PROBABILITY OF CONFORMAL FIELD THEORY

1. Abstracts

1.1. **Mini courses.**

Morris Ang and Xin Sun: Conformal Loop Ensembles and Conformal Field Theory

Abstract: The conformal loop ensemble (CLE) is a random collection of countably many non-crossing loops, which arises as the scaling limit of many critical loop models such as percolation, the Ising model, and (conjecturally) the $O(n)$ loop model. The CLE nesting function describes the joint conformal radii of loops surrounding three points on the sphere (reweighted by two-point functions). In the discrete, it corresponds to the $O(n)$ loop model where loops surrounding each of three points are given different weights. Ikhlef, Jacobsen and Saleur proposed that the CLE nesting function agrees with the imaginary DOZZ formula. We explain the mathematical proof of this conjecture via the coupling of Liouville quantum gravity (LQG) with CLE. Time-permitting, we may also discuss some details on our recent work with Gefei Cai (BICMR) and Baojun Wu (BICMR) concerning other types of CLE three-point functions that can be expressed in terms of the imaginary DOZZ formula, which will be reported by Baojun Wu.

Benoit Estienne: Introduction to minimal models.

Abstract: Minimal models are two-dimensional conformal field theories distinguished by a spectrum constructed from a finite number of irreducible representations of the Virasoro algebra. After outlining the relevant irreducible representations, we will discuss a method for classifying the spectrum and operator content through the modular bootstrap approach. This amounts to demand modular invariance of the torus partition function. Furthermore, we will explore how operator product expansion structure constants can be derived from the crossing symmetry of four-point functions on the sphere, via the conformal bootstrap approach. Finally, we will conclude by introducing an alternative approach for predicting these OPE structure constants using the Coulomb Gas.

Kale Kytola & Eveliina Peltola: Ising CFT: Questions about it and answers around it.

Abstract: What would a satisfactory Ising scaling limit CFT look like? How does it depend on the probabilistic questions that one asks, e.g., whether one considers correlation functions of locally defined random variables, or global configurations of interfaces? These lectures address the Ising CFT (and CFTs in general) mostly from the vertex operator algebraic axiomatization perspective. We discuss known aspects and open problems motivated by and related to, for example, Felder's BRST ("Coulomb gas") resolution of minimal models,

implications on braiding and single-valued correlations, solving crossing probabilities in terms of BPZ PDEs, and CFT structures already present in the lattice model.

Colin Guillarmou: The conformal blocks of Liouville conformal field theory.

Abstract: We will discuss the global construction of conformal blocks for Liouville CFT as holomorphic sections of a line bundle on Teichmüller space. The construction builds on probabilistic construction of Segal's amplitudes.

Ioan Manolescu: From the six-vertex model to FK-percolation and back.

Abstract: We will explore the connection between the six-vertex model and FK-percolation on the square lattice, and present some recent result that ensue. The eigenvalues of the six-vertex model transfer matrix may be computed using the Bethe Ansatz. As a consequence, it may be proved that FK-percolation has a continuous phase transition for the cluster weight q between 1 and 4, and a discontinuous one for $q > 4$. Furthermore, the critical FK-percolation (with q between 1 and 4) will be shown to be invariant under rotations at large scales. As a consequence, the same invariance may be deduced for the height function of the six-vertex model in the full plane. Finally, we will discuss how the rotational invariance of the six-vertex height function may be related to its conjectured convergence to the Gaussian Free Field.

FK and six vertex model: BKW correspondence. Continuity or not of phase transition. Rotational invariance of FK GFF convergence of six vertex

Remi Rhodes: Coulomb gas and compactified imaginary Liouville theory.

Abstract: Conformal Field Theories (CFT) play a central role in the description of statistical physics models undergoing a second order phase transition at their critical point. The recent development of the Liouville CFT, which is the scaling limit of random planar maps, has shed some light on the mathematical structure of CFT and has had many applications regarding the derivation of exact formulae for various statistical physics models. In this talk I will present the probabilistic construction of another CFT, called compactified imaginary Liouville CFT. In physics this path integral is conjectured to describe the scaling limit of critical loop models such as Q-Potts or O(n) models. I will explain the probabilistic construction and why it satisfies the axioms of CFT, in particular Segal's gluing axioms. This CFT has several exotic features: most importantly, it is non unitary and has conjecturally the structure of a logarithmic CFT. Therefore it provides a playground for the mathematical study of these concepts. Joint work with A. Kupiainen and C. Guillarmou.

Raoul Santachiarra: Dotsenko-Fateev integrals and their relation to rational and non-rational conformal field theories at central charge less than unity.

Abstract: In these lectures I review the Dotsenko-Fateev approach, which provides a representation in terms of Coulomb integrals of conformal blocks and multipoint correlation functions. I will discuss also certain relations (proven or conjectured) of this approach to some of the known bootstrap solutions in c¡=1, in particular to the minimal or to the generalized minimal models.

1.2. **Talks.**

Guillaume Baverez: Singular vectors in Liouville CFT

Abstract: The structure of degenerate modules is crucial to the study of a given CFT, as it encodes many analytic properties of conformal blocks. In Liouville CFT, we prove that certain modules are fully degenerate, as expected from physics. Joint with Baojun Wu.

Nathanael Berestycki:

Abstract:

Baptiste Cerclé: Equations of motion and singular vectors in Boundary Liouville CFT

Abstract: The notion of singular vector is fundamental in the understanding of two-dimensional (unitary) CFT. For example for Liouville CFT on the sphere, correlation functions containing a singular vector should satisfy a differential equation: a famous instance is the hypergeometric equation satisfied by certain four-point correlation functions and that led to the proof of the DOZZ formula. We will show that in the case of boundary Liouville CFT some singular vectors are actually non-zero in the sense that they yield so-called equations of motion rather than BPZ differential equations.

Franck Gabriel: Statistical Models and Unitary CFTs

Abstract: In this talk, we discuss the fundamental ideas of unitary conformal field theory and their connection to critical statistical models. After revisiting the origin of the unitarity property from the perspective of statistical models, we discuss the FQS non-unitarity theorem, which constrains the admissible central charges and scaling dimensions of unitary CFTs.

Jesper Jacobsen: Currents in the O(n) model

Abstract: Using methods from the conformal bootstrap, we study the properties of Noether currents in the critical $O(n)$ loop model. We confirm that they do not give rise to a Kac-Moody algebra (for $n \neq 2$), a result expected from the underlying lack of unitarity. By studying four-point functions in detail, we fully determine the current-current OPEs, and thus obtain several structure constants with physical meaning. We find in particular that the terms $: J\bar{J} :$ in the identity and adjoint channels vanish exactly, invalidating an argument made by Cardy in 1993 that adding orientation-dependent interactions to the model should lead to continuously varying exponents in self-avoiding walks. We also determine the residue of the identity channel in the JJ two-point function, finding that it coincides both with the result of a transfer-matrix computation for an orientation-dependent correlation function in the lattice model, and with an earlier Coulomb gas computation of Cardy. This is, to our knowledge, one of the first instances where the Coulomb gas formalism and the bootstrap can be successfully compared. (Work done in collaboration with Rongvoram Nivesvivat, Sylvain Ribault and Hubert Saleur.)

Antti Kupiainen: Wess-Zumino-Witten models and path integrals

Abstract: The Wess-Zumino-Witten (WZW) model is a 2 dimensional conformal quantum field theory where the field takes values in a Lie group G or its coset space. For a compact G this CFT is rational and its cosets G/H include for instance all unitary rational CFTs (e.g. the Ising model). WZW model has a formal path integral representation but it has

proved to be very hard to make mathematical sense of it and in fact most of its conjectured properties have been discussed using the representation theory of affine Lie algebras. In this talk I will review the basic facts about the path integral formulation of WZW models and then discuss the coset theory SL(2,C)/SU(2). This theory can be formulated in terms of field taking values in the 3-dimensional hyperbolic space and by the work of Ribault, Teschner and Schomerus it has been argued to have a mapping to the Liouville CFT. It has interesting connections to the 3d Chern-Simons topological QFT as well as a "quantum" deformation of the geometric and analytic Langlands correspondence. I will explain briefly how this theory can be formulated probabilistically using the Gaussian Multiplicative Chaos and how on a general Riemann surface the correlation functions of its primary fields can be mapped to those of the Liouville CFT.

Ellen Powell: Quantum length of SLE.

Abstract: I will discuss old and new constructions of natural length and quantum length of SLE curves. This will include recent joint work with Avelio Sepúlveda.

Sylvain Ribault: Correlation functions in loop models

Abstract: In the conformal bootstrap approach, loop models are characterized by: 1. A one-parameter family of degenerate fields, 2. Non-diagonal fields with fractional Kac indices, 3. Discrete operator product expansions, 4. Complex analyticity in the central charge. Correlation functions in loop models are related to sums over loop ensembles, parametrized by combinatorial maps. Using a numerical bootstrap approach, we can numerically compute solutions of crossing symmetry associated to simple maps, and deduce exact formulas for structure constants. These formulas involve a universal factor built from Barnes' double Gamma function, times polynomials in loop weights.

Baojun Wu: Proof of Delfino-Viti conjecture

Asbtract: In the context of random cluster models, the connectivity functions denoted as $P_n(x_1, x_2, ..., x_n)$ signify the probabilities associated with n points belonging to the same finite cluster. The initial conjecture by Delfino and Viti proposed that, at the critical point in the continuum limit, the ratio $R = P_3(x_1, x_2, x_3)/\sqrt{P_2(x_1, x_2)P_2(x_2, x_3)P_3(x_1, x_3)}$ converges to a universal constant solely dependent on κ. This dependence can be expressed through the imaginary DOZZ formula. For percolation, this constant approximates to 1.022. In this presentation, we elucidate the proof specifically for the percolation scenario. Additionally, we introduce analogous quantities within the conformal loop ensembles carpet/gasket measure, demonstrating their precise alignment with the imaginary DOZZ formula. The discussion will also delve into the statistical physics origin and its connections to conformal field theory, especially the recent work by Nivesvivat, Ribault and Jacobsen on exact solvability of critical loop model. This is based on the joint work with Morris Ang (Columbia), Gefei Cai (BICMR), and Xin Sun (BICMR).