THE MARKED LENGTH SPECTRUM OF ANOSOV MANIFOLDS

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Abstract. In all dimensions, we prove that the marked length spectrum of a Riemannian manifold (M, g) with Anosov geodesic flow and non-positive curvature locally determines the metric in the sense that two close enough metrics with the same marked length spectrum are isometric. In addition, we provide a completely new stability estimate quantifying how the marked length spectrum control the distance between the metrics. In dimension 2 we obtain similar results for general metrics with Anosov geodesic flows. We also solve locally a rigidity conjecture of Croke relating volume and marked length spectrum for the same category of metrics. Finally, by a compactness argument, we show that the set of negatively curved metrics (up to isometry) with the same marked length spectrum and with curvature in a bounded set of C^∞ is finite.

1. Introduction

Let (M, g) be a smooth closed Riemannian manifold. If the metric g admits an Anosov geodesic flow, the set of lengths of closed geodesics is discrete and is called the *length spectrum of* g. It is an old problem in Riemannian geometry to understand if the length spectrum determines the metric g up to isometry. Vigneras [Vi] found counterexamples in constant negative curvature. On the other hand we know that the closed geodesics are parametrised by the set $\mathcal{C}$ of free-homotopy classes, or equivalently the set of conjugacy classes of the fundamental group $\pi_1(M)$. Indeed, for each $c \in \mathcal{C}$, there is a unique closed geodesic γ_c of g in the class c. Particular examples of manifolds with Anosov geodesic flow are negatively curved compact manifolds. We can thus define a map, called the *marked length spectrum*

$$L_g : \mathcal{C} \to \mathbb{R}^+, \quad L_g(c) := \ell_g(\gamma_c) \tag{1.1}$$

where, if γ is a C^1-curve, $\ell_g(\gamma)$ denotes its length with respect to g.

We recall the following long-standing conjecture stated in Burns-Katok [BuKa] (and probably considered even before):

Conjecture 1. [BuKa, Problem 3.1] *If (M, g) and (M, g_0) are two closed manifolds with negative sectional curvature and same marked length spectrum, i.e $L_g = L_{g_0}$, then they are isometric, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.*

Note that if $\phi : M \to M$ is a diffeomorphism isotopic to the identity, then $L_{\phi^* g_0} = L_{g_0}$. The analysis of the linearised operator at a given metric g_0 is now well-understood, starting from the pionnering work of Guillemin-Kazhdan [GuKa], and pursued by the works of Croke-Sharafutdinov [CrSh], Dairbekov-Sharafutdinov [DaSh] and more recently by Paternain-Salo-Uhlmann [PSU, PSU2] and the first author [Gu1]. It is known that the linearised operator, the so called *X-ray transform*, is injective for non-positively curved manifolds with Anosov geodesic flows in all dimensions, and for all Anosov geodesic flows in dimension 2. These works imply the deformation rigidity result: there is no 1-parameter families of such metrics with the same marked length spectrum.

Concerning the non-linear problem (Conjecture 1), there are only very few results: in dimension 2 and non-positive curvature, the breakthrough was due to Otal [Ot] and Croke [Cr1] who solved that problem[1]. It was extended by Croke-Fathi-Feldman [CFF] to surfaces when one of the metrics has non positive curvature and the other has no conjugate points. In higher dimension, the only known result, to the best of our knowledge, is the result of Hamenstädt [Ha] which uses the famous entropy rigidity work by Besson-Courtois-Gallot [BCG]: if two negatively curved metrics (M, g) and (M, g_0) have the same marked length spectrum and their Anosov foliation is C^1, then $\mathrm{Vol}_g(M) = \mathrm{Vol}_{g_0}(M)$; since L_g determines the topological entropy, the result of [BCG] implies that if g_0 is a locally symmetric space, then g and g_0 are isometric. Besides these cases, the problem is largely open. The main difficulty is that the linearised operator takes values on functions on a discrete set and is not a very tractable operator to obtain non-linear results. We refer to the survey/lectures of Croke and Wilkinson [Cr1, Wi] for an overview of the subject.

The conjecture 1 actually also makes sense for Anosov geodesic flows, without the negative curvature assumptions, but it might be more reasonable to conjecture that only finitely many non isometric Anosov metrics have same marked length spectrum.

Our first result is a local rigidity statement that says that the marked length spectrum parametrises locally the isometry classes of metrics. This is the first (non-linear) progress towards Conjecture 1 in dimension $n \geq 3$, when g_0 is not a locally symmetric space.

Theorem 1. *Let (M, g_0) be:*

- *either a closed smooth Riemannian surface with Anosov geodesic flow,*
- *or a closed smooth Riemannian manifold of dimension $n \geq 3$ with Anosov geodesic flow and non-positive sectional curvature,*

and let $N > \frac{n}{2} + 8$. There exists $\epsilon > 0$ such that for any smooth metric g with same marked length spectrum as g_0 and such that $\|g - g_0\|_{C^N(M)} < \epsilon$, there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.

We actually prove a slightly stronger result in the sense that g can be chosen to be in the Hölder space $C^{N,\alpha}$ with $(N, \alpha) \in \mathbb{N} \times (0, 1)$ satisfying $N + \alpha > n/2 + 8$. Note also that $\epsilon > 0$ is chosen small enough so that the metrics g have Anosov geodesic flow too.

[1] Otal's work was in negative curvature and Croke in non-positive curvature. Note that Katok [Ka] already had a proof of conformally related metrics.

This result is new even if $\dim(M) = 2$ as we make no assumption on the curvature. If $\dim(M) > 2$ and g is Anosov, the same result holds outside a finite dimensional manifold of metrics, see Remark 4.1. This implies a general result supporting Conjecture 1:

Corollary 1.1. *Let (M, g_0) be an n-dimensional compact Riemannian manifold with negative curvature and let $N > \frac{n}{2}+8$. Then there exists $\epsilon > 0$ such that for any smooth metric with same marked length spectrum as g_0 and such that $\|g - g_0\|_{C^N(M)} < \epsilon$, there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.*

Since two C^0-conjugate Anosov geodesic flows that are close enough have the same marked length spectrum, we also deduce that for g_0 fixed as above, each metric g which is close enough to g_0 and has geodesic flow conjugate to that of g_0 is isometric to g.

To prove these results, a natural strategy would be to apply an implicit function theorem. The linearised operator I_2, called *X-ray transform*, consists in integrating 2-tensors along closed geodesics of g_0 (see Section 2.5). It is known to be injective under the assumptions of Theorem 1 by [CrSh, PSU, PSU2, Gu1], but as mentionned before, the main difficulty to apply this to the non-linear problem is that I_2 maps to functions on the discrete set $\mathcal{C}$ and it seems unlikely that its range is closed. To circumvent this problem, we use some completely new approach from [Gu1] that replaces the operator I_2 by a more tractable Fredholm one, that is constructed using microlocal methods in Faure-Sjöstrand [FaSj] and Dyatlov-Zworski [DyZw]. This new operator, denoted by Π_2, plays the same role as the normal operator $I_2^* I_2$ that is strongly used in the context of manifolds with boundary but Π_2 is not constructed from I_2. The additional crucial ingredient that allows us to relate the operators I_2 and Π_2 is a "positive Livsic theorem" due to Pollicott-Sharp [PoSh] and Lopes-Thieullen [LoTh]. We manage to obtain a sort of stability estimate for the X-ray operator with some loss of derivatives, but that is sufficient for our purpose. A corollary of this method is a completely new stability estimate for the X-ray transform on divergence-free tensors, that quantifies the smallness of a divergence-free symmetric m-tensor $f \in C^\alpha(M; S^m T^* M)$ (for $m \in \mathbb{N}$,$\alpha > 0$) in terms of the supremum of its integrals $\frac{1}{\ell(\gamma)} \int_\gamma f$ over all closed geodesics γ of g_0, see Theorem 5.

Combining these methods with some ideas developped by Croke-Dairbekov-Sharafutdinov [CDS] and the second author [Le] in the case with boundary, we are able to prove a new rigidity result which has similarities with the minimal filling volume problem appearing for manifolds with boundary and is a problem asked by Croke in [Cr2, Question 6.8].

Theorem 2. *Let (M, g_0) be as in Theorem 1 and let $N > \frac{n}{2} + 2$. There exists $\epsilon > 0$ such that for any smooth metric g satisfying $\|g - g_0\|_{C^N} < \epsilon$, the following holds true: if $L_g(c) \geq L_{g_0}(c)$ for all conjugacy class $c \in \mathcal{C}$ of $\pi_1(M)$, then $\mathrm{Vol}_g(M) \geq \mathrm{Vol}_{g_0}(M)$. If in addition $\mathrm{Vol}_g(M) = \mathrm{Vol}_{g_0}(M)$, then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.*

Again, in the proof, we actually just need $g \in C^{N,\alpha}$ with $(N, \alpha) \in \mathbb{N} \times (0, 1)$ satisfying $N+\alpha > n/2+2$. For two dimensional negatively curved surfaces, Croke-Dairbekov [CrDa] proved this result without the assumption that g is close to g_0, but our result holds in all dimensions and also without assumption on the curvature in dimension 2. To the best of our knowledge, this is the first result of that kind for closed manifolds in dimension $n > 2$.

In the next result, we get Hölder stability results quantifying how close are metrics with close marked length spectrum. In that aim we fix a metric g_0 with Anosov geodesic flow and define for g close to g_0 in some $C^N(M)$ norm

$$\mathcal{L}(g) \in \ell^\infty(\mathcal{C}), \quad \mathcal{L}(g) = \frac{L_g}{L_{g_0}}.$$

We are able to show: (here and below, $H^s(M)$ is the usual L^2-based Sobolev space of order $s \in \mathbb{R}$ on M)

Theorem 3. *Let (M, g_0) satisfy the assumptions of Theorem 1 and let $N > 3n/2+9$. For all $s > 0$ small there is a positive $\nu = \mathcal{O}(s)$ and a constant $C > 0$ such that the following holds: for all $\delta > 0$ small, there exists $\epsilon > 0$ small such that for all C^N metric g satisfying $\|g - g_0\|_{C^N} < \epsilon$, there is a diffeomorphism ϕ such that (here $\mathbf{1} = (1, ..., 1, ...)$)*

$$\|\phi^* g - g_0\|_{H^{-1-s}} \leq C\delta\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty}^{(1-\nu)/2} + C\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty}$$

We note that this the first quantitative result on the marked length rigidity problem. It is even new for negatively curved surfaces where the injectivity of $g \mapsto L_g$ (modulo isometry) is known by [Cr1, Ot].

We conclude by some finiteness results. On a closed manifold M, we consider for $\nu_1 \geq \nu_0 > 0$, $\theta_0 > 0$ and $C_0 > 0$ the set of smooth metrics g satisfying the assumptions of Theorem 1 and with Anosov geodesic flows satisfying the estimates (2.2) where the constants C, ν verify $C \leq C_0$, $\nu \in [\nu_0, \nu_1]$ and $d_G(E_s, E_u) \geq \theta_0$ where d_G denotes the distance in the Grassmanian of the unit tangent bundle SM induced by the Sasaki metric. We write $\mathcal{A}(\nu_0, \nu_1, C_0, \theta_0)$ for the set of such metrics. This is a closed set that consists of uniform Anosov geodesic flows. For example, metrics with curvatures contained in $[-a^2, -b^2]$ with $a > b > 0$ satisfy such property [**?**, Theorem 3.2.17]. In what follows, we denote by $\mathcal{R}_g$ the curvature tensor of g.

Theorem 4. *Let M be a smooth closed manifold and let $\nu_1 \geq \nu_0 > 0$, $C_0 > 0$ and $\theta_0 > 0$. For each sequence of positive numbers $B := (B_k)_{k\in\mathbb{N}}$, there is at most finitely many isometry classes of metrics g in $\mathcal{A}(\nu_0, \nu_1, C_0, \theta_0)$ satisfying the curvature bounds $|\nabla_g^k \mathcal{R}_g|_g \leq B_k$ and with the same marked length spectrum.*

Restricting to negatively curved metrics we get the finiteness results (new if $\dim M > 2$):

Corollary 1.2. *Let M be a compact manifold. Then, for each $a > 0$ and each sequence $B = (B_k)_{k\in\mathbb{N}}$ of positive numbers, there is at most finitely many smooth isometry classes of metrics with sectional curvature bounded above by $-a^2 < 0$, curvature tensor bounded by B (in the sense of Theorem 4) and same marked length spectrum.*

We remark that the C^∞ assumptions on the background metric g_0 in all our results and the boundedness assumptions on the C^∞ norms of the curvatures in Theorem 4 can be relaxed to C^k for some fixed k depending on the dimension.[2]

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2. The marked length spectrum and its linearisation

2.1. **Marked length spectrum.** We consider a smooth manifold M equipped with a smooth Riemannian metric g. We let $\pi_1(M)$ be the fundamental group of M and $\mathcal{C}$ be the set of conjugacy classes in $\pi_1(M)$. It is well-known that $\mathcal{C}$ corresponds to the set of free-homotopy classes of M. Assume now that the geodesic flow φ_t of g on the unit tangent bundle SM is Anosov, we will call *Anosov manifolds* such Riemannian manifolds and let

$$\mathcal{A} := \{g \in C^\infty(M; S^2_+T^*M); g \text{ has Anosov geodesic flow}\}.$$

We recall that φ_t with generating vector field X is called Anosov if there exists some constants $C > 0$ and $\nu > 0$ such that for all $z = (x, v) \in SM$, there is a continuous flow-invariant splitting

$$T_z(SM) = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{2.1}$$

where $E_s(z)$ (resp. $E_u(z)$) is the *stable* (resp. *unstable*) vector space in z, which satisfy

$$\begin{aligned} |d\varphi_t(z).\xi|_{\varphi_t(z)} \leq Ce^{-\nu t}|\xi|_z, \ \forall t \geq 0, \xi \in E_s(z) \\ |d\varphi_t(z).\xi|_{\varphi_t(z)} \leq Ce^{-\nu |t|}|\xi|_z, \ \forall t \leq 0, \xi \in E_u(z) \end{aligned} \tag{2.2}$$

The norm, here, is given in terms of the Sasaki metric of g. By Anosov structural stability [An, DMM], $\mathcal{A}$ is an open set. In particular, a metric $g \in \mathcal{A}$ has no conjugate points (see [Kl]) and there is a unique geodesic γ_c in each free-homotopy class $c \in \mathcal{C}$. We can thus define the *marked length spectrum* of g by (1.1).

It will be important for us to also consider the mapping $g \mapsto L_g$ from the space of metrics to the set of sequences. In order to be in a good functional setting and since we shall work

[2]The smoothness assumptions come from the fact we are using certain results based on microlocal analysis; it is a standard fact that only finitely many derivatives are sufficient for microlocal methods. It is likely that with some technical works one could improve the result to C^3 or C^4 regularity.

locally, we fix a smooth metric $g_0 \in \mathcal{A}$ and consider the metrics g in a neighborhood $\mathcal{U}_{g_0}$ of g_0 in $C^N(M; S^2_+T^*M)$ for some N large enough and which will be precised later. We can consider the map

$$\mathcal{L} : \mathcal{U}_{g_0} \to \ell^\infty(\mathcal{C}), \quad \mathcal{L}(g)(c) := \frac{L_g(c)}{L_{g_0}(c)}. \tag{2.3}$$

which we call the g_0-*normalized marked length spectrum*. We notice from the definition of the length that $\mathcal{L}(g) \in [0, 2]$ if $g \leq 2g_0$, justifying that $\mathcal{L}$ maps to $\ell^\infty(\mathcal{C})$.

Proposition 2.1. *The functional (2.3) is C^2 near g_0 if we choose the $C^3(M; S^2_+T^*M)$ topology. In particular, there is a neighborhood $\mathcal{U}_{g_0} \subset C^3(M; S^2_+T^*M)$ of g_0 and $C = C(g_0) > 0$ such that for all $g \in \mathcal{U}_{g_0}$*

$$\|\mathcal{L}(g) - \mathbf{1} - D\mathcal{L}_{g_0}(g - g_0)\|_{\ell^\infty(\mathcal{C})} \leq C\|g - g_0\|^2_{C^3}. \tag{2.4}$$

Proof. Let $\mathcal{M} := S_{g_0}M$ be the unit tangent bundle for g_0 and X_0 the geodesic vector field. We use the stability result in the work of De la Llave-Marco-Moryion [DMM, Appendix A] (see also the proof of [DGRS, Lemma 4.1]) which says that there is a neighborhood $\mathcal{V}_{X_0}$ in $C^2(\mathcal{M}; T\mathcal{M})$ of X_0 and a C^2 map $X \in \mathcal{V}_{X_0} \mapsto \theta_X \in C^0(\mathcal{M})$ such that for each $X \in \mathcal{V}_{X_0}$ and each fixed periodic orbit γ_{X_0} of X_0, there is a closed orbit γ_X freely-homotopic to γ_{X_0} and the period $\ell(\gamma_X)$ is C^2 as a map $X \in \mathcal{V}_{X_0} \mapsto \ell(\gamma_X) \in \mathbb{R}^+$ given by

$$\ell(\gamma_X) = \int_{\gamma_{X_0}} \theta_X.$$

In particular, we see that $X \in \mathcal{V}_{X_0} \mapsto \ell(\gamma_X)/\ell(\gamma_{X_0})$ is C^2 and its derivatives of order $j = 1, 2$ are bounded:

$$\|D^j\ell(\gamma_X)/\ell(\gamma_{X_0})\|_{C^2 \to \mathbb{R}} \leq \sup_{X \in \mathcal{V}_{X_0}} \|D^j\theta_X\|_{C^2 \to C^0} \leq C$$

for some C depending on $\mathcal{V}_{X_0}$. Now we fix $c \in \mathcal{C}$ and choose the geodesic $\gamma_c(g_0)$ for g_0 as being the element γ_{X_0} above, and we take $\mathcal{U}_{g_0}$ a small neighborhood of g_0 in the C^3 topology. The map $X : g \in \mathcal{U}_{g_0} \mapsto X_g \in C^2(\mathcal{M}; T\mathcal{M})$ is defined so that X_g is the geodesic vector field of g, where we used the natural diffeomorphism between $\mathcal{M} = S_{g_0}M$ and $S_gM := \{(x, v) \in TM; g_x(v, v) = 1\}$ obtained by scaling the fibers to pull-back the field on $\mathcal{M}$. It is a C^∞ map between the Banach space $C^3(M; S^2_+T^*M)$ and $C^2(\mathcal{M}; T(\mathcal{M}))$. Thus the composition $g \mapsto \ell(\gamma_{X_g})$, which is simply the map $g \mapsto L_g(c)$, is C^2 on $\mathcal{U}_{g_0}$ and the second derivative is uniformly bounded in $\mathcal{U}_{g_0}$. The inequality (2.4) follows directly. □

2.2. **The X-ray transform.** The central object on which stands our proof is the *X-ray transform* over symmetric 2-tensors, which is nothing more than the linearisation $D\mathcal{L}$ that appeared in Proposition 2.1. It is a direct computation, which appeared already in [GuKa], that for $h \in C^3(M; S^2T^*M)$

$$(D\mathcal{L}(g_0).h)(c) = \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} h_{\gamma_c(t)}(\dot\gamma_c(t), \dot\gamma_c(t))dt$$

where $\gamma_c(t)$ is the geodesic for g_0 homotopic to c and $\dot\gamma_c(t)$ its time derivative. This leads us to define the so-called *X-ray transform on* 2*-tensors* for g_0 as the operator

$$I_2^{g_0} : C^3(M; S^2T^*M) \to \ell^\infty(\mathcal{C}), \qquad I_2^{g_0} h(c) := \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} h_{\gamma_c(t)}(\dot\gamma_c(t), \dot\gamma_c(t))dt \tag{2.5}$$

Note that if φ_t is the geodesic flow for g_0 (the flow of X_{g_0}), this can be rewritten as

$$I_2^{g_0} h(c) = \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} \pi_2^* h(\varphi_t(z))dt$$

where $z \in \gamma_c$ is any point on the closed orbit and here, for $m \in \mathbb{N}$, we have denote by π_m^* is the natural continuous maps (for all $k \in \mathbb{N} \cup \{\infty\}$)

$$\pi_m^* : C^k(M, S^mT^*M) \to C^k(SM), \qquad f \mapsto (\pi_m^* f)(x, v) = f(x)(\otimes^m v)$$

where now we use SM as a notation for the unit tangent bundle for g_0. More generally, if we define the *X-ray transform* on SM by

$$I^{g_0} : C^0(SM) \to \ell^\infty(\mathcal{C}), \qquad I^{g_0} h(c) := \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} h(\varphi_t(z))dt \tag{2.6}$$

with $z \in \gamma_c$, we will also define the *X-ray transform on m-tensors* as the operator (for $m \in \mathbb{N}$) defined on $C^0(M; S^mT^*M)$ by

$$I_m^{g_0} := I^{g_0} \pi_m^*. \tag{2.7}$$

When the background metric is fixed, we will remove the g_0 index and just write I_m, I instead of $I_m^{g_0}, I^{g_0}$. There is also a dual operator acting on distributions

$$\pi_{m*} : C^{-\infty}(SM) \to C^{-\infty}(M, S^mT^*M), \qquad \langle \pi_{m*} u, f\rangle := \langle u, \pi_m^* f\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the distributional pairing. Let $H^s(SM)$ (resp. $H^s(M; S^mT^*M)$) denote the L^2-based Sobolev space of order $s \in \mathbb{R}$ on SM (resp. on m-tensors on M). We note that for all $s \in \mathbb{R}$, the following map are bounded

$$\pi_m^* : H^s(M; S^mT^*M) \to H^s(SM), \quad \pi_{m*} : H^s(SM) \to H^s(M; S^mT^*M). \tag{2.8}$$

Let us now explain the notion of *solenoidal injectivity* of the X-ray transform. If ∇ denotes the Levi-Civita connection of g_0 and $\sigma : \otimes^{m+1}T^*M \to S^{m+1}T^*M$ the symmetrisation operation, we define the symmetric derivative $D := \sigma \circ \nabla : C^\infty(M; S^mT^*M) \to C^\infty(M; S^{m+1}T^*M)$. The divergence operator is its formal adjoint given by $D^* f := -\mathrm{Tr}(\nabla f)$, where $\mathrm{Tr} : C^\infty(M; S^mT^*M) \to C^\infty(M; S^{m-2}T^*M)$ denotes the trace map defined by $\mathrm{Tr}(q)(v_1, ..., v_{m-2}) = \sum_{i=1}^n q(e_i, e_i, v_1, ..., v_{m-2})$, if $(e_1, ...e_n)$ is a local orthonormal basis of TM for g_0. If $f \in C^{k,\alpha}(M; S^m T^*M)$ with $(k,\alpha) \in \mathbb{N} \times (0,1)$, there exists a unique decomposition of the tensor f such that

$$f = f^s + Dp, \quad D^* f^s = 0, \tag{2.9}$$

where $f^s \in C^{k,\alpha}(M; S^m T^*M)$ and $p \in C^{k+1,\alpha}(M; S^{m-1}T^*M)$ (see [Sh, Theorem 3.3.2]). The tensor f^s is called the divergence-free part (or solenoidal part) of f. It is direct to see that for each $f \in C^k(M; S^m T^*M)$, we have $\pi_{m+1}^* Df = X\pi_m^* f$ and that for $u \in C^k(SM)$ with $k \geq 1$ we have $I(Xu) = 0$ if $X = X_{g_0}$ is the geodesic vector field for g_0 on SM. This implies that for $k \geq 1$

$$\forall f \in C^k(M; S^m T^*M), \quad I_{m+1}(Df) = 0.$$

Thus in general it is impossible to recover the exact part Dp of a tensor f from $I_m f$. We know recall some results about soleinoidal injectivity of I_m, defined as the property

$$\ker I_m \cap C^\infty(M; S^m T^*M) \cap \ker D^* = 0. \tag{2.10}$$

Proposition 2.2. *Let (M, g_0) be a smooth Riemannian manifold and assume that the geodesic flow of g_0 is Anosov. Then I_m is solenoidal injective in the sense (2.10) when:*

(1) $m = 0$ or $m = 1$, see [DaSh, Theorem 1.1 and 1.3]*,*
(2) $m \in \mathbb{N}$ and $\dim(M) = 2$, see [Gu1, Theorem 1.4]*,*
(3) $m \in \mathbb{N}$ and g_0 has non-positive curvature, see [CrSh, Theorem 1.3].

The case (2) with $m = 2$ was first proved in [PSU, Theorem 1.1].

3. The operator Π and stability estimates

In this section, we briefly review the results of the paper [Gu1] and in particular the operator Π defined there. As before we assume that (M, g_0) has Anosov geodesic flow and let $X = X_{g_0}$ be its geodesic vector field.

3.1. **The operator** Π. Since X preserves the Liouville measure μ, the operator $-iX$ is an unbounded self-adjoint operator on $L^2(SM) := L^2(SM, d\mu)$. The L^2-spectrum is then contained in $\mathbb{R}$ and the resolvents $R_\pm(\lambda) := (-X \pm \lambda)^{-1}$ are well-defined and bounded on $L^2(SM)$ for $\mathrm{Re}(\lambda) > 0$, they are actually given by

$$R_+(\lambda)f(z) = \int_0^{+\infty} e^{-\lambda t} f(\varphi_t(z))dt, \quad R_-(\lambda)f(z) = -\int_{-\infty}^0 e^{\lambda t} f(\varphi_t(z))dt$$

In [FaSj], Faure-Sjöstrand (see also [BuLi, DyZw]) proved that for Anosov flows, there exists a constant $c > 0$ such that for any $s > 0, r < 0$, one can construct a Hilbert space $\mathcal{H}^{r,s}$ such that $H^s(SM) \subset \mathcal{H}^{r,s} \subset H^r(SM)$ and $-X - \lambda : \mathrm{Dom}_{\mathcal{H}^{r,s}}(X) \to \mathcal{H}^{r,s}$ (with $\mathrm{Dom}_{\mathcal{H}^{r,s}}(X) := \{u \in \mathcal{H}^{r,s}; Xu \in \mathcal{H}^{r,s}\}$) is an unbounded Fredholm operator with index 0 on $\mathrm{Re}(\lambda) > -c\min(|r|, s)$; for $-X + \lambda$ the same holds with a Sobolev space $\mathcal{H}^{s,r}$ satisfying the same properties as above. Moreover, $-X \pm \lambda$ is invertible on these spaces for

$\mathrm{Re}(\lambda)$ large enough, the inverses coincide with $R_\pm(\lambda)$ when acting on $H^s(SM)$ and extend meromorphically to the half-plane $\mathrm{Re}(\lambda) > -c\min(|r|, s)$, with poles of finite multiplicity.

An Anosov geodesic flow is mixing [An], and $R_\pm(\lambda)$ has a simple pole at $\lambda = 0$ with rank 1 residue operator ([Gu1, Lemma 2.5]): one can then write the Laurent expansion:

$$R_+(\lambda) = \frac{1 \otimes 1}{\lambda} + R_0 + \mathcal{O}(\lambda), \;\; R_-(\lambda) = -\frac{1 \otimes 1}{\lambda} - R_0^* + \mathcal{O}(\lambda), \tag{3.1}$$

where $R_0, R_0^* : H^s(SM) \to H^r(SM)$ are bounded. The operator Π is then defined by:

$$\Pi := R_0 + R_0^* \tag{3.2}$$

The following Theorem was obtained by the first author in [Gu1]:

Proposition 3.1. [Gu1, Theorem 1.1] *The operator $\Pi : H^s(SM) \to H^r(SM)$ is bounded, for any $s > 0, r < 0$, with infinite dimensional range, dense in the space of invariant distributions $C^{-\infty}_{inv}(SM) := \{w \in C^{-\infty}(SM); Xw = 0\}$. It is a self-adjoint map $H^s(SM) \to H^{-s}(SM)$, for any $s > 0$, and satisfies:*

(1) $\forall f \in H^s(SM), X\Pi f = 0$,

(2) $\forall f \in H^s(SM)$ such that $Xf \in H^s(SM)$, $\Pi X f = 0$.[3]

If $f \in H^s(SM)$ with $\langle f, 1\rangle_{L^2} = 0$, then $f \in \ker \Pi$ if and only if there exists a solution $u \in H^s(SM)$ to the cohomological equation $Xu = f$, and u is unique modulo constants.

The link between the X-ray transform I and the operator Π is rather unexplicit and given by the Livsic theorem [Li]. For instance if $f \in C^\infty(SM)$ is in the kernel of the X-ray transform, i.e. $If = 0$, then we know by the smooth Livsic theorem that there exists $u \in C^\infty(SM)$ such that $f = Xu$ and thus $\Pi f = \Pi X u = 0$ by Theorem 3.1, (ii).

Remark 3.1. In the study of the X-ray transform on a manifold with boundary, it is natural to introduce the normal operator I^*I. It satisfies $XI^*Iu = 0$, for any $u \in C^\infty$ and $I^*IXu = 0$ if u vanishes on the boundary. For closed manifolds, the operator Π is the analogue of the operator I^*I used for manifolds with boundary (e.g. in [PeUh, Gu2, SUV, Le]).

3.2. **The operators Π_m.** For $m \in \mathbb{N}$, we introduce the operator $\Pi_m := \pi_{m*}\Pi\pi_m^*$ mapping $C^\infty(M; S^mT^*M)$ to $C^{-\infty}(M; S^mT^*M)$. In [Gu1], the first author studied the microlocal properties of Π_m by using in particular the works [FaSj, DyZw].

Proposition 3.2. [Gu1, Theorem 3.5, Lemma 3.6] *The operator Π_m is a pseudodifferential operator of order -1 which is elliptic on solenoidal tensors in the sense that there exists pseudodifferential operators Q, S, R of respective order $1, -2, -\infty$ such that:*

$$Q\Pi_m = \mathrm{Id} + DSD^* + R$$

[3] In [Gu1], it is shown that $\Pi X f = 0$ if $f \in H^{s+1}(SM)$, but this implies the result by a density argument and the approximation result [DyZw2, Lemma E.47].

Moreover for any $s > 0$, $\Pi\pi_m^* : H^{-s}(M; S^mT^*M) \cap \ker D^* \to H^{-s}(SM)$ is bounded. It is injective if I_m is solenoidal injective in the sense of (2.10).

This results implies the following stability estimates.

Lemma 3.3. *Assume that I_m is solenoidal injective in the sense (2.10). For all $s > 0$, there exists a constant $C > 0$ depending on g_0, s such that for all $f \in H^{-s}(M; S^mT^*M) \cap \ker D^*$*

$$\|f\|_{H^{-s-1}(M)} \leq C\|\Pi\pi_m^* f\|_{H^{-s}(M)} \tag{3.3}$$

Proof. This is actually a consequence of Proposition 3.2. We know that there exist pseudodifferential operators Q, S, R of respective order $1, -2, -\infty$ on M such that:

$$Q\Pi_m = \mathrm{Id} + DSD^* + R.$$

For each $f \in H^{-s}(M; S^mT^*M)$ with $D^*f = 0$, we have $\Pi\pi_2^* f \in H^{-s}(SM)$ by Theorem 3.2. Then then exists $C > 0$ (which may change from line to line) such that for all such f

$$\begin{aligned} \|f\|_{H^{-s-1}} &\leq C(\|Q\Pi_m f\|_{H^{-s-1}} + \|Rf\|_{H^{-s-1}}) \leq C(\|\pi_{m*}\Pi\pi_m^* f\|_{H^{-s}} + \|Rf\|_{H^{-s-1}}) \\ &\leq C(||\Pi\pi_m^* f\|_{H^{-s}} + \|Rf\|_{H^{-s-1}}). \end{aligned}$$

where we used (2.8) and the boundedness of pseudodifferential operators on Sobolev spaces. The proof now boils down to a standard argument of functional analysis. Assume (3.3) does not hold. Then, one can find a sequence of tensors $f_n \in H^{-s}(M; S^mT^*M) \cap \ker D^*$, such that $\|f_n\|_{H^{-s-1}} = 1$ and thus:

$$1 = \|f_n\|_{H^{-s-1}} \geq n\|\Pi\pi_m^* f_n\|_{H^{-s}},$$

that is $\Pi\pi_m^* f_n \to_{n\to\infty} 0$ in H^{-s}, and thus in particular in H^{-s-1}. Since R is compact and $(f_n)_{n\in\mathbb{N}}$ is bounded in H^{-s-1}, we can assume (up to extraction) that $Rf_n \to_{n\to\infty} v \in H^{-s-1}$. By the previous inequality, we deduce that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H^{-s-1}, which thus converges to an element $f \in H^{-s-1}(M; S^mT^*M) \cap \ker D^*$ such that $\|f\|_{H^{-s-1}} = 1$. The operator $\Pi\pi_m^* : H^{-s-1} \to H^{-s-1}$ is bounded by Proposition 3.2 so $\Pi\pi_m^* f_n \to_{n\to\infty} 0 = \Pi\pi_m^* f$ and it is also injective so $f \equiv 0$. This is a contradiction. □

Remark 3.2. With a bit more work, we can actually get a better estimate with a $-(s + 1/2)$ Sobolev exponent on the left-hand side of (3.3).

4. Proofs of the main results

As before, we fix a smooth Riemannian manifold (M, g_0) with Anosov flow and will shall consider metrics g with regularity $C^{N,\alpha}$ for some $N \geq 3, \alpha > 0$ to be determined later and such that $\|g - g_0\|_{\mathcal{C}^{N,\alpha}} < \epsilon$ for some $\epsilon > 0$ small enough so that g also has Anosov flow.

4.1. **Reduction of the problem.** The metric g_0 is divergence-free with respect to itself: $D^*g_0 = -\mathrm{Tr}(\nabla g_0) = 0$ where the Levi-Civita connection ∇ and trace Tr are defined with respect to g_0. By a standard argument, there is a slice transverse to the diffeomorphism action $(\phi, g) \mapsto \phi^* g$ at the metric g_0; here ϕ varies in the group of C^N-diffeomorphisms on M homotopic to the identity. We shall write $\mathrm{Diff}_0^{N,\alpha}(M)$ for the group of $C^{N,\alpha}(M)$ diffeomorphisms homotopic to Id, with $N \geq 2, \alpha \in (0,1)$. Since $\mathcal{L}(\phi^* g_0) = \mathcal{L}(g_0) = 1$ for all $\phi \in \mathrm{Diff}_0^{N,\alpha}(M)$, it suffices to work on that transverse slice to study the marked length spectrum. This is the content of the following:

Lemma 4.1. [CDS, Theorem 2.1] *Let $N \geq 2$ be an integer, $\alpha \in (0,1)$. For any $\delta > 0$ small enough, there exists $\epsilon > 0$ such that for any g satisfying $\|g - g_0\|_{C^{N,\alpha}} < \epsilon$, there exists $\phi \in \mathrm{Diff}_0^{N,\alpha}(M)$ that is $C^{N,\alpha}$ close to* Id *such that $g' := \phi^* g$ is divergence-free with respect to the metric g_0 and $\|g' - g_0\|_{C^{N,\alpha}} < \delta$.*

In the following, for the sake of simplicity, we will write g instead of g' (obtained from this Lemma) and reduce ourselves to the case where g is divergence-free with respect to the metric g_0. We introduce $f := g - g_0 \in C^N(M; S^2T^*M)$, which is, by construction, divergence-free and satisfies $\|f\|_{C^N} < \delta$. Our goal is to prove that $f \equiv 0$, if δ is chosen small enough.

4.2. **Geometric estimates.** We let g, g_0 be two C^3-metrics with Anosov geodesic flow.

Lemma 4.2. *Assume that for each $L_g(c) \geq L_{g_0}(c)$ for $c \in \mathcal{C}$. If γ_c denotes the unique geodesic freely homotopic to c for g_0, then*

$$I_2^{g_0} f(c) = \int_{\gamma_c} \pi_2^* f \geq 0.$$

Proof. We first write for γ_c' the g-geodesic in the free-homotopy class c. Then

$$\int_{\gamma_c} \pi_2^* f = \int_{\gamma_c} \pi_2^* g - \int_{\gamma_c} \pi_2^* g_0 = E_g(\gamma_c) - L_{g_0}(c)$$

where $E_g(\gamma_c) = \int_0^{\ell_{g_0}(\gamma_c)} g_{\gamma_c(t)}(\dot\gamma_c(t), \dot\gamma_c(t)) dt$ is the energy functional for g. By using Cauchy-Schwartz, $E_g(\gamma_c) \geq \ell_g(\gamma_c)^2/\ell_{g_0}(\gamma_c)$ and since γ_c is freely-homotopic to c, we get $\ell_g(\gamma_c) \geq \ell_g(\gamma_c')$. Since also $\ell_g(\gamma_c') \geq \ell_{g_0}(\gamma_c)$ by assumption, we obtain the desired inequality. $\square$

Next, we can use the following result

Lemma 4.3. *There exists $\epsilon > 0$ small such that if $\|g - g_0\|_{C^0} \leq \epsilon$ and $\mathrm{Vol}_g(M) \leq \mathrm{Vol}_{g_0}(M)$, then with $f = g - g_0$*

$$\int_{SM} \pi_2^* f \, d\mu \leq \frac{2}{3} \|f\|_{L^2}^2.$$

Proof. Let $g_\tau := g_0 + \tau f$ with $f \in C^3(M; S^2T^*M)$. A direct computation gives that $\int_M \mathrm{Tr}_{g_0}(f)\mathrm{dvol}_{g_0} = \int_{SM} \pi_2^* f \, d\mu$. Then the argument of [CrSh, Proposition 4.1] by Taylor expanding $\mathrm{Vol}_{g_\tau}(M)$ in τ gives directly the result. □

Finally, we conclude this section with the following:

Lemma 4.4. *Assume that $I_2^{g_0} f(c) \geq 0$ for all $c \in \mathcal{C}$. Then, there exists a constant $C = C(g_0) > 0$, such that:*

$$0 \leq \int_{SM} \pi_2^* f \, d\mu \leq C \left(\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})} + \|f\|_{C^3}^2 \right) \tag{4.1}$$

where $d\mu$ is the Liouville measure of g_0 and $g = g_0 + f$ as above.

Proof. For the Anosov geodesic flow of g_0, the Liouville measure is the unique equilibrium state associated to the potential given by $J^u(z) := -\partial_t \left(\det d\varphi_t(z)|_{E_u(z)} \right) |_{t=0}$ (the unstable Jacobian). By Parry's formula (see [Pa, Paragraph 3]), we have:

$$\forall F \in C^0(SM), \; \lim_{T \to \infty} \frac{1}{N(T)} \sum_{c \in \mathcal{C}, L_{g_0}(c) \leq T} \frac{e^{\int_{\gamma_c} J^u}}{L_{g_0}(c)} \int_{\gamma_c} F = \frac{1}{\mathrm{Vol}(SM)} \int_{SM} F \, d\mu \tag{4.2}$$

where as before γ_c is the g_0-geodesic in c and $N(T)$ is the constant of normalization corresponding to the sum when $F = 1$. The first inequality in (4.1) then follows from that formula and the assumption $I_2^{g_0} f \geq 0$. For the second inequality in (4.1) we use Proposition 2.1 with the fact that $D\mathcal{L}_{g_0} f = I_2^{g_0} f$ to deduce that there exists $C(g_0) > 0$ such that

$$\|I_2^{g_0} f\|_{\ell^\infty(\mathcal{C})} \leq \|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})} + C(g_0)\|f\|_{C^3}^2. \tag{4.3}$$

By the previous inequality (4.3), we get for any $T > 0$

$$\frac{1}{N(T)} \sum_{c \in \mathcal{C}, L_{g_0}(c) \leq T} e^{\int_{\gamma_c} J^u} I_2^{g_0} f(c) \leq \|I_2^{g_0} f\|_{\ell^\infty(\mathcal{C})} \leq \|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})} + C(g_0)\|f\|_{C^3}^2 \tag{4.4}$$

and the left-hand side converges to $\frac{1}{\mathrm{Vol}(SM)} \int_{SM} \pi_2^* f \, d\mu$ by Parry's formula (4.2), which is the sought result. □

We note that in the previous proof, the approximation of $\int_{SM} \pi_2^* f$ by $I_2^{g_0} f(c)$ could also be done using Birkhoff ergodic theorem and Anosov closing lemma to approximate $\int_{SM} \pi_2^* f$ by some $I_2^{g_0} f(c)$ for some $c \in \mathcal{C}$ so that $L_{g_0}(c)$ is large.

The following lemma is another key ingredient in the proof of our main results. It is a positive version of Livsic theorem which was proved independently by Pollicott-Sharp [PoSh] and Lopes-Thieullen [LoTh] (though the stronger version we use is actually that of [LoTh]). Here $\mathcal{M}_1$ denotes the Borel probability measures on SM which are invariant by the geodesic flow of g_0. Note that, by Sigmund [Si], the Dirac measures $u \mapsto \frac{1}{L_{g_0}(c)} \int_{\gamma_c} u$ on closed orbits are dense in $\mathcal{M}_1$.

Proposition 4.5. [LoTh, Theorem 1], [PoSh, Theorem 1] *Let $\alpha \in (0,1]$ and let X_0 be the geodesic vector field of g_0. There exists a constant $C = C(g_0) > 0$ and $\beta \in (0,1)$ such that for any $u \in C^\alpha(SM)$ satisfying*

$$\forall c \in \mathcal{C}, \ \int_{\gamma_c} u \geq 0,$$

there exists a constant $m \geq 0$, $h \in C^{\alpha\beta}(SM)$ and $F \in C^{\alpha\beta}(SM)$ such that $F \geq 0$ and $u + Xh = F$. Moreover $\|F\|_{C^{\alpha\beta}} \leq C\|u\|_{C^\alpha}$.

4.3. **Proof of Theorem 1 and Theorem 2.** We fix g_0 with Anosov geodesic flow on M and assume that either M is a surface or that g_0 has non-positive curvature in order to have that $I_2^{g_0}$ is solenoidal injective by Proposition 2.2. Fix $N \geq 3$ to be chosen later and $\alpha > 0$ small. As explained in Lemma 4.1, we take $\delta > 0$ small and $\epsilon > 0$ small so that $\|g - g_0\|_{C^{N,\alpha}} < \epsilon$ implies that there is $\phi \in \mathrm{Diff}_0^{N,\alpha}(M)$ with $\|\phi^* g - g_0\|_{C^{N,\alpha}} < \delta$ and $D^*(\phi^* g - g_0) = 0$.

We denote $f := \phi^* g - g_0$ and remark that the assumption $L_g \geq L_{g_0}$ implies $L_{\phi^* g} \geq L_{g_0}$ thus $I_2^{g_0} f(c) \geq 0$ for all $c \in \mathcal{C}$ by Lemma 4.2. By Proposition 4.5, we know that there exists $h \in C^\beta(SM)$ and $F \in C^\beta(SM)$ for some $0 < \beta < \alpha$ (depending on g_0 and linearly on α) such that $\pi_2^* f + Xh = F \geq 0$, with

$$\|\pi_2^* f + Xh\|_{C^\beta} \leq C\|\pi_2^* f\|_{C^\alpha} \leq C\|f\|_{C^\alpha}. \tag{4.5}$$

where $C = C(g_0)$. Take $0 < s \ll \beta$ very small (it will be fixed later) and let $\beta' < \beta$ very close to β. Thus we obtain (for some constant $C = C(g_0, s, \beta)$ that may change from line to line)

$$\begin{aligned}
\|f\|_{H^{-1-s}} &\leq C\|\Pi \pi_2^* f\|_{H^{-s}}, && \text{by Lemma 3.3} \\
&\leq C\|\Pi(\pi_2^* f + Xh)\|_{H^{-s}}, && \text{since } \Pi X h = 0 \\
&\leq C\|\pi_2^* f + Xh\|_{H^s}, && \text{by Theorem 3.1} \\
&\leq C\|\pi_2^* f + Xh\|_{L^2}^{1-\nu}\|\pi_2^* f + Xh\|_{H^{\beta'}}^{\nu}, && \text{by interpolation with } \nu = s/\beta'.
\end{aligned} \tag{4.6}$$

Note that by (4.5) we have a control:

$$\|\pi_2^* f + Xh\|_{H^{\beta'}} \leq C\|\pi_2^* f + Xh\|_{C^\beta} \leq C\|f\|_{C^\alpha}. \tag{4.7}$$

And we can once more interpolate between Lebesgue spaces so that:

$$\|\pi_2^* f + Xh\|_{L^2} \leq C\|\pi_2^* f + Xh\|_{L^1}^{1/2}\|\pi_2^* f + Xh\|_{L^\infty}^{1/2} \leq C\|\pi_2^* f + Xh\|_{L^1}^{1/2}\|f\|_{C^\alpha}^{1/2}. \tag{4.8}$$

Next, using that $\pi_2^* f + Xh \geq 0$, we have

$$\|\pi_2^* f + Xh\|_{L^1} = \int_{SM} (\pi_2^* f + Xh) d\mu = \int_{SM} \pi_2^* f \, d\mu \tag{4.9}$$

Now we will consider two cases: in case (1) we assume that $L_g = L_{g_0}$ while in case (2) we assume that $\mathrm{Vol}_g(M) \leq \mathrm{Vol}_{g_0}(M)$. Combining now with Lemma 4.2 and Lemma 4.4, we deduce that in case (1), we have

$$\|\pi_2^* f + Xh\|_{L^1} \leq C\|f\|_{C^3}^2,$$

while in case (2) we get by Lemma 4.3 that if $\epsilon > 0$ is small enough,

$$\|\pi_2^* f + Xh\|_{L^1} \leq C\|f\|_{L^2}^2.$$

These facts combined with (4.8) yield

$$\|\pi_2^* f + Xh\|_{L^2} \leq \begin{cases} C\|f\|_{C^3}.\|f\|_{C^\alpha}^{1/2}, & \text{case (1)} \\ C\|f\|_{L^2}.\|f\|_{C^\alpha}^{1/2}, & \text{case (2)} \end{cases}.$$

Thus we have shown

$$\|f\|_{H^{-1-s}} \leq \begin{cases} C\|f\|_{C^3}^{1-\nu}\|f\|_{C^\alpha}^{\frac{1+\nu}{2}}, & \text{case (1)} \\ C\|f\|_{L^2}^{1-\nu}.\|f\|_{C^\alpha}^{\frac{1+\nu}{2}}, & \text{case (2)} \end{cases}. \tag{4.10}$$

where $C = C(g_0, s, \beta)$. We choose α very small and $0 < s \ll \beta < \alpha$, $j \in \{\alpha, 3\}$ and $N_0 > n/2 + j + s$: by interpolation and Sobolev embedding we have

$$\|f\|_{C^j} \leq \|f\|_{H^{n/2+j+s}} \leq C\|f\|_{H^{-1-s}}^{1-\theta_j}\|f\|_{H^{N_0}}^{\theta_j} \tag{4.11}$$

with $\theta_j = \frac{n/2+j+1+2s}{N_0+s+1}$. If $N_0 > \frac{3}{2}n+8$, we see that $\gamma := \frac{1}{2}(1-\theta_\alpha)(1+\nu)+(1-\theta_3)(1-\nu) > 1$ if $s > 0$ and α are chosen small enough, thus in case (1) we get with $\gamma' := (1+\nu)\theta_\alpha/2+(1-\nu)\theta_3$

$$\|f\|_{H^{-1-s}} \leq C\|f\|_{H^{-1-s}}^{\gamma}\|f\|_{H^{N_0}}^{\gamma'}$$

and thus if $f \neq 0$ we obtain, if $\|f\|_{H^{N_0}} \leq \delta$

$$1 \leq C\|f\|_{H^{-1-s}}^{\gamma-1}\|f\|_{H^{N_0}}^{\gamma'} \leq C\|f\|_{H^{N_0}}^{\gamma-1+\gamma'} \leq C\delta^{\gamma-1+\gamma'}.$$

Since $\gamma - 1 + \gamma' > 0$, we see that by taking $\delta > 0$ small enough we obtain a contradiction, thus $f = 0$. This proves Theorem 1 by choosing $N \geq N_0$. In case (2) (corresponding to Theorem 2), this is the same argument except that we get a slightly better result due to the L^2 norm in (4.10): N_0 can be chosen to be any number $N_0 > n/2 + 2$. To conclude, we have shown that if $\|g - g_0\|_{C^{N,\alpha}} < \epsilon$ for $N \in \mathbb{N}$ with $N + \alpha > n/2 + 2$, then $L_g \geq L_{g_0}$ implies that either $\mathrm{Vol}_g(M) \leq \mathrm{Vol}_{g_0}(M)$ and $\phi^* g = g_0$ for some $C^{N,\alpha}$ diffeomorphism, or $\mathrm{Vol}_g(M) \geq \mathrm{Vol}_{g_0}(M)$. Note that in both cases, if g is smooth then ϕ is smooth.

4.4. **Stability estimates for X-ray transforms.** We end this article with the proof of new stability estimates for the X-ray transform. In the following, we will consider the X-ray transform I_m over divergence-free symmetric m-tensors.

Theorem 5. *Assume that (M, g_0) satisfies the assumptions of Theorem 1. Then for all $\alpha > 0$, there is $\beta \in (0, \alpha)$ depending linearly on α such that for all $s \in (0, \beta)$ and for all $\nu \in (s/\beta, 1)$, there exists a constant $C > 0$ such that for all $f \in C^\alpha(M; S^m T^* M) \cap \ker D^*$*

$$\|f\|_{H^{-1-s}} \leq C\|I_m^{g_0} f\|_{\ell^\infty}^{(1-\nu)/2}(\|f\|_{C^\alpha} + \|I_m^{g_0} f\|_{\ell^\infty})^{(1+\nu)/2}.$$

Proof. The proof is essentially the same as Theorem 1. By using Lemma 3.3 with $\pi_m^* f$ replaced by $\pi_m^* f + \|I_m^{g_0} f\|_{\ell^\infty}$ and Proposition 4.5, we have as in (4.6) that for all $0 < \alpha < 1$ small, there is $0 < \beta < \alpha$ depending on g_0 and linearly on α such that for all $0 < s < \beta' < \beta$, and for all $f \in C^\alpha(M; S^m T^* M) \cap \ker D^*$

$$\|f\|_{H^{-1-s}} \leq C\|\Pi\pi_m^* f\|_{H^{-s}} \leq C\|\Pi(\pi_m^* f + Xh)\|_{H^{-s}} \leq C\|\pi_m^* f + Xh\|_{L^2}^{1-\nu}\|\pi_m^* f + Xh\|_{H^{\beta'}}^{\nu}$$

for some C depending only on $(g_0, s, \beta, \beta', \alpha)$, $\nu := s/\beta'$ and where $\pi_m^* f + Xh = -\|I_m^{g_0} f\|_{\ell^\infty} + F$ with $h, F \in C^\beta$ such that $\|F\|_{C^\beta} \leq C(\|f\|_{C^\alpha} + \|I_m^{g_0} f\|_{\ell^\infty})$. Using (4.7), (4.8), (4.9), (4.4) with $I_2^{g_0} f$ replaced by $I_m^{g_0} f$, we get the result. $\square$

Remark 4.1. Note that ν and s can be chosen arbitrarily small in the estimate. In the general case of an Anosov manifold (without any assumption on the curvature), the s-injectivity of the X-ray transform is still unknown. However, it was proved in [DaSh, Theorem 1.5] and [Gu1, Lemma 3.6] that its kernel is finite-dimensional and contains only smooth tensors. The same arguments as above then show that Theorem 5 still holds for all f as above with the extra condition $f \perp \ker I_m$ with respect to the L^2 scalar product, and similarly for Theorem 1 if g is not in a finite dimensional manifold.

4.5. **Stability estimates for the marked length spectrum. Proof of Theorem 3.** We will apply the same reasoning as before to get a stability estimate for the non-linear problem (the marked length spectrum). We proceed as before and reduce to considering $f = \phi^* g - g_0$ where $\phi \in \mathrm{Diff}_0^{N,\alpha}(M)$ and $\|f\|_{C^{N,\alpha}} < \delta$. By Theorem 5, and using (4.3) we have for $0 < \alpha$ small, $0 < s \ll \alpha$ and β, ν as in Theorem 5 (in particular ν, α, s can be made arbitrarily small):

$$\begin{aligned}
\|f\|_{H^{-s-1}} &\leq C\|I_2^{g_0} f\|_{\ell^\infty}^{(1-\nu)/2}(\|f\|_{C^\alpha} + \|I_2^{g_0} f\|_{\ell^\infty})^{(1+\nu)/2} \\
&\leq C(\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty} + \|f\|_{C^3}^2)^{(1-\nu)/2}\|f\|_{C^\alpha}^{(1+\nu)/2} + C(\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty} + \|f\|_{C^3}^2) \\
&\leq C\left(\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty}^{(1-\nu)/2}\|f\|_{C^\alpha}^{(1+\nu)/2} + \|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty} + \|f\|_{C^3}^{1-\nu}\|f\|_{C^\alpha}^{(1+\nu)/2}\right)
\end{aligned}$$

We use the interpolation estimate (4.11) and for $N_0 > n/2 + 9$ we get

$$\|f\|_{H^{-s-1}} \leq C\left(\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty}^{(1-\nu)/2}\|f\|_{C^\alpha}^{(1+\nu)/2} + \|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty} + \|f\|_{H^{-s-1}}^{\gamma}\|f\|_{C^{N_0}}^{\gamma'}\right) \quad (4.12)$$

where $\gamma = \frac{1}{2}(1-\theta_\alpha)(1+\nu) + (1-\theta_3)(1-\nu) > 1$, $\gamma' > 0$, $\theta_3 = \frac{n/2+4+2s}{N_0+s+1}$, if $s > 0$ is chosen small enough. Assume δ is chosen small enough so that $C\delta^{\alpha/2} \leq 1/2$. Then:

$$\|f\|_{H^{-s-1}}^{\gamma}\|f\|_{C^{N_0}}^{\gamma'} \leq C\|f\|_{H^{-s-1}}\|f\|_{C^{N_0}}^{(\gamma-1)+\gamma'} \leq C\|f\|_{H^{-s-1}}\delta^{(\gamma-1)+\gamma'} \leq \tfrac{1}{2}\|f\|_{H^{-s-1}}$$

if $\delta > 0$ is chosen small enough depending on $C = C(g_0, s, \alpha, \beta, \nu)$ and $N + \alpha > N_0$. The sought result then follows from the previous inequality combined with (4.12).

4.6. **Compactness theorems and proof of Theorem 4.** We let M be a compact smooth manifold equipped with an Anosov geodesic flow. By the proof of [Kn, Theorem 4.8], the universal cover $\widetilde{M}$ and the fundamental group $\pi_1(M)$ are hyperbolic in the sense of Gromov [Gr]. We shall denote by $\mathcal{R}_g$ the curvature tensor associated to the metric g and by $\mathrm{inj}(g)$ the injectivity radius of g. We proceed by contradiction: let $(g_n)_{n\geq 0}$ be a sequence of smooth metrics on M in the class $\mathcal{A}(\nu_0,\nu_1,C_0,\theta_0)$ (defined in the Introduction) such that $L_{g_n}=L_{g_0}$ and such that for each $k\in\mathbb{N}$ there is $B_k>0$ such that $|\nabla^k_{g_n}\mathcal{R}_{g_n}|_{g_n}\leq B_k$ for all n, and we ssume that for each $n\neq n'$, g_n is not isometric to $g_{n'}$. Since the metrics have Anosov flows, they have no conjugate points and thus

$$\mathrm{inj}(g_n)=\tfrac12\min_{c\in\mathcal{C}}L_{g_n}(c)=\tfrac12\min_{c\in\mathcal{C}}L_{g_0}(c).$$

By Hamilton compactness result [Ham, Theorem 2.3], there is a family of smooth diffeomorphism ϕ_n on M such that $g'_n:=\phi_n^*g_n$ converges to $g\in\mathcal{A}(\nu_0,\nu_1,C_0,\theta_0)$ in the C^∞ topology (note that $\mathcal{A}(\nu_0,\nu_1,C_0,\theta_0)$ is invariant by pull-back through smooth diffeomorphisms). Denote by $\phi_{n*}\in\mathrm{Out}(\pi_1(M))$ the action of ϕ_n on the set of conjugacy classes $\mathcal{C}$. The universal cover $\widetilde{M}$ of M is a ball since M has no conjugate points, and $\pi_1(M)$ is a hyperbolic group thus we can apply the result of Gromov [Gr, Theorem 5.4.1] saying that the outer automorphism group $\mathrm{Out}(\pi_1(M))$ is finite if $\dim M\geq 3$. This implies in particular that there is a subsequence $(\phi_{n_j})_{j\in\mathbb{N}}$ such that $\phi_{n_j*}(c)=\phi_{n_0*}(c)$ for all $c\in\mathcal{C}$ and all $j\in\mathbb{N}$ where as before $\mathcal{C}$ is the set of conjugacy classes of $\pi_1(M)$. But $\phi_{n_0}^*g_{n_j}$ have same marked length spectrum as $\phi^*_{n_0}g_0$ for all j thus $L_{g'_{n_j}}=L_{\phi^*_{n_0}g_0}$ for all j. Since $g'_{n_j}\to g$ in C^∞, we have $L_g=L_{g'_{n_j}}$ for all j and by Theorem 1, we deduce that there is j_0 such that for all $j\geq j_0$, g'_{n_j} is isometric to g. This gives a contradiction.

Now, if $\dim M=2$, $\mathrm{Out}(\pi_1(M))$ is a discrete infinite group. We first show that for each $c\in\mathcal{C}$, the set of classes $(\phi_n^{-1})_*(c)\in\mathcal{C}$ is finite as n ranges over $\mathbb{N}$. Assume the contrary, then consider γ_n the geodesic for g_n in the class c, one has $L_{g_n}(c)=\ell_{g_n}(\gamma_n)=\ell_{g_0}(\gamma_0)$ by assumption. Now $\phi_n^{-1}(\gamma_n)$ is a g'_n geodesic in the class $(\phi_n^{-1})_*(c)$ with length $\ell_{g'_n}(\phi_n^{-1}(\gamma_n))=\ell_{g_n}(\gamma_n)=\ell_{g_0}(\gamma_0)$. We know that there are finitely many g-geodesics with length less than $\ell_{g_0}(\gamma_0)$, but we also have

$$L_g((\phi_n^{-1})_*(c))\leq\ell_g(\phi_n^{-1}(\gamma_n))\leq\ell_{g'_n}(\phi_n^{-1}(\gamma_n))(1+\epsilon)\leq\ell_{g_0}(\gamma_0)(1+\epsilon)$$

if $\|g'_n-g\|_{C^3}\leq\epsilon$ thus we obtain a contradiction for n large. The extended mapping class group[4] $\mathrm{Mod}(M)$ is isomorphic to $\mathrm{Out}(\pi_1(M))$ (see [FaMa, Theorem 8.1]). By [FaMa, Proposition 2.8][5], if M has genus at least 3, there is a finite set $\mathcal{C}_0\subset\mathcal{C}$ such that if $\phi_*\in\mathrm{Mod}(M)$ is the identity on $\mathcal{C}_0$ then ϕ is homotopic to Id, while if M has genus 2, the same condition implies that ϕ is either homotopic to Id or to an hyperelliptic involution

[4] extended in the sense that it includes orientation reversing elements.

[5] see also the proof of Theorem 3.10 in [FaMa]

h. In both cases we can thus extract a subsquence ϕ_{n_j} such that $\phi_{n_{j*}} = \phi_{n_0*}$ for all $j \geq 0$ and we conclude as for the higher dimensional case.

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