_{r-2} \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & & (x_{r-1}) \arrow[u, "\partial"]
\end{tikzcd}$$

FIGURE 3. An element $\mathbf{x} = x_0 + ... + x_{r-2} + (x_{r-1})$ in $S_r^{p,q}$

[2]For example for any z such that $\partial z = 0$, we have $(0, ..., 0, 0) = (0, ..., 0, z)$

$$
\begin{tikzcd}
x_0 & & & & \\
y_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_1 & & & \\
& y_2 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_2 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_{r-2} & \\
& & & y_{r-1} \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & (x_{r-1})
\end{tikzcd}
$$

FIGURE 4. The element $\mathbf{x} = x_0 + ... + x_{r-2} + (x_{r-1})$ is in $S_r^{p,q} \cap \mathrm{Im}(d)$ as it is the D-boundary of $\mathbf{y}$.

So we look at elements with boundary not zero, but of r orders of magnitude smaller than x_0. And we quotient these by the set of coboundaries that are at most r orders of magnitude larger, so that $y_0 \in K^{p,q}$ Note that one or more of the x_j could be taken equal to 0 (and that all unwritten elements are assumed to be zeros). The following is now obvious

LEMMA 8.17. *The module* $F_r^{p,n-p} = S^{p,n-p}/\mathrm{Im}(D)$ *is such that* $F_\infty^{p,n-p} = F^p H^n(K)$.

PROOF. This is obvious, since as r increases (in fact $r > p$ is enough), we have that $\mathbf{x}$ is actually closed. So $F_\infty^{p,n-p}$ will be the set of closed elements having a representative in K_p, modulo the boundaries, and this is exactly $F^p H^n(K)$. $\square$

We now set

DEFINITION 8.18. Set $E_r^{p,q} = F_r^{p,q} / F_r^{p+1,q-1}$. Then if $Z_r^{p,q}$ is the set of leading terms (i.e. the set of $x_0 \in K^{p,q}$ where $\mathbf{x} = x_0 + ... + x_{r-2} + (x_{r-1})$) of elements in $S_r^{p,q}$ and $B_r^{p,q}$ is the set of leading terms of boundaries in $S_r^{p,q}$.

Then $E_r^{p,q}$ is defined as the quotient of $Z_r^{p,q}$ by the subgroup $B_r^{p,q}$ of $Z_r^{p,q}$ of elements of the type $D(y_0 + ... + y_{r-1})$ represented as above. Again we do not worry about the value of $\overline{\delta} y_{r-1}$. We denote by $E_r^{p,q}$ the set of such equivalence classes of objects obtained with $x_0 \in K^{p,q}$.

Clearly a cohomology class of the total complex, yields by truncation, a class in $E_r^{p,q}$, and it is clear that for r large enough (namely $r \geq \min\{p,q\}$), an element of $E_r^{p,q}$ is nothing else than a D-cohomology class.

Our claim is that there is a differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $E_{r+1}^{p,q}$ is the cohomology of $(E_r^{p,q}, d_r)$. Let us first study the space $E_r^{p,q}$ for small values of r. We set, $E_0^{p,q} = K^{p,q}$ and easily check that for $r = 1$, $E_1^{p,q} = H^{p,q}(K^{\bullet,\bullet}, \partial)$. Then to x_0 such that $\partial x_0 = 0$ we associate $D(x_0 + x_1) = -\delta x_0$. This yields a map $d_1 : H^{p,q}(K^{\bullet,\bullet}, \partial) \to H^{p,q+1}(K^{\bullet,\bullet}, \partial)$, and for the class of x_0 to be in the kernel of this map, means that $\overline{\delta} x_1 \in \operatorname{Im}(\partial)$ so there exists x_1 such that $\partial x_1 = -\overline{\delta} x_0$, and so we may associate to it the element in $Z_2^{p,q}$

$$\begin{tikzcd}
0 & \\
x_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
 & (x_1) \arrow[u, "\partial"]
\end{tikzcd}$$

FIGURE 5. An element in $Z_2^{p,q}$.

the parenthesis around x_1 means, as usual, that the choice of x_1 is not part of the data defining the element, only its existence matters.

Then for any choice of x_1 as above, the element $\mathbf{x} = x_0 + (x_1)$ vanishes in $E_2^{p,q}$ if and only if there exists $\mathbf{y} = y_0 + y_1$ such that $\partial y_0 = 0, x_0 = \overline{\delta} y_0 + \partial y_1$, and $\overline{\delta} y_1 = x_1$ (note that this last equality can be taken as the choice of x_1 which automatically satisfies $\partial x_1 = -\overline{\delta} x_0$). This is clearly the definition of an element in $H_\delta^q H_\partial^p(K^{\bullet,\bullet})$, so we may indeed identify E_2 with the cohomology of (E_1, d_1), that is $H_\delta^q H_\partial^p(K^{\bullet,\bullet})$. The map d_2 is then defined as the class of $\overline{\delta} x_1$.

In the general case, we define the map d_r as follows. For the sequence $(x_0, ..., x_{r-1})$ we define its image by d_r to be the class of $-\overline{\delta} x_{r-1}$ in $E_r^{p+r,q-r+1}$. Note that x_{r-1} is only defined up to an element z in the kernel of ∂, but $\overline{\delta} x_{r-1}$ is well defined in $E_r^{p-r,q+r}$, since $\overline{\delta} z = D(z)$.

Clearly $\overline{\delta} x_{r-1} \in K^{p-r,q+r}$. We have to prove on one hand that if $\overline{\delta} x_{r-1}$ is zero (in the quotient space $E_r^{p-r,q+r+1}$) we may associate to $\mathbf{x}$ an element in $E_{r+1}^{p,q}$, and that this map is an isomorphism. Clearly if $\overline{\delta} x_{r-1} = 0$ in $E_r^{p-r,q+r}$ (not in $K^{p-r,q+r}$), this means we have the following two diagrams

$$
\begin{tikzcd}
0 & & & & \\
x_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & (x_{r-1}) \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & u
\end{tikzcd}
$$

FIGURE 6. The class u represents $d_r(\mathbf{x})$

$$
\begin{tikzcd}
0 & & & & & \\
y_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & u & & & & \\
& \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& & y_j \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
& & & y_{j+1} \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & & & y_{r-2} \arrow[u, "\partial"]
\end{tikzcd}
$$

FIGURE 7. Representing the vanishing of $d_r\mathbf{x}$ in $E_r^{p+r,q-r+1}$: a class $\mathbf{y}$ such that $D(\mathbf{y}) = u$.

Now claiming that u vanishes in the quotient $E_r^{p-r,q+r+1}$, means that we have a diagram of the above type In particular in the above case, u is not in the image of ∂, but of the form $\overline{\delta} y_j + \partial y_{j+1}$ with $\partial y_1 = 0$. Then the following sequence represents an element in $Z_{r+1}^{p,q}$:

$$
\begin{tikzcd}
0 & & & & \\
x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& x_2 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & x_{r-1} - y_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & & (-y_1) \arrow[u, "\partial"]
\end{tikzcd}
$$

FIGURE 8. How to make $\mathbf{x}$ into an element of $E_{r+1}^{p,q}$ assuming $d_r\mathbf{x} = 0$ in $E_r^{p+r,q-r+1}$.

However by substracting from $\mathbf{x}$ the above coboundary, as on Figure 8 we can get a strictly longer sequence, and then we get an element of $E_{r+1}^{p,q}$. Conversely, it is easy to see that an element in $E_{r+1}^{p,q}$ corresponds by truncation to an element $\mathbf{x}$ in $E_r^{p,q}$ with $d_r(\mathbf{x}) = 0$. □

REMARK 8.19. Because ∂ and δ play symmetric roles, there is also a spectral sequence from $H_\delta H_\partial(K^{\bullet,\bullet})$ converging to $H^{p+q}(Tot(K^{\bullet,\bullet}))$. This is often very useful in applications.

EXERCICE 8.20.
(1) Using spectral sequences reprove the snake lemma (Lemma 6.22) and the five lemma (Lemma 6.23). Prove that a short exact sequence of complexes yields a long exact sequence in homology.
(2) Prove that we may replace $D = d + \delta$ by $D_\varepsilon = d + \varepsilon\delta$ where ε is considered very small. Show that the spectral sequence is indeed an approximation scheme : we look for elements such that $D_\varepsilon(x) = O(\varepsilon^r)$....

PROPOSITION 8.21 (The canonical spectral sequence of a derived functor). *Let $A^\bullet \in$* **Chain**$(\mathscr{C})$*, and F a left-exact functor. Then there are two spectral sequences with respectively $E_2^{p,q} = H^p(R^qF(A))$ and $E_2^{p,q} = R^pF(H^q(A))$, converging to $R^{p+q}F(A)$.*

PROOF. Consider a Cartan-Eilenberg resolution of $A^\bullet$, and denote it by $(I^{p,q},\partial,\delta)$. Then, consider the complex $(F(I^{p,q}),F(\partial),F(\delta))$. By definition $R^mF(A^\bullet)$ is the cohomology of $(Tot(F(I^{p,q}),F(d))$. Now $H^q_\delta(F(I^{p,q}))=R^qF(A^\bullet)$, since the lines are injective resolutions of A^p, and so the cohomology of each line is $R^qF(A^\bullet)$. Thus the first spectral sequence has $E_2^{p,q}=H^p_\partial H^q_\delta(F(I^{\bullet,\bullet}))=H^p_\partial(R^qF(A^\bullet))$. Now consider the other spectral sequence. We must first compute $H_\partial(F(I^{p,q}))$. But by our assumptions the columns are injective, and have ∂ homology giving an injective resolution of $H^q(A^\bullet)$, so applying F and taking the δ cohomology, we get $R^pF(H^q(A))$. □

COROLLARY 8.22. *There is a spectral sequence with E_2 term $H^p(X,\mathscr{H}^q(\mathscr{F}^\bullet))$ and converging to $H^{p+q}(X,\mathscr{F}^\bullet)$. Similarly there is a spectral sequence from $E_2^{p,q}=H^p(X,\mathscr{F}^q)$ converging to $H^{p+q}(X,\mathscr{F}^\bullet)$.*

PROOF. Apply the above to the left-exact functor on **Sheaf (X)**, $F(\mathscr{F})=\Gamma(X,\bullet)$. □

The following result is often useful:

PROPOSITION 8.23 (Comparison theorem for spectral sequences). *Let $A^\bullet,B^\bullet$ be two objects in* **Chain**$(\mathscr{C})$*, $f^\bullet: A^\bullet\to B^\bullet$ a chain morphism. Let F be a left-exact functor, and assume that the map induced by $f^\bullet$ from $H^p(R^qF(A))$ to $H^p(R^qF(B))$ is an isomorphism. Then the induced map $RF(A)\to RF(B)$ is also an isomorphism.*

PROOF. This is a direct consequence of the fact that $f^\bullet$ induces a map $f_r: E_r^{p,q}(A)\longrightarrow E_r^{p,q}(B)$ which is a chain map for d_r., and the 5-lemma.

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cf [**Weib**] ?

□

Besides the above canonical spectral sequence, the simplest example of a spectral sequence is the following topological theorem, constructing the cohomology of the total space of a fibre bundle from the cohomology of the base and fiber. Indeed,

THEOREM 8.24 (Leray-Serre spectral sequence). *Let $\pi: E\to B$ be a smooth fibre bundle. Then there exists a spectral sequence with E_2 term $H^*(B,\mathscr{H}^q(F_x))$ and converging to $H^{p+q}(E)$.*

For the proof see page 118. Note that $\mathscr{H}^q(F_x)$ is a locally constant sheaf, i.e. local coefficients, with stalk $H^*(F)$, since $\mathscr{H}^q(\pi^{-1}(U))\simeq H^q(U\times F)=H^q(F)$ for U small enough and contractible. In particular when B is simply connected, and we take coefficients in a field, $H^*(B,\mathscr{H}^q(F_x))=H^*(B)\otimes H^*(F)$.

At the level of derived categories, this is even simpler. Let G be a left-exact functors, from $\mathscr{C}$ to $\mathscr{D}$ and F a left-exact functor from $\mathscr{D}$ to $\mathscr{E}$. We are interested in the derived functor $R(F\circ G)$

THEOREM 8.25 (Grothendieck's spectral sequence). *Assume the category $\mathscr{C}$ has enough injectives, and G transforms injectives into F-acyclic objects (i.e. such that $R^jF(A)=0$ for $j\geq 1$). Then*

$$R(F\circ G)=RF\circ RG$$

PROOF. Let $I^\bullet$ be an injective resolution of A. Then $G(I^\bullet)$ is a complex representing $RG(A)$. Since this is F-acyclic, it can be used to compute $RF(RG(A))$, and this is then represented by $FG(I^\bullet)$. But obviously this represents $R(F\circ G)(A)$. ☐

Note that this theorem could not be formulated if we only have the R^jF without derived categories, as was the case before Grothendieck and Verdier. Indeed, if we only know the R^jF there is no way of composing derived functors. The Grothendieck spectral sequences has the following important application:

THEOREM 8.26 (Cohomological Fubini theorem). *Let $f : X \to Y$ be a continuous map between compact spaces. Then , we have $R\Gamma(X,\mathscr{F}) = R\Gamma(Y, Rf_*(\mathscr{F}))$ hence, taking cohomology, $H^*(X,\mathscr{F}) = H^*(Y, Rf_*\mathscr{F})$.*

PROOF. Apply Grothendieck's theorem to $G = f_*$ and $F = \Gamma(Y,\bullet)$, use the fact that $\Gamma(X,\bullet) = \Gamma(Y,\bullet)\circ f_*$, and remember that $H^j(X,\mathscr{F}) = R^j\Gamma(X,\mathscr{F})$. We still have to check that f_* sends injective sheaves to $\Gamma(Y,\bullet)$ acyclic objects, but this is a consequence of corollary 7.26. The second statement follows from the first by taking homology. ☐

REMARKS 8.27.
(1) If there is a cofinal sequence of neighborhoods such that $\Gamma(f^{-1}(V),\mathscr{F}^\bullet)$ is acyclic, then $Rf_*(\mathscr{F}^\bullet)$ is acyclic, and $H^*(Y, Rf_*\mathscr{F}) = 0$. We thus get the Vietoris-Begle theorem. The Grothendieck spectral sequence looks like "three card monty" trick: there is no apparent spectral sequence, and the proof is essentially trivial. So what ? See the next theorem for an explanantion.
(2) Note that a priori we have not defined the cohomology of a an object in the derived category of sheaves. This does not even fall in the framework of sheaves with values in an abelian category, since the derived category is not abelian. However, $R\Gamma(X,) : D^b(\mathbf{Sheaf(X)}) \to D^b(\mathbf{Ab})$. Now taking homology does not lose anything, because any complex of abelian groups is quasi-isomorphic to its homology, since the category of abelian groups has homological dimension 1 ([**?**]). This fails for general modules, so in general, $R\Gamma(X, Rf_*(\mathscr{F}))$ is only defined in $D^b(\mathbf{R-mod})$, which is not well understood, except that any element has a well defined homology, so $R^p\Gamma(X,(Rf_*))$ is well defined. Need refer
(3) If c is the constant map, we get $H^*(X,\mathscr{F}^\bullet) = H^*(\{pt\},(Rc)_*(\mathscr{F}^\bullet))$, but $(Rc)_*(\mathscr{F}^\bullet)$ is a complex of sheaves over a point, that is just an ordinary complex. We thus associate a complex in $D^b(\mathbf{R-mod})$ to the cohomology of X with coefficients in $\mathscr{F}^\bullet$.
(4) If f is a diffeomorphism, then Rf_* and f^{-1} are inverse functors. This follows from the Grothendieck spectral sequence, but in a more elementary way, if $\mathscr{I}^\bullet$ is an Eilenberg-Cartan resolution of $\mathscr{F}^\bullet$ we have $f_*(\mathscr{I}^\bullet)$ is an injective complex, so $f^{-1}\circ Rf_*(\mathscr{F}^\bullet)$ is represented by $f^{-1}\circ f_*(\mathscr{I}^\bullet) = \mathscr{I}^\bullet$. Similarly using that

f^{-1} is exact (Proposition 7.21) and the fact that an exact functor sends an injective complex to an acyclic one (Lemma 7.35) and finally the fact that the derived functor can be computed using an acyclic resolution (Corollary 7.36)

Example: Let us consider the functor Γ_Z, then by Grothendieck's theorem, $\Gamma_Z(X,\mathscr{F}) = \Gamma(X,\Gamma_Z(\mathscr{F}))$ so that $R\Gamma_Z(X,\mathscr{F}) = R\Gamma(X,R\Gamma_Z(\mathscr{F}))$.

THEOREM 8.28 (Grothendieck's spectral sequence-cohomological version). *Under the assumptions of theorem 8.25, there is a spectral sequence from $E_2^{p,q} = R^pF \circ R^qG$ to $R^{p+q}(F\circ G)$.*

PROOF. Let $I^\bullet$ be an injective resolution of A, and consider $C^\bullet = G(I^\bullet)$.

Then one of the canonical spectral sequence of theorem 8.21 applied to RF and $C^\bullet$, has $E_2^{p,q}$ given by $R^pF(H^q(C^\bullet))$ and converges to $R^{p+q}F(C^\bullet)$. But since $H^q(C^\bullet) = R^qG(A)$ by definition, we get that this spectral sequence is $R^pF(R^qG(A))$, and converges to $R^{p+q}F(G(I^\bullet))$ that is the $p+q$ cohomology of $RF(G(I^\bullet)) = RF\circ RG(A)$. But we saw that $RF\circ RG(A) = R(F\circ G)(A)$, so the spectral sequence converges to $R^{p+q}(F\circ G)(A)$. □

Ideally, one should never have to construct a spectral sequence directly, any spectral sequence should be obtained from the Grothendieck's spectral sequence for some suitable pair fo functors F, G.

EXERCICE 8.29. Let F_1, F_2 be functors such that we have an isomorphism $RF_1 = RF_2$ on elements of $\mathscr{C}$. Then $RF_1 = RF_2$ on the derived category.

PROOF OF LERAY-SERRE. Let us see how this implies the Leray spectral sequence: take $\mathscr{C}$ =**Sheaves(X)**, $\mathscr{D}$ =**Sheaves(Y)**, $\mathscr{E}$ =**Ab** and $F = f_*$, $G = \Gamma_Y$. Since $\Gamma_Y \circ f_* = \Gamma_X$, we get $R\Gamma_X = R\Gamma_Y \circ Rf_*$, since f_* sends injectives to injectives (because f_* has an adjoint f^{-1}, see Proposition 7.22). So we get a spectral sequence $E_2^{p,q} = H^p(Y, R^qf_*(\mathscr{F}))$ to $H^{p+q}(X,\mathscr{F})$ is the sheaf associated to the presheaf $H^q(f^{-1}(U))$. If f is a fibration, this is a constant sheaf. Moreover the sheaf $R^qf_*(\mathscr{F})$ has stalk $\lim_{x\in U} H^q(f^{-1}(U))$ which is equal to $H^q(f^{-1}(x))$ if f is a fibration such that the $f^{-1}(U)$ form a fundamental basis of neighbourhoods of $f^{-1}(x)$. □

EXERCICE 8.30 (Čech cohomology for acyclic covers equals sheaf cohomology). Prove that if $\mathfrak{U}$ is a covering of X such that for all q and all sequences $(i_0, i_1, ..., i_q)$, we have $H^j(U_{i_0}\cap .... \cap U_{i_q},\mathscr{F}) = 0$ for $j \geq 1$, then the cohomology of the Čech complex(see 3.3, page 95), $\mathscr{C}(\mathscr{U},\mathscr{F})$ coincides with $H^*(X,\mathscr{F})$. Hint: consider an injective resolution of $\mathscr{F}$, $0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to ...$ and the double complex having as rows the Čech resolution of $\mathscr{I}^p$. Indeed, $K^{p,q} = \check{C}^q(\mathfrak{U},\mathscr{I}^p)$ and the map $\partial : K^{p,q} \longrightarrow K^{p+1,q}$ is induced by $\partial : \mathscr{I}^p \longrightarrow \mathscr{I}^{p+1}$ while δ is induced by the Čech differential. Now taking the ∂ differential of $K^{p,q}$ we get $\oplus_{(i_0,i_1,..i_q)} H^p(U_{i_0}\cap ... \cap U_{i_q},\mathscr{F})$ which vanishes for $p \neq 0$ by assumption, hence $H_\partial(K^{p,q}) = \sum_{i_0,...,i_q} \Gamma(U_{i_0}\cap ... \cap U_{i_q},\mathscr{F})$ while taking the δ cohomology of this yields the cohomology of the Čech complex. Conversely, taking first the δ cohomology of the double complex, we get using the following lemma, that it is nonzero only for $p = 0$ in which case it is equal to $\Gamma(X, \mathscr{I}^p)$ and taking the cohomology of the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{I}^0) \longrightarrow \Gamma(X, \mathscr{I}^1) \longrightarrow .... \longrightarrow \Gamma(X, \mathscr{I}^q) \longrightarrow$$

we get the cohomology of $\mathscr{F}$. By the remark following Theorem 8.15 these two spectral sequences have the same limit that is the cohomology of the total complex, and this proves our result.

LEMMA 8.31. *Let $\mathscr{I}$ be an injective sheaf on X. Then the Čech-cohomology, $\check{H}^j(\mathfrak{U}, \mathscr{I}) = 0$ for all $j \geq 1$ and $\check{H}^0(\mathfrak{U}, \mathscr{I}) = \Gamma(X, \mathscr{I})$*

PROOF. Since $\mathscr{I}$ is injective, its restriction to $U_{i_0} \cap ... \cap U_{i_q}$ is injective, hence $V \mapsto \Gamma(U_{i_0} \cap ... \cap U_{i_q} \cap V)$ is also an injective sheaf, and since the direct sum of injective sheaves is injective, we have that the Čech-resolution is an injective resolution, hence its cohomology coincides with the cohomology of $\mathscr{I}$, and since an injective sheaf is acyclic, this concludes the proof. □

3. Complements on functors and useful properties on the Derived category

3.1. Derived functors of operations and some useful properties of Derived functors. Consider the operations $\mathscr{H}om, \otimes, f_*, f^{-1}$. The operations f^{-1} is exact, so it is its own derived functor. The functor f_* is left exact, hence has a right-derived functor, Rf_*. The operation $\mathscr{H}om$ is covariant in the second variable and contravariant in the first. Considering it as a functor of the second variable it is left exact, so has a right-derived functor, $R\mathscr{H}om$. Finally the tensor product is right-exact,hence has a left derived functor denoted $\otimes^L$. Note that in the case of $\mathscr{H}om$ and $\otimes$, the symmetry of the functor is not really reflected, since for the moment one of the two factors must be a sheaf and not a chain complex of sheaves. For a satisfactory theory one would have to work with bifunctors, which we shall avoid (see [**K-S**], page 56). In particular we have as a complex of sheaves, $(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet)^r = \sum_{p+q=r} \mathscr{F}^p \otimes \mathscr{G}^q$ and $\mathscr{H}om(\mathscr{F}^\bullet, \mathscr{G}^\bullet)^r = \sum_{p+q=r} \mathscr{H}om(\mathscr{F}^p, \mathscr{G}^q)$.

Again acording to [**K-S**], under suitable assumptions, whether we consider $\mathscr{H}om$ as a bifunctor, or we consider the functor $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, \mathscr{G})$ (resp. $\mathscr{G} \to \mathscr{H}om(\mathscr{F}, \mathscr{G})$), their derived functors coincide.

REMARK 8.32.
(1) Let $\Gamma(\{x\}, \mathscr{F}) = \mathscr{F}_x$. This is an exact functor, since by definition a sequence is exact, if and only if the induced sequence at the stalk level is exact. So $R\Gamma(\{x\}, \mathscr{F}) = \mathscr{F}_x$.
(2) Be careful: there is no equality $\Gamma(\{x\}, \Gamma_Z(\mathscr{F})) = \Gamma(\{x\}, \mathscr{F})$, so we cannot use Grothendieck's theorem 8.25.
(3) As long as we are working over fields, and finite dimensional vector spaces, the tensor product and $\mathscr{H}om$ functors on the category k-**vect** are exact, so they coincide with their derived functors. We shall make this assumption whenever useful.

EXERCICE 8.33. We have for a complex $\mathscr{F}^\bullet$ that $(\mathscr{H}^\bullet(\mathscr{F}^\bullet))_x = H^*(\mathscr{F}^\bullet_x)$. This follows from the exactness of the functor $\mathscr{F} \longrightarrow \mathscr{F}_x$ from **Sheaves(X)** to **R − mod**.

3.2. More on Derived categories and functors and triangulated categories. There is no good notion of exact sequence in a derived category. Of course, the exact sequence of sheaves has a corresponding exact sequence of complexes of their injective resolution as the following extension of the Horseshoe lemma (Lemma 7.49) proves:

PROPOSITION 8.34. *Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be an exact sequence of complexes. There is an exact sequence of injective resolutions $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ and chain maps, a, b, c, which are quasi-isomorphisms*

$$
\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d, "a"] & B^\bullet \arrow[r] \arrow[d, "b"] & C^\bullet \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & I_A^\bullet \arrow[r] & I_B^\bullet \arrow[r] & I_C^\bullet \arrow[r] & 0
\end{tikzcd}
$$

PROOF. Indeed, if the complexes are reduced to single objects, this is just the Horseshoe lemma 7.49 applied to $0 \to A \to B \to C \to 0$. The general case follows from the theorem 8.6, (2), by replacing the double complexes by their total complex. ☐

Is this correct ?
Or only exact up to homotopy ?

EXERCICE 8.35. Let $0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ be an exact sequence of complexes. Let T^n be the mapping cone (see p. 91) of g, that is $T^n = B^n \oplus C^{n-1}, d_n = (\partial_B, \partial_C + (-1)^n g_n)$. Then the map $(f^\bullet, 0) : A^\bullet \longrightarrow T^\bullet$ is a quasi-isomorphism.

However, since the derived category does not have kernels or cokernels, the notion of exact sequence is not well defined. It is replaced by the notion of distinguished triangle, defined as follows.

DEFINITION 8.36. A distinguished triangle is a triangle

$$
\begin{tikzcd}
 & A^\bullet \arrow[dl, squiggly, "v"'] & \\
B^\bullet \arrow[rr, "f"] & & C^\bullet \arrow[ul, "u"']
\end{tikzcd}
$$

isomorphic to a triangle of the form

$$
\begin{tikzcd}
 & C(f)^\bullet \arrow[dl, squiggly, "v"'] & \\
M^\bullet \arrow[rr, "f"] & & N^\bullet \arrow[ul, "u"']
\end{tikzcd}
$$

associated to a map $f : M \to N$.

We now claim that to an exact sequence in $\mathbf{Chain^b}(\mathscr{C})$, we may associate a distinguished triangle in the derived category

Indeed, an exact sequence of injective sheaves $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ is split, so is isomorphic to $0 \to I_A^\bullet \to I_A^\bullet \oplus I_C \to I_C^\bullet \to 0$ and hence isomorphic to the above exact sequence for $M^\bullet = I_C^\bullet[-1]$, $N^\bullet = I_A$ and $f = 0$.

$0 \to I^\bullet_A \to I^\bullet_A \oplus I_C \to I^\bullet_C \to 0$ and this is isomorphic to $0 \to I^\bullet_A \to I^\bullet_B \to I^\bullet_C \to 0$
The following property will be useful in the proof of Proposition 9.3.

PROPOSITION 8.37 ([**Iv**], p.58). *Let F be a left exact functor from $\mathscr{C}$ to $\mathscr{D}$, where $\mathscr{C}, \mathscr{D}$ are categories having enough injectives. Then the functor $RF : D^b(\mathscr{C}) \to D^b(\mathscr{D})$ preserves distinguished triangles.*

PROOF. □

Let now F, G, H be left-exact functors, and λ, μ be transformations of functors from F to G and G to H respectively.

PROPOSITION 8.38 ([**K-S**] prop. 1.8.8, page 52). *Assume for each injective I we have an exact sequence $0 \to F(I) \xrightarrow{\lambda} G(I) \xrightarrow{\lambda} H(I) \to 0$. Then there is a transformation of functors ν and a distinguished triangle*

$$\to RF(A) \xrightarrow{R\lambda} RG(A) \xrightarrow{R\lambda} RH(A) \xrightarrow{\nu} RF(A)[1] \xrightarrow{R\lambda[1]} \ldots$$

PROOF. □

Example: We have an exact sequence $0 \to \Gamma_Z(\mathscr{F}) \to \mathscr{F} \to \mathscr{F}_{X-Z}$ that extends for $\mathscr{F}$ flabby to an exact sequence

$$0 \to \Gamma_Z(\mathscr{F}) \to \mathscr{F} \to \mathscr{F}_{X-Z} \to 0$$

therefore

COROLLARY 8.39. *There is a distinguished triangle*

$$R\Gamma_Z(\mathscr{F}) \to R\Gamma(\mathscr{F}) \to R\Gamma(\mathscr{F}_{X-Z}) \xrightarrow{[+1]} R\Gamma_Z(\mathscr{F})[1] \ldots$$

yielding a cohomology long exact sequence

$$\ldots \to H^j\Gamma_Z(\mathscr{F}) \to H^j(X, \mathscr{F}) \to H^j(X \setminus Z; \mathscr{F}) \to H^{j+1}_Z(\mathscr{F}) \to \ldots$$

REMARK 8.40. For each open U, we may consider $R\Gamma(U, \mathscr{F}^\bullet)$ that is an element in $D^b(\mathrm{R}-mod)$. We would like to put these toghether to make a sheaf. The only obstruction is that this would not be a sheaf in an abelian category, but only in a triangulated category. However, consider an injective resolution of $\mathscr{F}^\bullet$, $\mathscr{I}^\bullet$. Then $\mathscr{I}^\bullet(U)$ represents $R(\Gamma(U, \mathscr{F}^\bullet))$, so that $R\Gamma$ is just the functor associating to $\mathscr{F}^\bullet$ the injective resolution, which is the map so that we may define $R\Gamma(\mathscr{F}^\bullet) = \mathscr{I}^\bullet$ in the derived category, i.e. this is the functor D of Definiton 8.10. Then $R\Gamma(U, \mathscr{F}^\bullet) = R\Gamma(\mathscr{F}^\bullet)(U)$.

3.3. Čech cohomology of complexes of sheaves. Let $\mathscr{F}^\bullet$ be in $\mathbf{Chain^b}(\mathbf{Sheaves}(\mathbf{X}))$. Let $\mathfrak{U}$ be an open cover of X and consider the total complex $T^\bullet$ of the double complex $C^p(\mathfrak{U}, \mathscr{F}^\bullet)$. The the cohomology of T equals the cohomology $H^*(X, \mathscr{F}^\bullet)$ provided $H^p(U_{i_0} \cap \ldots \cap U_q; \mathscr{F}^j)$ for all $p \geq 1$.

3.4. Bifunctors and derived bifunctors. A bifunctor is a functor from the product category $\mathscr{C} \times \mathscr{C}'$ to a category $\mathbf{C}''$. For example $\mathscr{H}om$ and $\otimes$ (that will be sometimes denoted for convenience by T for tensor product) or $\boxtimes$ (that will be sometimes denoted for convenience by ET for external tensor product) are bifunctors from $\mathbf{Sheaves(X)} \times \mathbf{Sheaves(X)}$ to $\mathbf{Sheaves(X)}$ for the first two, and from $\mathbf{Sheaves(X)} \times \mathbf{Sheaves(Y)}$ to $\mathbf{Sheaves(X \times Y)}$ for the last one. The same theory, using the fact that we have enough injectives, and that for each of these bifunctors we can extend to bifunctors from $\mathbf{Chain^b}(\mathbf{Sheaves(X)}) \times \mathbf{Chain^b}(\mathbf{Sheaves(X)})$, where we use the functor

$$Tot : \mathbf{Chain^b}(\mathbf{Sheaves(X)}) \times \mathbf{Chain^b}(\mathbf{Sheaves(Y)}) \longrightarrow \mathbf{Chain^b}(\mathbf{Sheaves(X \times Y)})$$

given by $\mathscr{F}^\bullet \times \mathscr{G}^\bullet \longrightarrow \mathscr{K}^\bullet$ where $\mathscr{K}^n = \sum_{p+q=n} \mathscr{F}^p \times \mathscr{G}^q$. Then if F is a bifunctor, we set $Tot(F)$ the functor induced by $F \circ Tot$, i.e. $Tot(F)(\mathscr{F} \times \mathscr{G}) = F(Tot(\mathscr{F} \times \mathscr{G})$ in other words $Tot(F)(\mathscr{F} \times \mathscr{G})^n = \sum_{p+q=n} F(\mathscr{F}^p \times \mathscr{G}^q)$. We usually denote $Tot(F)$ by F, since there is no real ambiguity. The derived functor of F is then obtained as the derived functor of $Tot(F)$. In other words, replace $\mathscr{F}^\bullet$ by a quasi-isomorphic complex made of injective objects, $\mathscr{I}^\bullet$ and $\mathscr{G}^\bullet$ by $\mathscr{J}^\bullet$. Then $RF(\mathscr{F}^\bullet \times \mathscr{G}^\bullet) = Tot(F)(\mathscr{I}^\bullet \times \mathscr{G}^\bullet)$. For this to be well defined, we need that for any acyclic $\mathscr{F}^\bullet$ and any $\mathscr{G}^\bullet$, the complex $Tot(F)(\mathscr{F}^\bullet \times \mathscr{G}^\bullet)$ is acyclic. The Grothendieck spectral sequence yields the following

PROPOSITION 8.41. *Let $\mathscr{F}^\bullet, \mathscr{G}^\bullet$ in* $\mathbf{Chain^b}(\mathbf{Sheaves(X)})$. *There are spectral sequences from $H^p(X, \mathscr{F}^\bullet) \otimes H^q(Y, \mathscr{G}^\bullet)$ converging to $H^{p+q}(X \times Y, \mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet)$.*

PROOF. Let F be the bifunctor $\mathscr{F} \otimes \mathscr{G} \longrightarrow \mathscr{F} \times \mathscr{G}$ that is an exact functor. For $U \subset X, V \subset Y$ and $\mathscr{I}, \mathscr{J}$ injective sheaves, we have $\Gamma(U \times V, \mathscr{I} \boxtimes \mathscr{J}) = \Gamma(U, \mathscr{I}) \otimes \Gamma(V, \mathscr{J})$, because injective sheaves are fine. Let ET be the external tensor product bifunctor. Then we have $\Gamma_{U \times V} \circ ET(\mathscr{F}^\bullet, \mathscr{G}^\bullet)) = T \circ (\Gamma_U, \Gamma_V)(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$, hence $R\Gamma_{U \times V}(\mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet) = T(\Gamma_U(\mathscr{F}^\bullet), R\Gamma(cG))$, hence taking cohomologies we get the above spectral sequence. $\square$

4. $(\infty, 1)$-category theory

Let $\mathscr{C}$ be a category. Its nerve is the following Simplicial set that is a functor from the category **Simplicial** to the category **Sets** : the vertices $N(\mathscr{C})_0$ of $N(\mathscr{C})$ are the objects of $\mathscr{C}$, the edges $N(\mathscr{C})_1$ correspond to elements of $\mathrm{Mor}_{\mathscr{C}}(X, Y)$, and for $f \in N(\mathscr{C})_1 = \mathrm{Mor}_{\mathscr{C}}(X, Y)$, we have $d_0(f) = X, d_1(f) = Y$, and $s_0(X) = \mathrm{Id}_X$. Now $N(\mathscr{C})_n = \mathrm{Mor}_{sSet}(\Delta^n, N(\mathscr{C}))$ is the set of diagrams

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} ...X_{i-1} \xrightarrow{f_i} X_i \xrightarrow{f_{i+1}} .... \xrightarrow{f_n} X_n$$

The map d_i sends the above to

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} ...X_{i-1} \xrightarrow{f_{i+1} \circ f_i} X_{i+1} \xrightarrow{f_{i+2}} .... \xrightarrow{f_n} X_n$$

while s_i sends it to

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} ...X_{i-1} \xrightarrow{f_i} X_i \xrightarrow{\mathrm{Id}_{X_i}} X_i \xrightarrow{f_{i+1}} ... \xrightarrow{f_n} X_n$$

Finally, given a diagram $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$, the edge of $N(\mathscr{C})$ corresponding to $f_1 \circ f_0$ may be uniquely characterized by the fact that there exists a 2-simplex $\sigma \in N(\mathscr{C})_2$ with $d_2(\sigma) = f_0, d_0(\sigma) = f_1$ and $d_1(\sigma) = f_1 \circ f_0$.

DEFINITION 8.42. Let S be a simplicial set, and $\mathfrak{C}[S]$ be the simplicially enriched category such that objects are the objects of S, and $\mathrm{Mor}_{\mathfrak{C}[S]}(X,Y)_n$ is given by the sequences $(f_1,...,f_n)$ of composable functors $f_j \in \mathrm{Mor}_S(X_{j-1},X_j)$. The d_j^n are defined as above by $d_j^n(f_1,...,f_n) = (f_1,..,\hat{f}_j,...,f_n)$ and $s_j^n : (f_1,...,f_n) = (f_1,..f_{j-1},\mathrm{id},f_{j+1},....,f_n)$

Example: Let Δ^n be the category with objects $j \in \{0,...,n\}$ and morphisms $\mathrm{Mor}_{\Delta^n}(i,j) = q_{i,j}$ for $i \leq j$ and $\emptyset$ otherwise. By uniqueness, note that $q_{j,k} \circ q_{i,j} = q_{i,k}$. Then $\mathfrak{C}[\Delta^n]$ has the same objects as Δ^n, and $\mathrm{Mor}_{\mathfrak{C}[\Delta^n]}(i,j)_k$ is the set of sequences $i = i_0 \leq i_1 \leq ... \leq i_k = j$. In particular if $p_{i,j} \in \mathrm{Mor}_{\mathfrak{C}[\Delta^n]}(i,j)_1$, we don't have in general $p_{i_{r-1},j} \circ .... \circ p_{i_1,i_2} \circ p_{i,i_1} = p_{i,j}$. In fact, $\mathfrak{C}[\Delta^n]$ is the free category generated by the $p_{i,j}$.

DEFINITION 8.43. A homotopy coherent diagram from $\mathscr{C}$ to $\mathscr{D}$ is an ordinary functor

$$F_*(\mathscr{C}) = \mathfrak{C}[N(\mathscr{C})] \longrightarrow \mathscr{D}$$

Let N be a smooth manifold endowed with a triangulation. Let $W \subset V \subset U$, and $r_{U,V} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V), r_{V,W} : \mathscr{F}(V) \longrightarrow \mathscr{F}(W)$. We have $r_{V,W} \circ r_{V,W} \neq r_{U,W}$, but if $\mathscr{C} = \mathbf{Op}(\mathbf{X})^{op}$, we get a functor

$$F_*(\mathbf{Op}(\mathbf{X})^{op}) = \mathfrak{C}[N(\mathbf{Op}(\mathbf{X})^{op})] \longrightarrow \mathbf{Ch^b}$$

For this we must describe the structure of $\mathbf{Ch^b}$ as a simplicially enriched category. We have that

$$\mathrm{Mor}_{s\mathbf{Ch^b}}(A^\bullet,B^\bullet)_k = \mathrm{Mor}_{\mathbf{Ch^b}}(C_*(\Delta^k) \otimes A^\bullet, B^\bullet)$$

Now the theorem that tells us that such a category can be **rectified** tells us that

PROPOSITION 8.44. *Given a functor*

$$\mathscr{F} : F_*(\mathbf{Op}(\mathbf{X})^{op}) \longrightarrow \mathbf{Ch^b}$$

there is a sheaf on X, seen as a functor

$$\widehat{\mathscr{F}} : \mathbf{Op}(\mathbf{X})^{op} \longrightarrow \mathbf{Ch^b}$$

such that if we denote by $\pi : F_*(\mathbf{Op}(\mathbf{X})^{op}) \longrightarrow \mathbf{Op}(\mathbf{X})^{\mathbf{op}}$, then $\mathscr{F}$ is homotopy equivalent to $\widehat{\mathscr{F}} \circ \pi$.

We thus have to prove.

PROPOSITION 8.45. *The map $U \mapsto CF^*(L,\nu^*U)$ and the $r_{U,V}$ define a coherent homotopy from the category $\mathbf{Op}(\mathbf{X})^{op}$ to the category Ch^b. Its rectification, $\widehat{\mathscr{F}}_L^*$ defines a presheaf on N such that*

$$SS(\widehat{\mathscr{F}_L}) = \widehat{L}$$

PROOF. We denote by Γ the Dold-Kan equivalence, sending $\mathbf{Ch^b}$ to $s\mathscr{A}$, adjoint to N. We must prove that the $CF^*(L,\nu^*U)$ and the $r_{U,V}$ define a simplicial map from $\mathfrak{C}[N(\mathbf{Op}(\mathbf{X})^{op})]$ to $\mathbf{Ch^b} \cong s\mathscr{A}$. Now $\mathfrak{C}[N((\mathbf{Op}(\mathbf{X})^{op})]_k$ is defined as the set of sequences $U_1 \subset .... \subset U_k$, with $d_j^n(U_1,....,U_n) = (U_1,...,\hat{U}_j,...,U_n)$. A simplicial map to $s\mathscr{A}$, will send $\mathfrak{C}[N((\mathbf{Op}(\mathbf{X})^{op})]_k$ to $(s\mathscr{A})_k$ commuting with the d_j^n, s_j^n. in other words we send $U_1 \subset .... \subset U_k$ to $X_k(U_1,...,U_k) \in \mathscr{A}$ and we have $d_j^k X_k(U_1,....,U_k) = X_{k-1}(U_1,...,\hat{U}_j,....,U_k)$

U to $\Gamma\mathscr{F}(U)$, (U_1,U_2) to $\Gamma r_{U_2,U_1}$, and $(U_1 \subset .... \subset U_k)$ to $\Gamma(r_{U_k,U_{k-1}} \circ .... \circ r_{U_2,U_1})$ □

We shall consider sheaves on X (i.e. $\mathbf{Op}(\mathbf{X})^{op}$)) taking values in $\mathbf{Ch^b}(\mathscr{A})$.

Part 3

Applications of sheaf theory to symplectic topology

CHAPTER 9

Singular support in the Derived category of Sheaves.

1. Singular support

1.1. Definition and first properties. From now on, we shall denote by $D^b(X)$ the derived category of (bounded) sheaves over X, that is $D^b(\mathbf{Sheaf(X)})$.

Let U be an open set. The functor $\Gamma(U;\bullet)$ sends sheaves on X to R-modules, and has a derived functor $R\Gamma(U;\bullet)$. Its cohomology $R^j\Gamma(U;\mathscr{F}) = H^j(U,\mathscr{F})$. Now if Z is a closed set, we defined the functor Γ_Z in chapter 7, page 90 as the set of sections supported in Z, that is $\Gamma_Z(U,\mathscr{F})$ is the kernel of $\mathscr{F}(U) \longrightarrow \mathscr{F}(U \setminus Z)$. This is a sheaf, so Γ_Z is a functor from **Sheaf(X)** to **Sheaf(X)**. We checked (cf. Proposition 7.31) that this is left-exact, using the left-exactness of the functor $\mathscr{F} \to \mathscr{F}_{|X\setminus Z}$, where $\mathscr{F}_{|X\setminus Z}(U) = \mathscr{F}(U \setminus (Z \cap U))$. Hence we may define $R\Gamma_Z : D^b(X) \longrightarrow D^b(X)$. This is defined for example for a sheaf $\mathscr{F}$ as follows: construct an injective resolution $\mathscr{F}$, that is $0 \to \mathscr{F} \to \mathscr{I}_0 \to \mathscr{I}_1 \to \mathscr{I}_2 \to \mathscr{I}_3 \to ....$

Then the complex of sheaves

$$0 \to \Gamma_Z\mathscr{I}_0 \to \Gamma_Z\mathscr{I}_1 \to \Gamma_Z\mathscr{I}_2 \to \Gamma_Z\mathscr{I}_3 \to \Gamma_Z\mathscr{I}_4 \to ...$$

represents $R\Gamma_Z(\mathscr{F})$. The cohomology space $\mathscr{H}^j(R\Gamma_Z(\mathscr{F}))$ is an element in $D^b(X)$, often denoted $H^j_Z(\mathscr{F})$. Moreover we denote by $H^j_Z(X,\mathscr{F}) = H^j(R\Gamma_Z(X,\mathscr{F}))$.

DEFINITION 9.1. ([**K-S**]) Let $\mathscr{F}^\bullet$ be an element in $D^b(X)$. The singular support of $\mathscr{F}^\bullet$, $SS(\mathscr{F}^\bullet)$ is the closure of the set of (x,p) such that **there exists** a real function $\varphi : M \to \mathbb{R}$ such that $d\varphi(x) = p$, and we have

$$R\Gamma_{\{x|\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x \neq 0$$

Note that this is equivalent to the fact that $\lim_{x\in U} R\Gamma(U,\mathscr{F}^\bullet) \longrightarrow \lim_{x\in U} R\Gamma(U,\mathscr{F}^\bullet)$ is not an isomorphism, or the existence of j such that $R^j\Gamma_{\{x|\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x = H^j_{\{x|\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x \neq 0$.

An equivalent formulation is that $(x_0,\xi_0) \notin SS(\mathscr{F}^\bullet)$ if and only if there is a neighbourhood of (x_0,ξ_0) in T^*X such that for any (x,ξ) in this neighbourhood and any smooth function φ such that $d\varphi(x) = \xi$ we have $R\Gamma_{\{x|\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x = 0$. Or else that $\lim_{U\ni x} R\Gamma(U,\mathscr{F}^\bullet) \longrightarrow \lim_{U\ni x} R\Gamma(U \cap \{\varphi < 0\},\mathscr{F}^\bullet)$ is an isomorphism.

REMARKS 9.2. (1) Assume for simplicity that we are dealing with a single sheaf $\mathscr{F}$, rather than with a complex. The above vanishing can be restated by asking that the natural restriction morphism

$$\lim_{U\ni x} H^j(U;\mathscr{F}) \longrightarrow \lim_{U\ni x} H^j(U\cap\{\varphi<0\};\mathscr{F})$$

is an isomorphism for any $j\in Z$. This implies in particular ($j=0$) that "sections" of $\mathscr{F}$ defined on $U\cap\{\varphi<0\}$ uniquely extend to a neighborhood of x.

Indeed, let $\mathscr{I}^\bullet$ be a complex of injective sheaves quasi-isomorphic to $\mathscr{F}^\bullet$. Then we have an exact sequence

$$0\to\Gamma_Z\mathscr{I}^\bullet\to\mathscr{I}^\bullet\to\mathscr{I}^\bullet_{X\setminus Z}\to 0$$

where the surjectivity of the last map follows from the flabbiness of injective sheaves. This yields the long exact sequence

$$\to H^j_Z(U,\mathscr{F}^\bullet)\to H^j(U,\mathscr{F}^\bullet)\to H^j(U\setminus Z,\mathscr{F}^\bullet)\to H^{j+1}_Z(U,\mathscr{F}^\bullet)\to\ldots$$

so that the vanishing of $H^j_Z(U,\mathscr{F}^\bullet)=R\Gamma^j_Z(\mathscr{F}^\bullet)$ for all j is equivalent to the fact that $H^j(U,\mathscr{F}^\bullet)\to H^j(U\setminus Z,\mathscr{F}^\bullet)$ is an isomorphism.

(2) The set $SS(\mathscr{F})$ is a homogeneous subset in T^*X. Note that $SS(\mathscr{F})$ is in T^*X not $\mathring{T}^*X$.
(3) It is easy to see that $SS(\mathscr{F}^\bullet)\cap 0_X=\mathrm{supp}(\mathscr{F}^\bullet)$ where $\mathrm{supp}(\mathscr{F}^\bullet)=\overline{\{x\in X\mid \mathscr{H}^j(\mathscr{F}^\bullet)_x=0\}}$. Take $\varphi=0$, then $R\Gamma_{\{x\mid\varphi(x)\geq 0\}}(\mathscr{F})=R\Gamma(\mathscr{F})$, and $R^j\Gamma(\mathscr{F})_x=\mathscr{H}^j(\mathscr{F}_x)$. So if we are not interested in the support of $\mathscr{F}$, we could define $SSU(\mathscr{F})$ as $SS(\mathscr{F})\cap ST^*(N)\subset ST^*N$ and $SSJ^+(\mathscr{F})$ as $SS(\mathscr{F})\cap\{\tau=1\}\subset J^1(N,\mathbb{R})$. Note that since $SS(\mathscr{F})$ is only positively homogeneous, $SSJ^+(\mathscr{F})$ does not allow us to recover $SS(\mathscr{F})\setminus 0_N$, but only its positive part. However if we also know $SSJ^-(\mathscr{F})=SS(\mathscr{F})\cap\{\tau=-1\}$ we can recover $SS(\mathscr{F})$. Similarly the knowledge of $SSU(\mathscr{F})$ revcovers $SS(\mathscr{F})\setminus 0_N$.
(4) Clearly $(x,p)\in SS(\mathscr{F}^\bullet)$ only depends on $\mathscr{F}^\bullet$ near x. In other words if $\mathscr{F}^\bullet=\mathscr{G}^\bullet$ in a neighbourhood V of X, then

$$(x,p)\in SS(\mathscr{F}^\bullet)\Leftrightarrow(x,p)\in SS(\mathscr{G}^\bullet)$$

(5) It is also clear that $SS(\mathscr{F}^\bullet)$ is a diffeomorphism invariant. Indeed, if $f:X\longrightarrow Y$ is a diffeomorphism, φ a smooth fonction on Y, $T^*f:T^*X\longrightarrow T^*Y$ the map $(x,\xi)\mapsto(f(x),\xi\circ df(x)^{-1})$ we have

$$\begin{tikzcd}
H^j(U,f^{-1}\mathscr{F}^\bullet) \arrow[r] \arrow[d, "\simeq"] & H^j(U\cap\{\varphi>0\},f^{-1}\mathscr{F}^\bullet) \arrow[d, "\simeq"] \\
H^j(f(U),\mathscr{F}^\bullet) \arrow[r] & H^j(f(U)\cap\{\varphi\circ f^{-1}>0\},\mathscr{F}^\bullet)
\end{tikzcd}$$

Since $d(\varphi\circ f^{-1}) = d\varphi(f(x))df(x)^{-1}$ we have $(x,\xi)\in SS(f^{-1}\mathscr{F}^\bullet) \Leftrightarrow (f(x),\xi\circ df(x)^{-1})\in SS(\mathscr{F}^\bullet)$. Note that this will also be a consequence Proposition 9.5 (see page 131).

(6) We would like to check $R\Gamma_{\{x|\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x = 0$ for a single function φ with $d\varphi(x) = p\neq 0$. Is this possible ?

The first properties of $SS(\mathscr{F})$ are given by the following proposition

PROPOSITION 9.3. *The singular support has the following properties*

(1) $SS(\mathscr{F}^\bullet)$ is a conical subset of T^*X.

(2) If $\mathscr{F}_1^\bullet\to\mathscr{F}_2^\bullet\to\mathscr{F}_3^\bullet\xrightarrow{+1}\mathscr{F}_1^\bullet[1]$ is a distinguished triangle in $\mathscr{D}^b(X)$, then $SS(\mathscr{F}_i^\bullet)\subset SS(\mathscr{F}_j^\bullet)\cup SS(\mathscr{F}_k^\bullet)$ and $(SS(\mathscr{F}_i^\bullet)\setminus SS(\mathscr{F}_j^\bullet))\cup(SS(\mathscr{F}_j^\bullet)\setminus SS(\mathscr{F}_i^\bullet))\subset SS(\mathscr{F}_k^\bullet))$ for any i,j,k such that $\{i,j,k\}=\{1,2,3\}$.

(3) $SS(\mathscr{F}^\bullet)\subset\bigcup_j SS(\mathscr{H}^j(\mathscr{F}^\bullet))$.

PROOF. The first statement is obvious. For the second, we first notice that $SS(\mathscr{F}^\bullet) = SS(\mathscr{F}^\bullet[1])$. Now according to Proposition 8.37 (see page 121), $R\Gamma_Z$ maps a triangle as in (2) to a similar triangle, so that we get the following distinguished triangle $R\Gamma_Z(\mathscr{F}_1^\bullet)\to R\Gamma_Z(\mathscr{F}_2^\bullet)\to R\Gamma_Z(\mathscr{F}_3^\bullet)\xrightarrow{+1} R\Gamma_Z(\mathscr{F}_1^\bullet)[1]\to ...$
which yields
$...\to R\Gamma_Z(\mathscr{F}_1^\bullet)_x\to R\Gamma_Z(\mathscr{F}_2^\bullet)_x\to R\Gamma_Z(\mathscr{F}_3^\bullet)_x\xrightarrow{+1} R\Gamma_Z(\mathscr{F}_1^\bullet)_x[1]\to ...$
and in particular, taking $Z=\{y\mid\psi(y)\geq 0\}$ where $\psi(x)=0$ and $d\psi(x)=p$, if two of the above vanish, so does the third. This implies the first part of (2). Moreover if one of the above cohomologies vanish, for example $R\Gamma_Z(\mathscr{F}_1^\bullet)_x\simeq 0$, then the other two are isomorphic, hence vanish simultaneously. Thus $(x,p)\notin SS(\mathscr{F}_1^\bullet)$ implies that $(x,p)\notin SS(\mathscr{F}_2^\bullet)\Delta SS(\mathscr{F}_3^\bullet)$, where Δ is the symmetric difference. This implies the second part of (2).

Consider the canonical spectral sequence of Proposition 8.21 (see page 115) applied to $F=\Gamma_Z$. This yields a spectral sequence from $R^p\Gamma_Z(H^q(\mathscr{F}^\bullet))$, converging to $R^{p+q}\Gamma_Z(\mathscr{F}^\bullet)$. So if $\left(R^p\Gamma_Z(H^q(\mathscr{F}^\bullet))\right)_x$ vanishes we also have that $\left(R^{p+q}\Gamma_Z(\mathscr{F}^\bullet)\right)_x$ vanishes. □

Examples:

(1) An exact sequence of complexes of sheaves $0\to\mathscr{F}_1^\bullet\to\mathscr{F}_2^\bullet\to\mathscr{F}_3^\bullet\to 0$ is a special case of a distinguished triangle (or rather its image in the derived category is a distinguished triangle). So in this case, we have the inclusions $SS(\mathscr{F}_i^\bullet)\subset SS(\mathscr{F}_j^\bullet)\cup SS(\mathscr{F}_k^\bullet)$ and $(SS(\mathscr{F}_i^\bullet)\setminus SS(\mathscr{F}_j^\bullet))\cup(SS(\mathscr{F}_j^\bullet)\setminus SS(\mathscr{F}_i^\bullet))\subset SS(\mathscr{F}_k^\bullet))$ for any i,j,k such that $\{i,j,k\}=\{1,2,3\}$.

(2) If $\mathscr{F}$ is the 0-sheaf that is $\mathscr{F}_x=0$ for all x (hence $\mathscr{F}(U)=0$ for all U), we have $SS(\mathscr{F})=\emptyset$. Indeed, for all x and ψ, $R\Gamma_{\{\psi(x)\geq 0\}}(X,\mathscr{F})_x=0$, hence the result. It is easy to check that this if $SS(\mathscr{F})=\emptyset$, then $\mathscr{F}$ is equivalent to the zero sheaf (in $D^b(X)$), that is $\mathscr{F}$ is a complex of sheaves with exact stalks.

(3) Let k_X be the constant sheaf on X. Then $SS(k_X)=0_X$. Indeed, consider the deRham resolution of k_X,

$$0\to k_U\to\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\Omega^2\xrightarrow{d}\Omega^3\xrightarrow{d}...$$

and apply Γ_Z. We obtain

$$0\to\Gamma_Z\Omega^0\xrightarrow{d}\Gamma_Z\Omega^1\xrightarrow{d}\Gamma_Z\Omega^2\xrightarrow{d}\Gamma_Z\Omega^3\xrightarrow{d}...$$

where $\Gamma_Z\Omega^j$ is the set of j-forms vanishing on Z, and the cohomology of the above complex is obtained by considering closed forms, vanihing on Z, modulo differential of forms vanishing on Z.

But if Z is the set $\{y\mid\varphi(y)\geq 0\}$ where $p=d\varphi(y)\neq 0$, a chart reduces this to the case where Z is a half space. Then, Poincaré's lemma tells us that any closed form on a small ball, vanishing on the half ball is the differential of a form vanishing on the half ball. Thus $SS(k_X)$ does not intersect the complement of 0_X, and since the support of k_X is X, we get $SS(k_X)=0_X$.

Since SS is defined by a local property, $SS(F)=0_X$ for any locally constant sheaf on X.

(4) We have

$$SS(\mathscr{F}^\bullet\oplus\mathscr{G}^\bullet)=SS(\mathscr{F}^\bullet)\cup SS(\mathscr{G}^\bullet)$$

since $R\Gamma_Z(\mathscr{F}^\bullet\oplus\mathscr{G}^\bullet)=R\Gamma_Z(\mathscr{F}^\bullet)\oplus R\Gamma_Z(\mathscr{G}^\bullet)$.

(5) Let U be an open set with smooth boundary, ∂U and k_U be the constant sheaf over U. Then $SS(k_U)=\{(x,p)\mid x\in U, p=0, \text{ or } x\in\partial U, p=\lambda\nu(x),\lambda<0\}$ where $\nu(x)$ is the exterior normal.

Indeed, in a point outside ∂U the sheaf is locally constant, and the statement is obvious. If x is a point in U, then the singular support over T_x^*X is computed as in the case of the constant sheaf (since k_U is locally isomorphic to the constant sheaf) and we get that $SS(k_U)\cap T_x^*X=0_x$. For x in $X\setminus\overline{U}$, the same argument, but comparing to the zero sheaf, shows that $SS(k_U)\cap T_x^*X=\emptyset$. We must then consider the case $x\in\overline{U}\setminus U$.

Now let Ω_U^j be the sheaf defined by $\Omega_U^j(V)$ is the set of j-forms in $\Omega^j(U\cap V)$ supported in a closed subset of V. We then have an acyclic resolution

$$0\to k_U\to\Omega_U^0\xrightarrow{d}\Omega_U^1\xrightarrow{d}\Omega_U^2\xrightarrow{d}\Omega_U^3\xrightarrow{d}...$$

so that $R\Gamma_Z(k_U)$ is defined by

$$0\to\Gamma_Z\Omega_U^0\xrightarrow{d}\Gamma_Z\Omega_U^1\xrightarrow{d}\Gamma_Z\Omega_U^2\xrightarrow{d}\Gamma_Z\Omega_U^3\xrightarrow{d}...$$

where $Z=\{\varphi(x)\geq 0\}$ and $\Gamma_Z\Omega_U^j$ means the space of j-forms vanishing on the complement of Z. Now assume U and Z are half-spaces (respectively open and closed). Consider the closed forms in $(\Gamma_Z\Omega_U^k)$ modulo differentials of forms in $(\Gamma_Z\Omega_U^{k-1})$. But any closed form vanishing in a sector is the differential of a form vanishing in the same sector (by the proof of Poincaré's lemma 3.4, page 20)). There is an exception, of course, if the sector is empty and $k=0$, in which case the constant function is not exact. So at a point x of ∂U, $(R^j\Gamma_Z\Omega_U)_x=0$ unless $Z\cap U=\emptyset$, in which case $(R^0\Gamma_Z\Omega_U)_x=k_x=k$ and $d\varphi(x)$ is a positive multiple of the interior normal.

We may reduce to the above case by a chart of U, and using the locality of singular support.

(6) For U as above and $F=\overline{U}$, we have

$$SS(k_F)=\{(x,p)\mid x\in U,p=0,\text{ or } x\in\partial U,p=\lambda\nu(x),\lambda>0\}$$

This follows from (1) of the above proposition applied to the exact sequence (which is a special case of a distinguished triangle) $0\to k_{X\setminus F}\to k_X\to k_F\to 0$.

(7) Let k_Z be the constant sheaf on the closed submanifold Z. Then $SS(k_Z)=\nu_Z=\{(x,p)\mid x\in Z,p_{|T_xZ}=0\}$. This is the conormal bundle to Z. Faux ?

(8) We would like to check $R\Gamma_{\{x\mid\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x=0$ for a single function φ with $d\varphi(x)=p$. But this does not hold. Indeed, let us consider the situation above, where $\mathscr{F}=k_U$. We just saw that setting $Z=\{x\in X\mid\varphi(x)\geq 0\}$ we have $(R\Gamma_Z(k_U))_x=(\Gamma_Z(\Omega_U^\bullet))_x=0$ if and only if $Z\cap U\neq\emptyset$. But if this last condition . For example slet $\Omega=\{(x,y)\in\mathbb{R}^2\mid x<0\}$. Then let $\varphi_0(x,y)=x$ and $\varphi(x,y)=x-y^3$ that are tangent at $x=y=0$. Indeed let $\widetilde{SS}(\mathscr{F}^\bullet)$ be the interior of the set A compléte

$$\{(x,p)\mid\exists\,\varphi\in C^\infty(X,\mathbb{R}),d\varphi(x)=p,R\Gamma_{\{x\mid\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x=0\}$$

Then we claim $SS(\mathscr{F}^\bullet)=\widetilde{SS}(\mathscr{F}^\bullet)\cup\operatorname{supp}(\mathscr{F}^\bullet)$.

EXERCICE 9.4. Compute $SS(\mathscr{F})$ for $\mathscr{F}$ an injective sheaf defined by $\mathscr{F}(U)=\{(s_x)_{x\in U}\mid s_x\in\mathbb{C}\}$. What about the sheaf $\mathscr{F}_W(U)=\{(s_x)_{x\in U}\mid s_x\in\mathbb{C}\text{ for }x\in W,s_x=0\text{ for }x\notin W\}$

Let us now see how our operations on sheaves act on $SS(\mathscr{F}^\bullet)$.

PROPOSITION 9.5. *([**K-S**]) Let $f:X\to Y$ be a proper map on* $\operatorname{supp}(\mathscr{F}^\bullet)$. *Then*

$$SS(Rf_*(\mathscr{F}^\bullet))\subset\Lambda_f\circ SS(\mathscr{F}^\bullet)$$

and this is an equality if f is a closed embedding. We also have

$$SS(Rf_!(\mathscr{F}^\bullet))\subset\Lambda_f\circ SS(\mathscr{F}^\bullet)$$

If f is any submersive map,

$$SS(f^{-1}\mathscr{G}^\bullet)=\Lambda_f^{-1}\circ SS(\mathscr{G}^\bullet)$$

For L a Lagrangian, $\Lambda_f(L)$ is obtained as follows: consider $T^*X\times\overline{T^*Y}$ and the Lagrangian $\Lambda_f=\{(x,\xi,y,\eta)\mid y=f(x),\xi=\eta\circ df(x)\}$. This is a conical Lagrangian submanifold. Let $K_L=L\times\overline{T^*Y}$. This is a coisotropic submanifold, in $T^*X\times T^*Y$ and $\mathscr{K}_L^\omega(x,\xi,y,\eta)=L\times\{(y,\eta)\}$, so $K_L/\mathscr{K}_L^\omega\simeq\overline{T^*Y}$, and $\pi_Y(T^*f)^{-1}(L)=(\Lambda_f\cap K_L)/\mathscr{K}_L^\omega$.

PROOF. Let ψ be a smooth function on Y such that $\psi(f(x)) = 0$ and $\xi = d\psi(f(x))df(x)$. Assume we have $(x,p) \notin SS(\mathscr{F}^\bullet)$ for all $x \in f^{-1}(y)$. Then we have

$$R\Gamma_{\{\psi\circ f\geq 0\}}(\mathscr{F}^\bullet)_{|f^{-1}(y)} = 0$$

But

$$R\Gamma_{\{\psi\geq 0\}}(Rf_*(\mathscr{F}^\bullet))_y = Rf_*(R\Gamma_{\{\psi\circ f\geq 0\}}(\mathscr{F}^\bullet))_y =$$
$$R\Gamma(f^{-1}(y), R\Gamma_{\{\psi\circ f\geq 0\}}(\mathscr{F}^\bullet)) = 0$$

Here the first equality follows from the fact that $\Gamma_Z \circ f_* = f_* \circ \Gamma_{f^{-1}(Z)}$ so the same holds for the corresponding derived functors. The second equality follows from the fact that if f is proper on $\mathrm{supp}(\mathscr{F}^\bullet)$, we have $(f_*\mathscr{F}^\bullet)_y = \Gamma(f^{-1}(y), \mathscr{F}^\bullet_{|f^{-1}(y)})$, that we shall now prove.

Indeed, let $j : Z \to X$ be the inclusion of a closed set. We define $\Gamma(Z,\mathscr{F}^\bullet)$ as $\Gamma(Z, j^{-1}(\mathscr{F}^\bullet))$. We also have $\Gamma(Z,\mathscr{F}^\bullet) = \lim_{Z\subset U}\Gamma(U,\mathscr{F}^\bullet)$ according to remark 7.17 on page 86. Then $(f_*\mathscr{F}^\bullet)(U) = \mathscr{F}^\bullet(f^{-1}(U))$, so $(f_*\mathscr{F}^\bullet)_y = \lim_{U\ni y}\mathscr{F}^\bullet(f^{-1}(U))$ and since f is proper, $f^{-1}(U)$ is a cofinal family of neighbourhoods of $f^{-1}(y)$. This implies $(f_*\mathscr{F}^\bullet)_y \overset{def}{=} \Gamma(y, f_*\mathscr{F}^\bullet) = \Gamma(f^{-1}(y),\mathscr{F}^\bullet)$, hence taking the derived functors $(Rf_*\mathscr{F}^\bullet)_y = R\Gamma(y, Rf_*\mathscr{F}^\bullet) = R\Gamma(f^{-1}(y),\mathscr{F}^\bullet)$. Clearly if for all $x \in f^{-1}(y)$ we have $R\Gamma(x,\mathscr{F}^\bullet) = 0$, we will have $(Rf_*\mathscr{F}^\bullet)_y = 0$. We thus proved that $(x, \xi\circ df(x)) \notin SS(\mathscr{F}^\bullet)$ implies $(f(x),\xi) \notin SS(Rf_*(\mathscr{F}^\bullet))$.

If f is a closed embedding, $f^{-1}(y)$ is a discrete set of points, $R\Gamma(f^{-1}(y),\mathscr{F}^\bullet)$ vanishes if and only if for all x in $f^{-1}(y)$, the stalks $R\Gamma(\mathscr{F}^\bullet)_x$ vanish. In this case we have equality.

Now if f is submersive, it is an open map, so $\Gamma(U, f^{-1}\mathscr{F}^\bullet) = \lim_{V\supset f(U)}\Gamma(V,\mathscr{F}^\bullet) = \Gamma(f(U),\mathscr{F}^\bullet)$. So we must study the map

$$R\Gamma(f(U),\mathscr{F}^\bullet) \longrightarrow R\Gamma(f(U\cap\{\varphi>0\}),\mathscr{F}^\bullet)$$

But for U small enough, $f(U\cap\{\varphi>0\}) = f(U)$ unless $\ker(df(\varphi(x_0))) \subset \ker(d\varphi(x_0)) = 0$. So if $\ker(df(y)) \not\subset \ker(\xi)$, the above map is an isomorphism, and since this is an open condition, it implies $(x,\xi) \notin SS(f^{-1}(\mathscr{F}^\bullet))$. Otherwise, we can find $\varphi = \psi\circ f$ with $d\varphi(x_0) = \xi$ and then $f(U\cap\{\psi\circ f>0\}) = f(U)\cap\{\psi>0\})$, so the above is a quasi-isomorphism if and only if $(x_0,\eta_0) = (x_0, d\psi(x_0)) \notin SS(\mathscr{F}^\bullet)$. But $\{(x,\eta\circ df(x)) \mid (y,\eta) \in SS(\mathscr{F}^\bullet)\} = \Lambda_f^{-1}(SS(\mathscr{F}^\bullet))$.

□

Attempt to redefine $SS(\mathscr{F}^\bullet)$

LEMMA 9.6. *Let* $\mathscr{F}^\bullet \in D^b(X)$, $\varphi : X \longrightarrow \mathbb{R}$ *a smooth function such that* $d\varphi(x_0) = \xi_0 \neq 0$ *and* $\varphi(x_0) = 0$. *Assume for all open set with smooth boundary* Ω *such that* $\varphi_{\partial\Omega}$ *is a submersion near* $\varphi^{-1}(0)$, *we have* $(0,1) \notin SS(R\varphi_*(\mathscr{F}^\bullet\otimes k_\Omega))$. *Then* $(x_0,\xi_0) \notin SS(\mathscr{F}^\bullet)$.

PROOF. By a change of variables, and the fact that the result is local, it is enough to prove the case $X = \mathbb{R}^n, \varphi = x_1$ where $(x_1,..,x_n)$ are coordinates on $\mathbb{R}^n$, so $\xi_0 = e_1^*$.

□

The following continuity result is sometimes useful. Let $(\mathscr{F}^\bullet_\nu)_{\nu\geq 1}$ be a directed system of sheaves, i.e. there are maps $f_{\mu,\nu} : \mathscr{F}^\bullet_\mu \to \mathscr{F}^\bullet_\nu$ for $\mu \leq \nu$ satisfying the obvious compatibility conditions, and let $\mathscr{F}^\bullet = \lim_{\nu\to+\infty} \mathscr{F}^\bullet_\nu$ (we will assume the limit is a bounded complex, so the $\mathscr{F}^\bullet_\nu$ are uniformly bounded).

Now let S_ν be a sequence of closed sets in a metric space M. Then define $\lim_{\nu\to+\infty} S_\nu = S$ to mean that each point x in S is the accumulation point of some sequence of points $(x_\nu)_{\nu\geq 1}$ in S_ν. In other words

$$\lim_n S_\nu = \bigcap_{\nu\geq 1} \overline{\bigcup_{\mu\geq\nu} S_\mu}$$

With these notions at hand, we may now state

Lemma 9.7 (see [**K-S**] exercice V.7 page 246). *Let $(\mathscr{F}^\bullet_\nu)_{\nu\geq 1}$ be a directed system of sheaves. Then we have*

$$SS(\varinjlim_\nu \mathscr{F}^\bullet_\nu) \subset \lim_{\nu\to+\infty} SS(\mathscr{F}^\bullet_\nu)$$

Proof. Indeed, we must compute $R\Gamma_Z(\lim_{\nu\to+\infty} \mathscr{F}^\bullet_\nu)_x = \lim_{\nu\to+\infty} R\Gamma_Z(\mathscr{F}^\bullet_\nu)_x$ the equality follows from the fact that the direct limit is an exact functor, and thus commutes with Γ_Z (since it commutes with $\Gamma(U,\bullet)$). Set $Z = \{y \mid \psi(y) \geq 0\}$, where ψ is a function such that $\psi(x) = 0, d\psi(x) = p$. As a result $(x_0, p_0) \notin SS(\lim_{\nu\to+\infty} \mathscr{F}^\bullet_\nu)$ if **False ?**
$R\Gamma_Z(\lim_{\nu\to+\infty} \mathscr{F}^\bullet_\nu)_x = 0$ for all (x, p) in a neighbourhood of (x_0, p_0), and this implies our statement. □

As an application we may prove:

Proposition 9.8. *Let Ω be an open set, and $N^*(\Omega) \subset T^*X$ the union of the $N^*_x\Omega)$, the dual cone to the interior of the set $N_x\Omega = \{(x, v) \in TX \mid \exists \varepsilon > 0, x+]0,\varepsilon[\cdot v \subset \Omega\}$. Then $SS(k_\Omega) = N^*(\Omega)^a$.*

Il semble qu'une démonstration directe soit plus facile

We remind the reader that the dual cone to a convex cone $C \subset E$ is $C^o = \{\xi \in E^* \mid \forall v \in C, \langle\xi, v\rangle \geq 0\}$ and that $C^a = \{(x, -p) \mid (x, p) \in C\}$. Note that $N_x(\Omega) = T^*_x X$ if $x \in \Omega \cup \overset{\circ}{\widehat{(X \setminus \Omega)}}$, so $N^*_x(\Omega) = 0$.

Proof. Now we claim that $SS(\Omega) = SS(k_\Omega) \subset N^*(\Omega)$. Indeed this can be proved by approximating Ω by open sets with smooth boundary, and use the fact that if Ω_ν is a sequence such that $\lim_\nu \Omega_\nu = \Omega$ we have $\lim_\nu k_{\Omega_\nu} = k_\Omega$. Indeed, for an open set with smooth boundary, we have $N_x(\Omega)$ is the half-space defined by $T_x\partial\Omega$ and $N^*_x(\Omega) = \mathbb{R}_+\nu(x)$ where $\nu(x)$ is the outside normal. Then we know indeed that $SS(k_\Omega) = \nu^*\Omega$. Now we must prove that for a well chosen sequence Ω_k, we have

$$SS(\Omega) = SS(\lim_k \Omega_k) \subset \lim_k SS(\Omega_k) = N^*(\Omega)^a$$

□

1.2. The sheaf associated to a Generating function. Let $S(x,\xi)$ be a GFQI for a Lagrangian L, that is $L=\{(x,\frac{\partial}{\partial x}S(x,\xi))\mid \frac{\partial}{\partial \xi}S(x,\xi)=0\}$. We set $\Sigma_S=\{(x,\xi)\mid \frac{\partial}{\partial\xi}S(x,\xi)=0\}$, $\widehat{\Sigma}_S=\{(x,\xi,\lambda)\mid \frac{\partial S}{\partial \xi}(x,\xi)=0, \lambda=S(x,\xi)\}$, and $\widehat{L}=\{(x,\tau p,\lambda,\tau)\mid p=\frac{\partial S}{\partial x}(x,\xi), \frac{\partial S}{\partial\xi}(x,\xi)=0,\lambda=S(x,\xi)\}$. Set $U_S=\{(x,\xi,\lambda)\mid S(x,\xi)\leq\lambda\}\subset M\times\mathbb{R}^q\times\mathbb{R}$. Let $\mathscr{F}_S=R\pi_*(k_{U_S})$, where π is the projection $\pi: M\times\mathbb{R}^q\times\mathbb{R}\to M\times\mathbb{R}$.

We claim that $SS(\mathscr{F}_S)=\widehat{L}$. It is easy to prove that $SS(\mathscr{F}_S)\subset\widehat{L}$, since $\Lambda_\pi\circ SS(k_S)=\widehat{L}$. Indeed, the correspondence Λ_π corresponds to symplectic reduction by $p_\xi=0$, i.e. sends A to $\Lambda_\pi\circ A=A\cap\{p_\xi=0\}/(\xi)$.

To prove equality, we moreover assume the sets $\pi^{-1}(x,\lambda)\cap\widehat{\Sigma}_S$ are discrete sets. This is a generic condition. We then use the formula from the proof of the above proposition

$$R\Gamma_{\{\psi\geq 0\}}(R\pi_*(k_{U_S}))_{(x,\lambda)}=R\pi_*(R\Gamma_{\{\psi\circ\pi\geq 0\}}(k_{U_S}))_{(x,\lambda)}=$$
$$R\Gamma(\pi^{-1}(x,\lambda),R\Gamma_{\{\psi\circ\pi\geq 0\}}(k_{U_S}))=0$$

But $R\Gamma_{\{\psi\circ\pi\geq 0\}}(k_{U_S})_{(x,\xi,\lambda)}$ is non zero if and only if $(x,\xi,\lambda,d\psi(\pi(x,\xi,\lambda))d\pi(x,\xi,\lambda))\in SS(k_{U_S})$ that is $(x,d\psi(x,\lambda))\in\widehat{L}$. This is a discrete set by assumption (for (x,λ) fixed), thus $R\Gamma_{\{\psi\circ\pi\geq 0\}}(k_{U_S})_{|\pi^{-1}(x,\lambda)}$ has vanishing stalk except over the discrete set of points of $\widehat{\Sigma}_S\cap\pi^{-1}(x,\lambda)$. Note that such a sheaf is zero if and only if each of the stalks is zero. So we have that

$$R\Gamma(\pi^{-1}(x,\lambda),R\Gamma_{\{\psi\circ\pi\geq 0\}}(k_{U_S}))=0$$

if and only if for all $(x,\xi,\lambda)\in M\times\mathbb{R}^q\times\mathbb{R}$ we have $(x,\tau p,\lambda,\tau)\in\widehat{L}=\Lambda_\pi\circ SS(k_{U_S})$.

REMARKS 9.9. (1) If S is the generating function for Λ in $J^1(N,\mathbb{R})$ then $SSJ(\mathscr{F}_S)=\Lambda$.
(2) With the notations of the previous remark, note that if $\lim_{\nu\to+\infty}S_\nu=S$, where the limit is for the uniform C^0 convergence, we have for a suitable sequence ε_k converging to 0, so that $U_{S_\nu+\varepsilon_\nu}\subset U_{S_\mu+\varepsilon_\mu}$ for $\mu\geq\nu$, that $\lim_{\nu\to+\infty}U_{S_\nu+\varepsilon_\nu}=U_S$, and thus $\lim_{\nu\to+\infty}(k_{S_\nu+\varepsilon_\nu})=k_S$ (where we wrote k_S for k_{U_S}). Thus $SS(k_S)\subset\lim_{\nu\to+\infty}SS(k_{S_\nu})$. Thus the second assertion also holds in the non-generic case, i.e. without assuming the sets $\pi^{-1}(x,\lambda)\cap\widehat{\Sigma}_S$ to be discrete.

1.3. Uniqueness of the quantization sheaf of the zero section. The following plays the role of the uniqueness result for GFQI (see Theorem 5.19, page 47).

PROPOSITION 9.10. *Let $\mathscr{F}^\bullet$ in $D^b(X)$, be such that $SS(\mathscr{F}^\bullet)\subset 0_X$. Then $\mathscr{F}^\bullet$ is equivalent in $D^b(X)$ to a locally constant sheaf.*

PROOF. We start by proving the proposition for the case $X=\mathbb{R}$ (see [**K-S**] page 118, proposition 2.7.2 and lemma 2.7.3). First, since the support of $\Gamma_Z(\mathscr{F})$ is contained in Z, we have that $\Gamma_{\{t\geq s\}}\mathscr{F}(]-\infty,s+\varepsilon[)=\Gamma_{\{t\geq s\}}\mathscr{F}(]s-\varepsilon,s+\varepsilon[)$. Moreover this last space is the kernel of the map

$$\mathscr{F}(]-\infty,s+\varepsilon[)\to\mathscr{F}(]-\infty,s+\varepsilon[\setminus\{t\geq s\})=\mathscr{F}(]-\infty,s[)$$

so we have an exact sequence

$$0 \to \Gamma_{\{t\geq s\}}\mathscr{F}(]s-\varepsilon, s+\varepsilon[) \to \mathscr{F}(]-\infty, s+\varepsilon[) \to \mathscr{F}(]-\infty, s[)$$

which in the case of a flabby (and in particular for an injective) sheaf, $\mathscr{I}$ extends to

$$0 \to \Gamma_{\{t\geq s\}}\mathscr{I}(]s-\varepsilon, s+\varepsilon[) \to \mathscr{I}(]-\infty, s+\varepsilon[) \to \mathscr{I}(]-\infty, s[) \to 0$$

since the last map is surjective by flabbiness.

Thus, given an injective complex $0 \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \ldots$ quasi-isomorphic to $\mathscr{F}^\bullet$ we get a sequence

$$0 \to \Gamma_{\{t\geq s\}}\mathscr{I}^\bullet(]s-\varepsilon, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s[) \to 0$$

By definition, the complex $\Gamma_{\{t\geq s\}}\mathscr{I}^\bullet(]s-\varepsilon, s+\varepsilon[)$ represents $R\Gamma_{\{t\geq s\}}\mathscr{F}(]s-\varepsilon, s+\varepsilon[)$ which converges as ε goes to zero to $R\Gamma_{\{t\geq s\}}(\mathscr{F})_s$, which vanishes by assumption. Thus using the exactness of the direct limit, and this exact sequence we get a surjective quasi-isomorphism.

$$\varinjlim_{\varepsilon\to 0}\Gamma(]-\infty, s+\varepsilon[, \mathscr{I}^\bullet) \to \Gamma(]-\infty, s[, \mathscr{I}^\bullet)$$

Then by definition of a sheaf, $\varprojlim_{\varepsilon\to 0}\Gamma(]-\infty, s_1-\varepsilon[, \mathscr{I}^\bullet)$ is isomorphic to $\Gamma(]-\infty, s_1[, \mathscr{I}^\bullet)$.

We may thus apply the following lemma, due to Kashiwara in the case λ_t, μ_t are isomorphisms.

LEMMA 9.11 (Adapted from [**K-S**], lemma 1.12.6). *Let $(X_s^\bullet, \rho_{s,t})$ a family of complexes in $D^b(M)$ indexed by $s \in \mathbb{R}$, where $\rho_{s,t} : X_t^\bullet \longrightarrow X_s^\bullet$ are defined for $s < t$ and are chain map. Assume*

$$\lambda_t : X_t^\bullet \longrightarrow \varprojlim_{t>s} X_s^\bullet$$

is **an isomorphism** *and*

$$\mu_t : \varinjlim_{t<s} X_s^\bullet \longrightarrow X_t^\bullet$$

be a **surjective quasi-isomorphism**. *Then for all $s < t$ the map $\rho_{s,t}$ is a quasi isomorphism.*

PROOF. We first make the preliminary remark that μ_t is actually surjective on cycles, that is for any $z_t \in X_t^p$ with $dz_t = 0$, there exists some z_s for $s > t$ with $dz_s = 0$ and such that $\rho_{ts}(z_s) = z_t$. Indeed, because μ_t is surjective in homology, we can find $(z'_s)_{s>t}$ such that $dz'_s = 0$ $\rho_{s,s'}(z_{s'}) = z_s$ and $\rho_{ts}(z'_s) = z_t + dv$. But then since μ_t is onto, v_t corresponds to a sequence $(v_s)_{s>t}$ and then $z_s = z'_s - dv_s$ satisfies $\rho_{ts}(z_s) = z_t$ and of course $dz_s = dz'_s = 0$.

Now we first prove injectivity of $\rho_{s,t}$ in cohomology. Consider $x_{t_0} \in X_{t_0}^p$ such that $dx_{t_0} = 0$ and denote by $x_s = \rho_{st_0}(x_{t_0})$. We assume there exists some s such that x_s is exact. Let us consider the non-empty set S of pairs (s, y_s) such that $\rho_{s,t_0}(x_{t_0}) = dy_s$ with the order relation $(s, y_s) < (s', y_{s'})$ if and only if $s < s'$ and $\rho_{s,s'}y_{s'} = y_s$. If we have a totally ordered subset T of S it has an upper bound. Indeed, let $s_1 = \sup\{s \mid \exists (s, y_s) \in T\}$.

Then the family y_s defines an element in $\varprojlim_{s<s_1} X_s$ and since λ_{s_1} is an isomorphism this defines a unique element $y_{s_1} \in X_{s_1}$ and we have $dy_{s_1} = \rho_{s_1,t_0}(x_{t_0})$. So let (s_1, y_{s_1}) be an upper bound for T. Now according to Zorn's lemma, S has a maximal element. If this is of the form (t_0, y_{t_0}) we are finished, i.e. we proved x_{t_0} vanishes in cohomology. Otherwise, it is of the form (t_1, y_{t_1}) with $\rho_{t_1,t_0}(x_0) = dy_{t_1}$ but $\mu_{t_1} : \varinjlim_{t>t_1} X_t^\bullet \longrightarrow X_{t_1}^\bullet$ is injective in cohomology, so there exists $t > t_1$ and $y_t \in X_t^{p-1}$ such that $x_t = dy_t$. But then $d(\rho_{t_1,t}(y_t) - y_{t_1}) = \rho_{t_1,t}(x_t) - x_{t_1} = 0$, so $z_{t_1} = \rho_{t_1,t}(y_t) - y_{t_1}$ satisfies $dz_{t_1} = 0$. According to our preliminary remark, z_{t_1} is the image of a closed class z_t that is $\rho_{t_1,t}(z_t) = z_{t_1}$ for t close enough to t_1 and $dz_t = 0$. Then $x_t - d(y_t - z_t) = x_t - dy_t = 0$ and $\rho_{t_1,t}(y_t - z_t) = y_{t_1}$, so $(t, y_t - z_t) > (t_1, y_{t_1})$ which contradicts the maximality of (t_1, y_{t_1}).

We thus proved that the map induced in cohomology by $\rho_{s,t}$ is injective.

Now let us prove surjectivity. We argue by contradiction and let $x_{s_0} \in X_{s_0}^p$ such that $dx_{s_0} = 0$, and assume its cohomology class is not in the image of $\rho_{s_0,s}$ for some $s > s_0$. Consider the non-empty set

$$S = \{(t, z_t) \mid dz_t = 0, \exists y, \rho_{s_0,t}(z_t) = x_{s_0} + dy\}$$

We have an order relation on S given by $(t, z_t) < (t', z_t')$ if and only if $t < t'$ and $\rho_{t,t'}(z_{t'}) = z_t$. As above we prove S satisfies the assumptions of Zorn's lemma. Consider a totally ordered subset, T, and let us prove it has an upper bound. We first notice that for (t, z_t) in such a set, we may find y independent from t such that $\rho_{s_0,t}(z_t) = x_{s_0} + dy$. Indeed if $(t, z_t) < (t', z_t')$ and $\rho_{s_0,t}(z_t) = x_{s_0} + dy, \rho_{s_0,t'}(z_{t'}) = x_{s_0} + dy'$, we have $x_{s_0} + dy = \rho_{s_0,t}(z_t) = \rho_{s_0,t}\rho_{t,t'}(z_{t'}) = \rho_{s_0,t'}(z_{t'}) = x_{s_0} + dy'$. Now let $s_1 = \sup\{t \mid \exists (t, z_t) \in T\}$ which is finite by assumption. Then the family $(z_s)_{s<s_1}$ defines an element in $\varprojlim_{s_1>s} X_s^p$ which is isomorphic to $X_{s_1}^p$ since λ_{s_1} is an isomorphism. We thus get an element z_{s_1} in $X_{s_1}^p$ which is closed since the z_s are closed. Then $\rho_{s_0,s_1}(z_{s_1}) = x_{s_0} + dy$. As a result (s_1, z_{s_1}) is an upper bound for T. Now let (t_0, z_{t_0}) be a maximal element in S, which exists according to Zorn's lemma. Then according to our preliminary remark, z_{t_0} corresponds to a sequence $(z_s) \in X_s^p$ for $s > t_0$ such that $dz_s = 0$ and $\rho_{t_0,s}(z_s) = z_{t_0}$. Then $(s, z_s) \in S$ and $(t_0, z_{t_0}) < (s, z_s)$ which contradict the maximality of (t_0, z_{t_0}).

□

We thus proved that $R\Gamma(]-\infty, s[, \mathscr{F})$ is constant. Note that the same proof works if the family is only defined for s in some finite interval $]a, b[$. Now in the general case, we have to prove that if $\mathscr{F}^\bullet$ is in $D^b(X)$, it is locally constant. Let $B(x_0, R)$ be a small ball in X, that is of radius smaller than the injectivity radius of the manifold, so that the function $r(x) = d(x, x_0)$ has no critical point except 0. Then $SS(r_*\mathscr{F}^\bullet) \subset \Lambda_r \circ SS(\mathscr{F}^\bullet)$, but since $SS(\mathscr{F}) \subset 0_X$, and r has no positive critical value, we get $\Lambda_r \circ 0_X \subset 0_{\mathbb{R}} \cup \{(0, \tau) \mid \tau \in \mathbb{R}\}$, so that $R\Gamma(]-\infty, R[, Rf_*(\mathscr{F}^\bullet)) \longrightarrow R\Gamma(]-\infty, \varepsilon[, Rf_*(\mathscr{F}^\bullet))$ is an isomorphism. In other words, $R\Gamma(B(x_0, R), \mathscr{F}^\bullet) \longrightarrow R\Gamma(B(x_0, \varepsilon), \mathscr{F}^\bullet)$ is an isomorphism, and by going to the limit as ε goes to zero, we get $R\Gamma(B(x_0, R), \mathscr{F}^\bullet) \simeq R\Gamma(\mathscr{F}^\bullet)_{x_0}$. The same argument shows that for any open set U, diffeomorphic to a ball, we have $R\Gamma(U,\mathscr{F}^\bullet) \simeq R\Gamma(\mathscr{F}^\bullet)_{x_0}$ for any x_0 in U. Since M can be covered by open sets diffeomorphic to a ball and containing a fixed point x_0, this proves that $\mathscr{F}^\bullet$ is locally constant in $D^b(X)$.

□

REMARKS 9.12. (1) Note that in the case where X is the real line, we only need the sheaf $\mathscr{F}$ to be defined on open sets of the type $]-\infty,t[$, and that for an injective resolution $\mathscr{I}$, we have $\lim_{s\to t^+}\mathscr{I}(]-\infty,s[) \simeq \mathscr{I}(]-\infty,t[)$ to conclude that the $R\Gamma(-\infty,s[,\mathscr{F})$ are all isomorphic.

(2) One should not imagine that sheaves on contractible spaces have vanishing cohomology. Obviously if Z is a subspace of X, $H^*(X,k_Z) = H^*(Z,\mathbb{R})$ which does not vanish if X is contractible, but Z is not.

(3) Assume f is a smooth function, and $X_t = \Omega^*(\{f<t\})$. That λ_t is an isomorphism, follows from the fact that Ω^* defines a sheaf and that $\{f<t\} = \bigcup_{s<t}\{f<s\}$. Then μ_t is a quasi-isomorphism in particular if t is a regular value of f. The above result then reduces to the First Morse Lemma for functions defined on $\mathbb{R}$. This shall be more precise in Section 2 (see page 139).

EXERCICE 9.13. Compute the cohomology of the skyscraper sheaf at 0 in $\mathbb{R}$. Then compute its singular support.

1.4. The non-characteristic deformation lemma. We want to use the same argument as above to prove a propagation theorem: we have a condition under which $R\Gamma(U_1,\mathscr{F}^\bullet) \longrightarrow R\Gamma(U_0,\mathscr{F}^\bullet)$ is an isomorphism.

THEOREM 9.14 ([**K-S**], lemme 2.7.2 page 117). *Let $\mathscr{F}^\bullet$ be an element in $D^b(X)$ and $(U_t)_{t\in[0,1]}$ be an increasing family of open domains with smooth boundary ∂U_t such that*

(1) $U_t = \bigcup_{s<t} U_s$ and $\bigcap_{s>t}\overline{U}_s = \overline{U_t}$
(2) $\overline{(U_t\setminus U_s)}\cap \operatorname{supp}(\mathscr{F})$ is compact
(3) for all t, $SS(U_t)\cap SS(\mathscr{F})\cap\overline{\{(x,p)\mid x\in\bigcap_{s>t}U_s\setminus U_t\}}\subset 0_X$ (where $SS(U)$ is defined as $SS(k_U)$).

Then we have an isomorphism $R\Gamma(U_1,\mathscr{F}^\bullet)\longrightarrow R\Gamma(U_0,\mathscr{F}^\bullet)$.

PROOF. For the general proof we refer to [**K-S**]. Our proof is by "continuous induction": consider the set of t such that

$$H^*(U_t,\mathscr{F}^\bullet)\longrightarrow H^*(U_0,\mathscr{F}^\bullet)$$

is an isomorphism. We want to prove that this set is open and closed. We notice that our assumption implies that $R\Gamma_{X\setminus U_t}(\mathscr{F}^\bullet)_x = 0$ for all $x\in\delta U_t$, where we define $\delta U_t = \bigcap_{s>t}(U_s\setminus U_t)\subset\partial U_t$. Indeed, assume U_t is defined near x by $\{x\mid\varphi_t<0\}$ where φ_t has 0 as a regular value, then $X\setminus U_t$ is described locally as

$$\{x\mid\varphi_t(x)\geq 0\}$$

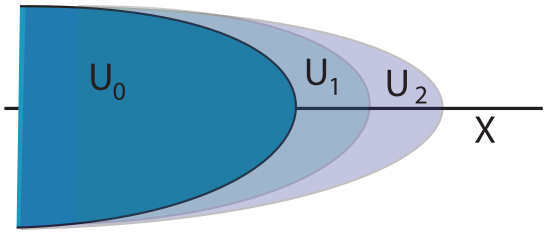

FIGURE 1. The sets U_t

and

$$SS(U_t) = \{(x, \lambda d\varphi_t(x)) \mid \lambda < 0, x \in \partial U_t\} \cup 0_{U_t}$$

Since by assumption for $x \in \delta U_t$, $(x, d\varphi_t(x)) \notin SS(\mathscr{F}^\bullet)$, we have $R\Gamma_{X\setminus U_t}(\mathscr{F}^\bullet))_x = 0$ hence for $\mathscr{I}^\bullet$ an injective complex quasi isomorphic to $\mathscr{F}^\bullet$ we have

$$0 \to \lim_{s\to t, s>t} \Gamma_{U_s\setminus U_t}(\mathscr{I}^\bullet) \longrightarrow \lim_{s\to t, s>t} \Gamma(U_s, \mathscr{I}^\bullet) \longrightarrow R\Gamma(U_t, \mathscr{I}^\bullet) \longrightarrow 0$$

By assumption we have $R\Gamma_{\delta U_t}(\mathscr{F}^\bullet) = 0$, since its stalk at each point vanishes, we have, using assumption (2), that $\lim_{s\to t, s>t} R\Gamma_{U_s\setminus U_t}(\mathscr{I}^\bullet)$ is quasi-isomorphic to 0. This implies that $\lim_{s\to t, s>t} \Gamma(U_s, \mathscr{I}^\bullet) \longrightarrow \Gamma(U_t, \mathscr{I}^\bullet)$ is a surjective quasi-isomorphism. Since by definition of a sheaf we have $\lim_{s\to t, s<t} R\Gamma(U_s, \mathscr{I}^\bullet) = R\Gamma(U_t, \mathscr{I}^\bullet)$, we see that using Lemma 9.11 (page 135), we have that $R\Gamma(U_s, \mathscr{I}^\bullet) \simeq R\Gamma(U_t, \mathscr{I}^\bullet)$. □

REMARK 9.15. One can remove assumption (2), but then the conclusion would be that we have an isomorphism $R\Gamma_c(U_s, \mathscr{I}^\bullet) \simeq R\Gamma_c(U_t, \mathscr{I}^\bullet)$. Indeed we used that if Z is compact, any family of open sets V_j such that $\bigcap V_j = Z$ is cofinal (for the set of open neighbourhoods of Z).

PROPOSITION 9.16 (A. Oancea). *Let $U_t = \varphi_t(U_0)$ be the image by an isotopy of U_0. Let $X_t(x) = \frac{d}{ds}\varphi_s(x)_{|s=t}$. Assume for all $x \in \partial U_0$ and all t, we have $\langle \xi, X_t(\varphi_t(x))\rangle = 0$ for all $(x,\xi) \in SS(\mathscr{F}^\bullet) \cap T^*_{\partial U_t}X$. Then we have an isomorphism*

$$R\Gamma(U_t, \mathscr{F}^\bullet) \longrightarrow R\Gamma(U_0, \mathscr{F}^\bullet)$$

Pb d enonce.
Est-ce vraiment utile
Missing proof

PROOF. Let $\tilde{\mathscr{F}}^\bullet$ be the element $k_{\mathbb{R}} \boxtimes \mathscr{F}^\bullet$ on $\mathbb{R}\times X$, and consider the map $\psi : (t,x) \longrightarrow (t, \varphi_t(x))$. Then $(R\psi)_*(\tilde{\mathscr{F}}^\bullet) = \mathscr{G}^\bullet$ is an element in $D^b(\mathbb{R}\times X)$ such that $j_t^{-1}(\mathscr{G}^\bullet) = R(\varphi_t^{-1})_*(\mathscr{F}^\bullet)$, where $j_t : X \longrightarrow \mathbb{R}\times X$ is the map $x \longrightarrow (t,x)$. Then $R\Gamma(U_t, \mathscr{F}^\bullet) = R\Gamma((R\pi)_*(\mathscr{G}))_{\{t\}}$ where $\pi(t,x) = t$. Now

$$SS((R\pi)_*(\mathscr{G})) \subset (\Lambda_\pi) SS(\mathscr{G}^\bullet) \subset (\Lambda_\pi)\circ\Lambda_{\psi^{-1}} SS(\mathscr{F}^\bullet) = \Lambda(\pi\circ\psi)$$

□

2. The sheaf theoretic Morse lemma and applications

The last paragraph in the proof of Proposition 9.10 (see page 134) can be generalized as follows.

PROPOSITION 9.17. *Let us consider a function $f : M \to \mathbb{R}$ proper on* $\mathrm{supp}(\mathscr{F})$. *Assume that* $\{(x, df(x)) \mid x \in f^{-1}([a,b])\} \cap SS(\mathscr{F})$ *is empty. Then for* $t \in [a,b]$ *the natural maps* $R\Gamma(\{x \mid f(x) \leq t\}, \mathscr{F}) \longrightarrow R\Gamma(\{x \mid f(x) \leq a\}, \mathscr{F})$ *are isomorphisms. In particular* $H^*(f^{-1}(a), \mathscr{F}) \simeq H^*(f^{-1}([a,b]), \mathscr{F})$.

PROOF. It follows from 9.14 on page 137 applied to $U_t = f^{-1}(]-\infty, t[)$, but we will provide a simpler and more direct proof. The proposition is equivalent to proving that the $R\Gamma(]-\infty, t], Rf_*(\mathscr{F}))$ are all canonically isomorphic for $t \in [a,b]$. But this follows from Proposition 9.10 see page 134), since $SS(Rf_*\mathscr{F}) \cap T^*([a,b]) \subset \Lambda_f \circ SS(\mathscr{F}) = \{(f(x), \tau) \mid x \in f^{-1}([a,b]), (x, \tau df(x)) \in SS(\mathscr{F})\}$ and this is contained in the zero section by our assumption. □

Note that the standard Morse lemma corresponds to the case $\mathscr{F} = k_M$.

LEMMA 9.18. *Let φ be a smooth function on X such that* 0 *is a regular level. Let $x \in \varphi^{-1}(0)$ and assume there is a neighbourhood U of x such that*

$$R\Gamma(U \cap \{\varphi(z) \leq t\}, \mathscr{F}^\bullet) \longrightarrow R\Gamma(U \cap \{\varphi(z) \leq 0\}, \mathscr{F}^\bullet)$$

is an isomorphism for all positive t small enough. Then $R\Gamma_{\{\varphi \geq 0\}}(\mathscr{F}^\bullet)_x = 0$.

PROOF. Again, we have $R\Gamma(]-\infty, t[, R\varphi_*(\mathscr{F}^\bullet)) \to R\Gamma(]-\infty, 0[, R\varphi_*(\mathscr{F}^\bullet))$ is an isomorphism. So if $\mathscr{G}^\bullet$ is a sheaf over $\mathbb{R}$, the fact that $R\Gamma(]-\infty, t[, \mathscr{G}^\bullet) \to R\Gamma(]-\infty, 0[, \mathscr{G}^\bullet)$ is an isomorphism implies $R\Gamma_{\{t \geq 0\}}(\mathscr{G}^\bullet)_{t=0} = 0$ since for $\mathscr{I}^\bullet$ an injective resolution of $\mathscr{G}^\bullet$ we have

$$0 \to \Gamma_{\{t \geq s\}} \mathscr{I}^\bullet(]s-\varepsilon, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s[) \to 0$$

□

2.1. Simple sheaves.

2.1. Simple sheaves. A sheaf is simple if for any point if its singular support, the change in cohomology is "as small as possible".

More precisely,

DEFINITION 9.19 (Simple complex of sheaves). Let $\mathscr{F}^\bullet \in D^b(X)$ and let φ be such that $\varphi(x_0) = 0, d\varphi(x_0) = \xi_0$. We assume $L_\varphi = \{(x, d\varphi(x)) \mid x \in M\}$ satisfies the following two conditions: $L_\varphi \cap SS(\mathscr{F}^\bullet) = (x_0, \xi_0)$ and L_φ is transverse to $SS(\mathscr{F}^\bullet)$. Then $\mathscr{F}^\bullet$ is simple if $\dim H^*_{\{x \mid \varphi(x)) \geq 0\}}(\mathscr{F}^\bullet)_{x_0} = 1$.

3. Some computations of Singular supports

Let f be a map from X to Y. Let B be a subset in T^*Y. We set

DEFINITION 9.20. We define $(\Lambda_f)^{\#}(B)$ to be the set of (x,ξ) such that there exists sequences $(x_n,\xi_n)\in T^*X, (z_n,\zeta_n)\in B$ such that $\lim_n x_n = x, \lim_n z_n = f(x), \zeta_n\circ df(x_n) \longrightarrow \xi$ and $d(z_n, f(x_n))\cdot|\zeta_n| \longrightarrow 0$.

Note that $(\Lambda_f)^{-1}(B)\subset(\Lambda_f)^{\#}(B)$, since $(\Lambda_f)^{-1}(B) = \{(x,\xi)\mid \exists(y,\eta)\in A, f(x)=y, \eta\circ df(x)=\xi\}$. However $(\Lambda_f)^{\#}(B)$ is closed, while $(\Lambda_f)^{-1}(B)$ is not necessarily so.

EXERCICE 9.21. Prove that $(\Lambda_f)^{\#}(B)$ is closed. Prove that it is not necessarily the smallest closed subset containing $(\Lambda_f)^{-1}(B)$.

Example: Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be the map $f(x)=x^2$. Then $(\Lambda_f)^{-1}(T^*\mathbb{R})\{(x,\xi)\mid y=x^2, \xi = 2\eta x\} = T^*(\mathbb{R}\setminus\{0\})\cup\{(0,0)\}$ while $(\Lambda_f)^{\#}(T^*\mathbb{R})\{(x,\xi)\mid \exists x_n, \lim_n x_n = x, \lim_n y_n = x^2, \xi = \lim_n 2\eta_n x_n, |y_n - x_n^2|\cdot\eta_n \longrightarrow 0\} = T^*(\mathbb{R})$, since for $x=0$ we may take the sequence $x_n = 1/n, y_n = 1/n^2, \eta_n = n\xi$.

PROPOSITION 9.22. *Let $\mathscr{F}^\bullet\in D^b(Y)$ and $f:X\longrightarrow Y$ be smooth map. Then*

$$SS(f^{-1}(\mathscr{F}^\bullet))\subset(\Lambda_f)^{\#}(SS(\mathscr{F}^\bullet))$$

The following lemma deals with a special case of an embedding of euclidean spaces.

LEMMA 9.23. *Let $X=\mathbb{R}^p, Y=\mathbb{R}^{p+q}, f(x_1,...,x_p) = (x_1,...,x_p,0,...,0)$. If $(0,e_1^*)\in SS(f^{-1}(\mathscr{F}^\bullet))$ then $(0,e_1^*)\in(\Lambda f)^{\#}(SS(\mathscr{F}^\bullet))$.*

PROOF. Let $x\in\mathbb{R}^p, y\in\mathbb{R}^q$ be coordinates in $\mathbb{R}^{p+q}$. Let φ be smooth function on $\mathbb{R}^p$ such that $d\varphi(0)=e_1^*$. Then after a change of variable, we may assume $\varphi = x_1$. Let $V_{\rho,\delta} = \{(x,y)\mid\delta(x_1-\rho(x_1))\geq|y|^2\}$. We assume $\rho(0)=\varepsilon, -\frac{1}{2}\leq\rho'(x)\leq 0$ with $\varepsilon\geq 0$ and $\rho(x)=0$ for $x\leq-\alpha$. We also denote $V_\delta = V_{0,\delta}$, $V^\alpha_{\varepsilon,\delta} = V_{\rho,\delta}$. Notice that $V^\alpha_{\varepsilon,\delta}\supset V_\delta$, and $\bigcap_{\delta>0}V_\delta = \mathbb{R}^*_-$ and $\bigcap_{\alpha,\varepsilon,\delta>0}V^\alpha_{\varepsilon,\delta} = \mathbb{R}_-$.

Now we claim that

$$R\Gamma(V^\alpha_{\varepsilon,\delta},\mathscr{F}^\bullet)\longrightarrow R\Gamma(V_\delta,\mathscr{F}^\bullet) \tag{*}$$

is an isomorphism. This follows from Proposition 9.14 (see page 137), applied to the deformation $t\mapsto V_t = V^\alpha_{t\varepsilon,\delta}$. Indeed, $\delta V_t\subset\{(x,y)\mid x_1\geq-\alpha\}$ and $SS(V_t)\cap\pi^{-1}(\delta V_t)$ is contained in

$$\{(x,\xi,y,\eta)\mid\delta(x_1-t\rho(x_1)) = -|y|^2, x_1\geq-\alpha;\ \xi=\lambda\delta(1-t\rho'(x_1))e_1^*, \eta=\lambda\cdot 2y\}$$

Assume that it intersects $SS(\mathscr{F}^\bullet)$ for all $\varepsilon,\delta,\alpha$ small enough. Then there exists a sequence $(z_n,\zeta_n) = (x_n,\xi_n,y_n,\eta_n)\in SS(\mathscr{F}^\bullet)$ such that $x_n\longrightarrow 0, y_n\longrightarrow 0, \xi_n = e_1^*, \eta_n = \frac{2y_n}{\delta(1-t\rho'(x_{n,1}))}$ where $x_{n,1}$ denotes the first coordinate of x_n, we have

$$y_n\eta_n = \frac{2y_n^2}{\delta(1-t\rho'(x_{n,1}))} = \frac{2\delta(x_{n,1}-t\rho(x_{n,1}))}{\delta(1-t\rho'(x_{n,1}))} = \frac{2(x_{n,1}-t\rho(x_{n,1}))}{(1-t\rho'(x_{n,1}))}$$

and this last sequence goes to zero with n, so $0 = \lim y_n\eta_n = \lim|z_n - x_n|\cdot|\zeta_n|$. Thus if $(0,e_1^*) \notin \Lambda_f^{\#}(SS(\mathscr{F}^\bullet))$ we have that (*) is an isomorphism. Since the families $V^{\alpha}_{\varepsilon,\delta}, V_\delta$ are cofinal families for neighbourhoods of $\mathbb{R}_-, \mathbb{R}_-^*$ this implies that for r small enough

$$\varinjlim_{U\supset\mathbb{R}_-} \Gamma(U\cap B(0,r),\mathscr{I}^\bullet) \longrightarrow \varinjlim_{V\supset\mathbb{R}_-^*} \Gamma(V\cap B(0,r),\mathscr{I}^\bullet)$$

is a quasi-isomorphism. Thus $\lim_{U\supset\mathbb{R}} R\Gamma_{\{x_1\geq 0\}}(U\cap B(0,r),\mathscr{I}^\bullet) = R\Gamma_Z(\mathscr{I}^\bullet)_0 = 0$. We thus proved $(0,e_1^*) \notin SS(f^{-1}(\mathscr{F}^\bullet))$. □

REMARK 9.24. (1) The above proves the proposition for f an embedding, since the singular support is local, and an embedding is always locally equivalent to an embedding of euclidean spaces. Note that is the main case we shall use.

(2) It seems that the sequences defining $\Lambda_f^{\#}$ can be refined to $\lim_n x_n = x, \lim_n z_n = f(x), \lim_n \zeta_n\circ df(x_n) = \xi$ and $d(z_n, f(x_n))|\zeta_n| \simeq d(x_n,x)$ as n goes to infinity.

LEMMA 9.25. *Let $f: X\longrightarrow Y, g: Y\longrightarrow Z$. Assume g is a submersion. Then $\Lambda_g^{\#} = \Lambda_g^{-1}$ and for any subset C of T^*Z,*

$$\Lambda_f^{\#}\circ\Lambda_g^{\#}(C) = \Lambda_f^{\#}\circ\Lambda_g^{-1}(C) \subset \Lambda_{g\circ f}^{\#}(C)$$

PROOF. Let $(y,\eta)\in T^*Y$ and assume we have a sequence such that $\lim_n y_n = y$, $\lim_n z_n = z = g(y)$, $\zeta_n\circ dg(y_n) \longrightarrow \eta$ and $d(z_n, g(y_n))\cdot|\zeta_n| \longrightarrow 0$. But since $dg(y_n)$ is surjective, ${}^t dg(y_n)$ is injective hence $|\zeta\circ dg(y_n))| \geq C|\zeta|$ for some postitive constant C, and y_n in a neighbourhood of y. So $\zeta_n\circ dg(y_n)\longrightarrow\eta$ implies $\zeta_n\longrightarrow\zeta$ and then $\zeta\circ dg(y) = \eta$ hence $(y,\eta)\in\Lambda_g^{-1}(C)$.

Let now $(x,\xi)\in T^*X$ and $(y,\eta)\in T^*Y$. We assume $(y,\eta)\in\Lambda_g^{-1}(C)$ and $(x,\xi)\in\Lambda_f^{\#}(y,\eta)$, so that $(x,\xi)\in\Lambda_f^{\#}\circ\Lambda_g^{-1}(C)$. Let $(x_n,\xi_n,y_n,\eta_n,z_n,\zeta_n)$ be a sequence in $T^*X\times T^*Y\times T^*Z$ such that $x_n\longrightarrow x, y_n\longrightarrow y = f(x)$ and $\eta_n\circ df(x_n)\longrightarrow\xi$ and $d(y_n, f(x_n))|\eta_n|\longrightarrow 0$. Let $(z_n,\zeta_n)\in C$ be such that $g(y_n) = z_n, \zeta_n\circ dg(y_n) = \eta_n$.

Then $\zeta_n\circ dg(f(x_n))df(x_n)\longrightarrow\xi_n$, $\lim_n z_n = gf(x)$ and
$d(z_n, gf(x_n))|\zeta_n| \leq Kd(y_n, f(x_n))|\eta_n|\longrightarrow 0$ where K is an upper bound for $|dg(y)|^2$ in a neighbourhood of y. As a result, $(x,\xi)\in\Lambda_{g\circ f}^{\#}(C)$. □

PROOF OF PROPOSITION. Any map $f: X\longrightarrow Y$ is the composition of the embedding $\gamma_f: X\longrightarrow X\times Y$ given by $x\mapsto(x,f(x))$ and of the sumbersion $\pi_Y: X\times Y\longrightarrow Y$ given by $(x,y)\mapsto y$. Then

$$SS(f^{-1}(\mathscr{F}^\bullet)) = SS((\pi_Y\circ\gamma_f)^{-1}(\mathscr{F}^\bullet)) = SS(\gamma_f^{-1}\circ\pi_Y^{-1}(\mathscr{F}^\bullet)) \subset \Lambda_{\gamma_f}^{\#}\circ\Lambda_{\pi_Y}^{-1}(SS(\mathscr{F}^\bullet)) \subset$$
$$\Lambda_{\pi_Y\circ\gamma_f}^{\#}(SS(\mathscr{F}^\bullet) = \Lambda_f^{\#}(SS(\mathscr{F}^\bullet))$$

□

Let C, D be two conic subsets in T^*M.

DEFINITION 9.26. Let C,D be two closed cones. Then $C\hat{+}D$ is defined as follows: $(z,\zeta)\in C\hat{+}D$ if and only if there are sequences $(x_n,\xi_n)\in C,(y_n,\eta_n)\in D$ such that $\lim_n x_n=\lim_n y_n=z$, $\lim_n(\xi_n+\eta_n)=\zeta$ and $\lim_n|x_n-y_n||\xi_n|=0$. We write $C\hat{+}D=(C+D)+C\underset{\infty}{\hat{+}}D$.

Note that $C+D$ is not necessarily a closed cone, but $C\hat{+}D$ is closed. Note also that $C\hat{+}D=(\Lambda_d)^{\#}(C\times D)$ where $d:X\longrightarrow X\times X$ is the diagonal map. We have the equality $C+D=C\hat{+}D$ if $C\cap\overline{D}\subset 0_X$, where $\overline{D}=\{(x,-\xi)\mid(x,\xi)\in D\}$. Indeed, it is enough to prove that if $\xi_n+\eta_n$ converges, then ξ_n and η_n converge. But for each x we may set $s(x)=\inf\{|\xi+\eta|\mid|\xi|+|\eta|=1,(x,\xi)\in C,(x,\eta)\in D\}$ and $s(x)>0$ so in a neighbourhood of x_0 there exists $\varepsilon>0$ such that $s(x)>\varepsilon$, hence $|\xi+\eta|\geq\varepsilon(|\xi|+|\eta|)$

PROPOSITION 9.27. *We have for* $\mathscr{F}^\bullet,\mathscr{G}^\bullet$ *in*

$$SS(\mathscr{F}\boxtimes^L\mathscr{G})\subset SS(\mathscr{F})\times SS(\mathscr{G})$$

$$SS(\mathscr{F}\otimes^L\mathscr{G})\subset SS(\mathscr{F})\hat{+}SS(\mathscr{G})$$

PROOF. Again, we limit ourselves to the situation of complexes of $\mathbb{C}$-modules sheaves, so that $\boxtimes^L,\otimes^L,RHom$ coincide with $\boxtimes,\otimes,\mathscr{H}om$, since vector spaces are always projective and injective. Note that the second equality follows from the first, since if $d:X\to X\times X$ is the diagonal map, we have $\mathscr{F}\otimes\mathscr{G}=d^{-1}(\mathscr{F}\boxtimes\mathscr{G})$, and

Pas sûr !

$$SS(d^{-1}\mathscr{F})=(\Lambda_d)^{\#}(SS(\mathscr{F})\times SS(\mathscr{G}))$$

but

$$\Lambda_d^{-1}=\{(x_1,\xi_1,x_2,\xi_2,x_3,\xi_3)\mid x_1=x_2=x_3,\xi_3=\xi_1+\xi_2\}$$

therefore $(\Lambda_d)^{\#}(SS(\mathscr{F})\times SS(\mathscr{G}))$ is equal to $SS(\mathscr{F})\hat{+}SS(\mathscr{G})$.

Let us now prove the first statement. Since the property is local, we may assume X,Y are euclidean spaces. Let $(x_0,\xi_0)\notin SS(\mathscr{F}^\bullet)$. We shall prove that for any $(y_0,\eta_0)\in T^*Y$, we have $(x_0,\xi_0,y_0,\eta_0)\notin SS(\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)$. We may replace $\mathscr{F}^\bullet,\mathscr{G}^\bullet$ by their injective Cartan-Eilenberg resolutions. Let $U\subset U'$ and $V\subset V'$ be neighbourhoods of x_0 and y_0 respectively, and A_t,B_t be families of open sets with smooth boundary parametrized by $t\in]-\varepsilon,\varepsilon[$ such that

(1) If $s<s'$ we have $A_s\subset A_{s'},B_s\subset B_{s'}$
(2) We have $A_s=A_{s'},B_s=B_{s'}$ for $(s,s')\in]-\infty,-\varepsilon[^2\cup]\varepsilon,+\infty[^2$
(3) $A_s\cap(X\setminus U')=A_{s'}\cap(X\setminus U')$ and $B_s\cap(X\setminus V')=B_{s'}\cap(X\setminus V')$
(4) $A_s\cap U=\{x\in U,\mid\langle\xi,x-x_0\rangle<s\}$ and $B_s\cap V=\{y\in V,\mid\langle\eta,y-y_0\rangle<s\}$
(5) $SS(A_s)\cap\pi_X^{-1}(U')\cap SS(\mathscr{F}^\bullet)=\emptyset$
(6) $A_{-\varepsilon}\cap U=\emptyset,A_\varepsilon\supset U$ and $B_{-\varepsilon}\cap V=\emptyset,B_\varepsilon\supset V$

We set $C_t=\bigcup_s A_{t-s}\times B_s$. Then $C_t\cap(U\times V)=\{(x,y)\in U\times V\mid\langle\xi,x-x_0\rangle+\langle\eta,y-y_0\rangle<t\}$ Therefore for $t>0$ small enough, and s close to zero, the inclusion $A_s\longrightarrow A_{s+t}$ induces a quasi-isomorphism $\Gamma(A_{s+t},\mathscr{F}^\bullet)\longrightarrow\Gamma(A_s,\mathscr{F}^\bullet)$, hence a quasi-isomorphism from $\Gamma(A_{t-s}\times B_s,\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)\longrightarrow\Gamma(A_{-s}\times B_s,\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)$

Notice that $A_{t-s}\times B_s \cap A_{t-s'}\times B_{s'} = A_{t-\sigma}\times B_{\sigma'}$ where $\sigma=\max\{s,s'\}$ and $\sigma'=\min\{s,s'\}$. We may thus apply the following lemma.

LEMMA 9.28. *Let $X=\bigcup_{s\in I}X_s$, and $Y\subset X$ such that $Y=\bigcup_{s\in I}Y_s$ where $Y_s\subset X_s$ and $H^*(X_s,Y_s;\mathscr{F}^\bullet)=0$. We assume for all finite sequences $s_1,\dots,s_r$ we have*

$$H^*\left(\bigcap_{i=1}^r X_{s_i},\bigcap_{i=1}^r Y_{s_i},\mathscr{F}^\bullet\right)=0$$

Then $H^*(X,Y,\mathscr{F}^\bullet)=0$.

We finish the proof of the proposition before proving the lemma. Indeed applying the lemma to $X=C_t, Y=C_0, X_s=A_{t-s}\times B_s$ and $Y_s=A_{-s}\times B_s$, and we get that we have a quasi-isomorphism from $\Gamma(C_t,\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)$ to $\Gamma(C_0,\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)$. Indeed, we have

$$\bigcap_{i=1}^r X_{s_i}=A_{t-\sigma}\times B_{\sigma'}$$

where $\sigma=\sup\{s_j\},\sigma'=\inf\{s_j\}$ and we have to check that

$$H^*(A_{t-\sigma}\times B_{\sigma'},A_{-\sigma}\times B_{\sigma'},\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)=0$$

But this follows by Kunneth's formula since there is a spectral sequence converging to

$$H^*(A_{t-\sigma}\times B_{\sigma'},A_{-\sigma}\times B_{\sigma'},\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)$$

starting from E_2 term

$$H^*(A_{t-\sigma},A_\sigma,\mathscr{F}^\bullet)\otimes H^*(B_{\sigma'},\mathscr{G}^\bullet)=0$$

Applying the lemma, we see that

$$H^*(C_t,C_0,\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)=0$$

and this implies that $(x_0,\xi_0,y_0,\eta_0)\notin SS(\mathscr{F}^\bullet\boxtimes\mathscr{G}^\bullet)$.

PROOF OF THE LEMMA. We may replace $\mathscr{F}^\bullet$ by an injective complex of sheaves.On each X_s the map $\Gamma(X_s,\mathscr{I}^\bullet)\longrightarrow\Gamma(Y_s,\mathscr{I}^\bullet)$ is a quasi-isomorphism. Let us consider the following statement

(*) The lemma holds for I of finite cardinal k.

We are going to prove this statement by induction. For $k=1$ the statement is just the assumption. Assume we proved the lemma when X is a union of $k-1$ of the X_s, and Y the union of the $k-1$ corresponding Y_s. Then let us prove it for a union of k such sets. So $X=\bigcup_{j=1}^{k+1}X_{s_j}=X'\cup X_{s_k}$ where $X'=\bigcup_{j=1}^k X_{s_j}$, and $Y=Y'\cup Y_{s_k}$ where $Y'=\bigcup_{i=1}^{k-1}Y_{s_j}$. We know that $H^j(X',Y',\mathscr{F}^\bullet)=0$ and that $H^j(X_s,Y_s,\mathscr{F}^\bullet)=0$. We also notice that $X'\cap X_{s_k}=\bigcup_{j=1}^{k-1}(X_{s_j}\cap X_{s_k})$ and since for each intersection $(X_{s_j}\cap X_{s_k})$ is an $X_{s'_j}$, we have that $X'\cap X_{s_k}$ (resp. $Y'\cap Y_{s_k}$) is the intersection of $k-1$ sets of the type X_s (resp. Y_s).

Now applying the Mayer-Vietoris (Proposition 7.13, page 85) exact sequence, we get

$$\begin{tikzcd} H^{j-1}(X_{s_k})\oplus H^{j-1}(X') \arrow[r] \arrow[d] & H^{j-1}(X'\cap X_{s_k}) \arrow[r] \arrow[d] & H^j(X'\cup X_{s_k}) \arrow[r] \arrow[d] & H^j(X_{s_k})\oplus H^j(X') \arrow[r] \arrow[d] & H^j(X'\cap X_{s_k})) \arrow[d] \\ H^{j-1}(Y_{s_k})\oplus H^{j-1}(Y') \arrow[r] & H^{j-1}(Y'\cap Y_{s_k})) \arrow[r] & H^j(Y'\cup Y_{s_k}) \arrow[r] & H^j(Y')\oplus H^j(Y') \arrow[r] & H^j(Y'\cap Y_{s_k}) \end{tikzcd}$$

we see that the vertical arrows other than the middle one are isomorphisms by induction assumption, and then according to the five lemma (see Lemma 6.23 page 67) the middle one is also an isomorphism.

Now, we see that our statement holds for any finite union of X_s. Now let Z_s be an increasing union of sets such that $\bigcup_s Z_s = X$ and that Z_s is a finite union of X_s, $Z_s = \bigcup_{t\in I_s} X_t$ and that similarly $W_s = \bigcup_{t\in I_s} Y_t$ is an increasing family with $\bigcup_s W_s = Y$. We just proved that $H^*(Z_s, W_s, \mathscr{F}^\bullet) = 0$. According to Proposition 7.44 on page 97, we have $H^*(X, Y, \mathscr{F}^\bullet) = 0$ hence thus the map $\Gamma(X,\mathscr{F}^\bullet) \longrightarrow \Gamma(Y,\mathscr{F}^\bullet)$ is a quasi-isomorphism. □

LEMMA 9.29 ([**K-S**], 2.6.6, p. 112, [**Iv**], p.320). *Let $f: X \to Y$ be a continuous map, and $\mathscr{F} \in D^b(X), G \in D^b(Y)$. Then*

$$Rf_!(\mathscr{F}^\bullet \otimes^L f^{-1}\mathscr{G}^\bullet) = Rf_!(\mathscr{F}^\bullet)\otimes^L \mathscr{G}^\bullet$$

PROOF. Again, we do not consider the derived tensor products, since we are dealing with $\mathbb{C}$-vector spaces. Then, there is a natural isomorphism from

$$f_!(\mathscr{F})\otimes\mathscr{G} \simeq f_!(\mathscr{F}\otimes f^{-1}(\mathscr{G}))$$

□

LEMMA 9.30 (Base change theorem ([**Iv**], p. 322). *Let us consider the following* **cartesian square** *of maps,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[d, "g"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

that is the square is commutative, and A is isomorphic to the fiber product $B\times_D C$. Then $Ru_! \circ f^{-1} = v^{-1}\circ Rg_!$

3.1. Resolutions of constant sheaves, the DeRham and Morse complexes. Let $W(f) = \{(x,\lambda) \mid f(x) \le \lambda\}$. We consider k_f the constant sheaf over $W(f)$, and we saw we have a quasi-isomorphism 3.1 (se page 94), between k_f and $\Omega_f^\bullet$ the set of differential forms on $W(f)$. Moreover according to LePeutrec-Nier-Viterbo ([**LeP-N-V**]), there is a quasi-isomorphism from $\Omega_f^\bullet$ to $BM_f^\bullet$ the Barannikov-Morse complex of f.

Also if f is a Morse function on X, we have that $Rf_*(k_X)$ is a sheaf over $\mathbb{R}$ and its singular support $SS(Rf_*(k_X))$ is of the form

$$0_{\mathbb{R}} \cup \bigcup_i \{c_i\} \times \mathbb{R}$$

4. Quantization of symplectic maps

We assume in this section that X, Y, Z are manifolds. Now we want to quantize symplectic maps in T^*X, that is to a homogeneous Hamiltonian symplectomorphism $\Phi : T^*X \to T^*Y$ we want to associate a map $\widehat{\Phi} : D^b(X) \to D^b(Y)$. There are (at least) two posibilites to do that, and one should not be surprised. In microlocal analysis, there are several possible quantizations from symbols to operators: pseudodifferential, Weyl, coherent state, etc...

Define $q_X : X \times Y \to X$ (resp. $q_Y : X \times Y \to Y$) and $q_{XY} : X \times Y \times Z \to X \times Y$ (resp. $q_{XZ} : X \times Y \times Z \to X \times Z$, $q_{XY} : X \times Y \times Z \to Y \times Z$) be the projections.

DEFINITION 9.31. Let $\mathscr{K} \in D^b(X \times Y)$. We then define the following operators: for $\mathscr{F} \in D^b(X)$ and $\mathscr{G} \in D^b(Y)$ define

$$\Psi_{\mathscr{K}}(\mathscr{F}) = (Rq_{Y*})(RHom(\mathscr{K}, q_X^!(\mathscr{F})))$$

$$\Phi_{\mathscr{K}}(\mathscr{G}) = (Rq_{X!})(\mathscr{K} \otimes^L q_Y^{-1}(\mathscr{G}))$$

Then $\Psi_{\mathscr{K}}, \Phi_{\mathscr{K}}$ are operators from $\mathscr{D}^b(X)$ to $D^b(Y)$ and $\mathscr{D}^b(Y)$ to $D^b(X)$ respectively.

REMARK 9.32. (1) The method is reminiscent of the definition of operators on the space of C^k functions using kernels.

(2) For the sake of completeness, we have used the derived functor language in all cases. However, for sheaves in the category of finite dimensional vector spaces, $R\mathscr{H}om = \mathscr{H}om$ and $\otimes^L = \otimes$. Also, if the projections are proper, i.e. if X, Y are compact, $R(q_{X!}) = R(q_{X*})$

(3) In the category of coherent sheaves over a projective algebraic manifold, the above definition extends to the Fourier-Mukai transform. Indeed if $\mathscr{K} \in D^b_{Coh}(X \times Y)$ is an element in the derived category of the coherent sheaves on the product of two algebraic varieties, the Fourier-Mukai transform from $D^b_{Coh}(X)$ to $D^b_{Coh}(Y)$ is defined as

$$\Phi_{\mathscr{K}}(\mathscr{G}) = (Rq_{X*})(K \otimes^L q_Y^{-1}(\mathscr{G}))$$

Consider Mirror symmetry as an equivalence of categories $\mathscr{M} : Fuk(T^*X) \longrightarrow D^b(X)$ sending $\mathrm{Mor}(L_1, L_2) = FH^*(L_1, L_2)$ to $\mathrm{Mor}_{D^b}(\mathscr{M}(L_1), \mathscr{M}(L_1))$. Moreover, let us consider the functor $SS : D^b(X) \longrightarrow Fuk(T^*X)$. This should send the element $\Phi_{\mathscr{K}} \in \mathrm{Mor}(D^b(X), D^b(Y))$ to the Lagrangian correspondence, $\Lambda_{SS(\mathscr{K})} : T^*X \longrightarrow T^*Y$. Vice-versa any such Lagrangian correspondence can be quantized, for example for each exact embedded Lagrangian L we can find $\mathscr{F}$ such that $SS(\mathscr{F}) = L$. We shall see this can be done using Floer homology. Can one use other methods, for example the theory of Fourier integral operators ?

(4) According to [**K-S**] proposition 7.1.8, the two functors $\Phi_{\mathscr{K}}, \Psi_{\mathscr{K}}$ are adjoint functors.

The sheaf $\mathscr{K}$ is called the **kernel** of the transform (or functor).We say that $\mathscr{K} \in D^b(X \times Y)$ is a **good kernel** if the map

$$SS(K) \longrightarrow T^*X$$

is proper. We denote by $N(X, Y)$ the set of good kernels. Note that any sheaf $\mathscr{F} \in D^b(X)$ can be considered as a kernel in $D^b(X) = D^b(X \times \{pt\})$, and it automatically belongs to $N(X, \{pt\})$, because $SS(\mathscr{F}) \to T^*X$ is trivially proper. We shall see that transforms defined by kernels can be composed, and, in the case of good kernels, act on the singular support in the way we expect. Let X, Y, Z three manifolds, and q_X (resp. q_Y, q_Z) be the projection of $X \times Y \times Z$ on X (resp. Y, Z) and q_{XY} (resp. q_{YZ}, q_{XZ}) be the projections on $X \times Y$ (resp. $Y \times Z, X \times Z$). Similarly π_{XY} etc... are the projections $T^*X \times T^*Y \times T^*Z \to T^*X \times T^*Y$.

We may now state

PROPOSITION 9.33. *Let* $\mathscr{K}_1 \in D^b(X \times Y)$ *and* $\mathscr{K}_2 \in D^b(Y \times Z)$. *Set*

$$\mathscr{K} = (Rq_{XZ})_!(q_{XY}^{-1}(\mathscr{K}_1) \otimes^L q_{YZ}^{-1}(\mathscr{K}_2))$$

Then $\mathscr{K} \in D^b(X \times Z)$, *and* $\Psi_{\mathscr{K}} = \Psi_{\mathscr{K}_2} \circ \Psi_{\mathscr{K}_1}$ *and* $\Phi_{\mathscr{K}} = \Phi_{\mathscr{K}_1} \circ \Phi_{\mathscr{K}_2}$. *We will denote* $\mathscr{K} = \mathscr{K}_2 \circ \mathscr{K}_1$.

PROOF. Consider the following diagram

$$\begin{tikzcd}
& X \times Y \times Z \arrow[dl, "q_{XY}"'] \arrow[d, "q_{XZ}"] \arrow[dr, "q_{YZ}"] & \\
X \times Y \arrow[d, "q_X^{XY}"'] \arrow[dr, "q_Y^{XY}"] & X \times Z \arrow[dl, "q_X^{XZ}"] \arrow[dr, "q_Z^{XZ}"] & Y \times Z \arrow[dl, "q_Y^{YZ}"] \arrow[d, "q_Z^{YZ}"] \\
X & Y & Z
\end{tikzcd}$$

Let $\mathscr{G} \in D^b(Z)$. We first claim that

$$(\star) \qquad (Rq_X^{XY})_!(\mathscr{K}_1 \otimes (q_Y^{XY})^{-1}((Rq_Y^{YZ})_!(\mathscr{K}_2 \otimes (q_Z^{YZ})^{-1}(\mathscr{G})))) = (Rq_X)_!(q_{XY}^{-1}(\mathscr{K}_1) \otimes q_{YZ}^{-1}(\mathscr{K}_2) \otimes q_Z^{-1}(\mathscr{G}))$$

The cartesian square with vertices $X \times Y \times Z, X \times Y, Y \times Z, Y$ and lemma 9.30, page 144 yields an isomorphism between the image of $\mathscr{K}_2 \otimes (q_Z^{YZ})^{-1}(\mathscr{G})$ by $(Rq_{XY})_! q_{YZ}^{-1}$ and its image by $(q_Y^{XY})^{-1}(Rq_Y^{YZ})_!$. The first image is

$$(Rq_{XY})_!q_{YZ}^{-1}(\mathcal{K}_2\otimes(q_Z^{YZ})^{-1}(\mathcal{G}))=(Rq_{XY})_!(q_{YZ}^{-1}(\mathcal{K}_2)\otimes q_{YZ}^{-1}\circ(q_Z^{YZ})^{-1}(\mathcal{G}))=$$
$$(Rq_{XY})_!(q_{YZ}^{-1}(\mathcal{K}_2)\otimes q_Z^{-1}(\mathcal{G}))$$

using for the last equality that $q_Z^{YZ}\circ q_{YZ}=q_Z$.

This is thus equal to

$$(q_Y^{XY})^{-1}(Rq_Y^{YZ})_!(\mathcal{K}_2\otimes(q_Z^{YZ})^{-1}(\mathcal{G})))$$

Apply now $\otimes\mathcal{K}_2$ and then $(Rq_X^{XY})_!$, we get

$$(Rq_X^{XY})_!(\mathcal{K}_1\otimes(q_Y^{XY})^{-1}(Rq_Y^{YZ})_!(\mathcal{K}_2\otimes(q_Z^{YZ})^{-1}(\mathcal{G})))$$

for the first term and

$$(Rq_X^{XY})_!(\mathcal{K}_1\otimes(Rq_{XY})_!(q_{YZ}^{-1}(\mathcal{K}_2)\otimes q_Z^{-1}(\mathcal{G}))))$$

for the second term.

Using lemma 9.29 applied to $f=q_{XY}$, we get

$$\mathcal{F}\otimes(Rq_{XY})!\mathcal{G}=(Rq_{XY})!(q_{XY}^{-1}(\mathcal{F})\otimes\mathcal{G})$$

hence applying $(Rq_X^{XY})_!$ and using the composition formula $(Rq_X^{XY})_!\circ(Rq_{XY})_!=(Rq_X)_!$, we get

$$(Rq_X^{XY})_!(\mathcal{K}_1\otimes(Rq_{XY})_!(q_{YZ}^{-1}(\mathcal{K}_2)\otimes q_Z^{-1}(\mathcal{G}))=(Rq_X)_!(q_{XY}^{-1}(\mathcal{K}_1)\otimes q_{YZ}^{-1}(\mathcal{K}_2)\otimes q_Z^{-1}(\mathcal{G})))$$

This proves our equality.

We must prove the right hand side above is equal to

$$(Rq_X^{XZ})!((Rq_{XZ})_!(q_{XY}^{-1}(\mathcal{K}_1)\otimes^L q_{YZ}^{-1}(\mathcal{K}_2))\otimes(q_Z^{XZ})^{-1}(\mathcal{G}))$$

But

$$(Rq_{XZ})_!(\mathcal{F}\otimes(q_{XZ})^{-1}(\mathcal{G}))=(Rq_{XZ})_!(\mathcal{F})\otimes\mathcal{G}$$

and $(Rq_X^{XZ})_!\circ(Rq_{XZ})_!=(Rq_X)_!$, so

$$(Rq_X^{XZ})!((Rq_{XZ})_!(q_{XY}^{-1}(\mathcal{K}_1)\otimes^L q_{YZ}^{-1}(\mathcal{K}_2))\otimes(q_Z^{XZ})^{-1}(\mathcal{G}))=$$
$$(Rq_X^{XZ})!(Rq_{XZ})_!(q_{XY}^{-1}(\mathcal{K}_1)\otimes^L q_{YZ}^{-1}(\mathcal{K}_2)\otimes q_{XZ}^{-1}(q_Z^{XZ})^{-1}(\mathcal{G}))=$$
$$(Rq_X)_!(q_{XY}^{-1}(\mathcal{K}_1)\otimes^L q_{YZ}^{-1}(\mathcal{K}_2))\otimes q_Z^{-1}(\mathcal{G})$$

$\square$

The next proposition tells us that Φ_K,Ψ_K act as expected on $SS(\mathcal{F})$.

PROPOSITION 9.34 ([**K-S**], Proposition 7.1.2). *We assume $\mathcal{K}\in D^b(X\times Y)$ and $\mathcal{L}\in D^b(Y\times Z)$ are good kernels. Then $\mathcal{K}\circ\mathcal{L}$ is a good kernel and*

$$SS(\mathcal{K}\circ\mathcal{L})=SS(\mathcal{K})\circ SS(\mathcal{L})$$

In particular,

$$SS(\Psi_{\mathcal{K}}(\mathcal{F})) \subset \pi_Y^a(SS(\mathcal{K}) \cap \pi_X^{-1}(SS(\mathcal{F})) = SS(\mathcal{K}) \circ SS(\mathcal{F})$$

$$SS(\Phi_{\mathcal{K}}(\mathcal{G})) \subset \pi_{XZ}(SS(\mathcal{K}) \times_{T^*Y} SS(\mathcal{G})) = SS(\mathcal{K})^{-1} \circ SS(\mathcal{G})$$

PROOF. We first notice that the properness assumption for good kernels implies that

$$(*) \qquad \pi_{XY}^{-1}(SS(\mathcal{K})) \underset{\infty}{\widehat{+}} \pi_{YZ}^{-1}(SS(\mathcal{L})) = \emptyset$$

Indeed, a sequence $(x_n, y_n, \xi_n, \eta_n)$ and $(y_n', z_n, \eta_n', \zeta_n)$ respectively in $SS(\mathcal{K})$ and $SS(\mathcal{L})$ such that

$$(9.1) \quad \lim_n x_n = x_\infty, \lim_n y_n = \lim_n y_n' = y_\infty, \lim_n z_n = z_\infty, \lim_n \xi_n = \xi_\infty, \lim_n (\eta_n + \eta_n') = \eta_\infty$$

By properness of the projection $SS(\mathcal{K}) \longrightarrow T^*X$, we have that the sequence η_n is bounded, hence η_n' is also bounded, and this proves $(*)$. Now we have

$$SS(q_{XY}^{-1}(\mathcal{K}) \otimes^L q_{YZ}^{-1}(\mathcal{L})) \subset \pi_{XY}^{-1}(SS(\mathcal{K})) + \pi_{XY}^{-1}(SS(\mathcal{L}))$$

Then

$$SS(R_{q_{XZ}!}(q_{XY}^{-1}(\mathcal{K}) \otimes^L q_{YZ}^{-1}(\mathcal{L}))) \subset \Lambda_{q_{XZ}}(SS((q_{XY}^{-1}(\mathcal{K}) \otimes^L q_{YZ}^{-1}(\mathcal{L}))) =$$
$$\Lambda_{q_{XZ}}(\pi_{XY}^{-1}(SS(\mathcal{K})) + \pi_{YZ}^{-1}(SS(\mathcal{L}))) = SS(\mathcal{K}) \circ SS(\mathcal{L})$$

□

REMARK 9.35. Assume $X = Y$ and $SS(\mathcal{K})$ be the graph of a symplectomorphism, then set $\mathcal{K}^a \in D^b(Y \times X)$ to be the direct image by $\sigma(x, y) = (y, x)$ of $\mathcal{K}$ (i.e. $\mathcal{K}^a = \sigma_*\mathcal{K}$). Then set for a Lagrangian in $T^*X \times \overline{T^*X}$, $L^a = \{(y, \eta, x, \xi) \mid (x, \xi, y, \eta) \in L\}$. Then $SS(\mathcal{K}^a) = SS(\mathcal{K})^a \subset T^*X \times \overline{T^*X}$, and $\Psi_{\mathcal{K}} \circ \Psi_{\mathcal{K}^a} = \Psi_{\mathcal{L}}$ where $SS(\mathcal{L}) = SS(\mathcal{K}) \circ SS(\mathcal{K}^a) = SS(\mathcal{K}) \circ SS(K)^a = SS(\mathrm{Id}) = \Delta_{T^*X}$.

From this we can prove the following result. Even though we technically do not use it in concrete questions (our singular support will be Lagrangian by construction), the following is an essential result, due to Kashiwara-Schapira ([**K-S**], theorem 6.5.4), Gabber [**Ga**] (for the general algebraic case)

PROPOSITION 9.36 (Involutivity theorem). *Let $\mathcal{F}^\bullet$ be an element in $D^b(X)$. Then $SS(\mathcal{F}^\bullet)$ is a coisotropic submanifold.*

Some remarks are however in order. Proving that $C = SS(\mathcal{F}^\bullet)$ is coisotropic is equivalent to proving that given any hypersurface Σ such that $C \subset \Sigma$, the characteristic vector field X_Σ of Σ is tangent to C. Besides, this is a local property, so we may asume we are in a neighbourhood of $0 \in \mathbb{R}^n$. Now consider the example $C \subset \Sigma = \{(q, p) \mid \langle v, q\rangle = 0\}$. Then $X_\Sigma = \mathbb{R}(0, v)$. Now remember that $C \cap 0_{\mathbb{R}^n} = \mathrm{supp}(\mathcal{F})$. Thus if $\mathcal{F}$ is nonzero near 0, since our assumption implies that $\mathrm{supp}(\mathcal{F}) \subset \{q \mid \langle v, q\rangle = 0\}$, whenever we move in the v direction, we certainly change $\mathcal{F}_x$, hence $\mathbb{R}(0, v) \subset C$.

Let us start with the case $M=\mathbb{R}^n$. We wish to prove that for a sheaf $\mathscr{F}$, $SS(\mathscr{F})$ cannot be contained in $\{q_1=p_1=0\}$. Indeed, let $f:\mathbb{R}^n\to\mathbb{R}$ be the projection on q_1. Then $\Lambda_f\circ SS(\mathscr{F})\subset\{0\}\subset T^*\mathbb{R}$. Thus $Rf_*\mathscr{F}$ is a sheaf on $\mathbb{R}$ with singular support contained in $\{0\}$.

Here we should rather consider the embedding $j:\mathbb{R}\to\mathbb{R}^n$ given by $j(x)=(x,0,...0)$, and $j^{-1}(SS(\mathscr{F}))$ has singular support $\Lambda_j^{-1}(SS(\mathscr{F}))\subset\{(0,0)\}$ and now use the fact that $SS(f^{-1}(\mathscr{F}))=\Lambda_f^{-1}(SS(\mathscr{F}))$. Assume we could find such a sheaf. Then $SS(\mathscr{F})$ being conic, locally, it either contains vertical lines, or is contained in a singleton. We may thus assume $SS(\mathscr{F})=\{0\}$ and find a contradiction. But locally $SS(\mathscr{F})\subset\{0\}$ implies $\mathrm{supp}(\mathscr{F})\subset\{0\}$ hence $F=\mathscr{F}_x$ is a sky-scraper sheaf at points of $f^{-1}(y)$, and $SS(\mathscr{F})=T_0^*\mathbb{R}$ a contradiction. A way to rephrase this is that the singular support can not be too small. In fact the proof can be reduced to the above.

LEMMA 9.37. *Let C_0 be a homogeneous submanifold of T^*X and $(x_0,p_0)\in C_0$. There is a homogeneous Lagrangian correspondence Λ, such that $C=\Lambda\circ C_0$ sends $T_{(x_0,p_0)}C_0$ to $T_{(x,p)}C$. If we moreover assume C_0 is not coisotropic, we may find local homogeneous coordinates $T_{(x,p)}C\subset\{(x,p)\mid x_1=p_1=0\}$*

PROOF. A space is coisotropic if and only if it is contained in no proper symplectic subspace. Let H be a hyperplane, ξ a vector transverse to H, C a subspace containing ξ. Assume ☐

PROOF OF THE INVOLUTIVITY THEOREM. Let us coinsider $C_0=SS(\mathscr{F})$ and assume we are at a smooth point (x_0,p_0) which is not coisotropic. Because the result is local, we may always assume we are working on $T^*\mathbb{R}^n$. Then there exists a local symplectic diffeomorphism, sending (x_0,p_0) to $(0,p_0)$ sending C_0 to C, such that $T_{(0,p_0)}C\subset\{x_1=p_1=0\}$. By applying a further C^1 small symplectic ap, we may assume Now let Λ be the correspondence in $T^*\mathbb{R}^n\times T^*\mathbb{R}^2$ given by $\{x_1=t_1,...,x_n=t_n,x_{n+1}=t_{n+1},p_1=t_{n+1},p_2=t_2,...,p_{n+1}=t_{n+1}\}$, and $\mathscr{K}$ the corresponding kernel. Then $\Lambda\circ C$ is obtained by projecting on $T^*\mathbb{R}$ the intersection $\Lambda\cap(C\times T^*\mathbb{R})$. We have near (x,p) that $\Lambda\cap C\times T^*\mathbb{R}\subset\{x_1=x_{n+1}=p_1=p_{n+1}=0\}$ so that the projection on $T^*\mathbb{R}^2$ is contained in $\{0,0\}$. But we proved that this is impossible, since this would mean that $\mathscr{K}\circ\mathscr{F}$ satisfies locally $SS(\mathscr{K}\circ\mathscr{F})\subset\{0\}$. ☐

PROOF OF THE LEMMA. Clearly if V is a proper symplectic subspace and $C\subset V$ be isotropic, we have $V^\omega\subset C^\omega$, but C^ω is isotropic, a contradiction.

☐

DEFINITION 9.38. A sheaf is **constructible** if and only if there is a stratification of X, such that $\mathscr{F}^\bullet$ is locally constant on each strata.

PROPOSITION 9.39. *If $\mathscr{F}^\bullet$ is constructible, then it is Lagrangian.*

PROOF. We refer to the existing literature, since we will not really use this proposition: our singular supports will be Lagrangian by construction. We can actually take this as the definition of constructible.

□

Example: Let us consider a contructible sheaf, $\mathscr{F}$ on $\mathbb{R}$. Then $SS(\mathscr{F})$ is Lagrangian. So it contains a piece of the zero section corresponding to the support of $\mathscr{F}$, and half lines $\{c_j\}\times\mathbb{R}^+$ or $\{c_j\}\times\mathbb{R}^-$. However the following turns out to be useful.

DEFINITION 9.40. We shall say that a sheaf on a metric space is **locally stable** if for any x there is a positive δ such that for all $\varepsilon\in]0,\delta[$, we have $H^*(B(x,\varepsilon),\mathscr{F}^\bullet)\to H^*(\mathscr{F}_x^\bullet)$ is an isomorphism.

PROPOSITION 9.41. *Constructible sheaves are locally stable*

5. Appendix: More on sheaves and singular support

5.1. The Mittag-Leffler property. The question we are dealing with here, is to whether $R\Gamma(V,\mathscr{F})=\varprojlim_{V\subset U}R\Gamma(U,\mathscr{F})$. Notice that by defintion of sheaves, we have

$$\Gamma(U,\mathscr{F})=\varprojlim_{V\subset U}\Gamma(U,\mathscr{F})$$

so our question deals with the commutation of inverse limit and cohomology. Note that on metric spaces all these limits can be taken to be limits for a countable sequence, and inverse limite being left-exact has a right-derived functor. One usually notes

$$\overset{n}{\varprojlim_{p}}=R^n\varprojlim_{p}.$$

The Mittag-Leffler property guarantees that a certain limit has $\overset{n}{\varprojlim_{p}}=0$ for $n\geq 1$.

DEFINITION 9.42 (Mittag-Leffler property). Let $f_{ij}:A_j\to A_i$ for $i\leq j$ be an inverse system. We say that the system satisfies the Mittag-Leffler property, if it is stationary: i.e. for each i, there is k such that for all $l\geq k$ we have $f_{ik}(A_k)=f_{il}(A_l)$.

5.2. Convolution of sheaves. Let $s:E\times E\to E$ be defined by $s(u,v)=u+v$. then $\Lambda_s\in T^*(E\times E\times E)$ is given by

$$\Lambda_s=\{(u,\xi,v,\eta,w,\zeta)\mid w=u+v,\zeta=\eta=\xi\}$$

DEFINITION 9.43 (Convolution). Let E be a real vector space, and $s:E\times E\to E$ be the map $s(u,v)=u+v$. We similarly denote by s the map $s:(X\times E)\times(Y\times E)\to(X\times Y)\times E$ given by $s(x,u,y,v)=(x,y,u+v)$. Let $\mathscr{F},\mathscr{G}$ be sheaves on $X\times E$ and $Y\times E$. We set

$$\mathscr{F}*\mathscr{G}=Rs_!(q_X^{-1}\mathscr{F}\boxtimes^L q_Y^{-1}\mathscr{G})$$

This is a sheaf on $D^b(X\times Y\times E)$. where $q_X:X\times Y\times E\to X\times E$ and $q_Y X\times Y\times E\to Y\times E$ are the projections.

EXERCICE 9.44 ([**K-S**] page 135-exercice II.20)).
(1) Prove that the operation $*$ is commutative and associative.

(2) Prove that $k_{\{0\}} * \mathscr{G} = \mathscr{G}$.
(3) Let $U(f) = \{(x,u) \in X \times \mathbb{R} \mid f(x) \leq u\}$, $V(g) = \{(y,v) \in Y \times \mathbb{R} \mid g(y) \leq v\}$, and $W(h) = \{(x,y,w) \in X \times \mathbb{R} \mid h(x,y) \leq w\}$. Then $k_{U(f)} * k_{V(g)} = k_{W(f\oplus g)}$ where $(f \oplus g)(x,y) = f(x) + g(y)$.
(4)

$$SS(\mathscr{F} * \mathscr{G}) = \Lambda_s \circ (SS(\mathscr{F}) \times SS(\mathscr{G})) =$$

$$\{(x,p_x,y,p_y,w,\eta) \mid \exists (x,p_x,u,\eta) \in SS(\mathscr{F}), \exists (x,p_x,v,\eta) \in SS(\mathscr{G}), w = u+v\}$$

As a consequence

$$SS(k_{U(f)}) * k_{V(g)}) = SS(k_{W(f\oplus g)})$$

(5) Prove that if $S : X \times \mathbb{R}^p \longrightarrow \mathbb{R}$ and $T : Y \times \mathbb{R}^q \longrightarrow \mathbb{R}$ are g.f.q.i. then $R\pi_*(k_{U_S} * k_{U_T})) = R\pi_*(k_{U_{S\oplus T}})$ where $(S \oplus T)(x,y,\xi,\eta) = S(x,\xi) + T(y,\eta)$.
(6) Let us consider a function $g(u,v)$ on $E\times E$ and assume $(u, \frac{\partial g}{\partial u}(u,v)) \to (v, -\frac{\partial g}{\partial v}(u,v))$ define a (necessarily Hamiltonian) map φ_g. Then, let Φ_g be the operator $\mathscr{F} \to k_{W(g)} * \mathscr{F}$. Prove that $SS(\Phi_g(\mathscr{F})) \subset \varphi_g(SS(\mathscr{F}))$.

Note that one can define the adjoint functor of the convolution, $RHom^*$ satisfying $\mathrm{Mor}(\mathscr{F} * \mathscr{G}, \mathscr{H}) = \mathrm{Mor}(F, RHom^*(\mathscr{G},\mathscr{H}))$.

DEFINITION 9.45. We set

$$R\mathscr{H}om^*(\mathscr{F},\mathscr{G}) = (Rq_X)_* R\mathscr{H}om(q_Y^{-1}\mathscr{F}, s^!\mathscr{G})$$

Do we have $H^0(X\times Y\times E, R\mathscr{H}om^*(\mathscr{F},\mathscr{G})) = RHom^*(\mathscr{G},\mathscr{H})$? Compute $R\mathscr{H}om^*(k_{U(f)}, k_{U(g)})$.

PROPOSITION 9.46. *We have*

$$SS(\mathscr{F} * \mathscr{G}) \subset SS(\mathscr{F}) \hat{*} SS(\mathscr{G})$$

where $A \hat{*} B = s_\# j^\#(A \times B)$

DEFINITION 9.47. Let $\mathscr{F},\mathscr{G} \in D^b(X \times E)$. Let d be induced by the diagonal map $d : X \times E \longrightarrow X \times X \times E$. Then, we set $\mathscr{F} \star \mathscr{G} = d^{-1}(\mathscr{F} * \mathscr{G})$.

EXERCICE 9.48. Prove that if $S : X \times \mathbb{R}^p \longrightarrow \mathbb{R}$ and $T : Y \times \mathbb{R}^q \longrightarrow \mathbb{R}$ are g.f.q.i. then $R\pi_*(k_{U_S} \star k_{U_T})) = R\pi_*(k_{U_{S\# T}})$ where $(S\# T)(x,\xi,\eta) = S(x,\xi) + T(x,\eta)$. More generally if $\mathscr{F} \in D^b(X \times \mathbb{R}), \mathscr{G} \in D^b(X \times \mathbb{R})$, we have $\mathscr{F} \star \mathscr{G}$.....

CHAPTER 10

The proof of Arnold's conjecture using sheaves.

1. Statement of the Main theorem

Here is the theorem we wish to prove

THEOREM 10.1 (Guillermou-Kashiwara-Schapira). *Let M be a (non-compact manifold) and N be a compact submanifold. Let Φ^t be a homogenous Hamiltonian flow on $T^*M \setminus 0_M$ and ψ be a function without critical point in M. Then for all t we have*

$$\Phi^t(\nu^* N) \cap \{(x, d\psi(x)) \mid x \in M\} \neq \emptyset$$

Of course, Φ^t can be identified with a contact flow $\hat{\Phi}^t$ on ST^*M, $\nu^* N \cap ST^*M = S\nu^*(N)$ is Legendrian, the set $L_\psi = \{(x, \frac{d\psi(x)}{|d\psi(x)|} \mid x \in M\}$ is co-Legendrian, and we get

COROLLARY 10.2. *Under the assumptions of the theorem, we have*

$$\hat{\Phi}^t(S\nu^*(N)) \cap L_\psi \neq \emptyset$$

Let us prove how this implies the Arnold conjecture, first proved on T^*T^n by Chaperon ([**Cha**]), using the methods of Conley and Zehnder ([**Co-Z**]), then in general cotangent bundles of compact manifolds by Hofer ([**Hofer**]) and simplified by Laudenbach and Sikorav ([**Lau-Sik**]), who established the estimate of the number of fixed points in the non-degenerate case (this was done in the general case in terms of cup-length in [**Hofer**]).

THEOREM 10.3. *Let φ^t be a Hamiltonian isotopy of T^*N, the cotangent bundle of a compact manifold.Then $\varphi^1(0_N) \cap 0_N \neq \emptyset$. If moroever the intersection points are transverse, there are at least $\sum_j \dim(H^j(N))$ of them.*

PROOF OF THEOREM 10.3 ASSUMING THEOREM 10.1. Consider $M = N \times \mathbb{R}$ and $\psi(z,t) = t$. Then $\varphi^s : T^*N \to T^*N$ can be assumed to be supported in a compact region containing $\bigcup_{s\in[0,1]} \varphi^s(L)$, so we may set $\Phi^s(q,p,t,\tau) = (x_s(x,\tau^{-1}p), \tau p_s(x,\tau^{-1}p), f_s(t,x,p,\tau),\tau)$ where $\varphi^s(x,p) = (x_s(x,p), p_s(x,p))$, and this is now a homogeneous flow on T^*M. We identify N to $N \times \{0\}$, and apply the main theorem: $\nu^*N = 0_N \times \{0\} \times \mathbb{R}$ and $L_\psi = \{(x,0,t,1) \mid (x,t) \in N \times \mathbb{R}\}$, so that $\Phi^s(\nu^*N) = \{(x_s(x,0), \tau p_s(x,0), f_s(0,x,0,\tau),\tau) \mid x \in N, \tau \in \mathbb{R}\}$ so that $\Phi^s(\nu^*N) \cap L_\psi = \{(x_s(x,0), p_s(x,0), f_s(0,x,0,\tau),\tau) \mid p_s(x,0) = 0, \tau = 1\} = \varphi^s(0_N) \cap 0_N$. According to the main theorem this is not empty, and this concludes the proof. □

2. The proof

PROOF OF THE MAIN THEOREM. We start with the sheaf $\mathbb{C}_N$, which satisfies $SS(\mathbb{C}_N) = \nu^* N$. We first consider a lift of Φ^t to $\widetilde{\Phi}: T^*(M\times I) \to T^*(M\times I)$ given by the formula

$$\widetilde{\Phi}:(q,p,t,\tau)\longrightarrow(\Phi^t(q,p),t,\tau+F(t,q,p))$$

where $\Phi^t(q,p) = (Q_t(q,p), P_t(q,p))$ and $F(t,q,p) = -P_t(q,p)\frac{\partial}{\partial t}Q_t(q,p)$ because denoting $\Phi^t(q,p) = (Q_t(q,p), P_t(q,p))$ the homogeneity of Φ^t and Proposition 4.39 imply that $P_t dQ_t = pdq$ and $F(t,q,p)$ is homogeneous in p. Let $\mathscr{K}$ be a kernel in $D^b(M\times I\times M\times I)$ such that $SS(\mathscr{K}) = graph(\widetilde{\Phi})$. The existence of such a kernel will be proved in Proposition 10.4. Then consider the sheaf $\mathscr{K}(\mathbb{C}_{N\times I}) \in D^b(M\times I)$. It has singular support given by

$$SS(\mathscr{K}(\mathbb{C}_{N\times I}))\subset\widetilde{\Phi}(SS(\mathbb{C}_{N\times I}))\subset\widetilde{\Phi}(\nu^*N\times 0_I)$$

Now consider the function $f(q,t) = t$ on $M\times I$. It satisfies $L_f = \{(q,t,0,1)\mid q\in M, t\in I\}\notin SS(\mathscr{K}(\mathbb{C}_{N\times I}))$ since this last set is contained in

$$\widetilde{\Phi}(\nu^*N\times 0_I) = \{(Q_t(q,p),P_t(q,p),t,F(t,q,p))\mid (q,t)\in N\times I, p=0 \text{ on } T_qN\}$$

If we had a point in $L_f\cap SS(\mathscr{K}(\mathbb{C}_{N\times I}))$ it should then satisfy $P_t(q,p) = 0$, but then we would have $F(t,q,p) = -P_t(q,p)\frac{\partial}{\partial t}Q_t(q,p) = 0$ which contradicts $\tau = 1$. We now denote by $\mathscr{K}_t \in D^b(M)$ the sheaf obtained by restricting $\mathscr{K}$ to $M\times\{t\}\times M\times\{t\}$.

The Morse lemma (cf. lemma 9.17) then implies that $H^*(M\times[0,t],\mathscr{K}(\mathbb{C}_{N\times I}))\longrightarrow H^*(M\times[0,s],\mathscr{K}(\mathbb{C}_{N\times I}))$ is an isomorphism for all $s<t$, and also that

$$H^*(M\times\{0\},\mathscr{K}(\mathbb{C}_{N\times I}))\simeq H^*(M\times\{t\},\mathscr{K}(\mathbb{C}_{N\times I}))$$

for all t. But on one hand

$$H^*(M\times\{0\},\mathscr{K}(\mathbb{C}_{N\times I}))\simeq H^*(M,\mathscr{K}_0(\mathbb{C}_N)) = H^*(M,\mathbb{C}_N)\simeq H^*(N,\mathbb{R})$$

on the other hand,

$$H^*(M,\mathscr{K}_1(\mathbb{C}_{N\times I})) = H^*(\mathbb{R},\psi_*(\mathscr{K}_1(\mathbb{C}_{N\times I})) = 0$$

the last equality follows from the fact that

$$SS(\psi_*(\mathscr{K}_1(\mathbb{C}_{N\times I}))\subset\Lambda_\psi\cap SS((\mathscr{K}_1(\mathbb{C}_{N\times I})) = \Lambda_\psi\cap\Phi(\nu^*N) = \emptyset,$$

$\psi_*(\mathscr{K}_1(\mathbb{C}_{N\times I})$ is compact supported and Proposition 9.10. This is a contradiction and concludes the proof modulo the next Proposition. ☐

PROPOSITION 10.4. *Let $\Phi: T^*X\to T^*X$ be a compact supported symplectic diffeomorphism C^1-close to the identity, and $\widetilde{\Phi}$ its homogeneous lift to $\mathring{T}^*(X\times\mathbb{R})\to\mathring{T}^*(X\times\mathbb{R})$, given by $\widetilde{\Phi}(q,p,t,\tau) = (Q(q,\tau^{-1}p),\tau P(q,\tau^{-1}p),F(q,p,t,\tau),\tau)$. Then there is a kernel $\mathscr{K}\in D^b(X\times\mathbb{R}\times X\times\mathbb{R})$ such that $SS(\mathscr{K}) = \Gamma_{\widetilde{\Phi}}$.*

PROOF ("TRANSLATED" FROM [**Bru**]). Because any Hamiltonian symplectomorphism is the product of C^1-small symplectomorphisms, thanks to the decomposition formula

$$\Phi_0^1 = \prod_{j=1}^{n} \Phi_{\frac{j-1}{N}}^{\frac{j}{N}}$$

we can restrict ourselves to the case where Φ is C^∞-small. Note also that $\widetilde{\Phi}$ is well defined by the compact support assumption: for τ close to zero,

$$(Q(q,\tau^{-1}p),\tau P(q,\tau^{-1}p)) = (q,\tau\tau^{-1}p) = (q,p)$$

Let $f(q,Q)$ be a generating function for Φ so that $p = \frac{\partial f}{\partial q}(q,Q), P = -\frac{\partial f}{\partial Q}(q,Q)$ defines the map Φ. Let $W = \{(q,t,Q,T) \mid f(q,Q) \le t - T\}$ and $\mathscr{F}_f = k_W \in D^b(X \times \mathbb{R})$. Then $SS(\mathscr{F}_f) = \Gamma_{\widetilde{\Phi}}$. Let us start with $X = Y = \mathbb{R}^n$, and $f_0(q,Q) = |q-Q|^2$. Then we get $\mathscr{K}_0$ with $SS(\mathscr{K}_0) = \Gamma_{\widetilde{\Phi}_0}$. Now if f is C^2 close to f_0, we will get any possible $\widetilde{\Phi}_f$, C^1-close to the map $(q,p) \to (q+p,p)$. Then $\widetilde{\Phi}_{f_1} \circ \widetilde{\Phi}_{f_2}$ where f_1 is close to f_0 and f_2 close to $-f_0$ will be C^1-close to the identity. Now since any time one of a Hamiltonian isotopy can be written as the decomposition of C^1-small symplectomorphisms, we get the general case.

Now let $i : N \to \mathbb{R}^n$ be an embedding. Then the standard Riemannian metric on $\mathbb{R}^l$ induces a symplectic embedding $\tilde{i} : T^*N \to T^*\mathbb{R}^n$ given by $(x,p) \mapsto (i(x), \widetilde{p}(i(x)))$ where $\widetilde{p}(i(x))$ is the linear form on $\mathbb{R}^l$ that equals p on T_xN and zero on $(T_xN)^\perp$. Now let Φ^t be a Hamiltonian isotopy of T^*N. We claim that it can be extended to $\widetilde{\Phi}^t$ such that

(1) $\widetilde{\Phi}^t$ preserves $\nu^*N = N \times (\mathbb{R}^l)^*$, and thus the leaves of this coisotropic submanifold. This implies that $\widetilde{\Phi}^t$ induces a map from the reduction of $N \times (\mathbb{R}^l)^*$ to itself, that is T^{*N}.
(2) we require that this map equals Φ^t.

The existence of $\widetilde{\Phi}^t$ follows from the following construction:

Assume Φ is the time one map of Φ^t associated to $H(t,x,p)$, where (x,p) is coordinates for T^*N. Locally, we can write (x,u,p,v) for points in $\mathbb{R}^l$ so that $N = \{u = 0\}$. We define

$$\widetilde{H}(t,x,u,p,v) = \chi(u)H(t,x,p),$$

where χ is some bump function which equals is 1 on N (i.e. $\{u = 0\}$) and 0 outside a neighborhood of N. By the construction, $X_{\widetilde{H}} = X_H$ on $N \times (\mathbb{R}^l)^*$. Then $\widetilde{\Phi} = \widetilde{\Phi}^1$, the time one flow of $\widetilde{H}$, is the map we need.

The theorem follows by noticing that if $\widetilde{\mathscr{K}}$ is such that $SS(\mathscr{K})$ is the graph of $\widetilde{\Phi}$, then the restriction $\widetilde{\mathscr{K}}_{|N\times N}$ has singular support the graph of Φ.

□

REMARK 10.5. The original proof of [**G-K-S**] follows a slightly different approach. Instead of decomposing a map into compositions of maps having a generating functions in the Jacobi sense, it uses the decomposition of Lagrange relations into relations given by the conormal of a smooth hypersurface. Indeed, if $\nu^* S_1$ is the conormal of a hypersurface, and φ a homogoneous Hamiltonian map C^1-close to the identity, then there exists a smooth hypersurface, S_2 such that $\varphi^* \nu^* S_1 = \nu^* S_1$ (see exercise 4.42 on page 40)

Exercice: Show that if L has a GFQI, then $\varphi(L)$ has a GFQI for $\varphi \in Ham(T^* N)$. **Hint.** If $S : N \times \mathbb{R}^k \to \mathbb{R}$ is a GFQI for L, then L is the reduction of $gr(dS)$.

REMARK 10.6. (1) We could have used directly that the graph of Φ has a GFQI.
(2) 0_N is generated by the zero function over the zero bundle over N, or less formally

$$\begin{array}{cccc} S: & N \times \mathbb{R} & \to & \mathbb{R} \\ & (x, \xi) & \mapsto & \xi^2 \end{array}$$

(3) There is no general upper bound on k (the minimal number of parameter of a generating functions needed to produce all Lagrangian.)
Reason: Consider a curve in $T^* S^1$