

LAGRANGIANS EMBEDDINGS AND SYMPLECTIC STRUCTURES

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0. Introduction

Arnold and Gromov have stated the following conjectures for $n \geq 2$:

Conjecture 1: For a closed n -manifold there is no lagrangian exact embedding in T^*R^n .

Conjecture 2: On R^{2n} there is an exotic symplectic structure with infinite volume.

Exotic means not symplectomorphic with T^*R^n , infinite volume means $\int_{R^{2n}} \omega^n = +\infty$. If we omit this last condition, conjecture 2 becomes a triviality, as the volume is clearly invariant by symplectomorphism if $i: R^{2n} \rightarrow T^*R^n$ is an embedding of R^{2n} in the 1-ball of T^*R^n , $i^*\omega_0$ is a symplectic form with finite volume. We are going to prove that conjecture 1 implies conjecture 2, in fact we prove that conjecture 1 restricted for instance to the n -torus T^n implies conjecture 2.

1. Statement and proof of the proposition

Proposition: ($n \geq 2$) Let (M^{2n}, ω) be a symplectic (resp. exact) open manifold, L^n a closed manifold, $i: L \rightarrow (M, \omega)$ a lagrangian immersion (resp. such that $[i]^*: H^1(M; R) \rightarrow H^1(L; R)$ is 0) regularly homotopic to an embedding. If there is no lagrangian (resp. exact) embedding of L in (M, ω) , then there is a symplectic form ω' on M with the following properties :

- (i) ω and ω' can be joined by a continuous family of cohomologous symplectic forms.
- (ii) (M, ω) and (M, ω') have the same volume.
- (iii) (M, ω) and (M, ω') are not symplectomorphic.

Recall that (M, ω) is said to be exact if $\omega = d\lambda$. We then say that i is exact if $[i]^*: H^1(M; \mathbb{R}) \rightarrow H^1(L; \mathbb{R})$ is 0 and $i^*\lambda$ is exact. By the first condition on i , the second does not depend on the choice of λ .

The proof of the proposition is based on the following lemmas:

Lemma 1: Let (M^{2n}, ω) be an exact symplectic manifold, $i: L \rightarrow (M, \omega)$ be a lagrangian immersion and $[i]^*: H^1(M; \mathbb{R}) \rightarrow H^1(L; \mathbb{R})$ be 0. Then i is regularly homotopic through lagrangian immersions to an exact immersion.

Lemma 2: Let M and W be open m -dimensional manifolds, L a compact submanifold of M with $m \geq \dim L + 2$ and $i_t: L \rightarrow W$ a regular homotopy. If i_0 can be extended to an immersion $j_0: M \rightarrow W$, then i_t can be extended to a regular homotopy $j_t: M \rightarrow W$. Moreover if Ω is a volume form on W , then we can assume $\int_M j_0^* \Omega = \int_M j_1^* \Omega$.

Proof of the proposition assuming the lemmas:

In the exact case, by lemma 1 we can take i to be exact. Let i_t be the regular homotopy between $i = i_1$ and an embedding i_0 . Via i_0 consider L as a submanifold of M . Then by lemma 2 applied to $W=M$, $j_0 = \text{id}_M$, we can extend i_t to j_t so that the following diagram is commutative:

$$\begin{array}{ccc} L & \xrightarrow{i_t} & M \\ \downarrow i_0 & \nearrow j_t & \\ M & & \end{array}$$

$$\text{and } \int_M \omega^n = \int_{M_0} \omega^n.$$

As i_1 is lagrangian $0 = i_1^* \omega = i_0^* (j_1^* \omega)$, taking for ω' $j_1^* \omega$ we are done:

- (i) $j_t \omega$ is a family of cohomologous symplectic forms
- (ii) $\int_M \omega^n = \int_M \omega'^n$
- (iii) i_0 is a lagrangian embedding of L into (M, ω') . in the exact case $[i_0]^* = [i_1]^* = 0$ and $i_0^* (j_1^* \lambda) = i_1^* \lambda$ is exact. Thus (M, ω) and (M, ω') can not be symplectomorphic.

Proof of lemma 1:

We consider a triangulation of L and let u_t be a lagrangian regular homotopy (i.e. every u_t is lagrangian) of a neighbourhood of the 1-skeleton, with $u_0 = i$ on the 1-skeleton and

$\int_A u_1^* \lambda = 0$ for every 1-simplex A . If we can extend u_t to a lagrangian regular homotopy i_t on L , then $i_1^* \lambda$ will be exact. By the Gromov- Lees theory we can extend u_t provided

$\int_{\partial \Delta} u_t^* \lambda = 0$ for every 2-simplex Δ (see [Lees] p. 219).
If we choose u_t so that $\int_A u_t^* \lambda = (1 - t) \int_A i_0^* \lambda$ on every 1-simplex, our condition is fullfilled since $\int_{\partial \Delta} i^* \lambda = \int_{\Delta} i^* \omega = 0$ by Stokes.

It now remains to define u_t :

Let A be a 1-simplex, a neighbourhood of A in L is diffeomorphic to $U = \{ (q_1, \dots, q_n) \in \mathbb{R}^n / |q_1| \leq 1 \text{ and } |q_i| \leq \varepsilon \text{ for } i \geq 2 \}$ and A is given by $q_2 = \dots = q_n = 0$. By Weinstein's theorem (see [Weinstein] p. 339) a neighbourhood of U in M is symplectomorphic to a neighbourhood V of the 0-section of T^*U .

Let now $(q_1^t, p_1^t) : [-1, 1] \longrightarrow \mathbb{R}^2$ be a regular homotopy

with:

- (i) $(q_1^0(x), p_1^0(x)) = (x, 0)$
- (ii) $(q_1^t(x), p_1^t(x))$ is independent of t for $|x| \geq 1/2$
- (iii) $(q_1^t(x), p_1^t(x))$ remains in some prescribed neighbourhood of $[-1, 1] \times \{0\}$
- (iv) $\int_{-1}^1 p_1^t dq_1^t = -t \int_A i_o^* \lambda$

For small values of t , we can take for (q_1^t, p_1^t) the graph of a function y^t with $\int_{-1}^1 y^t(x) dx = -t \int_A i_o^* \lambda$; but for large values of t $\int_A i_o^* \lambda$, (q_1^t, p_1^t) will be as suggested by the picture on fig. 1.

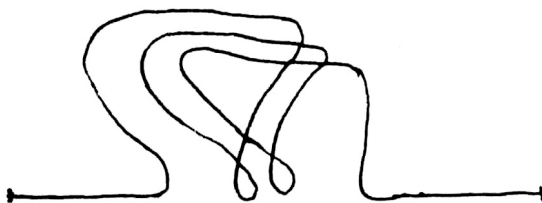


Fig. 1

Now set $u_t(q_1, \dots, q_n) = (q_1^t(q_1), q_2, \dots, q_n, p_1^t(q_1), 0, \dots, 0)$ (we read u_t in the coordinates of U and V). Let μ be the Liouville form on V , then $\int_A u_t^* \mu - u_o^* \mu = -t \int_A i_o^* \lambda$. The left hand side of this equality depends only on $d\mu$ by Stokes' theorem, so we can write $\int_A u_t^* \lambda - i_o^* \lambda = -t \int_A i_o^* \lambda$ or $\int_A u_t^* \lambda = (1-t) \int_A i_o^* \lambda$ as required.

Proof of lemma 2:

To prove our proposition, it is enough to have the lemma in the case $i_t = i_o$ for $0 \leq t \leq \varepsilon$ and $i_t = i_1$ for $1-\varepsilon \leq t \leq 1$; we will prove the lemma only in this case, but a slight change in our proof would easily make it cover the general case.

First, we triangulate M so that L is in the $m-2$ skeleton

of M . Write $\text{Im}(X, W)$ for the space of immersions of X in W and set $D^k = \{x \in \mathbb{R}^k / \|x\| \leq 1\}$, $D_{1/2}^k = \{x \in \mathbb{R}^k / 1/2 \leq \|x\| \leq 1\}$, then the map $\text{Im}(D^k D^{m-k}, W) \longrightarrow \text{Im}(D_{1/2}^k D^{m-k}, W)$ induced by restriction is for $k < m$ a Serre fibration (see [Haefliger] p. 135). Now we first extend i_t to a neighbourhood of L by the tubular neighbourhood theorem, and then by induction on the dimension of the simplexes to a neighbourhood N' of the $m-1$ skeleton of M . Call h_t this extension. As we will prove later, there is an isotopy f_t of M leaving some neighbourhood N of the $m-2$ skeleton fixed and with $f_1(M)$ contained in N' . We can then define

$$j_t = \begin{cases} j_0 \circ f_{t/\varepsilon} & \text{for } t \leq \varepsilon \\ h_t \circ f_1 & \text{for } t \geq \varepsilon \end{cases}$$

Moreover we can assume $\int_M j_{1-\varepsilon}^* \Omega = +\infty$. To see this, set $L' = L \cup \bigcup_{k \in N} V_k$ where the V_k are disjoint balls, contained in N , not intersecting L , and $i'_t = i_t$ on L . On V_k i'_t is an isotopy with $i'_0 = j_0$ and $i'_{1-\varepsilon}(V_k)$ contains some fixed ball B_0 in M (as all balls are isotopic i'_t exists). Then because L' is contained in N , we can as we did before find j'_t extending i'_t , and we have: $\int_M j'_{1-\varepsilon} \Omega \geq \sum_{k \in N} \int_{V_k} i'^*_{1-\varepsilon} \Omega \geq \sum_{k \in N} \int_{B_0} \Omega = +\infty$.

Let r_t be an isotopy of M fixed on the $m-2$ skeleton, such that:

- (i) $r_0 = \text{id}$
- (ii) $\int_{r_t(M)} j^*_{1-\varepsilon} \Omega = \rho(t)$ is continuous as a function with values in $[0, +\infty]$ and $\rho(1) < \int_M j_0^* \Omega$.

Then for some τ in $[0, 1]$ $\rho(\tau) = \int_M j_0^* \Omega$ and

$$j_t = j_t \text{ for } t \leq 1 - \varepsilon$$

$$j_{1-\varepsilon} \circ r_{(t-1+\varepsilon)\tau/\varepsilon} \text{ for } 1 - \varepsilon \leq t \leq 1$$

will be the required regular homotopy.

To complete our proof, we need to define f_t and r_t . We shall only construct r_t , the construction of f_t would be similar.

Let γ be a proper embedding of $[0, +\infty[$ in M , with $\gamma(k) = x_k$ (where x_k is the barycenter of some m -simplex Δ_k) such that $\gamma([0, +\infty[) \cap \Delta_k$ is the union of two segments having x_k as a common endpoint; and $\gamma([0, +\infty[)$ does not meet the $(m-2)$ -skeleton of M . It is clear how to construct γ (see fig. 2)

. Let U be some tubular neighbourhood of $\gamma([0, +\infty[)$ with finite volume (the volume form being $j_{1-\varepsilon}^* \Omega$) and such that $\Delta_k \cap U$ is starshaped with respect to x_k . The picture on fig. 3 then suggests an isotopy g_t with $g_0 = \text{id}$ and g_1 mapping M onto $M \setminus U$.

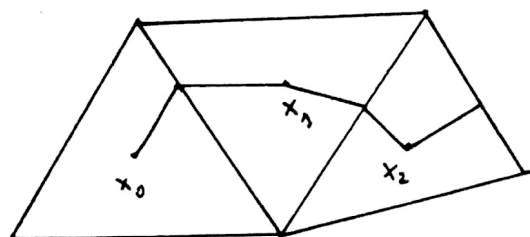


Fig. 2

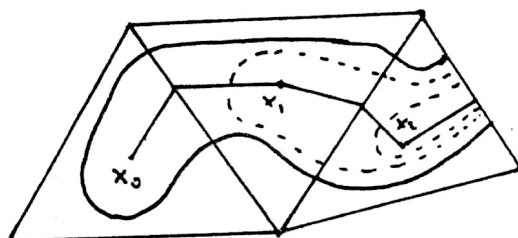


Fig. 3

As the volume of U is finite, $\int_{g_1(M)} j_{1-\varepsilon}^* \Omega$ is infinite. As $\Delta_k \cap U$ is starshaped with respect to x_k , we have a barycentric isotopy a_t^k , fixed on some neighbourhood of $\partial \Delta_k$, decreasing (i.e. $a_t^k(\Delta_k \setminus U) \subset a_s^k(\Delta_k \setminus U)$ for $t > s$) and such that setting $\rho_k(t) = \int_{a_t^k(\Delta_k \setminus U)} j_{1-\varepsilon}^* \Omega$ we have:

$$(i) \rho_k(t) = \rho_k(0) \text{ for } t \leq 1/k+2$$

$$(ii) \rho_k(1) \leq c/2^{i+1} \text{ for } t \geq 1/k+1 \text{ where } c \text{ is positive } < \int_M j_0^* \Omega$$

As a_t^k is decreasing so is $\rho_k(t)$. Then the series $\sum \rho_k(t)$ is normally convergent on $[1/n, 1]$ since on this interval $\rho_k(t) \leq \rho_k(1/n)$ and $\sum_k \rho_k(1/n)$ is convergent. So $\sum \rho_k(t)$ is continuous on $[0, 1]$ and it is easy to check that, as t goes to 0 $\lim \sum \rho_k(t) = +\infty = \sum \rho_k(0)$, and $\sum \rho_k(t)$ is continuous on $[0, 1]$ as a function with values in $[0, +\infty]$.

Now set:

$$r_t^1(x) = g_{2t}(x) \text{ for } t \leq 1/2$$

$$a_{2t-1}^k \circ g_1(x) \text{ for } g_1(x) \in \Delta_k \text{ and } t \geq 1/2$$

$$g_1(x) \text{ for } g_1(x) \notin \bigcup \Delta_k \text{ and } t \geq 1/2$$

Now, if $\bigcup \Delta_k = M$, $r_t = r_t^1$ is the required isotopy. Otherwise let $S_1 = \{\Delta_k\}$ then $r_1^1(M)$ is a neighborhood of $M^{(m-2)} \cup \bigcup_{\Delta \in S_1} \Delta$

Take Δ' to be an m -simplex having at least one face in common with some $\Delta \in S_1$; then we can find an isotopy r_t^2 of $r_1^1(M)$, fixed outside a neighbourhood of Δ' and such that $r_1^2 \circ r_1^1(M)$ is in a neighborhood of $M^{(m-2)} \cup \bigcup_{\Delta \in S_2} \Delta$ where $S_2 = S_1 \cup \{\Delta'\}$. (This is just engulfing;

see fig. 4).

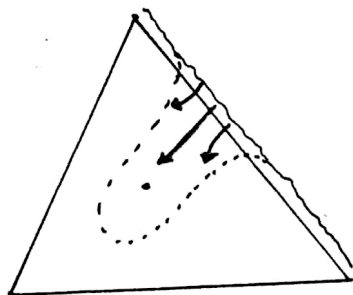


Fig. 4

Repeating this argument we get an increasing family S_j , with $\bigcup_j S_j =$ the set of m -simplexes of M , and r_t^j an isotopy of $r_1^{j-1} \circ r_1^{j-2} \circ \dots \circ r_1^1(M)$ such that $r_1^j \circ \dots \circ r_1^1(M)$ is a neighborhood of $M^{m-2} \cup \bigcup_{\Delta \in S_j} \Delta$. Moreover, r_t^j is fixed outside

of a neighborhood of the simplexes $\Delta \in S_j - S_{j-1}$, so we can define $r_t(x) = r_{\frac{2^{n+1}(1-t)-1}{2^{n+1}}}^j \circ r_1^{j-1} \circ \dots \circ r_1^1(x)$ for $t \in [1-1/2^n, 1-1/2^{n+1}]$

Now r_t is the required isotopy :

(i) $r_0 = r_0^1$ is indeed the identity map

(ii) as for $\text{vol}(r_t^1(M))$; we can make $\text{vol}(r_t(M))$ continuous and $\text{vol}(r_1(M))$ is the volume of some neighbourhood of M^{m-2} can be chosen as small as we please.

This completes the proof of lemma 2.

2. Proof that conjecture 1 implies conjecture 2

We are in the case $(M, \omega) = T^*R^n$ and L is the n -torus T^n . If i is an embedding of S^1 in T^*R , $i^n : T^n \longrightarrow T^*R^n$ defined by $i^n(\theta_1, \dots, \theta_n) = (i(\theta_1), \dots, i(\theta_n))$ is a lagrangian embedding of T^n into T^*R^n (of course non exact), and so applying our proposition we see that conjecture 1 restricted to the n -torus implies conjecture 2. In fact as Prof. Weinstein pointed out, if j is the exact immersion of S^1 in T^*R represented on fig. 5, then j^n is an exact immersion of T^n into T^*R^n regularly-homotopic to an embedding (because j is) and we do not need lemma 1 in this case.

Finally let us mention that in view of the Gromov-Lees theorem (see [Lees] p.217 Theorem 1), of the theorem of Weinstein we used in the proof of lemma 1, and a theorem of Hirsch (see [Hirsch] p. 273) ; it is easy to prove that any even dimensional manifold L , such that $TL \otimes \mathbb{C}$ is a trivial complex bundle and the Euler class of L vanishes has a lagrangian immersion in $T^*\mathbb{R}^n$ regularly homotopic to an embedding.

Note added in proof: M. Gromov has just proved conjecture 1 and also a different proof that conjecture 1 implies conjecture 2.

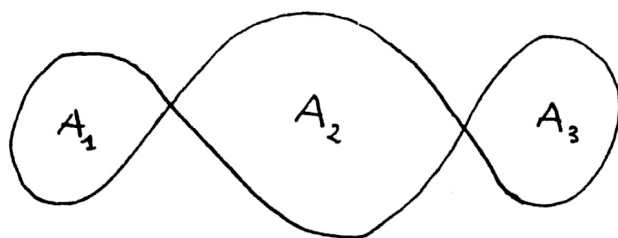


Fig. 5

$$A_1 + A_3 = A_2$$

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