Talk online Covid

5.1.21

How many electrons can dance on a Riemann surface (and on $\mathbb{CP}^2$) ?

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Laughlin state

$$\Psi_L(z_1,\dots,z_N) = C \cdot \prod_{n<m}^{N} (z_n - z_m)^\beta \cdot e^{-\frac{B}{4}\sum_{n=1}^{N}|z_n|^2}$$

$(z_1,\dots,z_N) \in \mathbb{C}^N$, $\qquad \Psi_L : \mathbb{C}^N \mapsto \mathbb{C}$

$\beta \in \mathbb{Z}_+$ $\qquad 1/\beta$ is "filling fraction"

$B > 0$ $\qquad$ "magnetic field"

Ψ_L is a wave function, i.e. given a configuration of N points $\{z_n\}$, $|\Psi_L(z_1,\dots,z_N)|^2$ is its probability.

I Laughlin state on a genus-g Riemann surface (w/ D. Zvonkine)

* definition
* topological degeneracy

II Laughlin state in complex dim two

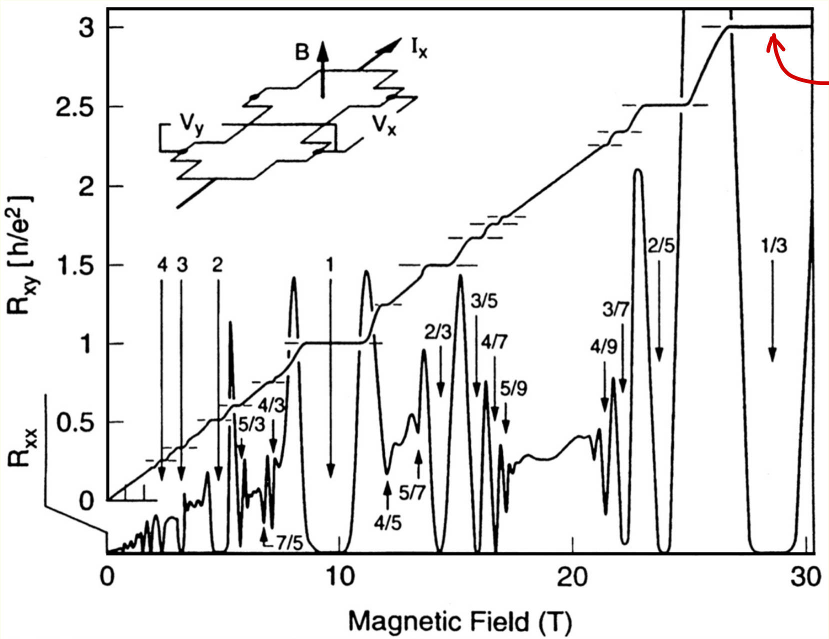
(w/ M. Douglas, J. Wang)

* definition
* first results

Quantum Hall effect (QHE)

Precise quantization of Hall conductance $\sigma_H = \frac{1}{R_{xy}}$

$R_{xy} = \frac{V_y}{I_x}$



Laughlin state corresponds to this plateau

$\sigma_H = \frac{1}{\beta}$

$= \frac{1}{3} = 0{,}3333\ldots$

Fractional QHE ($\sigma_H = \frac{1}{\beta}$) is a strongly-interacting system (Coulomb)

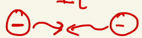
$$\Psi \in L^2(\mathbb{C}^N) \quad , \quad \left(\sum_{n=1}^{N} \bar{D}_n^{\dagger} \bar{D}_n + \sum_{n<m}^{N} V(z_n, z_m) \right) \Psi = 0$$

$$\bar{D} = \left(\partial_{\bar{z}} + \frac{B}{4} z \right)$$

Laughlin (1983): "trial state"

$$\Psi_L = \prod_{n<m}^{N} (z_n - z_m)^{\beta} \cdot e^{-\frac{B}{4} \sum_{n=1}^{N} |z_n|^2}$$

* holomorphic
* (anti) symmetric
* vanishing conditions $\Psi_L = 0$



* $\beta = 1$, $\prod_{n<m} (z_n - z_m) = \det \left(z_n^{m-1} \right) \Big|_{m,n=1}^{N}$

Laughlin state on a Riemann surface

Σ smooth genus-g Riemann surface

Σ^N its Nth power, $\pi_1, \dots, \pi_N$ projections from Σ^N to Σ.

$L \to \Sigma$ is a positive degree-d line bundle

$L^{\boxtimes N} = \pi_1^* L \otimes \dots \otimes \pi_N^* L$ is the line bundle on Σ^N

Def Laughlin state for filling fraction $\frac{1}{\beta}$, $\beta \in \mathbb{Z}_+$ is a holomorphic section Ψ of $L^{\boxtimes N}$

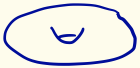
* vanishing to the order β along all diagonals $\Delta_{nm} = \{ z_n = z_m \} \subset \Sigma^N$
* completely symmetric (anti-symmetric) for β even (resp. odd)

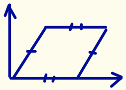
$$H = \sum_{n=1}^{N} \bar\partial_L^+ \bar\partial_L + \sum_{n<m}^{N} V(z_n, z_m)$$

$$\bar\partial_L : \; C^\infty(L) \to \Omega^{0,1}(L)$$

* Haldane- Rezayi '85

β- degeneracy of Laughlin states on torus



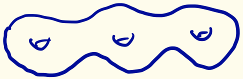


translational symmetry breaking

* Wen-Niu '90

<u>Topological degeneracy</u> on genus- g

β^g Laughlin states for $\sigma_H = 1/\beta$ (conjecture).



Topological phases of matter

Quantum optimal packing problem

Haldane 1983 $g=0$

Haldane - Rezayi '85 $g=1$

Wen - Niu '90 $g>1$

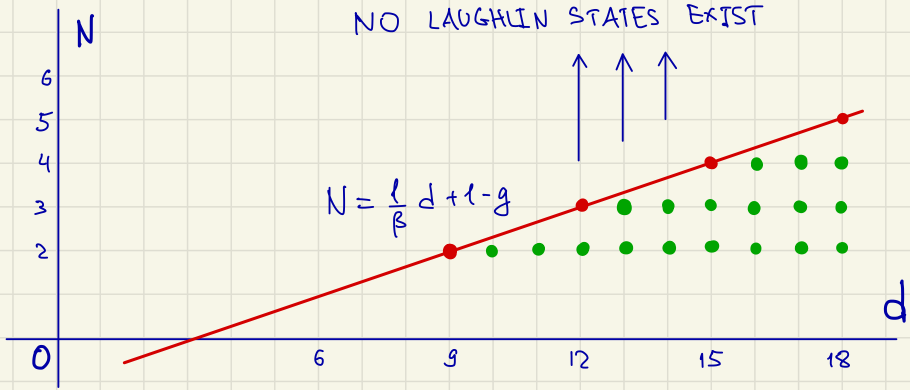
Dimension of the vector space $V_\beta(N,d)$ of Laughlin states :

- β^g - topological degeneracy completely filled (dense) states

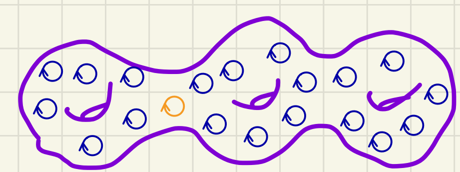
"Electron dancing patterns" (X.-G. Wen)

Iengo-Li '94 : $(\beta-1)g+1$ Laughlin states

- large (depends on N,d)



Illustrated for $\beta=3$, $g=2$



Warm-up : examples of Laughlin states

$*$ $\beta = 1$, take $N = d + 1 - g$ and $\{ s_n \} \in H^0(L)$ $\quad (d \geq 2g-1 \,,\; N \geq g)$

$$\Psi_1 = \det s_n(z_m) \Big|_{n,m=1}^{N}$$

("Slater determinant")

$*$ $\beta > 1$, take $Q \to \Sigma$, $Q^{\otimes \beta} = L$ and $\{ \sigma_n \} \in H^0(Q)$, $N = \frac{d}{\beta} + 1 - g$

$$\Psi_2 = \big(\det \sigma_n(z_m) \big)^{\otimes \beta}$$

but there are more...

Wen-Niu conjecture

Thm (D. Zvonkine, SK)

I. (Wen-Niu conjecture) Let $N \geq g$ and $N = \frac{d}{\beta} + 1 - g$ (i.e. $\beta | d$)

Dimension of the vector space of Laughlin states is $\dim V_\beta(N, d) = \beta^g$

II. Let $p = d - \beta(N+g-1) \geq 0$, then

$$\boxed{\dim V_\beta(N.d) = \sum_{k=0}^{g} \binom{g}{k} \binom{p+N-g}{p+k-g} \beta^k}$$

and $\dim V_\beta(N, d) = 0$ for $p < 0$.

(sketch of a) proof:

I. Introduce the notations $\Delta = \sum_{n<m} \Delta_{nm}$

$$\mathbb{L}(-\beta\Delta) = \pi_1^* L \otimes \dots \otimes \pi_N^* L(-\beta\Delta)$$

Hence Laughlin states can be identified with completely symmetric sections of $\mathbb{L}(-\beta\Delta)$ over Σ^N

Choose a line bundle Q, $Q^\beta = L$

then $\deg Q = N+g-1$

Consider line bundle $\mathbb{Q}(-\Delta) = \pi_1^* Q \otimes \ldots \otimes \pi_N^* Q\,(-\Delta)$

on Σ^N

Consider the map $f_Q : \Sigma^N \to \mathrm{Pic}_{g-1}(\Sigma)$ (in fact, $S^N\Sigma \to \mathrm{Pic}_{g-1}(\Sigma)$)

$$(z_1, \ldots, z_N) \to Q\left(-\sum_n z_n\right)$$

On $\mathrm{Pic}_{g-1}(\Sigma)$ there is a canonical bundle $\mathcal{O}(\Theta)$

The line bundles $\mathbb{Q}(-\Delta)$ and $f_Q^* \mathcal{O}(\Theta)$ are isomorphic

A useful way to show this is due to Fay (1973)
("bosonisation formula")

$$\operatorname{div}_{\Sigma^N} \Theta\left([Q] - \sum_n z_n - \Delta \right) = \operatorname{div}_{\Sigma^N} \frac{\det \sigma_n(z_m)}{\prod_{n<m}^{N} E(z_n, z_m)}$$

where $\Theta(u, \tau) = \sum_{m \in \mathbb{Z}^g} e^{\pi i \langle m, \tau m\rangle + 2\pi i \langle m, u\rangle}$ on $\mathbb{C}^g \times \mathcal{H}_g$

$\{\sigma_n\} \in H^0(Q, \Sigma)$

and Prime-form $E(z,w) = \dfrac{\Theta(w - z + \alpha)}{h_\alpha(z)\, h_\alpha(w)} \simeq \dfrac{z-w}{\sqrt{dz}\sqrt{dw}}$

α - odd spin structure and h_α is a certain section of $\sqrt{K_\Sigma}$.

We are interested in completely symmetric sections of

$$\mathbb{L}(-\beta\Delta) \simeq \left(Q(-\Delta)\right)^{\otimes\beta} \simeq \left(f^*\mathcal{O}(\Theta)\right)^{\beta}$$

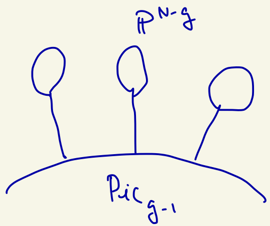
i.e. sections of line bundle $\mathbb{L}(-b\Delta)$ on $S^N\Sigma$.

The map $f_Q : S^N\Sigma \to \mathrm{Pic}_{g-1}(\Sigma)$

$$\{z_1, \dots, z_N\} \to Q(-\sum_n z_n)$$

has projective spaces $\mathbb{P}^{N-g}$ as fibers

(corresponding to the linear systems of Q)



Thus every global section of $\left(f_Q^* \mathcal{O}(\Theta)\right)^{\beta}$ is

constant on every fiber of f_Q and is equal

to pull-back of global section of $\mathcal{O}(\beta\Theta)$

Thus we identified the space of completely symmetric

sections of $\mathbb{L}(-\beta\Delta)$ and the space of global sections

of $\mathcal{O}(\beta\Theta)$. The latter has $\dim = \beta^g$

("level-β Riemann theta functions", see e.g. Mumford "Tata lectures")

$$\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_n z_n - [L] - \Delta \right) \cdot \prod_{n<m}^{N} E^{\beta}(z_n, z_m)$$

$r \in (1, \dots, \beta)^g$

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(u, \tau) = \sum_{n\in\mathbb{Z}^g} e^{\pi i \langle (m+a), \tau(m+a)\rangle + \langle m+b, u\rangle}$$

$a, b \in \mathbb{R}^g$

II. $p = d - \beta(N + g - 1) > 0$

strategy: - define Q as $L \simeq Q^{\otimes\beta}(p x_0)$ $x_0 \in \Sigma$

- first Chern class of $\mathbb{L}(-\beta\Delta)$ is $\beta\Theta + p\zeta$ where $\zeta \subset S^N\Sigma$ is the divisor of the configuration of N points w/ at least 1 pt coinciding w/ x_0.

- apply HRR thm

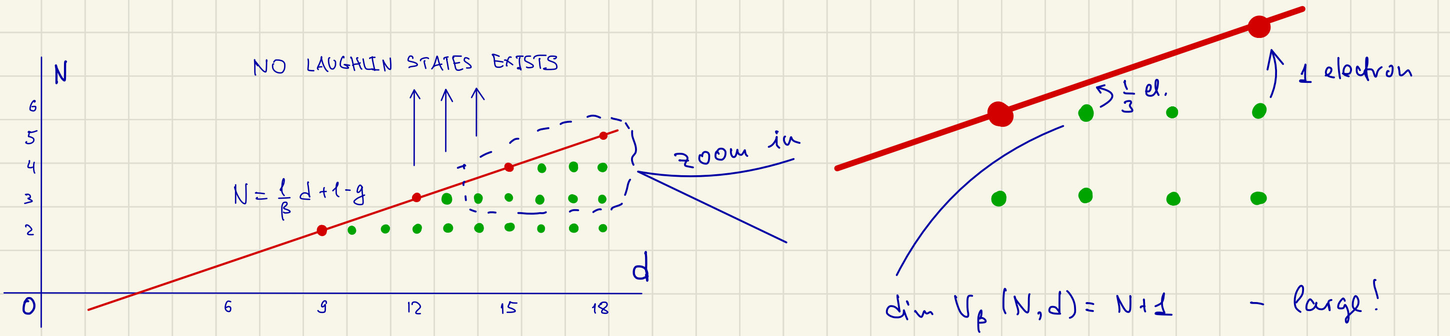
$$\sum_{i=0}^{N} (-1)^i \dim H^i(S^N\Sigma, \mathbb{L}(-\beta\Delta)) = \int_{S^N\Sigma} e^{\beta\theta + p\eta} \, td \, T_* S^N\Sigma$$

- show that $H^{i>0} = 0$ by Kodaira vanishing :

$$c_1\left(\mathbb{L}(-b\Delta) \otimes K^{-1}\right) > 0$$

$$\dim H^0(S^N\Sigma, \mathbb{L}(-\beta\Delta)) = \dim V_\beta(N,d) = \sum_{k=0}^{g} \binom{g}{k} \binom{p+N-g}{p+k-g} \beta^k$$

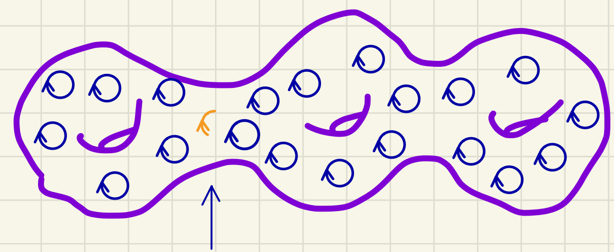
$\square$



Quasihole states

Line bundle $L \to \Sigma$ is replaced

by $L(+\omega_1 + \omega_2 + \dots + \omega_p) \to \Sigma$



One quasihole state

$$N = \frac{d-1}{\beta} + 1 - g$$

Hence the dimension of p-quasihole Laughlin states is again β^g!

$$\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_n z_n - [L] + w_1 + \dots + w_p + \Delta\right)$$

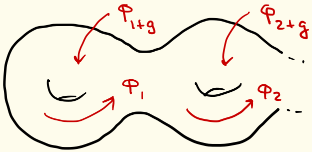
$$\cdot \prod_{n=1}^{N} E(z_n, w_1) \dots \prod_{n=1}^{N} E(z_n, w_p) \cdot \prod_{n<m}^{N} E(z_n, z_m)^{\beta}$$

$\rightsquigarrow$ family of β^g states over $S^p \Sigma$.

Families of Laughlin states over parameter spaces

Avron-Seiler-Zograf '95

* Hall conductance - transport on $L \otimes L_\phi \in \mathrm{Pic}_d$ (flat bundle: L_ϕ)

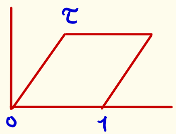


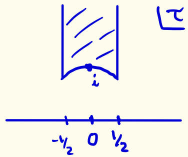
$$I_k = i \sum_{j=1}^{2g} \omega_{kj} \dot{\phi}_j \qquad \phi_j \in [0,2\pi]^{2g} = \mathrm{Jac}(\Sigma)$$

$$\omega = \sum_{j=1}^{g} d\varphi_j \wedge d\varphi_{j+g}$$

* transport on $\mathcal{M}_{1,1}$

("geometric adiabatic transport")





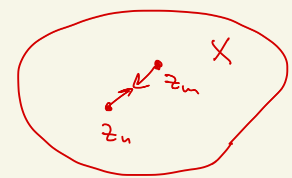
<u>Conjecture (N. Read '08)</u> Vector bundles of Laughlin states over Pic_d, $\mathcal{M}_{g,n}$, $S^p\Sigma$ are <u>projectively flat</u> (at least asymptotically as $N \to \infty$)

$$N = \frac{d}{\beta} + 1 - g$$

Laughlin states in complex dimension two.

* Let X be a compact complex manifold $\dim_{\mathbb{C}} X = 2$
* Let L be a positive line bundle over X and take its dth tensor power L^d
* Choose $\beta \in \mathbb{Z}_+$, N particles, X^N

Laughlin states are symmetric (for β even) or anti-symmetric (β odd) holomorphic section of $\pi_1^* X \otimes \cdots \otimes \pi_N^* X$ on X^N, which vanish to the order β, when $z_n = z_m$.



$$H\Psi = 0 \qquad H = \sum_{n=1}^{N} \bar{\partial}^{+}_{L^d,n} \bar{\partial}_{L^d,n} + \sum_{n} V(z_n, z_m)$$

In order to define vanishing, we consider N points $P_1, \ldots, P_N \in \mathbb{C}^2$

with coordinates $x_1, y_1, \ldots, x_N, y_N$; the polynomial ring $\mathbb{C}[x_1, y_1, \ldots, x_N, y_N]$

Let V_{nm} be a locus where $P_n = P_m$, i.e. a codim.-2 subspace in $(\mathbb{C}^2)^N$

where $x_n = x_m$ and $y_n = y_m$.

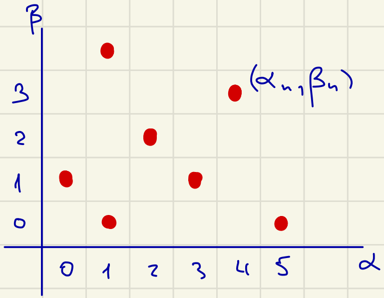
The locus $V = \bigcup_{n<m} V_{nm}$ is the zero locus of the radical ideal

$$I = \bigcap_{n<m} (x_n - x_m, y_n - y_m)$$

Order-β vanishing: $\Psi_L \in I^\beta$ in a local coordinate ring

Bivariate Vandermonde determinant

Let D be the set $D = \{ (\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N) \} \subseteq \mathbb{N} \times \mathbb{N}$



$$\Delta_D = \det \begin{vmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \cdots & x_1^{\alpha_N} y_1^{\beta_N} \\ \vdots & & \\ x_N^{\alpha_1} y_N^{\beta_1} & \cdots & x_N^{\alpha_N} y_N^{\beta_N} \end{vmatrix}$$

Thm (M. Haiman)

$$I = (\Delta_D : |D| = N)$$

$$I^\beta = \bigcap_{n<m} (x_n - x_m, y_n - y_m)^\beta = (\Delta_D : |D| = N)^\beta$$

We can projectivize Δ_D and get bivariate Vandermonde determinants on $X = \mathbb{CP}^2$

Let (x, y, t) be homogeneous coordinates on $\mathbb{CP}^2$ and take d large enough

$$D = \{ (\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N) \} \subseteq \mathbb{N} \times \mathbb{N}$$

$$\Delta_D = \det \begin{vmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \dots & x_1^{\alpha_N} y_1^{\beta_N} \\ \vdots & & \\ x_N^{\alpha_1} y_N^{\beta_1} & \dots & x_N^{\alpha_N} y_N^{\beta_N} \end{vmatrix} \rightarrow \det \begin{vmatrix} x_1^{\alpha_1} y_1^{\beta_1} t_1^{d-\alpha_1-\beta_1} & \dots & x_1^{\alpha_N} y_1^{\beta_N} t_1^{d-\alpha_N-\beta_N} \\ \vdots & & \\ x_N^{\alpha_1} y_N^{\beta_1} t_N^{d-\alpha_1-\beta_1} & \dots & x_N^{\alpha_N} y_N^{\beta_N} t_N^{d-\alpha_N-\beta_N} \end{vmatrix}$$

The matrix elements here a sections in $H^0(\mathbb{CP}^2, \mathcal{O}(k))$, $k>0$

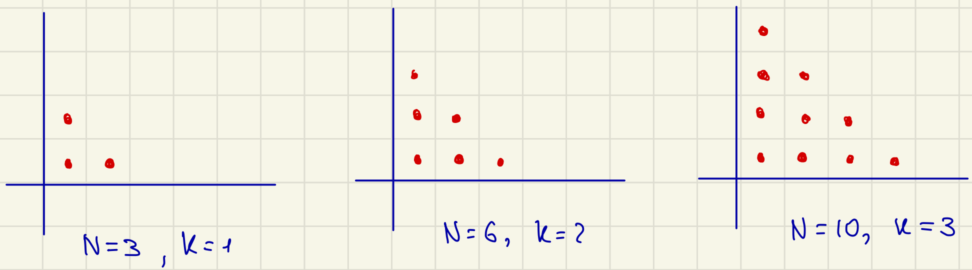
$$S_{(\alpha,\beta)} = \{ x^{\alpha} y^{\beta} t^{k-\alpha-\beta} , (\alpha,\beta) \in D \}$$

and thus $(\Delta_D)^{\beta}$ will be a Laughlin state for $L^d = \mathcal{O}(\beta k)$ $(d = \beta k)$

* Before we asked, for given β and d what is the <u>maximal N</u> for which a Laughlin state on X exists and how many states are there?

* We can equivalently ask, for given β and N, <u>find minimal d</u> for which a Laughlin state on X exists and how many states are there?

These optimally-packed states correspond to diagrams D with completely filled lower-left corner

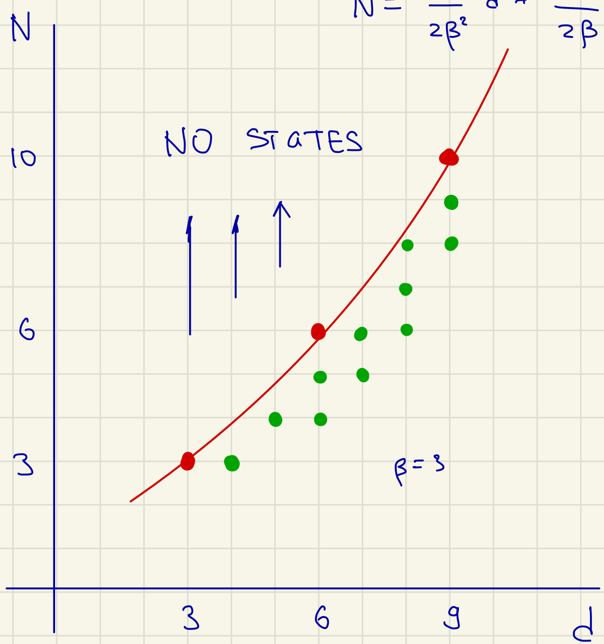


Take $\{s_n\}\Big|_{n=1}^{N}$ a full basis of $H^0(\mathbb{CP}^2, \mathcal{O}(k))$, $N = h^0 = \frac{1}{2}k^2 + \frac{3}{2}k + 1$

Then there is a unique Laughlin state for line bundle $\mathcal{O}(d = \beta k)$ and same number of particles N:
$$\Psi_L = \left(\det s_n(z_m)\Big|_{n,m=1}^{N}\right)^{\beta}$$

We checked this numerically for $k \leq 7$ and $\beta = 3$.

$$N = \frac{1}{2\beta^2} d^2 + \frac{3}{2\beta} d + 1$$



For general X we expect

$$N = c_2 d^2 + c_1 d + c_0 \qquad (*)$$

Questions

* Find c_0, c_1, c_2 as functions of L, X, β.
* How many Laughlin states are for (N,d) satisfying $(*)$?

An observation: Take (N,d) as above and consider the quasihole state at $(N, d+1)$.

It will have zeroes at d points $w_1, \ldots, w_d$ on X and the parameter space of these points will be $\mathrm{Hilb}_d(X)$ (not $S^d X$).

The End