Symplectic invariants and symplectic reduction Details of [V1] p. 706

Claude Viterbo

July 26, 1998

The aim of this note is to clarify the proof of the camel problem from [V1] p.706. I wish to thank David Théret for pointing out some shortcomings in the proof.

In this note we extend the notion of G.F.Q.I. by allowing functions $S : N \times (F \times \mathbb{R}^k) \longrightarrow \mathbb{R}$ coinciding with a nondegenerate quadratic form near infinity. Note that F is assumed to be compact. In this note the parentheses indicate the fibre variables.

Then

$$L_S = \{(x, \frac{\partial S}{\partial x}(x, y, \xi)) \mid \frac{\partial S}{\partial p}(x, y, \xi) = 0; \frac{\partial S}{\partial \xi}(x, y, \xi) = 0\}$$

Note that L has a G.F.Q.I. in this generalized sense if and only if it is the reduction of a submanifold of $T^*(N \times F)$, which has a G.F.Q.I. in the former sense, by $C_F = T^*N \times O_F = \{(x, y, X, Y) \mid Y = 0\}$ This is a coisotropic subspace and $(C_F)^\omega = \{(x, y, X, Y) \mid x = 0, X = 0, Y = 0\}$. It follows from this remark that a number of proofs go through, from our familiar setting, corresponding to $F = \{pt\}$, to the general case. In particular existence of a G.F.Q.I. is invariant by Hamiltonian isotopy as we see by extending the isotopy from T^*N to $T^*(N \times F)$. On the other hand, uniqueness is unclear, since we need to extend our notion of stable equivalence (see [Th] for the standard case).

Also we define $c(\alpha, S)$ to be the critical value obtained by minimax from the image by the Thom class of $\alpha \in H^*(N \times F)$. Note that $H^*(N \times F) = H^*(N) \otimes H^*(F)$, so α can be decomposed as $\rho \otimes \sigma$. Then we define $\gamma(S) = c(\mu \otimes \mu, S) - c(1 \otimes 1, S)$.

We may also apply proposition 5.1 from [V1]:

Proposition 0.1. *For $\alpha \in H^*(N \times F)$ we have*

$$c(\alpha \otimes 1, S) \leq \inf_w c(\alpha, S_w) \leq$$

$$\sup_w c(\alpha, S_w) \leq c(\alpha \otimes \mu, S)$$

Definition. Let L_0, L_1 be Lagrange submanifolds. We say that L_0 and L_1 are *gf-homotopic* if and only if there exists a continuous family S_t of functions quadratic at infinity, such that $L_1 = L_{S_1}$, $L_2 = L_{S_2}$

Remarks:

1. The homotopy can be made generic, so that for each t except a finite number of them, S_t is the generating function of some manifold (i.e. satisfies the transversality condition) we have that S_t generates a submanifold L_t. Thus L_0 and L_1 can be connected by a regular homotopy, modulo a finite number of singularities. We still will denote by L_t the set of points defined by S_t.

2. Note that the quadratic form is assumed to be nondegenerate for all t. We could also have assumed that S_t is a fixed quadratic form (independent from t) outside a compact set.

3. Note also that we only assume that L_0 and L_1 have a G.F.Q.I. , we do not require that it should be unique. So L_0 and L_1 need not be isotopic to the zero section.

4. Finally we remark that this property is invariant by symplectic reduction.

Let W be an open set in T^*N, we define an invariant by setting

Definition.

$$\tilde{c}(\alpha, W; 0_N) =$$

$$\sup\{c(\alpha, S) \mid L_S \text{ is gf-homotopic to } 0_N \text{ by a homotopy supported in } W\}$$

Remarks:

1. The support of the isotopy, $\mathrm{supp}(L_t)$, is defined as the closure of $\bigcup_{0\leq t\leq 1} L_t - 0_N$.

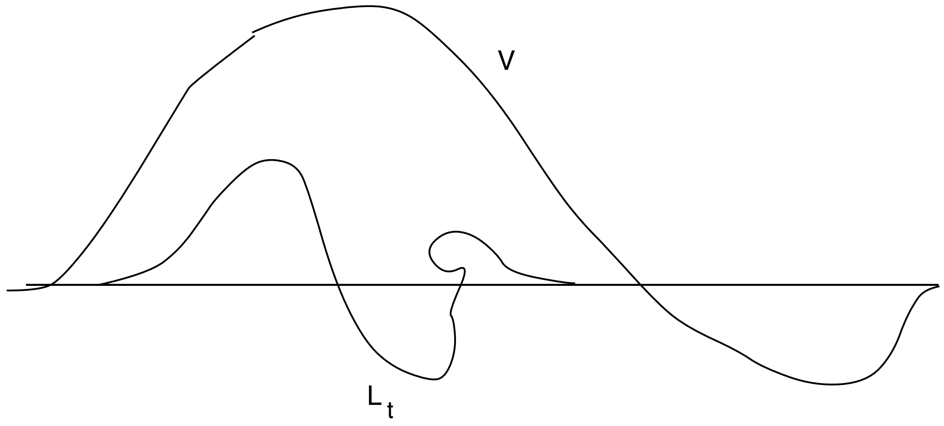

Figure 1:

2. Clearly we have that $\tilde{c}(\alpha, W; 0_N)$ is invariant by symplectic isotopies preserving the zero section.

3. For U an open set in $\mathbb{R}^{2n}$, we have the inequality $c(U) \leq \tilde{c}(\mu, U \times U; \Delta)$. Indeed the graphs of a symplectic map ϕ_t supported in U will be in $U \times U$. We denote by $\tilde{c}(\alpha, U) = \tilde{c}(\alpha, U \times U; \Delta)$. By the previous remark, this is a symplectic invariant.

Theorem 0.2. *Let S_t be a G.F.Q.I. for L_t, R be the G.F.Q.I. of V in T^*N such that $V \cap \operatorname{supp}(L_t) = \emptyset$. Then $c(\alpha, S_1) \leq c(\alpha, R) - c(1, R)$.*

Proof. We look as usual at $c(\alpha, R - S_t)$. This is independent of t, since critical points of $R - S_t$ correspond to points in $V \cap L_t$. Thus the number $c(\alpha, R - S_t)$ must be independent of t, and we have:

$$c(\alpha, R) = c(\alpha, R - S_0) = c(\alpha, R - S_1) \geq c(1, R) + c(\alpha, -S_1)$$

Thus

$$c(\alpha, S_1) \leq c(\alpha, R) - c(1, R).$$

□

We may now state

Proposition 0.3. *Let R be the G.F.Q.I. of N with $N \cap W = \emptyset$. Then*

$$\tilde{c}(\mu, W; O_N) \leq \gamma(R).$$

In particular since for N, graph of ψ with $\psi(U) \cap U = \emptyset$, and L_t supported in $U \times U$, we have $L_t \cap N = \emptyset$, we get:

Proposition 0.4. *If $\psi(U) \cap U = \emptyset$ then*

$$\tilde{c}(\mu, U) \leq \gamma(\psi)$$

Now we consider the following situation. Let U be a domain in $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$. The coordinates are denoted by z, q, p, with $z \in \mathbb{R}^{2n}$ $q \in \mathbb{R}^m$ $p \in (\mathbb{R}^m)^*$. Let $x \in \mathbb{R}^m$ and set

$$U_x = (U \cap (\mathbb{R}^{2n} \times \{x\} \times \mathbb{R}^m)/(\{0\} \times \{x\} \times \mathbb{R}^m)$$

We now make the following compactifications: $\mathbb{R}^{2n} \times \mathbb{R}^{2m} \times \overline{\mathbb{R}^{2n}} \times \overline{\mathbb{R}^{2m}}$ can be identified to $T^*\Delta_{\mathbb{R}^{2n}} \times T^*\Delta_{\mathbb{R}^{2m}}$, where Δ denotes the diagonal. The identification on the factor $\mathbb{R}^{2m} \times \overline{\mathbb{R}^{2m}}$ being given through $(q, p, Q, P) \Longrightarrow (q + Q, \frac{p+P}{2}, \frac{p-P}{2}, Q - q)$ and we compactify $\Delta_{\mathbb{R}^{2n}}$ to S^{2n} and $\Delta_{\mathbb{R}^{2m}}$ to T^{2m}.

We may then define $c(\rho \otimes \sigma, U)$ for $\rho \in H^*(S^{2n})$ and $\sigma \in H^*(T^{2m})$.

Theorem 0.5.

$$c(\mu \otimes 1, U) \leq \inf_x \gamma(U_x)$$

Proof. Let $S : (S^{2n} \times T^{2m}) \times \mathbb{R}^k$ be a G.F.Q.I. for $L \in T^*S^{2n} \times T^*T^{2m}$ with $L = 0_{S^{2n} \times T^{2m}}$ outside $U \times U$. More precisely we assume there is a gf-homotopy connecting L with the zero section, having this property. Let us consider S_x, the restriction of S to $S^{2n} \times (\{2x\} \times T^m \times \mathbb{R}^k)$. Note that the "base" was $S^{2n} \times T^{2m}$ and is now S^{2n}. Thus the factor $(\{x\} \times T^m)$ that we expect to be in the base, is now in the fibre.

The submanifold L_{2x} is the reduction of L by $T^*S^{2n} \times \nu^*(\{2x\} \times T^m) = T^*S^{2n} \times (\{2x\} \times T^m) \times (\mathbb{R}^m \times \{0\})$. This is the set $\{(z, Z, q, p, Q, P) \mid q + Q = 2x, q - Q = 0\}$ or else $\{(z, Z, q, p, Q, P) \mid q = Q = x\}$. Now we claim that if L coincides with the zero section outside $U \times U$, then L_{2x} coincides with the zero section outside $U_x \times U_x$, and there is a gf-homotopy connecting L_{2x} with the zero section. Indeed we have that $(z, Z) \in L_{2x} - (U_x \times U_x)$, if and only if there exist q_0, p_0, Q_0, P_0 such that $(z, Z, q_0, p_0, Q_0, P_0) \in L$ with $q_0 = Q_0 = x$ and for all p, P we have that $(z, Z, x, p, x, P) \notin U \times U\}$. This last assumption implies that $z = Z$, hence $(z, Z) \in 0_{S^{2n}}$. We just proved that L_{2x} coincides with the zero section away from $U_x \times U_x$. Thus $c(\alpha, S_x) \leq \tilde{c}(\alpha, U_x)$, hence

$$c(\alpha \otimes 1, S) \leq \inf_x \tilde{c}(\alpha, U_x) \leq \inf_x \gamma(\alpha, U_x)$$

As a result

$$c(\mu \otimes 1, U) \leq \inf_x \gamma(U_x).$$

$\square$

References

[Th] D. Théret. *Thèse de Doctorat.* Université de Paris 7, 1996.

[V1] C. Viterbo. Symplectic topology as the geometry of generating functions *Math. Annalen*, 292: 685-710, 1992.