CONFORMAL HARMONIC FORMS, BRANSON-GOVER OPERATORS AND DIRICHLET PROBLEM AT INFINITY.

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Abstract. For odd dimensional Poincaré-Einstein manifolds (X^{n+1}, g), we study the set of harmonic k-forms (for $k < \frac{n}{2}$) which are C^m (with $m \in \mathbb{N}$) on the conformal compactification $\bar{X}$ of X. This is infinite dimensional for small m but it becomes finite dimensional if m is large enough, and in one-to-one correspondence with the direct sum of the relative cohomology $H^k(\bar{X}, \partial\bar{X})$ and the kernel of the Branson-Gover [3] differential operators (L_k, G_k) on the conformal infinity $(\partial\bar{X}, [h_0])$. In a second time we relate the set of $C^{n-2k+1}(\Lambda^k(\bar{X}))$ forms in the kernel of $d + \delta_g$ to the conformal harmonics on the boundary in the sense of [3], providing some sort of long exact sequence adapted to this setting. This study also provides another construction of Branson-Gover differential operators, including a parallel construction of the generalization of Q curvature for forms.

1. Introduction

Let $(M, [h_0])$ be an n-dimensional compact manifold equipped with a conformal class $[h_0]$. The k-th cohomology group $H^k(M)$ can be identified with $\ker(d+\delta_h)$ for any $h \in [h_0]$ by usual Hodge-De Rham Theory. However, the choice of harmonic representatives in $H^k(M)$ is not conformally invariant with respect to $[h_0]$, except when n is even and $k = \frac{n}{2}$. Recently, Branson and Gover [3] defined new complexes, new conformally invariant spaces of forms and new operators to somehow generalize this $k = \frac{n}{2}$ case. More precisely, they introduce conformally covariant differential operators $L_k^{\mathrm{BG},\ell}$ of order 2ℓ on the bundle $\Lambda^k(M)$ of k-forms, for $\ell \in \mathbb{N}$ (resp. $\ell \in \{1, \dots, \frac{n}{2}\}$) if n is odd (resp. n is even). A particularly interesting case is the critical one in even dimension, this is

$$L_k^{\mathrm{BG}} := L_k^{\mathrm{BG}, \frac{n}{2}-k}. \tag{1.1}$$

The main features of this operator are that it factorizes under the form $L_k^{\mathrm{BG}} = G_{k+1}^{\mathrm{BG}} d$ for some operator

$$G_{k+1}^{\mathrm{BG}} : C^\infty(M, \Lambda^{k+1}(M)) \to C^\infty(M, \Lambda^k(M)) \tag{1.2}$$

and that G_k^{BG} factorizes under the form $G_k^{\mathrm{BG}} = \delta_{h_0} Q_k^{\mathrm{BG}}$ for some differential operator

$$Q_k^{\mathrm{BG}} : C^\infty(M, \Lambda^k(M)) \cap \ker d \to C^\infty(M, \Lambda^k(M)) \tag{1.3}$$

where δ_{h_0} is the adjoint of d with respect to h_0. This gives rise to an elliptic complex

$$\dots \xrightarrow{d} \Lambda^{k-1}(M) \xrightarrow{d} \Lambda^k(M) \xrightarrow{L_k^{\mathrm{BG}}} \Lambda^k(M) \xrightarrow{\delta_{h_0}} \Lambda^{k-1}(M) \xrightarrow{\delta_{h_0}} \dots$$

named the *detour complex*, whose cohomology is conformally invariant. Moreover, the pairs $(L_k^{\mathrm{BG}}, G_k^{\mathrm{BG}})$ and (d, G_k^{BG}) on $\Lambda^k(M) \oplus \Lambda^k(M)$ are graded injectively elliptic in the sense that $\delta_{h_0} d + dG_k^{\mathrm{BG}}$ and $L_k^{\mathrm{BG}} + dG_k^{\mathrm{BG}}$ are elliptic. Their finite dimensional kernel

$$\mathcal{H}_L^k(M) := \ker(L_k^{\mathrm{BG}}, G_k^{\mathrm{BG}}), \quad \mathcal{H}^k(M) := \ker(d, G_k^{\mathrm{BG}}) \tag{1.4}$$

are conformally invariant, the elements of $\mathcal{H}^k(M)$ are named *conformal harmonics*, providing a type of Hodge theory for conformal structure. The operator Q_k^{BG} above generalizes Branson Q-curvature in the sense that it satisfies, as operators on closed k-forms,

$$\hat{Q}_k^{\mathrm{BG}} = e^{\mu(2k-n)}(Q_k^{\mathrm{BG}} + L_k^{\mathrm{BG}} \mu)$$

if $\hat{h}_0 = e^{2\mu}h_0$ is another conformal representative.

The general approach of Fefferman-Graham [4] for dealing with conformal invariants is related to Poincaré-Einstein manifolds, roughly speaking it provides a correspondence between Riemannian invariants in the bulk (X, g) and conformal invariants on the conformal infinity $(\partial\bar{X}, [h_0])$ of (X, g), inspired by the identification of the conformal group of the sphere S^n with the isometry group of the hyperbolic space $\mathbb{H}^{n+1}$. A smooth Riemannian manifold (X, g) is said to be a *Poincaré-Einstein manifold* with conformal infinity $(M, [h_0])$ if the space X compactifies smoothly to $\bar{X}$ with boundary $\partial\bar{X} = M$, and if there is a boundary defining function of $\bar{X}$ and some collar neighbourhood $(0, \epsilon)_x \times \partial\bar{X}$ of the boundary such that

$$g = \frac{dx^2 + h_x}{x^2} \tag{1.5}$$

$$\mathrm{Ric}(g) = -ng + O(x^\infty) \tag{1.6}$$

where h_x is a one-parameter family of smooth metrics on $\partial\bar{X}$ such that there exist some family of smooth tensors h_x^j $(j \in \mathbb{N}_0)$ on $\partial\bar{X}$, depending smoothly on $x \in [0, \epsilon)$ with

$$\begin{cases} h_x \sim \sum_{j=0}^{\infty} h_x^j (x^n \log x)^j \text{ as } x \to 0 \text{ if } n+1 \text{ is odd} \\ h_x \text{ is smooth in } x \in [0, \epsilon) \text{ if } n+1 \text{ is even} \end{cases} \tag{1.7}$$

$$h_x|_{x=0} \in [h_0]. \tag{1.8}$$

The tensor h_0^1 is called *obstruction tensor* of h_0, it is defined in [4] and studied further in [9]. We shall say that (X, g) is a smooth Poincaré-Einstein manifold if $x^2 g$ extends smoothly on $\bar{X}$, i.e. either if $n+1$ is even or $n+1$ is odd and $h_x^j = 0$ for all $j > 0$. It is proved in [6] that $h_0^1 = 0$ implies that (X, g) is a smooth Poincaré-Einstein manifold.

The boundary $\partial\bar{X} = \{x = 0\}$ inherits naturally from g the conformal class $[h_0]$ of $h_x|_{x=0}$ since the boundary defining function x satisfying such conditions are not unique. A fundamental result of Fefferman-Graham [4], which we do not state in full generality, is that for any $(M, [h_0])$ compact that can be realized as the boundary of smooth compact manifold with boundary $\bar{X}$, there is a Poincaré-Einstein manifold (X, g) for $(M, [h_0])$, and h_x in (1.7) is uniquely determined by h_0 up to order $O(x^n)$ and up to diffeomorphism which restricts to the Identity on M. The most basic example is the hyperbolic space $\mathbb{H}^{n+1}$ which is a smooth Poincaré-Einstein manifold for the canonical conformal structure of the sphere S^n, as well as quotients of $\mathbb{H}^{n+1}$ by convex co-compact groups of isometries.

It has been proved by Mazzeo [16] that[1] for a Poincaré-Einstein manifold (X, g), the relative cohomology $H^k(\bar{X}, \partial\bar{X})$ is canonically isomorphic to the L^2 kernel $\ker_{L^2}(\Delta_k)$ of the Laplacian $\Delta_k = (d + \delta_g)^2$ with respect to the metric g, acting on the bundle $\Lambda^k(\bar{X})$ of k-forms if $k < \frac{n}{2}$. In other terms the relative cohomology has a basis of L^2 harmonic representatives. In this work, we give an interpretation of the spaces $\mathcal{H}^k, \mathcal{H}_L^k$ in terms of harmonic forms on the bulk X with a certain regularity on the compactification $\bar{X}$.

Theorem 1.1. *Let (X^{n+1}, g) be an odd dimensional Poincaré-Einstein manifold with conformal infinity $(M, [h_0])$ and let $\Delta_k = (d + \delta_g)^2$ be the induced Laplacian on k-forms on X where $0 \leq k < \frac{n}{2} - 1$. For $m \in \mathbb{N}$ and $0 < k < \frac{n}{2} - 1$, define*

$$K_m^k(\bar{X}) := \{\omega \in C^m(\bar{X}; \Lambda^k(\bar{X})); \Delta_k\omega = 0\},$$

[1]The class of manifold considered by Mazzeo is actually larger and does not require the asymptotic Einstein condition (1.6)

then $K^k_m(\bar{X})$ is infinite dimensional for $m < n-2k+1$ while it is finite dimensional for $m \in [n-2k+1, n-1]$ and there is a canonical short exact sequence

$$0 \longrightarrow H^k(\bar{X}, \partial\bar{X}) \xrightarrow{i} K^k_m(\bar{X}) \xrightarrow{r} \mathcal{H}^k_L(M) \longrightarrow 0 \tag{1.9}$$

where $\mathcal{H}^k_L$ is defined in (1.4) *and $H^k(\bar{X}, \partial\bar{X})$ is the relative cohomology space of degree k of $\bar{X}$, i denotes inclusion and r denotes pull back by the natural inclusion $\partial\bar{X} \to \bar{X}$. If in addition the Fefferman-Graham obstruction tensor of $(M, [h_0])$ vanishes, i.e. if (X, g) is a smooth Poincaré-Einstein manifold, then $K^k_{n-2k+1}(\bar{X}) = K^k_\infty(\bar{X})$.*
When $k = \frac{n}{2} - 1$, the same results hold by replacing $K^k_{n-2k+1}(\bar{X})$ by the set of harmonic forms in $C^{n-2k+1,\alpha}(\bar{X}, \Lambda^k(\bar{X}))$ for some $\alpha \in (0,1)$.
When $k = 0$, $K^0_m(\bar{X})$ is infinite dimensional for $m < n$ while $K^0_n(\bar{X})$ is finite dimensional and the exact sequence (1.9) *holds.*

In establishing this Theorem, we show that we can recover the Branson-Gover operators $L^{\mathrm{BG}}_k, G^{\mathrm{BG}}_k, Q^{\mathrm{BG}}_k$ from harmonic forms on a Poincaré-Einstein manifold with conformal infinity $(M, [h_0])$. Let us recall quickly and informally how the GJMS and Branson-Gover operators are defined in [11, 3]. The ambient metric is a Lorentzian metric on $\widetilde{\mathcal{Q}} := M \times (0,\infty)_t \times (-1,1)_\rho$, homogeneous of degree 2 in the t variable, which extends the tautological tensor $t^2 h_0$ at the cone $\mathcal{Q} = \{\rho = 0\}$ and with Ricci curvature vanishing to $n/2-1$ order (resp. infinite order) at $\mathcal{Q}$ when n is even (resp. n odd). The GJMS operators P_k are defined in two ways in [11]: for f a $(k - n/2)$-homogeneous function on $\mathcal{Q}$, take a homogeneous extension $\widetilde{f}$ on $\widetilde{\mathcal{Q}}$ and define $P_k f := [\widetilde{\Delta}^k \widetilde{f}]|_{\mathcal{Q}}$ where $\widetilde{\Delta}$ is the Laplacian for the ambient metric; the second equivalent way is to consider an extension $\widetilde{f}$ of f to $\widetilde{\mathcal{Q}}$ which satisfies $\widetilde{\Delta}\widetilde{f} = O(\rho^{k-1})$ and, up to mutliplicative constant, $P_k f = [(\rho t^2)^{1-k}\widetilde{\Delta} f]_{\mathcal{Q}}$. The second definition gives P_k as an obstruction to extend smoothly from the cone a harmonic homogeneous function f. The Branson-Gover operators defined in [3] are constructed following the first method in [11] but with many complications due to the fact that one works with bundle valued objects. Our approach is more closely related to the harmonic extension approach of GJMS [11]. We say that a k-form ω is *polyhomogeneous* on $\bar{X}$ if it is smooth on X and with an expansion at the boundary $M = \{x = 0\}$

$$\omega \sim \sum_{j=0}^{\infty} \sum_{\ell=0}^{\ell(j)} x^j \log(x)^\ell (\omega^{(t)}_{j,\ell} + \omega^{(n)}_{j,\ell} \wedge dx)$$

for some forms $\omega^{(t)}_{j,\ell} \in C^\infty(M, \Lambda^k(M))$ and $\omega^{(n)}_{j,\ell} \in C^\infty(M, \Lambda^{k-1}(M))$ and some sequence $j \in \mathbb{N}_0 \to \ell(j) \in \mathbb{N}_0$. We show that the Branson-Gover operators appear naturally in the resolution of the absolute or relative Dirichlet type problems for the Laplacian on forms on $\bar{X}$.

Theorem 1.2. *Let (X^{n+1}, g) be an odd-dimensional Poincaré-Einstein manifold with conformal infinity $(M, [h_0])$, let $k < \frac{n}{2}$ and $\alpha \in (0,1)$.*
(i) *For any $\omega_0 \in C^\infty(M, \Lambda^k(M))$, harmonic forms $\omega \in C^{\frac{n}{2}-k,\alpha}(\bar{X}, \Lambda^k(\bar{X}))$ with boundary value $\omega|_M = \omega_0$ exist, are unique modulo $\ker_{L^2}(\Delta_k)$ and are actually polyhomogeneous with an expansion at M at order $O(x^{n-2k+1})$ given by*

$$\omega = \omega_0 + \sum_{j=1}^{\frac{n}{2}-k} x^{2j}(\omega^{(t)}_j + \omega^{(n)}_j \wedge \frac{dx}{x}) + x^{n-2k}\log(x) L_k \omega_0$$
$$+ x^{n-2k+1}\log(x)(G_k\omega_0) \wedge dx + O(x^{n-2k+1})$$

where L_k, G_k are, up to a normalization constant, the Branson-Gover operators in (1.1), (1.2) *and $\omega^{(\cdot)}_j$ are forms on M.*
(ii) *For any closed form $\omega_0 \in C^\infty(M, \Lambda^{k-1}(M))$, harmonic forms ω such that $x\omega \in$*

$C^{\frac{n}{2}-k+1,\alpha}(\bar{X},\Lambda^k(\bar{X}))$ and $\omega = x^{-1}(\omega_0\wedge dx)+O(x)$ exist, are unique modulo $\ker_{L^2}(\Delta_k)$ and $x\omega$ is polyhomogeneous with expansion at M given by

$$\omega = \omega_0\wedge\frac{dx}{x} + \sum_{j=1}^{\frac{n}{2}-k} x^{2j}({\omega'_j}^{(t)} + {\omega'_j}^{(n)}\wedge\frac{dx}{x}) + x^{n-2k+1}\log(x)(Q_{k-1}\omega_0)\wedge dx + O(x^{n-2k+1})$$

where Q_{k-1} is, up to a normalization constant, the operator (1.3) of Branson-Gover and ${\omega'_j}^{(\cdot)}$ are smooth forms on M.

The Dirichlet problem for functions in this geometric setting is studied by Graham-Zworski [12] and Joshi-Sa Barreto [15]. In a more general setting (but again for functions), it was analyzed by Anderson [1] and Sullivan [20].

We also prove in Subsection 4.6 that, with Q_0 defined by the Theorem above,

$$Q_0 1 = \frac{n(-1)^{\frac{n}{2}+1}}{2^{n-1}\frac{n}{2}!(\frac{n}{2}-1)!}Q$$

where Q is Branson Q-curvature. So Q can be seen as an obstruction to find a harmonic 1-form ω with $x\omega$ having a high regularity at the boundary and value dx at the boundary.

In addition, this method allows to obtain the conformal change law of L_k, G_k, Q_k, the relations between these operators, and some of their analytic properties (e.g. symmetry of L_k and Q_k) see Subsection 4.4 and Section 4.6.

Next, we analyze the set of regular closed and coclosed forms on $\bar{X}$. Recall that on a compact manifold $\bar{X}$ with boundary, equipped with a smooth metric $\bar{g}$, there is an isomorphism

$$H^k(\bar{X}) \simeq \{\omega\in C^\infty(\bar{X},\Lambda^k(\bar{X})); d\omega = \delta_{\bar{g}}\omega = 0, i_{\partial_n}\omega|_{\partial\bar{X}} = 0\}$$

where ∂_n is a unit normal vector field to the boundary, and the absolute cohomology $H^k(\bar{X})$ is $\ker d/\mathrm{Im}\, d$ where d acts on smooth forms. Moreover, one has the long exact sequence in cohomology

$$(1.10)\qquad \ldots\to H^{k-1}(\partial\bar{X})\to H^k(\bar{X},\partial\bar{X})\to H^k(\bar{X})\to H^k(\partial\bar{X})\to H^{k+1}(\bar{X},\partial\bar{X})\to\ldots$$

and all these spaces are represented by forms which are closed and coclosed, the maps in the sequence are canonical with respect to $\bar{g}$. In our Poincaré-Einstein case (X,g), say when $k<\frac{n}{2}$, only the space $H^k(\bar{X},\partial\bar{X})$ in the long exact sequence has a canonical basis of closed and coclosed representatives with respect to g (the L^2 harmonic forms), in particular there is no canonical metric on the boundary induced by g but only a canonical conformal class. We prove

Theorem 1.3. *Let (X^{n+1},g) be an odd dimensional Poincaré-Einstein manifold with conformal infinity $(M,[h_0])$ and let $k\leq\frac{n}{2}$. Then the spaces*

$$Z^k(\bar{X}) := \{\omega\in C^{n-2k+1}(\bar{X},\Lambda^k(\bar{X})); d\omega=\delta_g\omega = 0\}$$

are finite dimensional and, if the obstruction tensor of $[h_0]$ vanishes, they are equal to $\{\omega\in C^\infty(\bar{X},\Lambda^k(\bar{X})); d\omega=\delta_g\omega=0\}$. Then, we have
(i) *For $k<\frac{n}{2}$ there is a canonical exact sequence*

$$0\to H^k(\bar{X},M)\to Z^k(\bar{X})\to\mathcal{H}^k(M)\to H^{k+1}(\bar{X},M)$$

where $\mathcal{H}^k(M)$ is the set of conformal harmonics defined in (1.4).
(ii) *Let $[Z^k(\bar{X})]$ and $[\mathcal{H}^k(M)]$ be respectively the image of $Z^k(\bar{X})$ and $\mathcal{H}^k(M)$ by the*

natural cohomology maps $Z^k(\bar{X}) \to H^k(\bar{X})$ and $\mathcal{H}^k(M) \to H^k(M)$. Then there exists a canonical complex with respect to g

(1.11)

$$0 \to \dots \xrightarrow{\iota^k} [Z^k(\bar{X})] \xrightarrow{r^k} [\mathcal{H}^k(M)] \xrightarrow{d_e^k} H^{k+1}(\bar{X}, M) \xrightarrow{\iota^{k+1}} [Z^{k+1}(\bar{X})] \to \dots \to H^{\frac{n}{2}}(\bar{X}, M)$$

whose cohomology vanishes except possibly the spaces $\ker \iota^k / \mathrm{Im}\, d_e^{k-1}$. Here ι^k, r^k and d_e^k denote respectively inclusion, restriction to the boundary and composition of d with harmonic extension (see Section 7).

(iii) *$[\mathcal{H}^k(M)] = H^k(M)$ if and only if $[Z^k(\bar{X})] = H^k(\bar{X})$ and $\ker \iota^{k+1} = \mathrm{Im}\, d_e^k$. If this holds for all $k \leq \frac{n}{2}$ this is a canonical realization of (half of) the long exact sequence* (1.10) *with respect to g.*

The surjectivity of the natural map $\mathcal{H}^k(M) \to H^k(M)$ is named $(k-1)$-*regularity* by Branson and Gover, while $(k-1)$-*strong regularity* means that the map is an isomorphism, or equivalently $\ker L_{k-1} = \ker d$ (see [3, Th.2.6]). Thus, $(k-1)$ regularity means that the cohomology group can be represented by conformally invariant representatives. If $H^{k+1}(X, M) = 0$, our result implies that $(k-1)$-regularity means that the absolute cohomology group $H^k(\bar{X})$ can be represented by $C^{n-2k+1}(\bar{X}, \Lambda^k(\bar{X}))$ forms in $\ker d + \delta_g$. We give a criteria for $(k-1)$-regularity:

Proposition 1.4. *Let $(M, [h_0])$ be a compact conformal manifold. If Q_k is a positive operator on closed forms in the sense that $\langle Q_k \omega, \omega \rangle_{L^2} \geq 0$ for all $\omega \in C^\infty(M, \Lambda^k(M)) \cap \ker d$, then $\mathcal{H}^k(M) \to H^k(M)$ is surjective.*

We should also remark that $(k-1)$-regularity holds for all $k = 1, \dots, \frac{n}{2}$ if for instance $(M, [h_0])$ contains an Einstein metric in $[h_0]$, this is a result of Gover and Silhan [7]. If $n = 4$, $L_{\frac{n}{2}-2} = L_0$ is the Paneitz operator (up to a constant factor) and using a result of Gursky [14], we deduce that if the Yamabe invariant $Y(M, [h_0])$ is positive and

$$\int_M Q \mathrm{dvol}_{h_0} + \frac{1}{6} Y(M, [h_0])^2 > 0$$

then $\mathcal{H}^1(M) \simeq H^1(M)$ and there is a basis of conformal harmonics of $H^1(M)$.

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2. Poincaré-Einstein manifolds and Laplacian on forms

2.1. **Poincaré-Einstein manifolds.** Let (X, g) be a Poincaré-Einstein manifold with conformal infinity $(M, [h])$. Graham-Lee and Graham [10, 8] proved that for any conformal representative $h_0 \in [h]$, there exists a boundary defining function x of $M = \partial \bar{X}$ in $\bar{X}$ such that

$$|dx|^2_{x^2 g} = 1 \text{ near } \partial \bar{X}, \quad x^2 g|_{TM} = h_0,$$

moreover x is the unique defining function near M satisfying these conditions. Such a function is called a *geodesic boundary defining function* and if ψ is the map $\psi : [0, \epsilon] \times M \to \bar{X}$ defined by $\psi(t, y) := \psi_t(y)$ where ψ_t is the flow of the gradient $\nabla^{x^2 g} x$, then ψ pulls the metric g back to

$$\psi^* g = \frac{dt^2 + h_t}{t^2}$$

for some one-parameter family of metrics on M with $h_0 = x^2 g|_{TM}$. In other words the special form (1.5) of the metric near infinity is not unique and corresponds canonically to a geodesic boundary defining function, or equivalently to a conformal representative of $[h_0]$.

We now discuss the structure of the metric near the boundary, the reader can refer to Fefferman-Graham [6, Th 4.8] for proofs and details. Let us define the endomorphism A_x on TM corresponding to $\partial_x h_x$ with respect to h_x, i.e. as matrices

$$A_x = h_x^{-1}\partial_x h_x.$$

Then the Einstein condition $\mathrm{Ric}(g) = -ng$ is equivalent to the following differential equations on A_x

$$\begin{gathered} x\partial_x A_x + (1-n+\frac{x}{2}\mathrm{Tr}(A_x))A_x = 2xh_x^{-1}\mathrm{Ric}(h_x) + \mathrm{Tr}(A_x)\mathrm{Id} \\ \delta_{h_x}(\partial_x h_x) = d\mathrm{Tr}(A_x) \\ \partial_x \mathrm{Tr}(A_x) + \frac{1}{2}|A_x|^2 = \frac{1}{x}\mathrm{Tr}(A_x) \end{gathered}$$

A consequence of these equations and (1.7) is that if $\mathrm{Ric}(g) = -ng + O(x^{n-2})$, then h_x has an expansion at $x = 0$ of the form

$$h_x = \begin{cases} h_0 + \sum_{j=1}^{\frac{n}{2}-1} x^{2j} h_{2j} + h_{n,1}x^n \log x + O(x^n) & \text{if } n \text{ is even} \\ h_0 + \sum_{j=1}^{(n-1)/2} x^{2j} h_{2j} + O(x^n) & \text{if } n \text{ is odd} \end{cases}$$

for some tensors h_{2j} and $h_{n,1}$ on M, depending in a natural way on h_0 and covariant derivatives of its Ricci tensor. When n is even, the tensor $h_{n,1}$ is the *obstruction tensor* of h_0 in the terminology of Fefferman-Graham [6], it is trace free (with respect to h_0) and so the first log term in A_x is $nh_0^{-1}h_{n,1}x^{n-1}\log(x)$. A smooth Poincaré-Einstein manifold such that h_x has only even powers of x in the Taylor expansion at $x = 0$ is called an *smooth even Poincaré-Einstein manifold.* If n is even and $h_{n,1} = 0$, the metric h_x is a smooth even Poincaré-Einstein manifold. When n is odd, the term $\partial_x^n h_x|_{x=0}$ is trace free with respect to h_0, which implies that A_x has an even Taylor expansion at $x = 0$ to order $O(x^{n-1})$. If $\partial_x^n h_x|_{x=0} = 0$, then h_x has an even Taylor expansion in powers of x at $x = 0$ with all coefficients formally determined by h_0. The equations satisfied by A_x easily give (see [4]) the first terms in the expansion

$$h_x = h_0 - x^2\frac{P_0}{2} + O(x^4), \quad \text{where } P_0 = \frac{1}{n-2}\Big(2\,\mathrm{Ric}_0 - \frac{\mathrm{Scal}_0}{n-1}h_0\Big), \tag{2.1}$$

P_0 is the Schouten tensor of h_0, Ric_0 and Scal_0 are the Ricci and scalar curvature of h_0.

2.2. **The Laplacian, d and δ.** Let $\Lambda^k(\bar{X})$ be the bundle of k-forms on $\bar{X}$. Since for the problem we consider it is somehow quite natural, we will also use along the paper the b-bundle of k-forms on $\bar{X}$ in the sense of [19], it will be denoted $\Lambda_b^k(\bar{X})$. This is the exterior product of the b cotangent bundle $T_b^*\bar{X}$, which is canonically isomorphic to $T^*\bar{X}$ over the interior X and whose local basis near a point of the boundary $\partial\bar{X}$ is given by $dy_1, \dots, dy_n, dx/x$ where $y_1, \dots, y_n$ are local coordinates on $\partial\bar{X}$ near this point. We refer the reader to Chapter 2 of [19] for a complete analysis about b-structures. Of course one can pass from $\Lambda^k(\bar{X})$ to $\Lambda_b^k(\bar{X})$ obviously when considering forms on X. The restriction $\Lambda_b^k(U_\epsilon)$ of $\Lambda^k(\bar{X})$ to the collar neighbourhood $U_\epsilon := [0,\epsilon] \times M$ of M in $\bar{X}$ can be decomposed as the direct sum

$$\Lambda_b^k(U_\epsilon) = \Lambda^k(M) \oplus (\Lambda^{k-1}(M) \wedge \frac{dx}{x}) =: \Lambda_t^k \oplus \Lambda_n^k.$$

In this splitting, the exterior derivative d and its adjoint δ_g with respect to g have the form

$$d = \begin{pmatrix} d & 0 \\ (-1)^k x\partial_x & d \end{pmatrix}, \quad \delta = \begin{pmatrix} x^2\delta_x & (-1)^k \star_x^{-1} x^{-2k+n+3}\partial_x x^{2k-n-2}\star_x \\ 0 & x^2\delta_x \end{pmatrix} \tag{2.2}$$

and the Hodge Laplace operator is given by

$$\begin{aligned} \Delta_k &= \begin{pmatrix} -(x\partial_x)^2 + (n-2k)x\partial_x & 2(-1)^{k+1}d \\ 0 & -(x\partial_x)^2 + (n-2k+2)x\partial_x \end{pmatrix} \\ &+ \begin{pmatrix} x^2\Delta_x - x\star_x^{-1}[\partial_x, \star_x]x\partial_x & (-1)^k x\big[d, \star_x^{-1}[\partial_x, \star_x]\big] \\ (-1)^{k+1}[x\partial_x, x^2\delta_x] & x^2\Delta_x - x\partial_x x\star_x^{-1}[\partial_x, \star_x] \end{pmatrix} \\ &= P + P'. \end{aligned} \tag{2.3}$$

where here, the subscript $\cdot_x$ means "with respect to the metric h_x on M" and d in the matrices is the exterior derivative on M. Note that P is the indicial operator of Δ_k in the terminology of [19].

If H is an endomorphism of TM, we denote $J(H)$ the operator on $\Lambda^k(M)$

$$J(H)(\alpha_1 \wedge \cdots \wedge \alpha_k) := \sum_{i=1}^{k} \alpha_1 \wedge \cdots \wedge \alpha_i(H) \wedge \cdots \wedge \alpha_k. \tag{2.4}$$

When H is symmetric, a straightforward computation gives $\star_0 J(H) + J(H)\star_0 = \mathrm{Tr}(H)\star_0$ and so

$$[\star_0, J(H)] = 2\star_0 J(H) - \mathrm{Tr}(H)\star_0 \tag{2.5}$$

Let us define the following operators on k-forms on M

$$E_1 = J(h_0^{-1}P_0) - \frac{\mathrm{Tr}(h_0^{-1}P_0)}{2}\mathrm{Id} = \frac{2J(h_0^{-1}\mathrm{Ric}_0)}{n-2} - \frac{n+2k-2}{2(n-1)(n-2)}\mathrm{Scal}_0\mathrm{Id}. \tag{2.6}$$

Using the approximate Einstein equation for g, we obtain

Lemma 2.1. *The operator Δ_k has a polyhomogeneous expansion at $x=0$ and the first terms in the expansion are given by*

$$\begin{aligned} \Delta_k &= P + \sum_{i=1}^{[\frac{n}{2}]} x^{2i} \begin{pmatrix} R_i + P_i x\partial_x & S_i \\ S_i' & R_i' + P_i' x\partial_x \end{pmatrix} \\ &+ nx^n\log(x) \begin{pmatrix} J(h_0^{-1}h_{n,1})x\partial_x & (-1)^{k+1}[d, J(h_0^{-1}h_{n,1})] \\ 0 & J(h_0^{-1}h_{n,1})(n + x\partial_x) \end{pmatrix} + O(x^n) \end{aligned} \tag{2.7}$$

where the operators $P_i, P_i', S_i, S_i', R_i$ and R_i' are universal differential operators on $\Lambda(M)$ that can be expressed in terms of covariant derivatives of the Ricci tensor of h_0. Moreover the operators R_i and R_i' are of order at most 2, the S_i, S_i' are of order at most 1 and the P_i, P_i' are of order 0. For instance, we have

$$\begin{pmatrix} R_1 + P_1 x\partial_x & S_1 \\ S_1' & R_1' + P_1' x\partial_x \end{pmatrix} = \begin{pmatrix} \Delta_0 - E_1 x\partial_x & (-1)^k[d, E_1] \\ 2(-1)^{k+1}\delta_0 & \Delta_0 - E_1(2 + x\partial_x) \end{pmatrix} \tag{2.8}$$

where A is defined in (2.6). *If $k=0$, the $x^n\log(x)$ coefficient vanishes. Finally, if (X,g) is smooth Poincaré-Einstein, then Δ_k is a smooth differential operator on $\bar{X}$, and if (X,g) is smooth even Poincaré-Einstein, then Δ_k has an even expansion.*

Proof: The polyhomogeneity comes from that of the metric g. It is moreover a smooth expansion if x^2g is smooth on $\bar{X}$. A priori, by (2.3) the first $\log x$ term in the expansion of Δ at $x=0$ appear at order (at least) $x^n\log x$ and it comes from the diagonal terms in P' in (2.3). Let us define $p = [\frac{n}{2}]$ so that the metric h_x has even powers in its expansion at $x=0$ up to order x^{2p+1}. We set D the Levi-Civita connection of the metric $x^2g = dx^2 + h_x$. Since $D_{\partial_x}\partial_x = 0$ and $D_{\partial x}\partial_{y_i} = \frac{1}{2}\sum_{jk}\partial_x h_{ij}h^{kj}\partial_{y_k}$, the matrix O_x of the parallel transport along the geodesic $x \mapsto (x,y)$ (with respect to the basis (∂_{y_i})) satisfies $D_{\partial_x} O_x(\partial_{y_i}) = 0$, hence $\partial_x O_x = -\frac{1}{2} A_x \times O_x$ where A_x is the endomorphism $h_x^{-1}\partial_x h_x$. Note that A_x has a Taylor expansion with only odd powers of x up to x^{2p} and the first log term is $nh_0^{-1}h_{n,1}x^{n-1}\log(x)$. We infer that O_x is polyhomogeneous in the x variable and has only even powers of x in its Taylor expansion up to x^{2p}, the first log term is $-\frac{h_0^{-1}h_{n,1}}{2}x^n\log(x)$. By (2.1), we have $\partial_x^2 h|_{x=0} = -P_0$, hence

$$A_x = -xh_0^{-1}P_0 + O(x^2), \quad O_x = \mathrm{Id} + \frac{1}{4}x^2 h_0^{-1}P_0 + O(x^3).$$

We note also O_x the parallel transport map. Now the operator $I_x(\alpha_1\wedge\cdots\wedge\alpha_k) = \alpha_1(O_x)\wedge\cdots\wedge\alpha_k(O_x)$ is an isometry from $\Lambda^k(M,h_x)$ to $\Lambda^k(M,h_0)$. So we have $\star_x = I_x^{-1}\star_0 I_x$ and we infer that $\star_x$ itself is an operator with a polyhomogeneous expansion in x and with only even powers of x in its taylor expansion up to x^{2p}, the first log term being $\frac{1}{2}x^n\log(x)[J(h_0^{-1}h_{n,1}),\star_0] = -x^n\log(x)\star_0 J(h_0^{-1}h_{n,1})$ by (2.5). Since we have

$$[\partial_x,\star_x] = \partial_x(\star_x), \quad \partial_x(\star_x)|_{x=0} = [\star_0, \partial_x I_x|_{x=0}] = 0 \quad \text{and} \quad \partial_x^2(\star_x)|_{x=0} = [\star_0, \partial_x^2 I_x|_{x=0}]$$

we get that $[\partial_x,\star_x]$ is polyhomogeneous with only odd powers of x up to order x^{2p}, with first log term $-nx^{n-1}\log(x)\star_0 J(h_0^{-1}h_{n,1})$, and that

$$[\partial_x,\star_x] = \partial_x(\star_x) = x\star_0\big(J(h_0^{-1}P_0) - \frac{\mathrm{Scal}_0}{2(n-1)}\mathrm{Id}\big) + O(x^2).$$

Since $\delta_x = (-1)^k\star_x^{-1} d\star_x$, the operators $x\delta_x$ and $x^2[\star^{-1}[\partial_x,\star_x],\delta_x]$ are odd in x up to $O(x^{2p+2})$. By the same way, $x^2[d,\star^{-1}[\partial_x,\star_x]]$ is odd up to order x^{2p+2} and the operators $\star_x^{-1}[\partial_x,\star_x]x(k-x\partial_x)$, $x^2\Delta_x$ and $(k-\partial_x x)x\star^{-1}[\star_x,\partial_x]$ are even in x up to $O(x^{2p+1})$. This achieves the proof by gathering all these facts. □

2.3. **Indicial equations.** We give the indicial equations satisfied by Δ_k, which are essential to the construction of formal power series solutions of $\Delta_k\omega = 0$.

Notation: If f is a function on $\bar{X}$ and ω a k-form defined near the boundary, we will say that ω is a $O_n(f)$ (resp. $O_t(f)$) if its Λ_n^k (resp. Λ_t^k) components are $O(f)$.

For $\lambda\in\mathbb{C}$, the operator $x^{-\lambda}\Delta_k x^\lambda$ can be considered near the boundary as a family of operators on $\Lambda_t^k\oplus\Lambda_n^k$ depending on (x,λ), and for any $\omega\in C^\infty(U_\epsilon,\Lambda_t^k\oplus\Lambda_n^k)$ one has

$$x^{-\lambda}\Delta_k(x^\lambda\omega) = P_\lambda\Big(\omega_0^{(t)} + \omega_0^{(n)}\wedge\frac{dx}{x}\Big) + O(x) \tag{2.9}$$

where $P_\lambda := x^{-\lambda}Px^\lambda$, $\omega_0^{(t)} = (i_{x\partial_x}(\omega\wedge\frac{dx}{x}))|_{x=0}$ and $\omega_0^{(n)} := (i_{x\partial_x}\omega)|_{x=0}$. The operator P_λ is named indicial family and is a one-parameter family of operators on $\Lambda_n^k\oplus\Lambda_t^k$ viewed as a bundle over M, its expression is

$$P_\lambda = \begin{pmatrix} -\lambda^2 + (n-2k)\lambda & 2(-1)^{k+1}d \\ 0 & -\lambda^2+(n-2k+2)\lambda \end{pmatrix} \tag{2.10}$$

The *indicial roots* of Δ_k are the $\lambda\in\mathbb{C}$ such that P_λ is not invertible on the set of smooth sections of $\Lambda_t^k\oplus\Lambda_n^k$ over M, i.e. on $C^\infty(M,\Lambda^k(M)\oplus\Lambda^{k-1}(M))$. In our case, a simple computation shows that these are given by $0, n-2k, 0, n-2k+2$. The first two roots are roots in the Λ_t^k component and the last two are roots in the Λ_n^k component. In particular, this proves that for j not a root, and $(\omega_0^{(t)},\omega_0^{(n)})\in\Lambda^k(M)\oplus\Lambda^{k-1}(M)$, there exists a unique pair $(\alpha_0^{(t)},\alpha_0^{(n)})\in\Lambda^k(M)\oplus\Lambda^{k-1}(M)$ such that near M

$$\Delta_k\Big(x^j\alpha_0^{(t)} + x^j\alpha_0^{(n)}\wedge\frac{dx}{x}\Big) = x^j\Big(\omega_0^{(t)}+\omega_0^{(n)}\wedge\frac{dx}{x}\Big) + O(x^{j+1})$$

More precisely, and including coefficients with log terms, we have for $l \in \mathbb{N}^*$ (resp. $l = 0$)

$$\Delta_k x^j \log^l(x) \begin{pmatrix} \omega_0^{(t)} \\ \omega_0^{(n)} \end{pmatrix} = x^j \log^l(x) \begin{pmatrix} j(n-2k-j)\omega_0^{(t)} + 2(-1)^{k+1} d\omega_0^{(n)} \\ j(n-2k+2-j)\omega_0^{(n)} \end{pmatrix} + O(x^j \log^{l-1}(x)) \quad \big(\text{resp. } + O(x^{j+1})\big) \tag{2.11}$$

if $\omega_0^{(t)}, \omega_0^{(n)} \in C^\infty(M, \Lambda^k(M) \oplus \Lambda^{k-1}(M))$, and in the critical cases, for any $l \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$

(2.12)

$$\begin{aligned}
\Delta_k(\log^l(x)\omega_0^{(t)}) &= l(n-2k)\log^{l-1}(x)\omega_0^{(t)} - l(l-1)\log^{l-2}(x)\omega_0^{(t)} + O(x^2 \log x)\\
\Delta_k(x^{n-2k}\log^l(x)\omega_0^{(t)}) &= l(2k-n)x^{n-2k}\log^{l-1}(x)\omega_0^{(t)} - l(l-1)x^{n-2k}\log^{l-2}(x)\omega_0^{(t)}\\
&\quad + O(x^{n-2k+2}\log^l(x))\\
\Delta_k\big(x^{n-2k+2}\log^l(x)\omega^{(n)} \wedge \frac{dx}{x}\big) &= l(2k-2-n)x^{n-2k+2}\log^{l-1}(x)\omega^{(n)} \wedge \frac{dx}{x}\\
&\quad - l(l-1)x^{n-2k+2}\log^{l-2}(x)\omega^{(n)} \wedge \frac{dx}{x} + O(x^{n-2k+3}\log^l(x)).
\end{aligned}$$

3. Absolute and relative Dirichlet problems

The goal of this section is to solve the Dirichlet type problems for Δ_k when $k < \frac{n}{2}$ for the two natural boundary conditions. Note that the vector field $x\partial_x$ can be seen as the unit, normal, inward vector field to M in $\bar{X}$. A k-form $\omega \in \Lambda_b^k(\bar{X})$ is said to satisfy the absolute (resp. the relative) boundary condition if

$$\lim_{x\to 0} i_{x\partial_x}\omega = 0 \qquad\qquad (\text{resp. } \lim_{x\to 0} i_{x\partial_x}(\frac{dx}{x} \wedge \omega) = 0).$$

We denote $C^{p,\alpha}(\bar{X}, \Lambda_b^k(\bar{X}))$ the sections of $\Lambda_b^k(\bar{X})$ which are $C^{p,\alpha}$, equivalently $i_{x\partial_x}\omega$ and $i_{x\partial_x}\big(\frac{dx}{x} \wedge \omega\big)$ are $C^{p,\alpha}$ on $\bar{X}$.

3.1. Absolute boundary condition.

Proposition 3.1. *Let $k < n/2$, $\alpha \in (0,1)$ and $\omega_0 \in C^\infty(M, \Lambda^k(M))$.*
(i) *There exists a solution ω to the following absolute Dirichlet problem:*

$$\begin{cases} \omega \in C^{n-2k-1,\alpha}(\bar{X}, \Lambda^k(\bar{X})),\\ \Delta_k \omega = 0 \text{ on } X,\\ \omega|_M = \omega_0, \lim\limits_{x\to 0} i_{x\partial_x}\omega = 0. \end{cases} \tag{3.1}$$

Moreover, this solution is unique modulo the L^2 kernel of Δ_k.
(ii) *The solution ω is smooth in $\bar{X}$ when n is odd, while it is polyhomogeneous when n is even with an expansion at order x^n of the form*

$$\begin{aligned}\omega = &\sum_{j=0}^{n-1} x^j \omega_j^{(t)} + \sum_{j=2}^{n-1} x^j \omega_j^{(n)} \wedge \frac{dx}{x} + \log x \Big(\sum_{j=n-2k}^{n-1} x^j \omega_{j,1}^{(t)} + \sum_{j=n-2k+2}^{n} x^j \omega_{j,1}^{(n)} \wedge \frac{dx}{x} \Big)\\ &+ \begin{cases} O_t(x^n \log x) + O_n(x^{n+1}\log x) & \text{if } k > 0\\ O(x^n) & \text{if } k = 0 \end{cases}\end{aligned} \tag{3.2}$$

as $x \to 0$, where $\omega_j^{(\cdot)}, \omega_{j,1}^{(\cdot)}$ are smooth forms on M. Moreover, we have

$$\omega_j^{(t)} = P_j^{(t)}\omega_0 \text{ for } j < n-2k, \quad \omega_j^{(n)} = P_j^{(n)}\omega_0 \text{ for } j < n-2k+2$$
$$\omega_{n-2k,1}^{(t)} = P_{n-2k,1}^{(t)}\omega_0$$

where $P_j^{(t)}, P_j^{(n)}, P_{n-2k,1}^{(t)}$ are universal smooth differential operators on $\Lambda(M)$ depending naturally on covariant derivatives of the curvature tensor of h_0.
(iii) *If n is even and (X, g) is a smooth Poincaré-Einstein manifold, then we have $\omega = \omega_1 + x^{n-2k}\log(x)\omega_2$ for some forms $\omega_1, \omega_2 \in C^\infty(\bar{X}, \Lambda_b^k(\bar{X}))$ with $\omega_2 = O(x^\infty)$ if and only*

if $\omega^{(t)}_{n-2k,1}=\omega^{(n)}_{n-2k+2,1}=0$.
(iv) ω *satisfies* $\delta_g\omega=0$. *If in addition* ω_0 *is closed, then* $d\omega\in\ker_{L^2}(\Delta_{k+1})$ *(and* $d\omega=0$ *when* $k=\frac{n-1}{2}$*).*

3.1.1. *Proof of Proposition 3.1.* To prove this Proposition, we first need a result of Mazzeo [16] (note that the ambiant manifold has dimension n in [16] and $n+1$ in this paper):

Theorem 3.2 (Mazzeo). *For* $k<n/2$, *the operator* Δ_k *is Fredholm and there exists a pseudodifferential inverse* E, *bounded on* $L^2(X)$, *such that* $\Delta_kE=I-\Pi_0$ *where* Π_0 *is the projection on the finite dimensional space* $\ker_{L^2}(\Delta_k)$. *This implies an isomorphism between* $\ker_{L^2}(\Delta_k)$ *and the relative cohomology* $H^k(\bar X,\partial\bar X)$ *of* $\bar X$. *Moreover any* L^2 *harmonic form* α *is polyhomogeneous with an expansion near* $\partial\bar X$ *of the form*

$$\alpha\sim x^{n-2k}\sum_{j=0}^{\infty}\sum_{l=0}^{l(j)}(\alpha^{(t)}_{j,l}x^j\log(x)^l+x^{j+2}\log(x)^l\alpha^{(n)}_{j,l}\wedge\frac{dx}{x})\tag{3.3}$$

for some $\alpha^{(t)}_{j,l}\in C^\infty(M,\Lambda^k(M)),\alpha^{(n)}_{j,l}\in C^\infty(M,\Lambda^{k-1}(M))$ *and some sequence* $l:\mathbb{N}_0\to\mathbb{N}_0$. *In addition* E *maps the space* $\{\omega\in C^\infty(\bar X,\Lambda^k(\bar X));\omega=O(x^\infty)\}$ *into polyhomogeneous forms on* $\bar X$ *with a behaviour like* (3.3) *near* M.

Remark: By using duality through the Hodge star operator $\star_g$, one obtains trivially a corresponding result for the case $k>\frac{n}{2}+1$. In particular, this gives $\ker_{L^2}(\Delta_k)\simeq H^k(\bar X)$ for $k>\frac{n}{2}+1$. It sould be noticed that in the case $k=n/2,(n+1)/2$ and $n/2+1$, [16] does not give a bounded pseudodifferential inverse and actually the Laplacian is not Fredholm in these cases: for $k=n/2$ or $k=n/2+1$ the range is not closed while for $k=(n+1)/2$ it has infinite dimensional kernel.

We can make the second part of this theorem more precise thanks to the indicial identities obtained by (2.3).

Corollary 3.3. *For* $k<n/2$, *any* L^2 *harmonic* k*-form* α *on* (X,g) *is polyhomogeneous and has an expansion at order* $x^n\log x$ *of the form*

$$\alpha=x^{n-2k+2}\Big(\sum_{j=0}^{n-1}x^j\alpha^{(t)}_j+\sum_{j=0}^{n-1}x^j\alpha^{(n)}_j\wedge\frac{dx}{x}+O(x^n\log x)\Big)$$

where $\alpha^{(\cdot)}_j$ *are smooth forms on* M. *If in addition the metric* (X,g) *is a smooth Poincaré-Einstein manifold, then* $\alpha\in x^{n-2k+2}C^\infty(\bar X,\Lambda^k(\bar X))$ *and* E *maps*

$$E:\{\omega\in C^\infty(\bar X,\Lambda^k(\bar X));\omega=O(x^\infty),\Pi_0\omega=0\}\longrightarrow x^{n-2k}C^\infty(\bar X,\Lambda^k(\bar X)).$$

Proof: Note that if

$$\alpha\sim x^{n-2k}\sum_{j=0}^{\infty}\sum_{l=0}^{l(j)}(\alpha^{(t)}_{j,l}x^j\log(x)^l+x^{j+2}\log(x)^l\alpha^{(n)}_{j,l}\wedge\frac{dx}{x})\quad\text{and }\Delta_k\alpha=O(x^\infty),$$

then the indicial equations in Subsection 2.3 and Lemma 2.1 imply that $l(0)=0$ and $l(j)\leq 1$ for all $j=1,\dots,n-1$ (and for all $j>0$ if h_x is smooth in x). Moreover since $d\alpha=0$ for any $\alpha\in\ker_{L^2}(\Delta_k)$, we first obtain from (2.2) that $\alpha^{(t)}_{0,0}=0$ and so, by (2.12) that $l(j)=0$ for all $j=0,\dots,n-1$ (and for all $j>0$ if h_x is smooth). The mapping property of E is straightforward by the same type of arguments and the fact that $\Delta_kE\omega=O(x^\infty)$ for $\omega=O(x^\infty)$ such that $\Pi_0\omega=0$.

We will now use the relations (2.9), (2.11) and (2.12) to show that the jet of a solution ω to the Dirichlet problem in Proposition 3.1 is partly determined. Let $\omega_0\in$

$C^\infty(M,\Lambda^k(M))$. Using (2.9) and the form (2.7) of Δ, we can construct a smooth form ω_{F_1} on $\bar{X}$, solution to the problem

$$\left\{\begin{array}{l}\Delta_k\omega_{F_1}=O_t(x^{n-2k})+O_n(x^{n-2k+2})\\ \omega_{F_1}|_M=\omega_0\end{array}\right. \tag{3.4}$$

it can be taken as a polynomial in x

$$\omega_{F_1}=\sum_{2j=0}^{n-2k-1}x^{2j}\omega_{2j}^{(t)}+\sum_{2l=2}^{n-2k+1}x^{2l}\omega_{2l}^{(n)}\wedge\frac{dx}{x} \tag{3.5}$$

and it is the unique solution of (3.4) modulo $O_t(x^{n-2k})+O_n(x^{n-2k+2})$. Moreover, by (2.7) and parity arguments, we see that when n is odd, the remaining term in (3.4) can be repaced by $O_t(x^{n-2k+1})+O_n(x^{n-2k+3})$ (recall also that h_x is smooth in that case). By construction, the $\omega_j^{(t)},\omega_n^{(n)}$ are forms on M which can be expressed as a differential operators $P_j^{(t)},P_j^{(n)}$ on M acting on ω_0, determined by the expansion of P given in (2.7), i.e. by h_0 and the covariant derivatives of its curvature tensor.

The indicial factor in (2.9) vanishes if and only if $j=n-2k, l=n-2k+2$ and n is even. Therefore, if n is odd, we can continue the construction and there is a formal series

$$\omega_\infty=\sum_{j=0}^{\infty}x^j(\omega_j^{(t)}+\omega_j^{(n)}\wedge dx)$$

such that $\Delta_k\omega_\infty=O(x^\infty)$. The formal form ω_∞ can be realized by Borel's Lemma[2], in the sense that there exists a form $\omega'_\infty\in C^\infty(\bar{X},\Lambda^k(\bar{X}))$ with the same asymptotic expansion than ω_∞ at all order and then $\Delta_k\omega'_\infty=O(x^\infty)$.

Now for n even, we need to add log terms to continue the parametrix: by (2.12) one can modify ω_{F_1} to

$$\omega_{F_2}=\omega_{F_1}+x^{n-2k}\log(x)\omega_{n-2k,1}^{(t)},\quad \omega_{n-2k,1}^{(t)}=\frac{1}{n-2k}\Big[x^{-n+2k}\Delta_k\omega_{F_1}\Big]_{|x=0} \tag{3.6}$$

such that $\Delta_k\omega_{F_2}=O(x^{n-2k+2}\log x)$. Actually, using (2.7) and parity arguments once more, we see that

$$\Delta_k\omega_{F_2}=2(-1)^{k+1}x^{n-2k+2}\log x(\delta_0\omega_{n-2k,1}^{(t)})\wedge\frac{dx}{x}+O_t(x^{n-2k+2}\log x)+O_n(x^{n-2k+2}). \tag{3.7}$$

Now we want to show

Lemma 3.4. *The k-form $\omega_{n-2k,1}^{(t)}$ on M satisfies $\delta_0\omega_{n-2k,1}^{(t)}=0$.*

Proof: From (3.7), and the expression of δ, we obtain

$$\delta_g\Delta_k\omega_{F_2}=-2x^{n-2k+2}\,\delta_0\omega_{n-2k,1}^{(t)}+O(x^{n-2k+3}\log x).$$

But $\delta_g\Delta_k\omega_{F_2}=\Delta_{k-1}\delta_g\omega_{F_2}$ and

$$\delta_g\omega_{F_2}=\sum_{j=2}^{n-2k+2}x^j\omega_j'^{(t)}+\sum_{l=3}^{n-2k+3}x^l\omega_l'^{(n)}\wedge\frac{dx}{x}+x^{n-2k+2}\log(x)\,\delta_0\omega_{n-2k,1}^{(t)}+O(x^{n-2k+3}\log x)$$

for some forms $\omega_j'^{(.)}$ on M, so by uniqueness of (3.4) and the fact that $\delta_g\omega_{F_2}=O(x^2)$ we deduce that

$$\delta_g\omega_{F_2}=x^{n-2k+2}\omega_{n-2k+2}'^{(t)}+x^{n-2k+2}\log(x)\,\delta_0\omega_{n-2k,1}^{(t)}+O(x^{n-2k+3}\log x).$$

[2]Borel's Lemma states that if $(f_{k,l})_{l,k\in\mathbb{N}_0}$ is a given sequence of smooth functions on $\partial\bar{X}$ such that, for each k, $f_{k,l}(y)=0$ for all but finitely many l, then there exists a smooth function f in X with an asymptotic expansion at $\partial\bar{X}=\{x=0\}$ of the form $f(x,y)\sim\sum_{k=0}^\infty\sum_{l=0}^\infty f_{k,l}(y)x^k(\log x)^l$.

Using now (2.9) and (2.12), we obtain $\Delta_{k-1}\delta_g\omega_{F_2} = (2k-n-2)x^{n-2k+2}\delta_0\omega^{(t)}_{n-2k,1} + O(x^{n-2k+3}\log x)$, and since $k<\frac{n}{2}$ this implies $\delta_0\omega^{(t)}_{n-2k,1}=0$. $\square$

We infer that there is no term of order $x^{n-2k+2}\log x$ in the Λ^k_n part of $\Delta_k\omega_{F_2}$ and we can continue to solve the problem modulo $O(x^\infty)$ using formal power series with log terms using the indicial equations. The formal solution when n is even will be given by

$$\begin{aligned}\omega_\infty = &\sum_{j=0}^{\frac{n}{2}-1} x^{2j}\omega^{(t)}_{2j} + \sum_{j=1}^{\frac{n}{2}} x^{2j}\omega^{(n)}_{2j}\wedge\frac{dx}{x} + \sum_{j=\frac{n}{2}-k}^{\frac{n}{2}-1} x^{2j}\log(x)\omega^{(t)}_{2j,1}\\ &+\sum_{j=\frac{n}{2}-k+1}^{\frac{n}{2}} x^{2j}\log(x)\omega^{(n)}_{2j,1}\wedge\frac{dx}{x} + x^n\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}(\omega^{(t)}_{n+j,l} + x\omega^{(n)}_{n+j,l}\wedge\frac{dx}{x})x^j(\log x)^l\end{aligned} \tag{3.8}$$

which again is realized through Borel's Lemma to have $\Delta_k\omega_\infty = O(x^\infty)$. Notice that when the metric h_x is smooth, the second line in (3.8) has $\omega^{(t)}_{j,l} = \omega^{(n)}_{j,l} = 0$ for $l>1$ since these terms come from the log terms of the expansion of h_x in (1.7) (and thus of Δ_k). The terms $(\omega^{(t)}_j)_{j<n-2k}$, $(\omega^{(n)}_j)_{j<n-2k+2}$ and $\omega^{(t)}_{n-2k,1}$ are formally determined by ω_0 and are expressed as a differential operator on M acting on ω_0, the terms $\omega^{(t)}_{n-2k},\omega^{(n)}_{n-2k+2}$ are formally undetermined, the remaining terms are formally determined by $\omega_0,\omega^{(t)}_{n-2k}$ and $\omega^{(n)}_{n-2k+2}$.

So we have proved

Proposition 3.5 (Formal solution). *Let $\omega_0, v^{(t)}, v^{(n)} \in C^\infty(M,\Lambda(M))$, then there exists a form $\omega_\infty\in C^{n-2k-1,\alpha}(\bar{X},\Lambda^k_b(\bar{X}))$ with $\alpha\in[0,1)$, unique modulo $O(x^\infty)$, which is smooth on $\bar{X}$ when n is odd and with a polyhomogeneous expansion at $\partial\bar{X}$ of the form* (3.8) *when n is even, such that $\Delta_k\omega_\infty = O(x^\infty)$, $\omega_\infty|_{\partial\bar{X}} = \omega_0$, $\omega^{(t)}_{n-2k} = v^{(t)}$ and $\omega^{(n)}_{n-2k+2} = v^{(n)}$ in the expansion* (3.8).

To correct the approximate solution and obtain a true harmonic form, we add $-E\Delta_k(\omega_\infty)$ to ω_∞ and so

$$\Delta_k(\omega_\infty - E\Delta_k\omega_\infty) = \Pi_0\Delta_k\omega_\infty.$$

We want to prove that $\Pi_0\Delta_k\omega_\infty = 0$ or equivalently that $\langle\Delta_k\omega_\infty,\alpha\rangle = 0$ for any $\alpha\in\ker_{L^2}(\Delta_k)$. For that, we use Green's formula on $\{x\geq\varepsilon\}$ and let $\varepsilon\to 0$, together with the asymptotic $\alpha = O(x^{n-2k+1})$ obtained from Theorem 3.2, $d\alpha=0$ and $\delta\alpha=0$:

$$\int_{x\geq\varepsilon}\langle\Delta\omega_\infty,\alpha\rangle \mathrm{dvol}_g = (-1)^n\int_{x=\varepsilon}(\star_g d\omega_\infty)\wedge\alpha - (\star_g\alpha)\wedge\delta\omega_\infty = O(\varepsilon)\to_{\varepsilon\to 0} 0.$$

In view of the mapping properties of E from Theorem 3.2, we have thus proved that $\omega = \omega_\infty - E\Delta_k\omega_\infty$ is a harmonic k-form of X such that $\omega_{|M} = \omega_0$, with an asymptotic of the form (3.8) when n is even and smooth on $\bar{X}$ when n is odd, such that

$$\omega - \omega_{F_2} = O_t(x^{n-2k}) + O_n(x^{n-2k+2})$$

and with $C^{n-2k-1,\alpha}(\bar{X},\Lambda^k(\bar{X}))$ regularity.

Let us now consider the problem of uniqueness. If one assumes polyhomogeneity of the solution of $\Delta_k\omega = 0$ with boundary condition $\omega = \omega_0 + o(x)$, the construction above with formal series arguments and indicial equations shows that ω is unique up to $O_t(x^{n-2k}) + O_n(x^{n-2k+2})$, i.e. the first positive indicial roots, then of course two such solutions would differ by an L^2 harmonic form if $k<\frac{n}{2}$. Indeed, an easy computation shows that

Remark 3.6. *A polyhomeogeneous k-form in $O_t(x^{\frac{n}{2}-k+\epsilon}) + O_n(x^{\frac{n}{2}-k+1+\epsilon})$ for some $\epsilon>0$ is in $L^2(X)$.*

This gives

Lemma 3.7. *Polyhomogeneous forms satisfying $\Delta_k\omega$ and $\omega = \omega_0 + o(x)$ are unique modulo the L^2 kernel of Δ_k.*

Here, since we want a sharp condition on regularity for uniqueness, i.e. we do not assume polyhomogeneity but $C^{n-2k-1,\alpha}$ regularity, we first need a preliminary result. Let $H^s(\Lambda^k(M))$ be the Sobolev space of order $s \in \mathbb{Z}$ with k-forms values, which we will also denote by $H^s(M)$ to simplify. The sections of the bundle $\Lambda^k_t \oplus \Lambda^k_n$ over M are equipped with the natural Sobolev norm $||.||_{H^s(M)}$ induced by $H^s(\Lambda^k(M) \oplus \Lambda^{k-1}(M))$. The following property is proved by Mazzeo [18, Th. 7.3][3]

Lemma 3.8 (**Mazzeo**)**.** *Let $k < n/2$ and let $\omega \in x^\alpha L^2(\Lambda^k(X), dvol_g)$ with $\alpha < -\frac{n}{2}$ such that $\Delta_k\omega = 0$, then for all $N \in \mathbb{N}$, there exist some forms $\omega^{(t)}_{j,l}, \omega^{(n)}_{j,l} \in H^{-N}(M)$ for $j, l \in \mathbb{N}_0$ and some sequence $l : \mathbb{N}_0 \to \mathbb{N}_0$ such that*

$$\Big|\Big|\omega - \sum_{j=0}^{N-3}\sum_{l=0}^{l(j)} x^j(\log x)^l(\omega^{(t)}_{j,l} - \omega^{(n)}_{j,l} \wedge \frac{dx}{x})\Big|\Big|_{H^{-N}(M)} = O(x^{N-2-\varepsilon}) \tag{3.9}$$

for all $\varepsilon > 0$.

Let ω, ω' be two harmonic forms which are $C^{n-2k-1,\alpha}(\bar{X}, \Lambda^k(\bar{X}))$ and which coincide on the boundary, we want to show that their Taylor expansions at $x = 0$ coincide to order $n - 2k - 1$. Using Lemma 3.8 with N large enough, we see that the arguments used above on formal series (based on the indicial equations) also apply by considering norms $|| \cdot ||_{H^{-N}(M)}$ on $\Lambda^k_t \oplus \Lambda^k_n$, in particular that $l(j) = 0$ for $j = 0, \dots, n - 2k - 1$ in (3.9) for both ω and ω', and that their coefficients of x^j for $j = 0, \dots, n-2k-1$ in the weak expansion (3.9) are the same for ω and ω', these are given by $\omega^t_{j,0} = P^{(t)}_j\omega_0$ and $\omega^{(n)}_j = P^{(n)}_j\omega_0$ (and are then continuous on M since $\omega \in C^{n-2k+1}(\bar{X}, \Lambda^k(\bar{X}))$). But by uniqueness of the expansion (3.9) and the regularity assumption on ω, ω', this implies that $\omega^{(t)}_j, \omega^{(n)}_j$ are the coefficients in the Taylor expansion of both ω and ω' to order $n-2k-1$. The extra Hölder regularity then gives that $||\omega - \omega'||_{L^\infty(M)} = O(x^{n-2k-1+\alpha})$, but then this implies that $\omega - \omega' \in \ker_{L^2}(\Delta_k)$ thus it is in the L^2 kernel of Δ_k, so our construction is unique modulo $\ker_{L^2}(\Delta_k)$. This ends the proof of the solution of (3.1).

Now to deal with (iv), we notice that $d\omega$ is solution of the problem (3.1) for $(k+1)$-forms with the additional condition that the boundary value is $d\omega_0 = 0$. When $k+1 < \frac{n}{2}$, we can then apply Proposition 3.1 (i), when $k \geq \frac{n}{2} - 1$, we have $d\omega = O(x^2)$ and $\Delta_{k+1}d\omega = 0$. However, the discussion below in Subsections 3.1.2 and 3.1.3 about the solutions of $\Delta_{k+1}\omega = O(x^\infty)$ gives the same result, namely that $d\omega \in \ker_{L^2}(\Delta_{k+1})$ if ω is a solution of (3.1) with $k = \frac{n}{2} - 1$ or $k = \frac{n-1}{2}$.

We conclude the proof of (iv) Proposition 3.1 using

Proposition 3.9. *The forms ω_{F_1} of (3.4) and ω of Proposition 3.1 satisfy*

$$\delta_g\omega = 0, \quad \delta_g\omega_{F_1} = O_t(x^{n-2k+2}) + O_n(x^{n-2k+4}).$$

Proof: Let ω be the exact solution of $\Delta_k\omega = 0$, $\omega|_{x=0} = \omega_0$ in Proposition 3.1. Since $\delta_g\Delta_k = \Delta_{k-1}\delta_g$, we deduce that $\omega' := \delta_g\omega$ is solution of $\Delta_k\omega' = 0$ with $\omega'|_{x=0} = 0$ and moreover it is polyhomogeneous since ω is polyhomogeneous, so Proposition 3.1 and Lemma 3.8 imply that $\delta_g\omega \in \ker_{L^2}(\Delta_{k-1})$ and thus $\delta_g\omega = O(x^{n-2k+3})$ by Corollary

[3]Notice that the result of Mazzeo is stated for 0-elliptic operators with smooth coefficients and acting on functions, but it is straightforward to check that it applies on bundles and with polyhomogeneous coefficients like this is the case for even-dimensionnal Poincaré-Einstein manifolds.

3.3. Since an L^2 harmonic form is closed, then $\delta_g\omega$ is closed and integration by parts on $\{x\geq \epsilon\}$ shows, by letting $\epsilon\to 0$ in

$$\int_{x\leq \epsilon}|\delta_g\omega|^2\mathrm{dvol}_g=-\int_{x=\epsilon}\langle \iota_{x\partial_x}\omega,\delta_g\omega\rangle \mathrm{dvol}_g=O(\epsilon)$$

that $\langle\delta_g\omega,\delta_g\omega\rangle=0$. The part with ω_{F_1} is also based on $\delta_g\Delta_k=\Delta_{k-1}\delta_g$ and the uniqueness of the solution of (3.4) up to $O_t(x^{n-2k+2})+O_n(x^{n-2k+4})$ on $(k-1)$-forms. □

3.1.2. *The case $k=\frac{n}{2}$.* In this case one only intends to solve the equation $\Delta_k\omega=O(x^\infty)$, say in the set of almost bounded forms ($\log x$ times bounded). The indicial equation tells us that 0 is a double indicial root for the Λ^k_t part, while $0,2$ are the two simple roots for the Λ_n part. By a double root, we mean a root $\lambda=\lambda_0$ of order 2 of one of the eigenvalues of P_λ in (2.10). In this case, a straightforward inspection shows that an additional power of $\log(x)$ must come in the formal expansion of solutions. Since the discussion of this case is not fundamental in our analysis, we prefer to give the result without details. For $\omega_0,\omega_1\in\Lambda^{\frac{n}{2}}(M)$ and $\omega_2\in\Lambda^{\frac{n}{2}-1}(M)$ one can construct, using (2.11) and (2.12), a polyhomogeneous form

$$\begin{aligned}\omega_F=&\,\omega_2\wedge\frac{dx}{x}+\omega_1\log(x)+\omega_0\\ &+x^2\log x\Big(-(\log x)^2\frac{\delta_0 d\omega_2}{3}+\log x\big(\frac{\delta_0 d\omega_2}{2}+\frac{(-1)^{\frac{n}{2}+1}}{2}\delta_0\omega_1\big)+(-1)^{\frac{n}{2}+1}\delta_0\omega_0\\ &+(-1)^{\frac{n}{2}}\frac{1}{2}\delta_0\omega_1-\frac{1}{2}d\delta_0\omega_2-A(\omega_2)\Big)\wedge\frac{dx}{x}+O_t(x^2(\log x)^2)+O_n(x^2)\end{aligned}$$

such that $\Delta_k\omega_F=O(x^\infty)$ and it is unique modulo $O(x^\infty)$ if the order x^2 coefficient in the Λ_n component is assumed to be 0.

3.1.3. *The case $k=(n+1)/2$.* The indicial equation tells us that $-1,0$ are the roots for the Λ^k_t part, while $0,1$ are the roots for the Λ_n part. For $\omega_0\in\Lambda^k(M)$ and $\omega_1,\omega_2\in\Lambda^{k-1}(M)$, one can construct, using (2.11) and (2.12), a polyhomogeneous form such that

$$\omega_F=\omega_1\wedge\frac{dx}{x}+\omega_0+x\,\omega_2\wedge\frac{dx}{x}+O(x^2)\text{ and }\Delta_k\omega_F=O(x^\infty)$$

and it is unique modulo $O(x^\infty)$. So if ω is a solution of Problem 3.1 with $k=\frac{n-1}{2}$ and boundary value ω_0 closed, then $d\omega=O(x^2)$ and $\Delta_{\frac{n+1}{2}}d\omega=0$ so $d\omega=O(x^\infty)$. But the unique continuation theorem of Mazzeo [17] implies that $d\omega=0$.

We also recall a result proved by Yeganefar [21, Corollary 3.10].

Proposition 3.10. *For an odd dimensional Poincaré-Einstein manifold (X^{n+1},g), there is an isomorphism between $\ker_{L^2}(\Delta_{\frac{n}{2}})$ and $H^{\frac{n}{2}}(\bar{X},\partial\bar{X})$ and between $\ker_{L^2}(\Delta_{\frac{n}{2}+1})$ and $H^{\frac{n}{2}+1}(\bar{X})$.*

3.2. **Relative boundary condition.**

Proposition 3.11. *Let $0<k<\frac{n}{2}$, x be a geodesic boundary defining function and $\omega_0\in C^\infty(M,\Lambda^{k-1}(M))$ be a closed form. Then there exists a unique, modulo $\ker_{L^2}(\Delta_k)$, form ω such that, for all $\alpha\in[0,1)$,*

$$\left\{\begin{array}{l}\omega\in C^{n-2k,\alpha}(\bar{X},\Lambda^k_b(\bar{X})),\\ \Delta_k\omega=0\ \textit{on}\ X,\\ \omega|_M=0,\lim\limits_{x\to 0}i_{x\partial_x}\omega=\omega_0.\end{array}\right.\tag{3.10}$$

Moreover ω is closed, smooth on $\bar{X}$ when n is odd, while it is polyhomogeneous when n is even with an expansion at order $O(x^{n-1}\log x)$ of the form

$$\begin{aligned}\omega =&\Big(\sum_{j=0}^{n-1} x^j\omega_j^{(n)}\wedge\frac{dx}{x}+\sum_{j=1}^{n-2}x^j\omega_j^{(t)}\Big)\\ &+\log x\Big(\sum_{j=n-2k+2}^{n-1}x^j\omega_{j,1}^{(n)}\wedge\frac{dx}{x}+\sum_{j=n-2k+2}^{n-2}x^j\omega_{j,1}^{(t)}\Big)+O(x^{n-1}\log x)\end{aligned}\tag{3.11}$$

for some forms $\omega_j^{(.)},\omega_{j,1}^{(.)}$ on M.

Proof: the proof is similar to that of Proposition 3.1, so we do not give the full details but we shall use the same notations. We search a formal solution ω'_∞ of $\Delta_k\omega'_\infty=0$ with $\omega'_\infty=\omega_0\wedge\frac{dx}{x}+O(x)$. Using the indicial equations in Subsection 2.3 and the form of Δ_k in Lemma 2.1, we can construct the exponents in the formal series as long as the exponent is not a solution of the indicial equation. Since $d\omega_0=0$ by assumption, we have

$$\Delta_k(\omega_0\wedge\frac{dx}{x})=2(-1)^{k+1}d\omega_0+O(x^2)=O(x^2)$$

and so we can continue the construction of ω'_∞ until the power x^{n-2k} in the tangential part Λ^k_t and x^{n-2k+2} in the Λ^k_n part. At that point, since x^{n-2k} and x^{n-2k+2} are solutions of the indical equation of Δ_k in respectively the Λ^k_t and Λ^k_n components, there is a $x^{n-2k}\log x$ term to include in the Λ^k_t part. Using in addition that Δ_k begins with a sum of even powers of x, we see like in Proposition 3.1 that when n is odd, a formal series ω'_∞ with no log terms can be constructed to solve $\Delta_k\omega'_\infty=O(x^\infty)$, while when n is even we can first construct

$$\omega'_{F_2}=\sum_{2j=0}^{n-2k}x^{2j}\omega_{2j}^{(n)}\wedge\frac{dx}{x}+\sum_{2j=2}^{n-2k-2}x^{2j}\omega_{2j}^{(t)}+x^{n-2k}\log(x)\,\omega_{n-2k,1}^{(t)}\tag{3.12}$$

with $\omega_0^{(n)}=\omega_0$ so that $\Delta_k\omega'_{F_2}=O(x^{n-2k+2}\log x)$, and the coefficients are uniquely determined by ω_0. First observe that $d\omega'_{F_2}=O(x^2)$ satisfies $\Delta_{k+1}d\omega'_{F_2}=O(x^{n-2k+2}\log x)$ and since the indicial root in $[1,n-2k]$ for Δ_{k+1} are $n-2k-2$ in the Λ^{k+1}_t part and $n-2k$ in the Λ^{k+1}_n part, we deduce that $d\omega'_{F_2}=O_t(x^{n-2k-2})+O_n(x^{n-2k})$ and so

$$\begin{aligned}d\omega'_{F_2}=&\sum_{2j=2}^{n-2k-2}d\omega_{2j}^{(t)}x^{2j}+\sum_{2j=2}^{n-2k-2}x^{2j}((-1)^k2j\omega_{2j}^{(t)}+d\omega_{2j}^{(n)})\wedge\frac{dx}{x}+x^{n-2k}d\omega_{n-2k}^{(n)}\wedge\frac{dx}{x}\\ &+x^{n-2k}\log(x)\big(d\omega_{n-2k,1}^{(t)}+(-1)^k(n-2k)\omega_{n-2k,1}^{(t)}\wedge\frac{dx}{x}\big)+x^{n-2k}(-1)^k\omega_{n-2k,1}^{(t)}\wedge\frac{dx}{x}\\ =&x^{n-2k}\log(x)\big(d\omega_{n-2k,1}^{(t)}+(-1)^k(n-2k)\omega_{n-2k,1}^{(t)}\wedge\frac{dx}{x}\big)\\ &+x^{n-2k}\big(d\omega_{n-2k}^{(n)}+(-1)^k\omega_{n-2k,1}^{(t)}\big)\wedge\frac{dx}{x}.\end{aligned}\tag{3.13}$$

Note that we have used that $(-1)^k(n-2k-2)\omega_{n-2k-2}^{(t)}=-d\omega_{n-2k-2}^{(n)}$. With these simplifications, we get

$$\Delta_{k+1}d\omega'_{F_2}=(n-2k)x^{n-2k}\big((-1)^{k+1}(n-2k)\omega_{n-2k,1}^{(t)}\wedge\frac{dx}{x}+\log(x)d\omega_{n-2k,1}^{(t)}\big)+O(x^{n-2k+1}).$$

But since $d\Delta_k\omega'_{F_2}=O(x^{n-2k+2}\log(x))$, we infer that $\omega_{n-2k,1}^{(t)}$ must vanish, and we obtain

$$\Delta_k\omega'_{F_2}=O(x^{n-2k+2})$$

Since the order x^{n-2k+2} is a solution of the indicial equation in the normal part Λ^k_n, we need to add a $x^{n-2k+2}\log(x)$ normal term to continue the construction of the formal solution. Since all the subsequent orders are not solution of the indicial equation for Δ_k, we can construct, using Borel lemma, a polyhomogeneous k-form on X with expansion to order $x^{n-1}\log(x)$ of the form given by (3.11), which coincides with ω_{F_2} at order $O_n(x^{n-2k+2}\log x)+O_t(x^{n-2k})$. To obtain an exact solution of (3.10), we can correct ω'_∞ by setting $\omega=\omega'_\infty - E\Delta_k\omega'_\infty$ where E is defined in Proposition 3.2.

The argument for the uniqueness modulo $\ker_{L^2}\Delta_g$ is similar to that used in the proof of Proposition 3.1.

To prove that ω is closed, it suffices to observe that $\omega=\omega_{F_2}+O_n(x^{n-2k+2}\log x)+O_t(x^{n-2k})$ and so $d\omega = x^{n-2k}d\omega^{(n)}_{n-2k}\wedge\frac{dx}{x}+O(x^{n-2k})$. By remark 3.6, we have $d\omega\in\ker_{L^2}(\Delta_{k+1})$. Then $\delta_g d\omega=0$ and, considering the decay of $d\omega$ and ω at the boundary, we see by integration by parts that $d\omega=0$. □

Remarks: it is important to remark that the solution ω of the problem (3.10) depends on ω_0 but also on the choice of x. Note also that the form ω solution of (3.10) satisfies $x\omega\in C^{n-2k+1,\alpha}(\bar{X},\Lambda^k(\bar{X}))$ for all $\alpha\in(0,1)$.

4. L_k, G_k and Q_k operators

In this section we suppose that M has an even dimension n.

4.1. Definitions.

The operators L_k, G_k derive from the solution of the absolute Dirichlet problem:

Definition 4.1. *For $k<\frac{n}{2}$, the operators $L_k : C^\infty(M,\Lambda^k(M))\to C^\infty(M,\Lambda^k(M))$ and $G_k : C^\infty(M,\Lambda^k(M))\to C^\infty(M,\Lambda^{k-1}(M))$ are defined by $L_k\omega_0 =: \omega^{(t)}_{n-2k,1}$ and $G_k\omega_0 := \omega^{(n)}_{n-2k+2,1}$ where $\omega^{(t)}_{n-2k,1},\omega^{(n)}_{n-2k+2,1}$ are given in the expansion* (3.2). *When $k=\frac{n}{2}$, we define $G_{\frac{n}{2}} := (-1)^{\frac{n}{2}+1}\delta_0$.*

The operator Q_k derives from the solution of the relative Dirichlet problem:

Definition 4.2. *Let n be even and $k<\frac{n}{2}$, the operator $Q_{k-1} : (C^\infty(M,\Lambda^{k-1}(M))\cap\ker d)\to C^\infty(M,\Lambda^{k-1}(M))$ is defined by $Q_{k-1}\omega_0 := \omega^{(n)}_{n-2k+2,1}$ where $\omega^{(n)}_{n-2k+2,1}$ is given in the expansion* (3.11).

By Corollary 3.3, L_k, G_k and Q_k do not depend on the choice of the solution ω in Propositions 3.1 or 3.11, though L_k depends only on the boundary $(M,[h_0])$, the operators G_k and Q_k may well depend on the whole manifold (X,g) and not only on the conformal boundary. We will see that they actually depend only on $(M,[h_0])$ and that they are differential operators.

4.2. A formal construction.

We show that the definition of L_k, G_k, Q_k can be done using only the formal series solutions. Let us first define

Definition 4.3. *For $k<\frac{n}{2}$, the operators $B_k, C_k : C^\infty(M,\Lambda^k(M))\to C^\infty(M,\Lambda^{k-1}(M))$ and $D_k : C^\infty(M,\Lambda^k(M))\cap\ker d\to C^\infty(M,\Lambda^k(M))$ are defined by*

$$\begin{aligned} B_k\omega_0 &:= \Big(x^{-n+2k-2}i_{x\partial_x}\Delta_k\omega_{F_1}\Big)|_{x=0},\\ C_k\omega_0 &:= \Big(x^{-n+2k-2}i_{x\partial_x}(\frac{dx}{x}\wedge\delta_g\omega_{F_1})\Big)|_{x=0} \qquad (4.1)\\ D_k\omega_0 &:= \Big(x^{-n+2k}i_{x\partial_x}d\omega_{F_1})\Big)|_{x=0}\end{aligned}$$

where ω_{F_1} solves (3.4).

Remark: from the indicial equations and Lemma 3.4, $B_k\omega_0$ is $(-1)^k(n-2k+2)$ times the $x^{n-2k+2}\log(x)$ coefficient in the Λ^k_n part of ω'_∞ defined in Proposition 3.5 when $v^{(t)}=0$, this is a differential operator on M of order $n-2k+1$ since by construction, ω_{F_1} contains only derivatives of order at most $n-2k-1$ with respect to ω_0. The operator C_k is well defined thanks to Proposition 3.9, and it is a differential operator of order $n-2k$. As they come from the expansion of Δ_k, δ_g, they are natural differential operators depending only on h_0 and the covariant derivatives of its curvature tensor.

4.2.1. *The case of L_k.* It is clear from the proof of Proposition 3.1 that $L_k\omega_0$ is also the coefficient of the $x^{n-2k}\log x$ term in the expansion of ω_{F_2} defined in (3.6) and of the formal solution ω_∞ defined in Proposition 3.8. The indicial equation shows that

$$L_k\omega_0 := \frac{1}{n-2k}\Big(x^{2k-n} i_{x\partial_x}(\frac{dx}{x}\wedge \Delta_k\omega_{F_1})\Big)|_{x=0} \tag{4.2}$$

where ω_{F_1} solves (3.4).

4.2.2. *The case of G_k.* Let us return to the construction of the formal series solution in the proof of Proposition 3.1. Now let ω_{F_2} defined in (3.6) and

$$\omega_{F_2} := \omega_{F_1} + x^{n-2k}v^{(t)} + x^{n-2k}\log(x)\omega^{(t)}_{n-2k,1}$$

where $v^{(t)}\in C^\infty(M,\Lambda^k(M))$ is an arbitrary form. By construction of $\omega_{F_1},\omega_{F_2}$, the fact that $n-2k$ is an indicial root in the Λ^k_t component and Lemma 2.1, we have

$$\Delta_k\omega_{F_2} = (-1)^{k+1}x^{n-2k+2}(B_k\omega_0 + 2\delta_0 v^{(t)})\wedge\frac{dx}{x} + O_t(x^{n-2k+2}\log x) + O_n(x^{n-2k+4}\log x)$$

to solve away the x^{n-2k-2} term in Λ^k_n we need to define

$$\omega_{F_3} := \omega_{F_2} + \frac{(-1)^{k+1}}{n+2-2k}x^{n-2k}\log(x)(B_k\omega_0 + 2\delta_0 v^{(t)})\wedge\frac{dx}{x} \tag{4.3}$$

so that $\Delta_k\omega_{F_3} = O_n(x^{n-2k+4}\log(x)) + O_t(x^{n-2k+2}\log(x))$. Since $v^{(t)}$ can be chosen arbitrarily, the coefficient of $x^{n-2k+4}\log(x)$ in the Λ^k_n component of the formal solution ω_{F_3} does not determine a natural operator in term of the initial data ω_0, contrary to the $x^{n-2k}\log(x)$ coefficient in Λ^k_t. In the definition of G_k above, we used an exact solution on X to fix the $v^{(t)}$ term through the Green function, which a priori makes G_k depend on (X,g) and not only on $(M,[h_0])$. However there is an equivalent way of fixing $\delta_0 v^{(t)}$ without solving a global Dirichlet problem but by adding an additional condition:

Proposition 4.4. *Let $\omega_0\in C^\infty(M,\Lambda^k(M))$, then there is a polyhomogeneous k-form ω_F such that*

$$\begin{cases} \Delta_k\omega_F = O_t(x^{n-2k+1}) + O_n(x^{n-2k+3}) \\ \delta_g\omega_F = O(x^{n-2k+3}) \\ \omega = \omega_0 + O(x) \end{cases} . \tag{4.4}$$

It is unique modulo $O_t(x^{n-2k}) + O_n(x^{n-2k+2})$ and has an expansion of the form

$$\begin{aligned}\omega_F = &\sum_{j=0}^{\frac{n}{2}-k-1} x^{2j}\omega^{(t)}_{2j} + \sum_{j=1}^{\frac{n}{2}-k} x^{2j}\omega^{(n)}_{2j}\wedge\frac{dx}{x} \\ &+ x^{n-2k}\log(x)\Big(L_k\omega_0 + x^2\frac{(-1)^{k+1}}{n-2k}(B_k\omega_0 - 2C_k\omega_0)\wedge\frac{dx}{x}\Big).\end{aligned} \tag{4.5}$$

Proof: First consider the uniqueness. By the discussion above, the condition on $\Delta_k\omega_F$ implies that ω_F is necessary of the form $\omega_F = \omega_{F_3}$ defined in (4.3) for some $v^{(t)}$. Now we notice that $\delta_g\omega_{F_3} = O(x^2)$ satisfies in particular $\Delta_{k-1}\delta_g\omega_{F_3} = \delta_g\Delta_k\omega_{F_3} = O(x^{n-2k+3})$, and again by the indicial equation this implies that $\delta_g\omega_{F_3} = O(x^{n-2k+2})$ since the first positive indicial root for Δ_{k-1} is $n-2k+2$. Using that $\delta_0 L_k\omega_0=0$ and the form of δ_g we obtain

$$\delta_g\omega_{F_3}=\delta_g\omega_{F_1}+x^{n-2k+2}\Big(\delta_0 v^{(t)}-\frac{1}{n+2-2k}(B_k\omega_0+2\delta_0 v^{(t)})\Big)+O(x^{n-2k+3}).$$

By Proposition 3.9, $\delta_g\omega_{F_1}=O_t(x^{n-2k+2})+O_n(x^{n-2k+4})$ and from the definition of C_k, a necessary condition to have $\delta_g\omega_F=O(x^{n-2k+3})$ is

$$(n-2k)\delta_0 v^{(t)}=B_k\omega_0-(n-2k+2)C_k\omega_0.$$

Writing now $\delta_0 v^{(t)}$ in terms of B_k, C_k in (4.3) proves the uniqueness and the form of the expansion. Now for the existence, one can take the form in Proposition (3.1). Another way, which again is formal, is first to construct a polyhomogeneous $(k+1)$-form ω'_F such that

$$\begin{cases} \Delta_{k+1}\omega'_F=O_t(x^{n-2k-1})+O_n(x^{n-2k+1}) \\ \omega'_F=\frac{2(-1)^{k+1}}{n-2k}\log(x)d\omega_0+\omega_0\wedge\frac{dx}{x}+O(x), \end{cases}$$

which can be done as in Proposition 3.11 by using the indicial equations, and then to set $\omega_F:=\delta_g\omega'_F$. It is easy to see that this form is a polyhomogeneous solution of (4.4). □

Since the exact solution in Proposition 3.1 is coclosed, we deduce from Proposition 4.4 the

Corollary 4.5. *The operator G_k is a natural differential operator of order $n-2k+1$ which is given by*

$$G_k=(-1)^{k+1}\frac{B_k-2C_k}{n-2k}$$

and depends only on h_0 and the covariant derivatives of its curvature tensor.

Remark 4.6. *Problem* (4.4) *and Corollary 4.5 allow to define L_k and G_k on any even manifold M^{2m} with no need of cobordism assumption (as in Proposition 3.1). We just have to work on $X=M\times[0,\epsilon[$.*

4.2.3. *The operator Q_k.* Following the ideas used above for G_k, we shall show how to construct Q_k from a formal solution ω_{F_1}. We start by

Definition 4.7. *For $1\leq k\leq\frac{n}{2}$, define the operators $B'_{k-1}:C^\infty(M,\Lambda^{k-1}(M))\cap\ker d\to C^\infty(M,\Lambda^{k-1}(M))$ and $D'_{k-1}:C^\infty(M,\Lambda^{k-1}(M))\cap\ker d\to C^\infty(M,\Lambda^{k}(M))$ by*

$$B'_{k-1}\omega_0:=\Big(x^{-n+2k-2}i_{x\partial_x}\Delta_k\omega'_{F_1}\Big)|_{x=0},\qquad (4.6)$$
$$D'_{k-1}\omega_0:=\Big(x^{-n+2k}i_{x\partial_x}d\omega'_{F_1}\Big)|_{x=0}$$

where ω'_{F_1} is the form in (3.12) *such that $\Delta_k\omega'_{F_1}=O(x^{n-2k+2})$ and $\omega'_{F_1}=\omega_0\wedge\frac{dx}{x}+O(x^2)$.*

Let us now set $\omega'_{F_2}:=\omega'_{F_1}+v^{(t)}x^{n-2k}$ for some arbitrary smooth form $v^{(t)}$ on M, we obtain

$$\Delta_k\omega'_{F_2}=(-1)^{k+1}x^{n-2k+2}(B'_{k-1}\omega_0+2\delta_0 v^{(t)})\wedge\frac{dx}{x}+O_t(x^{n-2k+2})+O_n(x^{n-2k+3}).$$

so to solve away the x^{n-2k+2} normal coefficient, we need to define

$$\omega'_{F_3}:=\omega'_{F_2}+\frac{(-1)^{k+1}}{n-2k+2}x^{n-2k+2}\log(x)\Big(B'_{k-1}\omega_0+2\delta_0 v^{(t)}\Big)\wedge\frac{dx}{x}\qquad (4.7)$$

which satisfies $\Delta_k\omega'_{F_3}=O_t(x^{n-2k+2}\log(x))+O_n(x^{n-2k+3})$. Like for G_k, the term $v^{(t)}$ is arbitrary and so we have to impose an additional condition to fix this term (or at least to fix $\delta_0 v^{(t)}$).

Proposition 4.8. *Let $\omega_0 \in C^\infty(M, \Lambda^{k-1}(M))$ be closed, then there is a polyhomogeneous k-form ω'_F which satisfies*

$$\begin{cases} \Delta_k \omega'_F = O_t(x^{n-2k+1}) + O_n(x^{n-2k+3}) \\ d\omega'_F = O(x^{n-2k+1}) \\ \omega'_F = \omega_0 \wedge \frac{dx}{x} + O(x^2) \end{cases}, \tag{4.8}$$

which is unique modulo $O_t(x^{n-2k}) + O_n(x^{n-2k+2})$ and has an expansion of the form

$$\begin{aligned} \omega'_F = &\sum_{j=1}^{\frac{n}{2}-k-1} x^{2j}\omega^{(t)}_{2j} + \sum_{j=0}^{\frac{n}{2}-k} x^{2j}\omega^{(n)}_{2j} \wedge \frac{dx}{x} - x^{n-2k}\frac{1}{n-2k}D'_{k-1}\omega_0 \\ &+ \frac{(-1)^{k+1}}{n-2k+2}x^{n-2k+2}\log(x)\Big(B'_{k-1}\omega_0 - \frac{2\delta_0 D'_{k-1}\omega_0}{n-2k}\Big) \wedge \frac{dx}{x}. \end{aligned} \tag{4.9}$$

Proof: (i) Take $\omega'_F = \omega'_{F_3}$ defined in (4.7), then $\Delta_k\omega'_F = O_t(x^{n-2k+1}) + O_n(x^{n-2k+3})$ by construction. Moreover, since ω_0 is closed, one has $d\omega'_F = O(x^2)$ and $\Delta_{k+1} d\omega'_F = O(x^{n-2k+1})$. Since the indicial roots for Δ_{k+1} in $[2, n-2k+1]$ are $n-2k-2$ in the Λ^{k+1}_t part and $n-2k$ in the Λ^{k+1}_n part, this implies that $d\omega'_F = O_t(x^{n-2k-2}) + O_n(x^{n-2k})$. Then, using (3.13), we obtain

$$\begin{aligned} d\omega'_F =& x^{n-2k}\Big(dv^{(t)} + \big((-1)^k(n-2k)v^{(t)} + d\omega^{(n)}_{n-2k}\big) \wedge \frac{dx}{x}\Big) + O(x^{n-2k+1}) \\ =& x^{n-2k}\Big(dv^{(t)} + \big((-1)^k(n-2k)v^{(t)} + (-1)^k D'_{k-1}\omega_0\big) \wedge \frac{dx}{x}\Big) + O(x^{n-2k+1}). \end{aligned}$$

So $d\omega'_F = O(x^{n-2k+1})$ if and only if $v^{(t)} = -D'_{k-1}\omega_0/(n-2k)$. ◻

The first corollary is

Corollary 4.9. *For $k < \frac{n}{2}$, the operator Q_k is a natural differential operator of order $n-2k$ which is given by*

$$Q_k = \frac{(-1)^k}{n-2k}\Big(B'_k - \frac{\delta_0 D'_k}{\frac{n}{2}-k-1}\Big)$$

and it depends only on h_0 and the covariant derivatives of its curvature tensor.

Remark 4.10. *Here also this corollary allows to define the operator Q_k on any even manifold by considering Problem* (4.8).

As a corollary of Propositions 4.4 and 4.8, we also have

Corollary 4.11. *If ω_0 is a closed k-form on M, then there is a polyhomogeneous k-form ω_F on $\bar{X}$ such that*

$$\begin{cases} d\omega_F = O(x^{n-2k+1}) \\ \delta_g\omega_F = O(x^{n-2k+3}) \\ \omega_F = \omega_0 + O(x) \end{cases} \tag{4.10}$$

It is unique modulo $O_t(x^{n-2k+1}) + O_n(x^{n-2k+2})$ and it has an expansion

$$\begin{aligned} \omega_F = &\sum_{j=0}^{\frac{n}{2}-k-1} x^{2j}\omega^{(t)}_{2j} + \sum_{j=1}^{\frac{n}{2}-k} x^{2j}\omega^{(n)}_{2j} \wedge \frac{dx}{x} - \frac{1}{n-2k}D_k\omega_0 x^{n-2k} \\ &+ x^{n-2k+2}\log(x)\Big(\frac{(-1)^{k+1}}{n-2k}(B_k\omega_0 - 2C_k\omega_0) \wedge \frac{dx}{x}\Big). \end{aligned}$$

Proof: for the existence, take ω'_F in Proposition 4.8 (ω'_F is $k+1$ form now since $\omega_0 \in \Lambda^k(M)$) and consider $\omega_F := (-1)^{k+1}/(2k-n)\delta_g\omega'_F$. It is easy to see that $\omega_F = \omega_0 + O(x^2)$ and that $\Delta_k\omega_F = O_t(x^{n-2k+1}) + O_n(x^{n-2k+3})$. Since $d\delta_g\omega'_F = -\delta_g d\omega'_F +$

$O_t(x^{n-2k-1})+O_n(x^{n-2k+1})$, we deduce that $d\omega_F=O_t(x^{n-2k-1})+O_n(x^{n-2k+1})$. But from the Proposition 4.4, $\omega_F=\omega_{F_1}+v^{(t)}x^{n-2k}+O(x^{n-2k+1})$ (note that $L_k\omega_0=0$ by Proposition 4.12) for some k-form $v^{(t)}$ on M and so we conclude that

$$\begin{aligned}d\omega_F=&\sum_{2j=2}^{n-2k-2}d\omega^{(t)}_{2j}x^{2j}+\sum_{2j=2}^{n-2k-2}x^{2j}((-1)^k2j\omega^{(t)}_{2j}+d\omega^{(n)}_{2j})\wedge\frac{dx}{x}\\&+x^{n-2k}(d\omega^{(n)}_{n-2k}+(-1)^k(n-2k)v^{(t)})\wedge\frac{dx}{x}+x^{n-2k}dv^{(t)}+O(x^{n-2k+1})\\=&O(x^{n-2k+1})\end{aligned}$$

so $v^{(t)}$ has to be $(-1)^{k+1}d\omega^{(n)}_{n-2k}/(n-2k)$ to get $d\omega_F=O_t(x^{n-2k-1})+O_n(x^{n-2k+1})$. But clearly this argument also implies that $d\omega_{F_1}=x^{n-2k}d\omega^{(n)}_{n-2k}\wedge\frac{dx}{x}$ and the expansion of ω_F is then a consequence of this fact together with the expansion (4.5) in Proposition 4.4 and the definition of D_k. □

Remark: in Proposition 4.4, 4.8 and Corollary 4.11, we do not really need to take $\omega_0\in C^\infty(M,\Lambda(M))$. Indeed, for an ω_0 in $L^2(\Lambda(M))$, the arguments would work in a similar fashion except that the expansion in power of x and $\log(x)$ have coefficients in some $H^{-N}(\Lambda(M))$ with N large enough, as we discussed in the proof of Proposition 3.1.

4.3. Factorizations.

Proposition 4.12. *For any $k<\frac{n}{2}-1$, the following identities hold*

$$\begin{aligned}G_k=&(-1)^k\frac{\delta_{h_0}Q_k}{n-2k}\qquad\textit{on closed forms,}\\L_k=&\frac{(-1)^k}{(n-2k)}G_{k+1}d=-\frac{\delta_{h_0}Q_{k+1}d}{(n-2k)(n-2k-2)}.\end{aligned}\tag{4.11}$$

while for $k=\frac{n}{2}-1$

$$L_{\frac{n}{2}-1}=\frac{1}{2}\delta_{h_0}d.\tag{4.12}$$

Proof: Let ω be a solution of Problem (4.8) with initial data ω_0 closed. Then its first log term is $x^{n-2k+2}\log(x)Q_{k-1}\omega_0\wedge\frac{dx}{x}$ and thus the first normal log term of $\delta_g\omega$ is $x^{n-2k+4}\log(x)(\delta_0Q_{k-1}\omega_0)\wedge\frac{dx}{x}$. But $\delta_g\omega$ is a solution of Problem (4.4) with boundary term $\delta_g\omega=(-1)^k(2k-n-2)\omega_0+O(x)$. Thus, the form $\delta_g\omega$ has for first normal log term $(-1)^k(2k-n-2)x^{n-2k+4}(G_{k-1}\omega_0)\wedge\frac{dx}{x}$.

Let ω be a solution of 3.1 with initial data ω_0. Since $\Delta_{k+1}d=d\Delta_k$ and $\delta_gd\omega=\Delta_{k+1}d\omega-d\delta_g\omega$, the form $\omega':=d\omega$ is a solution Problem (4.4) with initial data $d\omega_0$ and first log term $(-1)^k(n-2k)x^{n-2k}\log(x)L_k\omega_0\wedge\frac{dx}{x}$, which gives (4.11).

To compute $L_{\frac{n}{2}-1}$ we use Equation (4.2). Using Relations (2.9), we get $\omega_{F_1}=\omega_0-x\frac{(-1)^{\frac{n}{2}}}{2}\delta_0\omega_0\wedge dx$, therefore $\Delta_{\frac{n}{2}-1}\omega_{F_1}=x^2\delta_0d\omega_0+o(x^2)$. □

Remark: Note that this implies that L_k is zero on closed forms and G_k has its range in co-closed forms.

4.4. **Conformal properties.** A priori our construction of L_k,G_k,Q_k depends on the choice of geodesic boundary defining function x, i.e. on the choice of conformal representative in $[h_0]$. In order to study the conformal properties of these operators, we need to compare the splittings of the differential forms associated to different conformal representatives.

A system of coordinates $y = (y_i)_{i=1,\dots,n}$ on M near a point $p \in M$ give rise to a system of coordinates (x, y) in $\bar{X}$ near the boundary point p through the diffeomorphism $\psi : (x, y) \to \psi_x(y)$ where ψ_t is the flow of the gradient $\nabla^{x^2 g} x$ of x with respect to $x^2 g$. Such a system (x, y) is called a system of *geodesic normal coordinates associated to* h_0.

Lemma 4.13. *Let (x, y) and $(\hat{x}, \hat{y})$ be two systems of geodesic normal coordinates associated respectively to h_0 and $\hat{h}_0 = e^{2\varphi_0} h_0$. If $\hat{\omega}$ (resp. $\hat{\omega} \wedge d\hat{x}$) is a k-form tangential (resp. normal) in the coordinates $(\hat{x}, \hat{y})$ with $\hat{\omega}|_{\hat{x}=0} = \omega_0$, then we have*

$$\hat{\omega} = \omega_0 + (-1)^{k+1} x^2 (i_{\nabla \varphi_0} \omega_0) \wedge \frac{dx}{x} + O_t(x^2) + O_n(x^3),$$
$$\hat{\omega} \wedge \frac{d\hat{x}}{\hat{x}} = \omega_0 \wedge \frac{dx}{x} + \omega_0 \wedge d\varphi_0 + O_t(x) + O_n(x^2).$$

Proof: By the proof of Lemma 2.1 in [13], if $\hat{h}_0 = e^{2\varphi_0} h_0$ is another conformal representative, a geodesic boundary defining function $\hat{x}$ associated to $\hat{h_0}$ satisifies $\hat{x} = e^{\varphi} x$ with $\varphi = \varphi_0 + O(x^2)$ at least C^{n-1} and $\hat{y}_i(x, y) = y_i + \frac{x^2}{2} \sum_{j=1}^n h^{ij} \partial_{y_j} \varphi_0 + O(x^3)$. Hence $d\hat{y}_i = dy_i + x \sum_j h^{ij} \partial_{y_j} \varphi_0 dx$ and $d\hat{x} = x e^{\varphi_0} d\varphi_0 + e^{\varphi_0} dx + O(x^2)$, which gives the relations above. □

This implies the following corollary:

Corollary 4.14. *Under a conformal change $\hat{h}_0 = e^{2\varphi_0} h_0$, the associated operators $\hat{L}_k$, $\hat{H}_k$ and $\hat{Q}_k$ are given by*

$$\hat{L}_k = e^{(2k-n)\varphi_0} L_k, \quad \hat{G}_k = e^{(2k-2-n)\varphi_0} \big(G_k + (-1)^k i_{\nabla \varphi_0} L_k \big)$$
$$\hat{Q}_k \omega_0 = e^{\varphi_0 (2k-n)} \Big(Q_k \omega_0 + (n-2k) L_k (\varphi_0 \omega_0) \Big) \tag{4.13}$$

where $\omega_0 \in C^\infty(M, \Lambda^k(M))$ is any closed form. Thus L_k is conformally covariant and G_k is conformally covariant on the kernel of L_k (hence on closed forms).

Proof: Let ω be a solution of Problem (4.4) with respect to (x, y) a system associated to h_0. Then by Lemma 4.13, ω is also a solution of Problem (4.4) with respect to $(\hat{x}, \hat{y})$ a system associated to $\hat{h}_0$. Now, when we change x to $\hat{x}$ the first $\log x$ term (i.e. the $x^{n-2k} \log x$ term) in the expansion of ω changes by a multiplication by $e^{(2k-n)\varphi_0}$. As for the $x^{n-2k+2} \log x$ term in the normal part, we have a similar effect but the tangential $x^{n-2k} \log x$ term gives rise to a $\hat{x}^{n-2k+2} \log \hat{x}$ normal term which gives the term $i_{\nabla \varphi_0} L_k$.

Let ω be a solution of Problem (4.8) in the variable x with initial data $\omega_0 \wedge \frac{dx}{x}$, $\hat{\omega}_1$ be a solution of Problem (4.8) in the variable $\hat{x}$ with initial data $\omega_0 \wedge \frac{d\hat{x}}{\hat{x}}$ and $\hat{\omega}_2$ be a solution of Problem (4.10) in the variable $\hat{x}$ with initial data $-\omega_0 \wedge d\varphi_0$. Using Lemma 4.13 get that $\hat{\omega}_1 + \hat{\omega}_2$ satisifies Problem 4.8 in the variable x with initial data ω_0. So $\omega = \hat{\omega}_1 + \hat{\omega}_2$ modulo $O_t(x^{n-2k}) + O_n(x^{n-2k+2})$ and the $x^{n-2k+2} \log x$ normal terms must be the same. Using 4.13, we get

$$Q_{k-1} \omega_0 \wedge \frac{dx}{x} = e^{\varphi_0 (n-2k+2)} \big(\hat{Q}_{k-1} \omega_0 - \hat{G}_k (\omega_0 \wedge d\varphi_0) \big) \wedge \frac{dx}{x}.$$

Now we use the transformation formula of $\hat{G}_k$ and (4.11) with $d\omega_0 = 0$ to see that

$$\begin{aligned} e^{\varphi_0 (n-2k+2)} \hat{G}_k (\omega_0 \wedge d\varphi_0) =& (-1)^{k+1} G_k d(\varphi_0 \omega_0) + (-1)^k i_{\nabla \varphi_0} L_k d(\varphi_0 \omega_0) \\ =& (n-2k+2) L_{k-1} (\varphi_0 \omega_0) \end{aligned}$$

This ends the proof of the transformation law of Q_{k-1} by conformal change. □

Remark: while Q_k on $\ker d$ is not conformally invariant (by Proposition 4.14), the pairing $\langle Q_k u, u \rangle_{L^2(\mathrm{dvol}_{h_0})}$ for the metric h_0 is conformally invariant for $u \in \ker d$. Indeed, using

(4.13), a conformal change of metric $\hat{h}_0 = e^{2\varphi_0}h_0$ gives

$$\int_M \langle \hat{Q}_k u, u\rangle_{\hat{h}_0} \mathrm{dvol}_{h_0} = \int_M \langle Q_k u, u\rangle_{h_0} + \frac{\langle \delta_0 Q_{k+1} d(\varphi_0 u), u\rangle_{h_0}}{2k+2-n} \mathrm{dvol}_{h_0}$$

which by integration by parts and $du = 0$ gives the $\langle \hat{Q}_k u, u\rangle_{L^2(\mathrm{dvol}_{\hat{h}_0})} = \langle Q_k u, u\rangle_{L^2(\mathrm{dvol}_{h_0})}$. Of course, when we restrict this form to exact forms, this is given by

$$\langle Q_k du, du\rangle = \langle L_{k-1}u, u\rangle$$

which is real and conformally invariant.

4.5. **Analytical properties.**

Proposition 4.15. *For any $k < \frac{n}{2}$ we have*

$$Q_k = \frac{(-1)^{\frac{n}{2}+k+1}(n-2k)(\Delta_0)^{\frac{n}{2}-k}}{2^{n-2k}[(\frac{n}{2}-k)!]^2} + \textit{lower order terms in } \partial^j_{y_i}$$
$$L_k = \frac{(-1)^{\frac{n}{2}+k+1}(n-2k)(\delta_0 d)^{\frac{n}{2}-k}}{2^{n-2k}[(\frac{n}{2}-k)!]^2} + \textit{lower order terms in } \partial^j_{y_i}$$
$$G_k = \frac{(-1)^{\frac{n}{2}+1}(\delta_0 d)^{\frac{n}{2}-k}\delta_0}{2^{n-2k}[(\frac{n}{2}-k)!]^2} + \textit{lower order terms in } \partial^j_{y_i}$$

Proof: We first review the computation of ω_{F_1} which solves (3.4). By Lemma 2.1, ω_{F_1} has the form $\omega_{F_1} = \sum_{i=0}^{\frac{n}{2}-k-1} x^{2i}\omega_{2i}^{(t)} + \sum_{i=1}^{\frac{n}{2}-k} x^{2i}\omega_{2i}^{(n)} \wedge \frac{dx}{x}$, where the $\omega_i^{(*)}$ are images of ω_0 by differential operators on M. We compute the principal part of these operators by recurrence.

The decomposition (2.7) of Δ_k and the identity $\Delta_k\omega_{F_1} = O_t(x^{n-2k}) + O_n(x^{n-2k+1})$ give

$$\sum_{i=1}^{\frac{n}{2}-k} x^{2i}\Big(-4i(k+i-\frac{n}{2}-1)\omega_{2i}^{(n)} + \sum_{j=1}^{i} S'_j\omega_{2i-2j}^{(t)} + \sum_{j=1}^{i-1}\big(R'_j + 2(i-j)P'_j\big)\omega_{2i-2j}^{(n)}\Big) \wedge \frac{dx}{x} +$$
$$\sum_{i=1}^{\frac{n}{2}-k-1} x^{2i}\Big(-4i(k+i-\frac{n}{2})\omega_{2i}^{(t)} - (-1)^k d\omega_{2i}^{(n)} + \sum_{j=1}^{i}\big(R_j + 2(i-j)P_j\big)\omega_{2i-2j}^{(t)} + \sum_{j=1}^{i-1} S_j\omega_{2i-2j}^{(n)}\Big) = 0$$

This determines uniquely the $\omega_i^{(*)}$.

Let us write LOT for lower order term operators on M. Then we get

$$\omega_2^{(n)} = \frac{(-1)^{k+1}}{2k-n}\delta_0\omega_0, \quad \omega_2^{(t)} = \Big(\frac{d\delta_0}{2(2k-n)} + \frac{\delta_0 d}{2(2k+2-n)}\Big)\omega_0$$

and given the order in ∂_{y_i} of the $R_i, R'_i, \bar{R}_i, \bar{R}'_i, Q_i$ and Q'_i, we have

$$\omega_{2i}^{(t)} = \frac{1}{2i(2k+2i-n)}\Big(2(-1)^{k+1}d\omega_{2i}^{(n)} + \Delta_0\omega_{2i-2}^{(t)}\Big) + \mathrm{LOT}(\omega_0)$$
$$\omega_{2i+2}^{(n)} = \frac{1}{2(i+1)(2k+2i-n)}\Big(2(-1)^{k+1}\delta_0\omega_{2i}^{(t)} + \Delta_0\omega_{2i}^{(n)}\Big) + \mathrm{LOT}(\omega_0)$$

So we have

$$\omega_{2i}^{(t)} = \Big(a_{2i}(\delta_0 d)^i + b_{2i}(d\delta_0)^i + \mathrm{LOT}\Big)\omega_0$$
$$\omega_{2i+2}^{(n)} = \Big(a_{2i+1}(\delta_0 d)^l\delta_0 + \mathrm{LOT}\Big)\omega_0$$

where the sequences (a_i) and (b_{2i}) satisfy the relations

$$a_{2i} = \frac{a_{2i-2}}{2i(2k+2i-n)}, \qquad a_{2i+1} = \frac{2(-1)^{k+1} b_{2i}}{2(i+1)(2k+2i-n)} + \frac{a_{2i-1}}{2(i+1)(2k+2i-n)}$$
$$b_{2i} = \frac{2(-1)^{k+1} a_{2i-1}}{2i(2k+2i-n)} + \frac{b_{2i-2}}{2i(2k+2i-n)}$$

and $a_1 = \frac{(-1)^{k+1}}{2k-n}$, $a_2 = \frac{1}{2(2k+2-n)}$, $b_2 = \frac{1}{2(2k-n)}$. By uniqueness of the solution of this equation we find

$$a_{2i} = \frac{1}{2^i i! \prod_{j=1}^{i} (2k+2j-n)}, \qquad a_{2i+1} = \frac{(-1)^{k+1}}{2^i i! \prod_{j=0}^{i} (2k+2j-n)},$$
$$b_{2i} = \frac{1}{2^i i! \prod_{j=0}^{i-1} (2k+2j-n)}$$

for all $i \le \frac{n}{2} - k - 1$. We infer the equality

(4.14)
$$\begin{aligned}
\Delta_k \omega_{F_1} =& x^{n-2k} \Big(a_{n-2k-2} (\delta_0 d)^{\frac{n}{2}-k} + (b_{n-2k-2} + 2(-1)^{k+1} a_{n-2k-1})(d\delta_0)^{\frac{n}{2}-k} + \mathrm{LOT} \Big) \omega_0 \\
&+ x^{2k-n+2} \Big(a_{n-2k-1} (\delta_0 d)^{\frac{n}{2}-k} \delta_0 + \mathrm{LOT} \Big) \omega_0 \wedge \frac{dx}{x} + o(x^{n-2k+1}) \\
=& x^{n-2k} \Big(\frac{(\delta_0 d)^{\frac{n}{2}-k}}{2^{\frac{n}{2}-k-1} (\frac{n}{2}-k-1)! \prod_{j=1}^{\frac{n}{2}-k-1} (2k+2j-n)} + \mathrm{LOT} \Big) \omega_0 \\
&+ x^{2k-n+2} \Big(\frac{(-1)^{k+1} (\delta_0 d)^{\frac{n}{2}-k} \delta_0}{2^{\frac{n}{2}-k-1} (\frac{n}{2}-k-1)! \prod_{j=0}^{\frac{n}{2}-k-1} (2k+2j-n)} + \mathrm{LOT} \Big) \omega_0 \wedge \frac{dx}{x} \\
&+ o(x^{n-2k+1})
\end{aligned}$$

so we have

$$L_k = \frac{-(\delta_0 d)^{\frac{n}{2}-k}}{2^{\frac{n}{2}-k-1} (\frac{n}{2}-k-1)! \prod_{j=0}^{\frac{n}{2}-k-1} (2k+2j-n)} + \mathrm{LOT}$$
$$B_k = \frac{(\delta_0 d)^{\frac{n}{2}-k} \delta_0}{2^{\frac{n}{2}-k-1} (\frac{n}{2}-k-1)! \prod_{j=0}^{\frac{n}{2}-k-1} (2k+2j-n)} + \mathrm{LOT}$$

Note also that δ_g is of order 1 so C_k has no contribution to the principal part of G_k and we get

$$G_k = \frac{(-1)^{k+1} (\delta_0 d)^{\frac{n}{2}-k} \delta_0}{2^{\frac{n}{2}-k} (\frac{n}{2}-k)! \prod_{j=0}^{\frac{n}{2}-k-1} (2k+2j-n)} + \mathrm{LOT}.$$

The proof is the same (and even easier) for Q_k. We could have deduced the principal parts of L_k and G_k from the one of Q_k, but a slight generalization of the proof above will allow to compute the principal part of the non-critical L_k^l in the next section. $\square$

We finally prove that the operators L_k and Q_k are symmetric on $C^\infty(M, \Lambda(M))$:

Proposition 4.16. *For $k \le \frac{n}{2} - 1$, the operators L_k are symmetric on $C^\infty(M, \Lambda^k(M))$ while for $k < \frac{n}{2} - 1$, the operators Q_k are symmetric on $C^\infty(M, \Lambda^k(M)) \cap \ker d$.*

Proof: The proof for L_k is done in Proposition 5.4 which covers the non-critical cases. The proof for Q_k is quite similar, we let ω_0, ω_0' be two closed k-forms on M and ω, ω' the forms constructed in the proof of Proposition 3.11 with respective initial conditions ω_0

and ω_0'. Then integration by part and the fact that $d\omega = d\omega' = 0$ gives

$$0 = \int_{x\geq\epsilon} (\langle \Delta_k\omega, \omega'\rangle_g - \langle \Delta_k\omega', \omega\rangle_g)\mathrm{dvol}_g =$$

$$\int_{x=\epsilon} \Big(\langle i_{x\partial_x}\omega, \delta_g\omega'\rangle_{h_x} - \langle i_{x\partial_x}\omega', \delta_g\omega\rangle_{h_x}\Big)x^{-n}\mathrm{dvol}_{h_x}.$$

But a straightforward analysis and the fact that $L_k(\omega_0) = L_k(\omega_k') = 0$ give that the second line has an expansion of the form

$$a_{-2\ell}\epsilon^{-2\ell} + \cdots + a_{-2}\epsilon^{-2} + L\log(\epsilon) + O(1)$$

$$\text{with } L := (-1)^{k+1}(2k-n)\Big(\langle Q_k\omega_0, \omega_0'\rangle_{L^2(\mathrm{dvol}_{h_0})} - \langle \omega_0, Q_k\omega_0'\rangle_{L^2(\mathrm{dvol}_{h_0})}\Big)$$

This achieves the proof. □

4.6. **Branson Q-curvature.** We conclude this section by the observation that Q_0 is the Q-curvature of Branson.

Proposition 4.17. *The operator Q_0 of Definition 4.2 satisfies*

$$Q_0 1 = \frac{n(-1)^{\frac{n}{2}+1}}{2^{n-1}\frac{n}{2}!(\frac{n}{2}-1)!}Q$$

where Q is Branson Q-curvature defined in [2].

Proof: Since the operator Q_0 and the function Q are local on $(M, [h_0])$ and do not depend on the chosen Poincaré-Einstein manifold with conformal infinity $(M, [h_0])$, it suffices to consider the cylinder $X = (-1, 1) \times M$ equipped with a Poincaré-Einstein metric with conformal metric $[h_0]$ on the boundary $M \sqcup M$. In [5], Fefferman and Graham showed that the Q-curvature of Branson is the function Q on M such that if $U \in C^\infty(X)$ is a solution of

$$\begin{cases} \Delta_g U = n \\ U = \log(x) + A + x^n B\log(x) \text{ with } A, B \in C^\infty(\bar{X}) \\ A|_{x=0} = 0 \end{cases}$$

then $B|_{x=0} = (-1)^{\frac{n}{2}+1}(2^{n-1}\frac{n}{2}!(\frac{n}{2}-1)!)^{-1}Q$. Consider dU, clearly it is a harmonic 1-form and it is given by

$$dU = \frac{dx}{x} + dA + nx^n B\log(x)\frac{dx}{x} + O(x^n)$$

and by uniqueness of the solution in Proposition 3.11 and the decay of L^2 harmonic 1-forms (of order x^n), we deduce that $Q_0 1 = nB|_{x=0}$, this proves the claim (note that the log term in the development of Δ_k does not interfer since it acts trivially on normal zero forms). □

5. The non-critical case

Let (X, g) be a Poincaré-Einstein manifold with conformal infinity $(M, [h_0])$. We assume $k \leq (n+1)/2$ and n may be odd or even in this section, and we let ℓ be an integer in $[1, \frac{n}{2} - k]$ in general, and $\ell \in \mathbb{N}$ if n is odd and (X, g) is an even Poincaré-Einstein manifold. We want to construct the operators L_k^ℓ of [3] by solving the following equation

$$\begin{gathered}\Big(\Delta_k - (\frac{n}{2} - k + \ell)(\frac{n}{2} - k - \ell)\Big)\omega = O_t(x^{\frac{n}{2}-k+\ell}) + O_n(x^{\frac{n}{2}-k+\ell+1}) \\ \text{with } \omega = x^{\frac{n}{2}-k-\ell}\omega_0 + o(x^{\frac{n}{2}-k-\ell}) \text{ as } x \to 0.\end{gathered} \tag{5.1}$$

where O_n, O_t are defined in the proof of Proposition 3.1 and where $\omega_0 \in C^\infty(M, \Lambda^k(M))$. This can be done essentially like in the critical case, using the indicial equations of Subsection 2.3. Indeed, the indicial roots of $\Delta_k - (\frac{n}{2} - k + \ell)(\frac{n}{2} - k - \ell)$ can be computed rather easily, these are

$$\frac{n}{2} - k \pm \ell \quad \text{in the } \Lambda_t^k \text{ component}$$
$$\frac{n}{2} - k + 1 \pm \sqrt{\ell^2 + n + 1 - 2k} \quad \text{in the } \Lambda_n^k \text{ component.}$$

Notice that there are no indicial roots in $(\frac{n}{2} - k - \ell, \frac{n}{2} - k + \ell)$ when $\ell \leq n/2 - k$, but there is one root in this interval when $\ell > n/2 - k$ and n odd, it is given by

$$\frac{n}{2} - k + 1 - \sqrt{\ell^2 + n + 1 - 2k} \in (\frac{n}{2} - k - \ell, \frac{n}{2} - k - \ell + 1] \tag{5.2}$$

and thus is not in $n/2 - k - \ell + 2\mathbb{N}_0$. We obtain

Lemma 5.1. *For $\omega_0 \in C^\infty(M, \Lambda^k(M))$ fixed, there exists a series*

$$\omega_{F_1} = x^{\frac{n}{2}-k-\ell}\Big(\sum_{2j=0}^{2l-2} x^{2j}\omega_{2j}^{(t)} + \sum_{2j=2}^{2l} x^{2j}(\omega_{2j}^{(n)} \wedge \frac{dx}{x})\Big) \tag{5.3}$$

such that $\omega_0^{(t)} = \omega_0$ and

$$\Big(\Delta_k - (\frac{n}{2} - k + \ell)(\frac{n}{2} - k - \ell)\Big)\omega_{F_1} = O_t(x^{\frac{n}{2}-k+\ell}) + O_n(x^{\frac{n}{2}-k+\ell+2}) \tag{5.4}$$

where the forms $\omega_j^{(.)}$ on M are uniquely determined by ω_0 and the expansion of Δ_k in powers of x given by Lemma 2.1.

Note that the condition $\ell \leq \frac{n}{2}$ when n is even insures that that the first $\log(x)$ coefficient coming from the metric does not show up in (5.1). Remark that when $\ell > n/2 - k$ and n odd, the fact that the indicial root (5.2) is not in $n/2 - k - \ell + 2\mathbb{N}_0$ does not affect the construction. Since $(\frac{n}{2} - k + \ell)$ is an indicial root in the Λ_t^k component, we can then define

$$\begin{gathered}\omega_{F_2} = \omega_{F_1} + x^{\frac{n}{2}-k+\ell}\log(x)\omega^{(t)}_{n-k+\ell,1}, \\ \text{with } \omega^{(t)}_{n-k-\ell,1} = \frac{1}{2\ell}\Big[x^{-\frac{n}{2}+k-\ell}\Big(\Delta_k - (\frac{n}{2} - k + \ell)(\frac{n}{2} - k - \ell)\Big)\omega_{F_1}\Big]_{|x=0}\end{gathered} \tag{5.5}$$

which satisfies

$$\Big(\Delta_k - (\frac{n}{2} - k + \ell)(\frac{n}{2} - k - \ell)\Big)\omega_{F_2} = O(x^{\frac{n}{2}-k+\ell+2}\log x). \tag{5.6}$$

Remark: we could continue the construction to get a solution ω of

$$(\Delta_k - (\frac{n}{2} - k - \ell)(\frac{n}{2} - k + \ell))\omega = O(x^\infty)$$

and even an exact solution (with no $O(x^\infty)$) using the resolvent of Δ_k. However, since the mapping properties of $(\Delta_k - (\frac{n}{2} - k - \ell)(\frac{n}{2} - k + l))^{-1}$ is not explicitly available in the literature when $\ell \neq \frac{n}{2} - k$, we do not discuss this case further.

Like we did for L_k, we can then define an operator on M as follows:

Definition 5.2. *For $k \leq (n+1)/2$, we let ℓ be an integer in $[1, \frac{n}{2} - k]$ if n is even and in $\mathbb{N}$ if n is odd. The operator $L_k^\ell : C^\infty(M, \Lambda^k(M)) \to C^\infty(M, \Lambda^k(M))$ is defined by $L_k^\ell\omega_0 := \omega^{(t)}_{n-k-\ell,1}$ where $\omega^{(t)}_{n-k-\ell,1}$ is given in (5.5).*

Remark: clearly, we have $L_k^{\frac{n}{2}-k} = L_k$ when n is even.

Lemma 5.3. *The form ω_{F_1} of (5.3) satisfies $\delta_g\omega_{F_1} = O(x^{\frac{n}{2}-k+\ell+2})$.*

Proof: by (5.4) and $\delta_g\Delta_k = \Delta_{k-1}\delta_g$, the form $\delta_g\omega_{F_1}$ solves

$$\Big(\Delta_{k-1} - (\frac{n}{2} - k + \ell)(\frac{n}{2} - k - \ell)\Big)\delta_g\omega_{F_1} = O(x^{\frac{n}{2}-k+\ell+2}) \tag{5.7}$$

and with $\delta_g\omega_{F_1}=O(x^{\frac{n}{2}-k-\ell+2})$. The Taylor series T of $x^{-\frac{n}{2}+k+\ell}\delta_g\omega_{F_1}$ to order $O(x^{2\ell+2})$ is such that $x^{\frac{n}{2}-k-\ell}T$ solves (5.7), and moreover T is even by Lemma 2.1. A short computation shows that the indicial roots of $(\Delta_{k-1}-(\frac{n}{2}-k+\ell)(\frac{n}{2}-k-\ell))$ are

$$\begin{aligned}&\frac{n}{2}-k+1\pm\sqrt{\ell^2+n+1-2k}\quad\text{in } \Lambda^{k-1}_t \text{ component },\\ &\frac{n}{2}-k+2\pm\sqrt{\ell^2+2(n-2k+2)}\quad\text{in } \Lambda^{k-1}_n \text{ component }.\end{aligned}$$

Thus if $\ell\leq n/2-k$, the indicial roots are not contained in $[\frac{n}{2}-k-\ell+2,\frac{n}{2}-k+\ell+2]$ except when $2k=n+1$ where $\frac{n}{2}-k+\ell+1$ is a root in the Λ_t component, this implies that the Taylor series of $\delta_g\omega_{F_1}$ vanishes to order $O(x^{\frac{n}{2}-k+\ell+1})$ except maybe when $2k=n+1$. However in the last case, by parity of T, we see that there is no $\frac{n}{2}-k+\ell+1$ term in the expansion of $\delta_g\omega_{F_1}$, this ends the proof for n even. When n is odd and $\ell>n/2-k$, the only indicial root in the interval of interest is $n/2-k+1+\sqrt{\ell^2+n+1-2k}$ and it is in $(\frac{n}{2}-k+\ell+1,\frac{n}{2}-k+\ell+2)$ thus not in $n/2-k-\ell+\mathbb{N}_0$, which shows that the argument used for $\ell\leq n/2-k$ applies similarly. □

By an obvious integration by part, we have the

Proposition 5.4. *The operators L^ℓ_k are symmetric on $C^\infty(M,\Lambda^k(M))$.*

Proof: Consider $\omega^1_{F_2}$ and $\omega^2_{F_2}$ like in (5.5) with respective boundary values ω^1_0 and ω^2_0, they are well defined form in some collar neighbourhood $X_1:=(0,\epsilon_0)_x\times M$ of M in $\bar{X}$. Let $\varphi\in C_0^\infty((-\epsilon_0,\epsilon_0))$ be a cut-off function which equals 1 near 0 and $\widetilde{\omega}^i:=\varphi(x)\omega^i_{F_2}$ for $i=1,2$. Then using Lemma 5.3 we have $\delta_g\widetilde{\omega}^i=O(x^{\frac{n}{2}-k+\ell+1})$, but since $i_{x\partial_x}\widetilde{\omega}^i=O(x^{\frac{n}{2}-k-\ell+2})$, the Green formula gives for small $\epsilon>0$

$$\int_{x\geq\epsilon}(\langle\Delta_k\widetilde{\omega}^1,\widetilde{\omega}^2\rangle_g-\langle\Delta_k\widetilde{\omega}^2,\widetilde{\omega}^1\rangle_g)\mathrm{dvol}_g=(-1)^n\int_{x=\epsilon}(\star_g d\widetilde{\omega}^1)\wedge\widetilde{\omega}^2-(\star_g d\widetilde{\omega}^2)\wedge\widetilde{\omega}^1+O(\epsilon).$$

But the first line is a $O(1)$ as $\epsilon\to0$ by (5.6), and a straightforward analysis gives that the second line has an expansion of the form

$$\begin{aligned}&a_{-2\ell-1}\epsilon^{-2\ell-1}+\dots+a_{-1}\epsilon^{-1}+L\log(\epsilon)+O(1)\\ &\text{with } L:=(-1)^n(\frac{n}{2}-k+\ell)\int_M(\star_0L^\ell_k\omega^1_0)\wedge\omega^2_0-(\star_0L^\ell_k\omega^2_0)\wedge\omega^1_0\end{aligned}$$

and this implies $L=0$ by comparing the $\log(\epsilon)$ terms. □

Proposition 5.5. *We have $L^l_k=\frac{(-1)^{l+1}l}{2^{2l-1}(l!)^2}\Big[(\delta_0d)^l+\frac{n-2k-2l}{n-2k+2l}(d\delta_0)^l\Big]+\mathrm{LOT}$.*

Proof: We define T by $\omega_{F_1}=x^{\frac{n}{2}-k-l}T$, $\lambda=(\frac{n}{2}-k+l)(\frac{n}{2}-k-l)$ and $P=x^{k+l-\frac{n}{2}}(\Delta-\lambda)x^{\frac{n}{2}-k-l}$.

Then we have $T=\sum_{i=0}^{l-1}x^{2i}\omega^{(t)}_{2i}+\sum_{i=1}^{l}x^{2i}\omega^{(n)}_{2i}\wedge\frac{dx}{x}$ and P admits the same decomposition as Δ_k in Lemma 2.1 but with indicial operator equal to

$$\begin{pmatrix}2lx\partial_x-(x\partial_x)^2 & 2(-1)^{k+1}d\\ 0 & -(x\partial_x)^2+2(l+1)x\partial_x+n-2k-2l\end{pmatrix}$$

The equation $PT=O_t(x^{2l})+O_n(x^{2l+1})$ gives then

$$\omega^{(n)}_{2i+2}=\big(a_{2i+1}(\delta_0d)^i\delta_0+\mathrm{LOT}\big)\omega_0\qquad\omega^{(t)}_{2i}=\big(a_{2i}(\delta_0d)^i+b_{2i}(d\delta_0)^i+\mathrm{LOT}\big)\omega_0$$

with

$$a_1=\frac{(-1)^k}{\frac{n}{2}-k+l},\qquad a_2=\frac{-1}{4(l-1)},\qquad b_2=\frac{-(\frac{n}{2}-k+l-2)}{4(l-1)(\frac{n}{2}-k+l)}$$

and

$$a_{2i+2} = \frac{a_{2i}}{4(i+1)(i+1-l)}, \qquad a_{2i+1} = \frac{2(-1)^{k+1}b_{2i} + a_{2i-1}}{4(i+1)(i-l) + 2k - n + 2l},$$

$$b_{2i+2} = \frac{b_{2i} + 2(-1)^{k+1}a_{2i+1}}{4(i+1)(i+1-l)}.$$

The solutions of these equations are

$$b_{2i} = \frac{(-1)^i(n-2k+2l-4i)(l-i-1)!}{4^i i!(l-1)!(n-2k+2l)} \qquad a_{2i+1} = \frac{(-1)^{k+i}(l-i-1)!}{2^{2i-1}i!(l-1)!(n-2k+2l)}$$

$$a_{2i} = \frac{(-1)^i(l-i-1)!}{4^i i!(l-1)!}$$

Since the equation (5.5) reads

$$L_k^l = \Big[\frac{x^{-2l}}{2l} i_{x\partial_x}\big(\frac{dx}{x} \wedge PT\big)\Big]_{|x=0}$$

we get the result. □

6. Relation with Branson-Gover operators

First we recall a few fact on the ambient metric of Fefferman-Graham, see [4, 6] for details. If $(M, [h_0])$ is a compact manifold equipped with a conformal class, we call

$$\mathcal{Q} = \{t^2 h_0(m); t > 0, m \in M\} \subset S^2T^*M$$

the conformal bundle, it is identified with $(0,\infty)_t \times M$. Let $\widetilde{\mathcal{Q}} = (-1,1) \times \mathcal{Q}$ be the ambient space with the inclusion $\iota : \mathcal{Q} \to \widetilde{Q}$ defined by $z \to (0, z)$. There are dilations $\delta_s : (t, m) \to (st, m)$ of $\mathcal{Q}$ which extends naturally to $\widetilde{\mathcal{Q}}$. The functions on $\mathcal{Q}$ which are w-homogeneous in the sense

$$f(st, m) = s^w f(t, m)$$

are the section of a bundle denoted $E[w]$, they extend naturally on $\widetilde{\mathcal{Q}}$. We denote by $\widetilde{h}$ the ambient metric of Fefferman-Graham [4] on $\widetilde{\mathcal{Q}}$. This is a smooth Lorentzian metric on $\mathcal{Q}$ such that

(1) $\delta_s^*\widetilde{h} = s^2\widetilde{h}, \forall s > 0,$

(2) $\iota^*\widetilde{h}$ is the tautological tensor t^2h_0 on $\mathcal{Q}$,

(3*) $\mathrm{Ric}(\widetilde{h})$ vanishes to infinite order at $\mathcal{Q}$ if n is odd,

(3**) $\mathrm{Ric}(\widetilde{h})$ vanishes to order $\frac{n}{2} - 1$ at $\mathcal{Q}$ if n is even.

We let T be the vector field which generates the dilations δ_s, and let

$$Q = \widetilde{h}(T, T), \quad \rho := -t^{-2}Q/2, \quad x = \sqrt{2\rho}, \quad u = xt$$

so that Q is homegeneous of degree 2 with respect to δ_s, u and t are homogeneous of degree 1 and x of degree 0, moreover Q, ρ are smooth defining function of $\mathcal{Q}$, x, u are defining function of $\mathcal{Q}$ in $\{Q \le 0\}$ for some finer smooth structure on $\{Q \le 0\}$. Let us define $\mathcal{C} := \{Q = -1, \rho < \epsilon\}$ for some small fixed ϵ, then $\mathcal{C}$ can be identified with a collar $(0,\epsilon)_\rho \times M$ and there is a system of coordinates $(u, m) \in (0, 1] \times \mathcal{C}$ that covers the part $\{0 > Q \le -1, \epsilon > \rho > 0\}$ which is a neighbourhood of the cone $\mathcal{Q}$ near $t = \infty$. The metric $\widetilde{h}$ has the model form (see [4]) in this neighbourhood

$$\widetilde{h} = -du^2 + u^2 g$$

where $g = (dx^2 + h_x)/x^2$ is a Poincaré-Einstein metric on the collar $\mathcal{C}$.

The space $\mathcal{T}^k[s]$ is the space of k-form tractors which are homogeneous of degree s, i.e. these are restrictions to the null cone $\mathcal{Q}$ of k-forms on $\widetilde{\mathcal{Q}}$ and such that $\widetilde{\nabla}_T F = sF$ where $T = t\partial_t = u\partial_u$ is the generator of dilations in the cone fibers, $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{\mathcal{Q}}$. Since $\widetilde{\nabla}_T * = *$ for $* = T, \partial_x, \partial_{m_i}$, we have $\mathcal{L}_T = \widetilde{\nabla}_T + k$, on $\mathcal{T}^k[s]$, where $\mathcal{L}$ denotes Lie derivative. The bundle $\mathcal{E}^k[s]$ is the bundle which consists of the s-homogeneous k forms on M, in the sense that they are the sections of $\Lambda^k T^*M \otimes E[s]$ and thus satisfy $\mathcal{L}_T\omega = s\omega$. We can view $\mathcal{E}^k[s]$ as a subspace of $\mathcal{T}^k[s-k]$. We let $\mathcal{G}_k[s]$ be the subundle of $\mathcal{T}^k[s+k-n]$ consisting of forms which are annihilated by the interior product i_T. It has a conformally invariant projection onto $\mathcal{E}^k[s+2k-n]$ denoted by q^k, this is given for instance by $i_{\partial_\rho} d\rho\wedge$.

If $\widetilde{\Delta}$ is the ambient Laplacian on $\widetilde{\mathcal{Q}}$ associated to $\widetilde{h}$, if $\omega_0 \in \mathcal{E}^k[k+\ell-\frac{n}{2}]$ and $\widetilde{\omega}_0$ is an homogeneous extension of ω_0 to $\widetilde{\mathcal{Q}}$, then it is proved in [3, Prop. 4.3] that the operator defined by the formula

$$\mathbf{L}_k^\ell\omega_0 = \Big[\iota_T\Big(\widetilde{d}(n+2\widetilde{\nabla}_T-2)+\frac{1}{2}\widetilde{d}Q\wedge\widetilde{\Delta}\Big)\widetilde{\Delta}^\ell\widetilde{\omega}_0\Big]_{|\mathcal{Q}} = \Big[\iota_T\widetilde{d}(n+2\widetilde{\nabla}_T-2)\widetilde{\Delta}^\ell\widetilde{\omega}_0\Big]_{|\mathcal{Q}} \tag{6.1}$$

can be viewed as a conformally invariant operator $\mathbf{L}_k^\ell : \mathcal{E}^k[k+\ell-\frac{n}{2}] \to \mathcal{G}_k[\frac{n}{2}-k-\ell]$. Here $\widetilde{d}$ denotes the exterior differential on $\widetilde{\mathcal{Q}}$. They also define the operators (see Proposition 4.4 and Theorem 4.5 in [3])

$$L_k^{\mathrm{BG},\ell} := q^k\mathbf{L}_k^\ell : \mathcal{E}^k[k+\ell-\frac{n}{2}] \to \mathcal{E}^k[k-\frac{n}{2}-\ell],$$

$$G_k^{\mathrm{BG}} := q^{k-1}i_Y\mathbf{L}_k^{\frac{n}{2}-k} : \mathcal{E}^k[0] \to \mathcal{E}^{k-1}[2k-2-n] \tag{6.2}$$

where $Y = -\frac{\partial_\rho}{t^2}$ is a vector field dual to $\widetilde{dt}/t$ via $\widetilde{h}$, it satisfies in particular $\widetilde{d}Q(Y) = 2$. Finally the operator Q_k^{BG} acting on a closed k-form ω_0 is defined as follows

$$Q_k^{\mathrm{BG}}\omega_0 := -2(\frac{n}{2}-k+1)q^k\Big[i_Y i_T\widetilde{\Delta}^{\frac{n}{2}-k}(\frac{dQ}{2}\wedge\frac{\widetilde{dt}}{t}\wedge\widetilde{\omega}_0)\Big]|_{\mathcal{Q}}.$$

where $\widetilde{\omega}_0$ is any homogeneous extension of ω_0 to $\widetilde{\mathcal{Q}}$.

We now prove a Lemma whose proof is essentially the same as the for functions in [11].

Lemma 6.1. *Let $\omega \in \mathcal{T}^{k'}[-\alpha]$ and $j\in\mathbb{N}$, then we have*

$$\widetilde{\Delta}(Q^j\omega) = 4j(\alpha-\frac{n}{2}-j)Q^{j-1}\omega + Q^j\widetilde{\Delta}\omega.$$

Proof: Using $\widetilde{\nabla}Q = 2T$, we have $[\widetilde{\Delta}, Q] = -2(2\widetilde{\nabla}_T+n+2)$ and so we can compute

$$\begin{aligned}\widetilde{\Delta}(Q^jL\omega_0) &= \sum_{m=0}^{j-1} Q^m[\widetilde{\Delta},Q]Q^{j-1-m}\omega + Q^j\widetilde{\Delta}\omega\\ &= -2Q^{j-1}\sum_{m=0}^{j-1}\big(2(-2m+2j-2-\alpha)+n+2\big)\omega + Q^j\widetilde{\Delta}\omega\\ &= 4Q^{j-1}j(\alpha-\frac{n}{2}-j)\omega + Q^j\widetilde{\Delta}\omega\end{aligned}$$

which achieves the proof. □

As a consequence, and using Lemma 6.1, we get the

Theorem 6.2. ***(i)*** *Let L_k^ℓ, L_k and G_k be the operators of Definition 5.2 and 4.1, and let $c_k^\ell := (-4)^\ell(\ell-1)!(\ell+1)!(k-\frac{n}{2}-\ell)$. Then the following identity holds*

$$L_k^{\mathrm{BG},\ell} = c_k^\ell L_k^\ell.$$

In the critical case $\ell = \frac{n}{2}-k$, if G_k^{BG} is the Branson-Gover operator of (6.2) *we have*

$$L_k^{\mathrm{BG}} = c_kL_k, \quad G_k^{\mathrm{BG}} = (-1)^k c_k G_k$$

with $c_k := (-1)^{\frac{n}{2}-k-1}2^{n-2k+1}((\frac{n}{2}-k)!)^2(\frac{n}{2}-k+1) = c_k^{\frac{n}{2}-k}$.
(ii) *Let* Q_k *be the operator of Definition 4.2, then*

$$Q_k^{\mathrm{BG}} = (2k-n-2)c_{k+1}Q_k.$$

Proof: (i) For $\omega_0 \in \Lambda^k(M)$, we consider the form ω_{F_1} of Lemma 5.1 of the previous section and we extend it homogeneously in a smooth k-form of degree $k-\frac{n}{2}+\ell$ by

$$\begin{aligned}\widetilde{\omega}_F = u^{k-\frac{n}{2}+\ell}\omega_{F_1} &= u^{k-\frac{n}{2}+\ell}x^{\frac{n}{2}-k-\ell}\sum_{i=0}^{\ell} x^{2i}\big(\omega_i^{(t)} + x^2\omega_i^{(n)} \wedge \frac{dx}{x}\big)\\ &= t^{k-\frac{n}{2}+\ell}\sum_{i=0}^{\ell}(-Q)^i t^{-2i}\big(\omega_i^{(t)} + \omega_i^{(n)} \wedge d\rho\big).\end{aligned}$$

In the coordinates u, x, y representing a neighbourhood $\{-1 \le Q < 0, \rho < \epsilon\}$ and in the k-form bundle decomposition $\Lambda^k(\mathcal{C}) \oplus \Lambda^{k-1}(\mathcal{C}) \wedge \frac{du}{u}$, the exterior derivative, its dual and the form Laplacian of $\widetilde{h}$ are given by

$$\widetilde{d} = \begin{pmatrix} d & 0\\ (-1)^k u\partial_u & d\end{pmatrix}, \quad \widetilde{\delta} = u^{-2}\begin{pmatrix}\delta_g & (-1)^{k+1}(n+2-2k+u\partial_u)\\ 0 & \delta_g\end{pmatrix} \tag{6.3}$$

and

$$\widetilde{\Delta} = u^{-2}\begin{pmatrix}(u\partial_u)(u\partial_u+n-2k)+\Delta_k & 2(-1)^{k+1}d\\ 2(-1)^k\delta_g & (u\partial_u-2)(u\partial_u+n-2k+2)+\Delta_{k-1}\end{pmatrix}. \tag{6.4}$$

So, using the properties of ω_{F_1} in Lemma 5.1 and Lemma 5.3, we have (where $s = k-\frac{n}{2}+\ell$)

$$\begin{aligned}\widetilde{\Delta}\widetilde{\omega}_F &= u^{s-2}\big(\Delta_k + s(s+n-2k)\big)\omega_{F_1} + 2(-1)^k u^{s-3}\delta_g\omega_{F_1} \wedge du\\ &= 2\ell u^{s-2}x^{\ell-k+\frac{n}{2}}\left(L_k^\ell\omega_0 + O_t(x^2)\right) + u^{s-2}x^{\ell-k+\frac{n}{2}+2}\big(B \wedge \frac{dx}{x} + O_n(x^2)\big)\\ &\quad + 2(-1)^k u^{s-3}x^{\ell-k+\frac{n}{2}+2}\big(C + O(x^2)\big) \wedge du\\ &= (-Q)^{\ell-1}t^{k-\ell-\frac{n}{2}}\Big(2\ell L_k^\ell\omega_0 + (B+2(-1)^kC)\wedge d\rho\Big) + O(Q^\ell)\end{aligned}$$

for some $(k-1)$-forms B, C on M. We can now apply $\ell-1$ times Lemma 6.1 and get

$$\widetilde{\Delta}^\ell\widetilde{\omega}_F = \widetilde{\Delta}^{\ell-1}\widetilde{\Delta}\omega_F = (-4)^{\ell-1}[(\ell-1)!]^2t^{k-\ell-\frac{n}{2}}\Big(2\ell L_k^\ell\omega_0 + (B+2(-1)^kC)\wedge d\rho\Big) + O(Q).$$

Since $(n+2\widetilde{\nabla}_T-2)$ acts on homogeneous k-forms of degree $k-\ell-\frac{n}{2}$ by multiplication by $-2(\ell+1)$ and $i_T\widetilde{d} = \mathcal{L}_T$ on $\mathcal{G}_k[\frac{n}{2}-k-\ell]$, we get

$$\mathbf{L}_k^\ell = (\ell+1)(k-\ell-\frac{n}{2})(-4)^\ell[(\ell-1)!]^2t^{k-\ell-\frac{n}{2}}\Big(\ell L_k^\ell\omega_0 + (\frac{B}{2}+(-1)^kC)\wedge d\rho\Big)$$

Note that by definition of B_k, C_k, G_k we have, in the case $\ell = \frac{n}{2}-k$,

$$\frac{B}{2} + (-1)^kC = (-1)^{k-1}\big(\frac{B_k}{2} - C_k\big)\omega_0 = \ell G_k\omega_0.$$

(ii) Similarly, for $\omega_0 \in \Lambda^k(M)$ closed, we set $\widetilde{\omega}_F := \omega'_{F_1} \wedge \frac{1}{2}\widetilde{d}Q$ in $\{Q < 0, \rho \le \epsilon\}$ where the form ω'_{F_1} is the 0-homogeneous expansion of $\omega'_{F_1} \in \Lambda^{k+1}(\mathcal{C})$ given by (3.12). Since $\frac{\widetilde{d}Q}{2} = -t^2d\rho + Q\frac{dt}{t}$, we have

$$\begin{aligned}\widetilde{\omega}_F &= \sum_{j=0}^{\frac{n}{2}-k} x^{2j}\big(\omega_{2j}^{(n)} \wedge \frac{dx}{x} + x^2\omega_{2j}^{(t)}\big) \wedge \frac{\widetilde{d}Q}{2}\\ &= \sum_{j=0}^{\frac{n}{2}-k} -(-Q)^j t^{-2(j-1)}\omega_{2j}^{(n)} \wedge d\rho \wedge \frac{dt}{t} + (-Q)^{j+1}t^{-2j-2}\omega_{2j}^{(t)} \wedge (-t^2d\rho + Q\frac{dt}{t})\end{aligned}$$

and so $\widetilde{\omega}_F$ is a smooth $(k+2)$ form. By (6.4) and the definition of B'_k, D'_k we have

$$\begin{aligned}
\widetilde{\Delta}\widetilde{\omega}_F =& \widetilde{\Delta}(-u^2\omega'_{F_1}\wedge\frac{du}{u}) = 2(-1)^k d\omega'_{F_1} - (\Delta_{k+1}\omega'_{F_1})\wedge\frac{du}{u}\\
=& -x^{n-2k-2}2D'_k\omega_0\wedge\frac{dx}{x} + (-1)^{k+1}x^{n-2k}\big(B'_k\omega_0\wedge\frac{dx}{x}+\omega_1\big)\wedge\frac{du}{u} + O(Q^{\frac{n}{2}-k})\\
=& (-1)^{\frac{n}{2}-k-1}2Q^{\frac{n}{2}-k-2}t^{2k-n+4}D'_k\omega_0\wedge d\rho\\
&+ (-1)^{\frac{n}{2}+1}Q^{\frac{n}{2}-k-1}t^{2k-n+2}\big(B'_k\omega_0\wedge\frac{dt}{t}+(-1)^k\omega_1\big)\wedge d\rho + O(Q^{\frac{n}{2}-k})
\end{aligned}$$

for some form ω_1 on M, the value of which is not important for our purpose. By Lemma 6.1, we have

$$\begin{aligned}
\widetilde{\Delta}^2\widetilde{\omega}_F =& (-1)^{\frac{n}{2}-k-1}2Q^{\frac{n}{2}-k-2}t^{2k-n+4}\widetilde{\Delta}(D'_k\omega_0\wedge d\rho)\\
&+ 4(\frac{n}{2}-k-1)(-1)^{\frac{n}{2}+1}Q^{\frac{n}{2}-k-2}t^{2k-n+2}\big(B'_k\omega_0\wedge\frac{dt}{t}+(-1)^k\omega_1\big)\wedge d\rho + O(Q^{\frac{n}{2}-k-1})
\end{aligned}$$

and by (6.4), we have

$$\begin{aligned}
\widetilde{\Delta}(D'_k\omega_0\wedge d\rho) =& \widetilde{\Delta}(x^2D'_k\omega_0\wedge\frac{dx}{x})\\
=& u^{-2}\Delta_{k+2}(x^2D'_k\omega_0\wedge\frac{dx}{x}) + 2u^{-2}(-1)^k\delta_g(x^2D'_k\omega_0\wedge\frac{dx}{x})\wedge\frac{du}{u}\\
=& 2(-1)^k t^{-2}\delta_0D'_k\omega_0\wedge d\rho\wedge\frac{dt}{t} + 2(2k-n-4)t^{-2}D'_k\omega_0\wedge\frac{dt}{t}
\end{aligned}$$

where we have used (2.2), (2.3) and $dD'_k\omega_0=0$. We thus have

$$\begin{aligned}
\widetilde{\Delta}^2\widetilde{\omega}_F =& Q^{\frac{n}{2}-k-2}t^{2k-n+2}\Big[4(-1)^{\frac{n}{2}-1}\Big(\delta_0D'_k\omega_0-(\frac{n}{2}-k-1)B'_k\omega_0\Big)\wedge d\rho\wedge\frac{dt}{t}+\omega'_1\wedge\frac{dt}{t}\\
&+\omega'_2\wedge d\rho\Big] + O(Q^{\frac{n}{2}-k-1})\\
=& Q^{\frac{n}{2}-k-2}t^{2k-n+2}\Big[4(n-2k)(\frac{n}{2}-k-1)(-1)^{\frac{n}{2}+k}Q_k\omega_0\wedge d\rho\wedge\frac{dt}{t}+\omega'_1\wedge\frac{dt}{t}\\
&+\omega'_2\wedge d\rho\Big] + O(Q^{\frac{n}{2}-k-1})
\end{aligned}$$

by Corollary 4.9, and where ω'_1,ω'_2 are forms in $\Lambda^{k+1}(M)$. By iterative use of Lemma 6.1, we get

$$\begin{aligned}
\widetilde{\Delta}^{\frac{n}{2}-k}\widetilde{\omega}_F =& \widetilde{\Delta}^{\frac{n}{2}-k-2}\widetilde{\Delta}^2\widetilde{\omega}_F\\
=& t^{2k-n+2}\Big[2^{n-2k-1}(\frac{n}{2}-k)[(\frac{n}{2}-k-1)!]^2(-1)^{\frac{n}{2}+k}Q_k\omega_0\wedge d\rho\wedge\frac{dt}{t}\\
&+\omega'_1\wedge\frac{dt}{t}+\omega'_2\wedge d\rho\Big] + O(Q)
\end{aligned}$$

we infer from the definition of $Q_k^{\rm BG}$ that

$$Q_k^{BG} = (-1)^{\frac{n}{2}+k+1}2^{n-2k}(\frac{n}{2}-k+1)!(\frac{n}{2}-k-1)!Q_k$$

$\square$

7. Proof of the main results

We start with the proof of Theorem 1.2.

Proof of Theorem 1.2: the existence of ω in (i) is proved in Proposition 3.1. The fact that the log terms L_k, Q_k coincide with the Branson-Gover operators follows from Theorem 6.2. The uniqueness of the solution is rather clear by construction: using the arguments used in the proof of Proposition 3.1, a solution in $C^{\frac{n}{2}-k,\alpha}(\bar{X},\Lambda^k(\bar{X}))$ would have its first $\frac{n}{2}-k$ Taylor coefficients uniquely (and locally) determined by the boundary value ω_0 and then two such solutions with same boundary data would have a difference in $O_t(x^{\frac{n}{2}-k+\alpha})+O_n(x^{\frac{n}{2}-k+1+\alpha})$ and would then be in $\ker_{L^2}(\Delta_k)$ (see Remark 3.6). We conclude that any $C^{\frac{n}{2}-k,\alpha}$ solution is in fact a solution of Proposition 3.1. The proof of (ii) is similar and follows from Proposition 3.11 and Theorem 6.2. □

Proof of Theorem 1.1: The infinite dimensionality of $K^k_m(\bar{X})$ for $m<n-2k+1$ follows from Proposition 3.1. Indeed for $m<n-2k$ this is clear since the solutions of (3.1) are parametrized by $C^\infty(M,\Lambda^k(M))$. If $m=n-2k$, one can use that there is an infinite set of $\omega_0\in C^\infty(M,\Lambda^k(M))$ such that $G_k\omega_0\neq 0$ and $L_k\omega_0=0$ since $\ker L_k$ is infinite dimensional and $\ker G_k\cap\ker L_k$ is finite dimensional by ellipticity of dG_k+L_k. Solutions of (3.1) are then in $C^{n-2k}(\bar{X},\Lambda^k(\bar{X}))$.

The finite dimensionality for $m=n-2k+1$ is a little more involved. Let ω be a harmonic form in $C^{n-2k+1}(\bar{X},\Lambda^k(\bar{X}))$, then Taylor expanding, there exist some forms $\omega_j^{(n)},\omega_j^{(t)}\in C^{n-2k+1-j}(M,\Lambda(M))$ so that

$$\omega-\sum_{j=0}^{n-2k}x^j(\omega_j^{(t)}+\omega_j^{(n)}\wedge dx)\in x^{n-2k+1}L^\infty(\Lambda^k(\bar{X})),$$

and $L_k\omega_0=0$. Now by Lemma 3.8 we know that ω has a weak expansion to order x^N with values in $H^{-N}(M)$ like in (3.9) for any $N>0$ large. Moreover $\delta_g\omega$ is also a harmonic form in $C^{n-2k}(\bar{X},\Lambda^{k-1}(M))$, moreover using (2.2) after decomposing the form in $\Lambda_t^{k-1}\oplus\Lambda_n^{k-1}$, we see that it is a $O(x)$ and has an expansion to order x^N with values in $H^{-N-1}(M)$ for any N. Now, using the indicial equation like in the proof of Proposition 3.1, the weak expansion of $\delta_g\omega$ vanishes to order x^{n-2k+2}, so in particular we obtain $\delta_g\omega\in x^{n-2k}L^\infty(\Lambda^{k-1}(\bar{X}))$ from the regularity of ω. Then $\delta_g\omega\in L^2(\Lambda^{k-1}(X))$ for $k<\frac{n}{2}-1$, while for $k=\frac{n}{2}-1$ it is in L^2 if we assume in addition that $\omega\in C^{n-2k+1,\alpha}(\bar{X},\Lambda^k(\bar{X}))$ for some $\alpha>0$ (since then $\delta_g\omega\in x^{n-2k+\alpha}L^\infty(\Lambda^k(\bar{X}))$). But as shown in Corollary 3.3, $\delta_g\omega$ is then smooth and vanishing to order x^{n-2k+3}. Then, like in the proof of Proposition 3.9, an integration by part on $||\delta_g\omega||^2_{L^2}$ shows that $\delta_g\omega=0$. Now we can apply the result of Proposition 4.4 (see the Remark below Corollary 4.11), which gives $G_k\omega_0=0$. Since dG_k+L_k is elliptic, $\ker L_k\cap\ker G_k$ is finite dimensional and contains only smooth forms, so ω_0 is smooth. Then ω is polyhomogeneous and is the solution of Proposition 3.1, up to an element of $\ker_{L^2}(\Delta_k)$, it is then in $C^{n-1}(\bar{X},\Lambda^k(\bar{X}))$ in general and in $C^\infty(\bar{X},\Lambda^k(\bar{X}))$ if (X,g) smooth Poincaré-Einstein manifold.

Let $m\in[n-2k+1,n-1]$ be an integer. The exact sequence (1.9) is defined by inclusion of $\iota:H^k(\bar{X},\partial\bar{X})\to K^k_m(\bar{X})$ and restriction to the boundary $r:K^k_m(\bar{X})\to\mathcal{H}^k_L(M)$, here of course we use the identification $H^k(\bar{X},\partial\bar{X})\simeq\ker_{L^2}(\Delta_k)$ and the regularity of harmonic L^2 forms in Theorem 3.2. The injectivity of ι is clear, the surjectivity of r comes from Proposition 3.1, the definition of $\mathcal{H}^k_L$ and Theorem 6.2. The kernel of r is composed of those forms of $K^k_m(\bar{X})$ which vanish at M, but by Proposition 3.1, these are L^2, and thus in the image of $H^k(\bar{X},\partial\bar{X})$ by the map ι. □

Proof of Theorem 1.3: First note that the space $Z^k(\bar{X})$ in Theorem 1.3 is included in $K^k_{n-2k+1}(\bar{X})$, and thus of finite dimension and composed of forms in $C^{n-1}(\bar{X},\Lambda^k(\bar{X}))$ (even in the case $k=\frac{n}{2}$ by the arguments above).

(i) the maps in the complex

$$0\to H^k(\bar{X},\partial\bar{X})\xrightarrow{\iota}Z^k(\bar{X})\xrightarrow{r}\mathcal{H}^k(\partial\bar{X})\xrightarrow{d_e}H^{k+1}(\bar{X},\partial\bar{X})$$

are defined as follows: ι is given by inclusion where $H^k(\bar{X},\partial\bar{X})\simeq\ker_{L^2}(\Delta_k)$, this is well defined since L^2 harmonic forms are closed, coclosed and in $C^{n-2k+1}(\bar{X},\Lambda^k(\bar{X}))$; r is defined as restriction at the boundary and it maps in $\mathcal{H}^k(M)$ since $r(\omega)\in\ker L_k\cap\ker G_k$ by the discussion above and $d\omega=0$ implies $dr(\omega)=0$; the last map d_e is the composition

$d_e = d\circ\Phi$ where $\Phi : C^\infty(M,\Lambda^k(M)) \to C^\infty(X,\Lambda^k(X))/\ker_{L^2}(\Delta_k)$ is defined by $\Phi(\omega_0) = \omega$ where ω is the solution of (3.1) in Proposition 3.1. Note that Φ is only defined modulo $\ker_{L^2}(\Delta_k)$ and is linear by uniqueness of the solution in (3.1) modulo $\ker_{L^2}(\Delta_k)$. Applying d kills the indeterminacy with respect to $\ker_{L^2}(\Delta_k)$ since L^2 harmonic forms are closed. Then $d\Phi(\omega_0)$ is harmonic and since the boundary value of $\Phi(\omega_0)$ is closed, then $d\Phi(\omega_0) = O(x)$, and by Proposition 3.1 it is in L^2. For the exactness of the sequence, first note that $\ker r$ is composed of closed and coclosed forms which are $O(x)$, this implies that those forms are L^2 by Proposition 3.1, so $\mathrm{Im}\,\iota = \ker r$ since also L^2 harmonic forms vanish at the boundary. Now $\omega_0 \in \ker d_e$ if $\Phi(\omega_0)$ is closed, but it is also coclosed and in $C^{n-2k+1}(\bar{X},\Lambda^k(\bar{X}))$ by Proposition 3.1 and the fact that $\omega_0 \in \ker d\cap\ker G_k \subset \ker L_k\cap\ker G_k$, therefore $\Phi(\omega_0)\in Z^k(\bar{X})$ and $\omega_0 \in \mathrm{Im}\, r$. Moreover by Proposition 3.1 we have $\Phi(r(\omega)) - \omega \in \ker_{L^2}(\Delta_k)$, this implies $\mathrm{Im}\, r \subset \ker d_e$, this proves exactness of the sequence.

(ii) the map in the complex (1.11) are defined similarly: first $\iota : H^k(\bar{X},\partial\bar{X}) \to [Z^k(\bar{X})]$ is the composition of the inclusion $\ker_{L^2}(\Delta_k) \to Z^k(\bar{X})$ with the natural map $Z^k(\bar{X}) \to [Z^k(\bar{X})]$ obtained by taking cohomology class. The map $r : [Z^k(\bar{X})] \to [\mathcal{H}^k(\partial\bar{X})]$ is the map induced by the restriction map $Z^k(\bar{X}) \to \mathcal{H}^k(\partial\bar{X})$ used in (i). This is well defined since if $d\alpha \in Z^k(\bar{X})$, then $r(d\alpha) = d\alpha_0$ where $\alpha_0 = \alpha|_{\partial\bar{X}}$, and so $[r(d\alpha)] = 0$ if $[\cdot]$ denotes cohomology class in $H^k(\partial\bar{X})$. The last map $d_e : [Z^k(\partial\bar{X})] \to H^{k+1}(\bar{X},\partial\bar{X})$ is the map induced by d_e defined in (i), i.e. $d_e = d\circ\Phi$ where Φ maps ω_0 to the solution of (3.1). Note that it is well defined since for $d\alpha_0 \in \mathcal{H}^k(\partial\bar{X})$, we have $d_e(d\alpha_0) = d\Phi(d\alpha_0)$ and, by uniqueness of the solution of (3.1), $\Phi(d\alpha_0) - d\Phi(\alpha_0) \in \ker_{L^2}(\Delta_{k+1})$ thus $d\Phi(d\alpha_0) = 0$.

To show that $\ker r = \mathrm{Im}\,\iota$, we need to show that if $\omega \in Z^k(\bar{X})$ is a representative in $[Z^k(\bar{X})]$ such that $r(\omega) = d\alpha_0$ for some smooth α_0, then there is $\omega' \in \ker_{L^2}(\Delta_k)$ such that $\omega-\omega'$ is exact. But as said above, we have $\Phi(d\alpha_0)-d\Phi(\alpha_0) \in \ker_{L^2}(\Delta_k)$ and $\Phi(r(\omega))-\omega \in \ker_{L^2}(\Delta_k)$ thus $\omega - d\Phi(\alpha_0) \in \ker_{L^2}(\Delta_k)$ and we are done. To show that $\ker d_e = \mathrm{Im} r$, we need to prove that for $\omega_0 \in \mathcal{H}^k(\partial\bar{X})$ a representative in $[\mathcal{H}^k(\partial\bar{X})]$ then $\Phi(\omega_0)$ is closed if and only if there exists $\omega \in Z^k(\bar{X})$ so that $r(\omega) - \omega_0$ is exact. But $\Phi(\omega_0)$ is in $Z^k(\bar{X})$ if $d\Phi(\omega_0) = 0$, thus $\ker d_e \subset \mathrm{Im}\, r$; conversely if there is $\omega \in Z^k(\bar{X})$ with $\omega = \omega_0+d\alpha_0+O(x)$, then $\omega - \Phi(\omega_0+d\alpha_0) \in \ker_{L^2}(\Delta_k)$ and so $d\Phi(\omega_0) = 0$ since $\Phi(d\alpha_0)-d\Phi(\alpha_0) \in \ker_{L^2}(\Delta_k)$. To conclude, we need to prove that $\mathrm{Im}\, d_e \subset \ker\iota$. But this is clear since $d_e\omega_0 = d\Phi(\omega_0)$ is an exact $(k+1)$-form in L^2 with $\Phi(\omega_0) \in C^{n-2k+1}(\bar{X},\Lambda^k(\bar{X}))$. Note that in the case $k = \frac{n}{2}$, we make use of Proposition 3.10.

(iii) Suppose that $[\mathcal{H}^k(\partial\bar{X})] = H^k(\partial\bar{X})$. If $\omega \in \ker\iota$, it is a k-form in $\ker_{L^2}(\Delta_k)$ which can be written $\omega = d\alpha$ with α smooth. Moreover if $\alpha_0 = \alpha|_{\partial\bar{X}}$, then $d(\Phi(\alpha_0)-\alpha) \in \ker_{L^2}(\Delta_k)$ and $\Phi(\alpha_0)-\alpha = O(x)$, an easy integration by parts shows that $d\Phi(\alpha_0) = d\alpha = \omega$. Here α_0 is closed since $\omega = O(x)$, but by assumption there is a $\alpha_0' \in \mathcal{H}^k(\partial\bar{X})$ such that $\alpha_0 - \alpha_0' = d\beta$ for some smooth β. Since now $d\Phi(d\beta) = d[\Phi,d]\beta = 0$, we have $d_e\alpha_0' = \omega$ and $\omega \in \mathrm{Im}\, d_e$, which gives $\ker\iota = \mathrm{Im}\, d_e$. Eventually, the equality $[Z^k(\bar{X})] = H^k(\bar{X})$ is clear from the discussion above since $[Z^k(\bar{X})] \subset H^k(\bar{X})$ and

$$H^k(\bar{X},\partial\bar{X}) \xrightarrow{\iota} [Z^k(\bar{X})] \xrightarrow{r} H^k(\partial\bar{X}) \xrightarrow{d_e} H^{k+1}(\bar{X},\partial\bar{X})$$
$$H^k(\bar{X},\partial\bar{X}) \xrightarrow{\iota} H^k(\bar{X}) \xrightarrow{r} H^k(\partial\bar{X}) \xrightarrow{d_e} H^{k+1}(\bar{X},\partial\bar{X})$$

are both exact sequences.

As for the converse, if $\ker\iota^{k+1} = \mathrm{Im}\, d_e^k$ and $[Z^k(\bar{X})] = H^k(\bar{X})$, then we have the exact sequences

$$H^k(\bar{X}) \xrightarrow{r} [\mathcal{H}^k(M)] \xrightarrow{d_e} H^{k+1}(\bar{X},M) \xrightarrow{\iota} [Z^{k+1}(\bar{X})]$$
$$H^k(\bar{X}) \xrightarrow{r} H^k(M) \xrightarrow{d_e} H^{k+1}(\bar{X},M) \xrightarrow{\iota'} H^{k+1}(\bar{X})$$

and since $[Z^{k+1}(\bar{X})] \subset H^{k+1}(\bar{X})$, we obviously have $\ker \iota = \ker \iota' = \operatorname{Im} d_e$ and so $[\mathcal{H}^k(M)] = H^k(M)$ (recall $[\mathcal{H}^k(M)] \subset H^k(M)$). $\square$

Proof of Proposition 1.4: Assume $\langle Q_k v, v\rangle \geq 0$. To show surjectivity of $\mathcal{H}^k(M) \to H^k(M)$, we need to prove that for all $\omega_0 \in C^\infty(M, \Lambda^k(M))$ closed, there exists an exact form $d\alpha$ (with $\alpha \in C^\infty(M, \Lambda^k(M))$) such that $G_k(\omega_0 + d\alpha) = 0$. Consider $\square := \delta_0 Q_k d + (d\delta_0)^{\frac{n}{2}-k+1}$ which is elliptic, self-adjoint and non-negative if $Q_k \geq 0$. Its kernel is finite dimensional (containing $\ker(d + \delta_0)$) and all $v \in \ker \square$ are smooth by elliptic regularity, and satisfy $\langle \delta_0 Q_k dv, v\rangle_{L^2} = 0$, which implies $\langle Q_k dv, dv\rangle_{L^2} = 0$. Let $\mathbf{H} \subset L^2(\Lambda^k(M))$ be the L^2 completion of the set $C^\infty(M, \Lambda^k(M)) \cap \ker d$ of smooth closed forms and let us define $\mathbf{Q}$ the symmetric form $\mathbf{Q}(v, v) := \langle Q_k v, v\rangle_{L^2}$ on $\mathbf{H}$, it is a non-negative form induced by $\Pi_{\mathbf{H}} Q_k$ on $\mathbf{H}$ where $\Pi_{\mathbf{H}}$ denotes orthogonal projection from $L^2(\Lambda^k(M))$ to $\mathbf{H}$. The form has a domain $D(\mathbf{Q})$ and Friedrichs extension theorem implies that there exists a self adjoint operator $Q_k^{\mathrm{Fr}} : \mathbf{H} \to \mathbf{H}$ with domain $D(Q^{\mathrm{Fr}})$ such that $\langle Q_k^{\mathrm{Fr}} u, u\rangle = \mathbf{Q}(u, u)$ for $u \in D(\mathbf{Q}) \cap D(Q^{\mathrm{Fr}})$. But clearly $d(C^\infty(M, \Lambda^{k-1}(M))) \subset D(Q_k^{\mathrm{Fr}})$ and so $\Pi_{\mathbf{H}} Q_k dv = Q_k^{\mathrm{Fr}} dv$ for v smooth. Using now the spectral theorem for Q_k^{Fr}, we see that $Q_k^{\mathrm{Fr}} dv = 0$ with v smooth if and only if $\langle Q_k dv, dv\rangle = 0$ and v is smooth, thus in particular if $v \in \ker \square$. Thus $Q_k dv \perp \omega$ for all $\omega \in \mathbf{H}$ if $v \in \ker \square$. Now this implies that, with ω closed and smooth, we have $\langle v, G_k \omega\rangle = \langle Q_k dv, \omega\rangle = 0$ for $v \in \ker \square$ since Q_k is symmetric on closed forms, and so $G_k \omega$ is in the range of $\square$ and there exists α such that $\square \alpha = -G_k \omega$, but since $\operatorname{Im} G_k \subset \operatorname{Im} \delta_0$ which is orthogonal to $\operatorname{Im} d$, we deduce that $(d\delta_0)^{\frac{n}{2}-k+1}\alpha = 0$ and this achieves the proof. Note in particular that in this case $\{d\varphi; L_{k-1}\varphi = 0\} = \{d\varphi; Q_k d\varphi \in \operatorname{Im} \delta_0\}$, see Corollaries 2.12 and 2.13 of [3] for discussions about these spaces. $\square$

8. Computations in some special cases

In this section we compute the operator L_k, G_k and Q_k in dimension 4 and 6.

Proposition 8.1. *Let (M^4, h) a four dimensional Riemannian manifold and define for any symmetric 2-tensor H the map $j(H) := J(h^{-1}H)$ where J is defined in (2.4). Then we have*

$$L_1 = \frac{1}{2}\delta d, \qquad G_1 = -\frac{1}{4}\delta\Big(\Delta - 2j(\mathrm{Ric}) + \frac{2}{3}\mathrm{Scal}\Big), \qquad Q_1 = \frac{1}{2}\Big(\Delta - 2j(\mathrm{Ric}) + \frac{2}{3}\mathrm{Scal}\Big),$$
$$L_0 = -\frac{1}{16}\delta\Big(\Delta - 2j(\mathrm{Ric}) + \frac{2}{3}\mathrm{Scal}\Big)d, \quad G_0 = 0, \quad Q_0 = -\frac{1}{24}\Big(\Delta\mathrm{Scal} - 3|\mathrm{Ric}|^2 + \mathrm{Scal}^2\Big)$$

where Ric *is the Ricci tensor of h and* Scal *its scalar curvature*

Remark: If $n = 4$, $L_{\frac{n}{2}-2}$ is the Paneitz operator (up to a constant factor). The result of Gursky and Viaclovsky [14] says that if the Yamabe invariant $Y(M, [h_0])$ is positive and

$$\int_M Q\mathrm{dvol}_{h_0} + \frac{1}{6}Y(M, [h_0])^2 > 0$$

then L_0 is a non-negative operator with kernel reduced to constants. Combining with Theorem 2.6 of Branson-Gover[3], we have that $\mathcal{H}^1(M) \simeq H^1(M)$ and there is a conformally invariant basis of $H^1(M)$ with respect to $[h_0]$ made of conformal harmonics.

Corollary 8.2. *Let M^4 be a four dimensional manifold and $\lambda_1(x) \geq \cdots \geq \lambda_4(x)$ the eigenvalues of its Ricci curvature at x. If $\lambda_2(x) + \lambda_3(x) + \lambda_4(x) \geq 0$ for all $x \in M$ then $\mathcal{H}^1(M) \to H^1(M)$ is surjective.*

Proof: Let D be the Levi-Civita connection of the metric h and ω be a closed 1-form. By decomposing orthogonally the bilinear tensor $D\alpha$ in antisymmetric part $\frac{d\alpha}{2}$, symmetric trace free part S_0 and trace part $\frac{-\delta\alpha}{4}h$, we get (recall that ω is closed)

$$|D\alpha|^2 = \frac{|d\alpha|^2}{2} + |S_0|^2 + \frac{(\delta\alpha)^2}{4} \geq \frac{(\delta\alpha)^2}{4}.$$

Now, by the Bochner formula, we get

$$\langle \Delta\omega,\omega\rangle = \|D\omega\|_2^2 + \mathrm{Ric}(\omega,\omega) \geq \|\delta\omega\|^2/4 + \int_M \mathrm{Ric}(\omega,\omega) = \frac{\langle\Delta\omega,\omega\rangle}{4} + \int_M \mathrm{Ric}(\omega,\omega),$$

and so $\langle\Delta\omega,\omega\rangle \geq \frac{4}{3}\int_M \mathrm{Ric}(\omega,\omega)$. Therefore,

$$\langle Q_1\omega,\omega\rangle = \frac{1}{2}\langle\Delta\omega,\omega\rangle - \int_M \mathrm{Ric}(\omega,\omega) + \frac{\mathrm{Scal}}{3}|\omega|^2 \geq \frac{1}{3}\int_M \mathrm{Scal}|\omega|^2 - \mathrm{Ric}(\omega,\omega),$$

$$\int_M (\lambda_1+\lambda_2+\lambda_3+\lambda_4)|\omega|^2 - \lambda_1|\omega|^2 \geq \int_M (\lambda_2+\lambda_3+\lambda_4)|\omega|^2 \geq 0,$$

and we conclude by using Proposition 1.4. □

Proposition 8.3. *Let (M^6,h) be a six dimensional manifold. Let j be defined like in Lemma 8.1 and* $\mathrm{tr}(H)$ *denotes the trace of a symmetric tensor H with respect to h, then we have*

$$L_2 = \frac{1}{2}\delta d, \quad G_2 = \frac{1}{4}\delta\Big(\Delta - j(\mathrm{Ric}) + \frac{2}{5}\mathrm{Scal}\Big), \quad Q_2 = \frac{1}{2}\Big(\Delta - j(\mathrm{Ric}) + \frac{2}{5}\mathrm{Scal}\Big),$$

$$L_1 = -\frac{1}{16}\delta\Big(\Delta - j(\mathrm{Ric}) + \frac{2}{5}\mathrm{Scal}\Big)d,$$

$$G_1 = \frac{(\delta d)^2\delta}{64} + \frac{\delta d E_1\delta}{32} - \frac{\delta E_1\delta d}{16} - \frac{\delta E_1 d\delta}{16} - \frac{\delta d\delta E_1}{16} + \frac{\delta E_2}{16} + \frac{\delta E_1^2}{8},$$

$$Q_1 = -\frac{\Delta^2}{16} + \frac{d\delta E_1}{4} + \frac{\delta d E_1}{4} - \frac{dE_1\delta}{8} + \frac{E_1 d\delta}{4} - \frac{2E_1^2 + E_2}{4},$$

$$L_0 = \frac{1}{384}(\delta d)^3 + \frac{\delta d E_1\delta d}{192} - \frac{\delta E_1 d\delta d}{96} - \frac{\delta d\delta E_1 d}{96} + \frac{\delta E_2 d}{96} + \frac{\delta E_1^2 d}{48},$$

$$Q_0 1 = -\frac{1}{1920}\Delta^2\mathrm{Scal} - \frac{h_0(\mathrm{Ric},\mathrm{Hess\,Scal})}{320} + \frac{\Delta|\mathrm{Ric}|^2}{768} - \frac{29}{38400}\Delta\mathrm{Scal}^2 - \frac{|d\mathrm{Scal}|^2}{640}$$
$$-\frac{51}{64000}\mathrm{Scal}^3 - \frac{1}{256}\mathrm{tr}\big((h_0^{-1}\mathrm{Ric})^3\big) + \frac{9}{2560}|\mathrm{Ric}|^2\mathrm{Scal} - \frac{1}{32}h_0(\mathrm{Ric},B),$$

where $E_2 := J(P^2+2B) - |P|^2/2$ and $E_1 := J(P) - \mathrm{tr}(P)/2$, B denotes the Bach tensor of h, P the Schouten tensor, Ric *the Ricci tensor and* Scal *the scalar curvature.*

Proposition 8.4. *For any $n \geq 4$, we have the identities*

$$G_{\frac{n}{2}-1} = (-1)^{\frac{n}{2}+1}\Big(\frac{\delta d\delta}{4} - \frac{\delta j(P)}{2} + \delta\frac{Tr(P)\mathrm{Id}}{4}\Big) = (-1)^{\frac{n}{2}+1}\Big(\frac{\delta d\delta}{4} - \delta\Big(\frac{j(\mathrm{Ric})}{n-2} - \frac{\mathrm{ScalId}}{2(n-1)}\Big)\Big),$$

$$Q_{\frac{n}{2}-1} = \Big(\frac{\Delta}{2} - \frac{2j(\mathrm{Ric})}{n-2} + \frac{\mathrm{Scal\,Id}}{n-1}\Big),$$

$$L_{\frac{n}{2}-2} = -\delta\Big(\frac{d\delta}{16} - \frac{j(\mathrm{Ric})}{4(n-2)} + \frac{\mathrm{Scal\,Id}}{8(n-1)}\Big)d,$$

$$G_{\frac{n}{2}-2} = (-1)^{\frac{n}{2}+1}\Big(\frac{(\delta d)^2\delta}{64} + \frac{\delta d E_1\delta}{32} - \frac{\delta E_1\delta d}{16} - \frac{\delta E_1 d\delta}{16} - \frac{\delta d\delta E_1}{16} + \frac{\delta E_2}{16} + \frac{\delta E_1^2}{8}\Big),$$

$$Q_{\frac{n}{2}-2} = -\frac{\Delta^2}{16} + \frac{d\delta E_1}{4} + \frac{\delta d E_1}{4} - \frac{dE_1\delta}{8} + \frac{E_1 d\delta}{4} - \frac{2E_1^2+E_2}{4},$$

$$L_{\frac{n}{2}-3} = \frac{(\delta d)^3}{384} + \frac{\delta d E_1\delta d}{192} - \frac{\delta E_1 d\delta d}{96} - \frac{\delta d\delta E_1 d}{96} + \frac{\delta E_2 d}{96} + \frac{\delta E_1^2 d}{48},$$

where $E_2 := J(P^2 + \frac{4B}{n-4}) - |P|^2/2$ and $E_1 := J(P) - \mathrm{tr}(P)/2$, B denotes the Bach tensor of h, P the Schouten tensor, Ric *the Ricci tensor and* Scal *the scalar curvature.*

For the non critical case, we have

Proposition 8.5. *We set $j^\sharp(H) = 2j(H) - \mathrm{tr}(H)\,\mathrm{Id}$. For any $n \geq 3$, we have*

$$L_k^1 \;=\; \frac{\delta d}{2} + \frac{(n-2k-2)d\delta}{2(n-2k+2)} + \frac{(n+k-2)(n-2k-2)}{8(n-1)(n-2)}\mathrm{Scal} - \frac{(n-2k-2)j(\mathrm{Ric})}{2(n-2)}$$

which generalizes the conformal Laplacian on functions,

$$\begin{aligned} L^2_k \quad &= \quad -\frac{n-2k-4}{16}\Big(\frac{(d\delta)^2}{n-2k+4}+\frac{(\delta d)^2}{n-2k-4}+\frac{2dj^\sharp(P)\delta}{n-2k+4}-\frac{2\delta j^\sharp(P)d}{n-2k-4} \\ &\quad -\frac{j(P)\Delta+\Delta j^\sharp(P)}{2}+j^\sharp(P^2+\frac{B}{n-4})+\frac{(n-2k)j^\sharp(P)^2}{4}\Big) \end{aligned}$$

which generalizes the Paneitz-Branson operator on functions.

Proofs of Propositions 8.1, 8.3, 8.4 and 8.5: This is a quite tedious computation, therefore we do not give the full details. By [6, Eq. (3.18)], we have

$$h_0^{-1}h_x = \big(I - x^2\frac{P}{2} + x^4\frac{h_2}{8} - x^6\frac{h_3}{48} + o(x^6)\big),$$

whith $P = \frac{1}{n-2}\big(2h_0^{-1}\mathrm{Ric} - \frac{\mathrm{Scal}}{n-1}I\big)$, $h_2 = \frac{P^2}{2}$ for $n = 4$ and $h_2 = -\frac{2h_0^{-1}B}{n-4} + \frac{P^2}{2}$ and $\mathrm{tr}(h_3) = -\frac{8\mathrm{tr}\big(P(h_0^{-1}B)\big)}{n-4}$ for $n = 6$ (where B is the Bach tensor); note that we have ignored the first log term in the metric expansion (i.e. the obstruction tensor) in dimension 4 and 6 since, as it is clear from Lemma 2.1, they do not show up in the construction the L^ℓ_k, G_k, Q_k. We set $B' = 0$ for $n = 4$ and $B' = \frac{2h_0^{-1}B}{n-4}$ for $n = 6$. Using the relations

$$L^{-1} = I - A_1x^2 - (A_2 - A_1^2)x^4 - (A_3 + A_1^3 - A_1A_2 - A_2A_1)x^6 + o(x^6),$$

$$L^{-1}QL = I + A_1'x^2 + (A_2' + [A_1', A_1])x^4 + (A_3' + [A_1', A_2] + [A_2', A_1] + A_1[A_1, A_1'])x^6 + o(x^6),$$

for $L = \big(I + A_1x^2 + A_2x^4 + A_3x^6 + o(x^6)\big)$ and $Q = \big(I + A_1'x^2 + A_2'x^4 + A_3'x^6 + o(x^6)\big)$,

and the notations of the proof of Lemma 2.1, we get

$$\begin{aligned} h_x^{-1}h_0 &= I + x^2\frac{P}{2} + x^4\frac{3P^2+2B'}{16} + x^6\frac{h_3+6P^2+3B'P+3PB'}{48} + o(x^6), \\ A_x &= -Px - \frac{P^2+2B'}{4}x^3 - \frac{2h_3+P^3+4PB'+2B'P}{16} + o(x^6), \\ O_x &= I + x^2\frac{P}{4} + x^4\frac{P^2+B'}{16} + x^6\frac{2h_3+3P^3+5PB'+4B'P}{192} + o(x^6), \end{aligned}$$

$$\begin{aligned} I_x &= I + x^2\frac{J(P)}{4} + x^4\frac{J(P^2+2B')+J(P)^2}{32}, \\ &\quad + x^6\Big(\frac{J(4h_3+2P^3+10PB'+2B'P)+3J(2B'+P^2)J(P)+J(P)^3}{384}\Big) + o(x^6), \\ I_x^{-1} &= I - x^2\frac{J(P)}{4} + x^4\frac{-J(P^2+2B')+J(P)^2}{32}, \\ &\quad + x^6\Big(\frac{-J(4h_3+2P^3+10PB'+2B'P)+3J(P)J(2B'+P^2)-J(P)^3}{384}\Big) + o(x^6). \end{aligned}$$

Then we obtain

$$\begin{aligned} \star_x = \star_0 - x^2\frac{[J(P),\star_0]}{4} &+ x^4\Big(\frac{\big[J(P),[J(P),\star_0]\big]-[J(P^2+2B'),\star_0]}{32}\Big) \\ &+ x^6\Big(-\frac{\big[J(P)^3+J(4h_3+2P^3+10PB'+2B'P),\star_0\big]}{384} \\ &\qquad + \frac{J(P)[J(P),\star_0]J(P)+\big[J(P),[J(2B'+P^2),\star_0]\big]}{128}\Big) + o(x^6) \end{aligned}$$

from which we infer that

$$\star_x^{-1}\big[\partial_x,\star_x\big] = x\frac{\star_0^{-1}[\star_0,J(P)]}{2} + x^3\frac{\star_0^{-1}[\star_0,J(P^2+2B')]+[J(P),\star_0^{-1}J(P)\star_0]}{8} + x^5\frac{C}{64}$$

where

$$C := -2\star_0^{-1}\big[J(P)^3 + J(2h_3 + P^3 + 5PB' + B'P), \star_0\big] - \big[J(P), [J(P), \star_0^{-1}J(P)\star_0]\big]$$
$$+2[\star_0^{-1}J(P)\star_0 + 2J(P), \star_0^{-1}J(2B'+P^2)\star_0 - J(2B'+P^2)]$$

using the relations $\star_0 J(H) + J(H)\star_0 = \mathrm{tr}(H)\star_0$ and $[J(H), J(H')] = J([H,H'])$, we get

$$\star_x^{-1}\big[\partial_x, \star_x\big] = xE_1 + x^3\frac{E_2}{4} + x^5\frac{C'}{32}$$

with

$$C' := -\star_0^{-1}\big[J(P)^3 + J(2h_3 + P^3 + 3PB' + 3B'P), \star_0\big], \quad E_2 := J(P^2 + 2B') - \frac{\mathrm{tr}(P^2)}{2},$$
$$\delta_x = \delta_0 + x^2\frac{[\delta_0, E_1]}{2} + x^4\frac{D}{16}, \text{ where } D := [\delta_0, E_2] + 2\big[[\delta_0, E_1], E_1\big].$$

Therefore,

$$\Delta_k = \begin{pmatrix} -(x\partial_x)^2 + (n-2k)x\partial_x & 2(-1)^{k+1}d \\ 0 & -(x\partial_x)^2 + (n-2k+2)x\partial_x \end{pmatrix}$$
$$+x^2\begin{pmatrix} \Delta_0 - E_1x\partial_x & (-1)^k[d, E_1] \\ 2(-1)^{k+1}\delta_0 & \Delta_0 - E_1(2 + x\partial_x) \end{pmatrix}$$
$$+x^4\begin{pmatrix} A_1 - A_2x\partial_x & (-1)^k[d, A_2] \\ 2(-1)^{k+1}[\delta_0, E_1] & A_1 - A_2(4 + x\partial_x) \end{pmatrix}$$
$$+x^6\begin{pmatrix} A_3 & A_4 \\ A_5 & A_6 \end{pmatrix} + o(x^6)$$

where $A_1 := \frac{d[\delta_0,E_1]+[\delta_0,E_1]d}{2}$, $A_2 := \frac{E_2}{4}$ (since the Bach tensor is trace-free), $A_5 := -(-1)^k\frac{3D}{8}$ and, when $k=1$, $A_61 := -\frac{3}{8}A1 = \frac{3}{2000}(\mathrm{Scal})^3 + \frac{3}{16}\mathrm{tr}(P^3) + \frac{3}{8}h_0(h_0P, B)$.

For $n=6$ and $k=1$ we follow the formal method of Subection 4.2.3. We first have

$$\omega'_{F_1} = \frac{dx}{x} - x^2\Big(\frac{d\mathrm{Scal}}{80} + \frac{\mathrm{Scal}dx}{40x}\Big) + x^4\Big(\frac{\Delta\mathrm{Scal}}{160} + \frac{\mathrm{Scal}^2}{800} - \frac{|P|^2}{16}\Big)\frac{dx}{x}$$

and so, by computing $d\omega_{F_1}$ and $\Delta_k\omega_{F_1}$, one finds

$$\begin{aligned}
D'_01 &= -\frac{d\Delta\mathrm{Scal}}{160} - \frac{d\mathrm{Scal}^2}{800} + \frac{d|P|^2}{16} = -\frac{d\Delta\mathrm{Scal}}{160} - \frac{11d\mathrm{Scal}^2}{3200} + \frac{d|\mathrm{Ric}|^2}{64}\\
-B'_01 &= A_61 + \frac{[E_1,\delta_0]d\mathrm{Scal}}{40} - \frac{(A_1 - 6A_2)\mathrm{Scal}}{40} + \big(\Delta_0 - 6E_1\big)\Big(\frac{\Delta\mathrm{Scal}}{160} + \frac{\mathrm{Scal}^2}{800} - \frac{|P|^2}{16}\Big)\\
&= \frac{3}{2000}\mathrm{Scal}^3 + \frac{3}{16}\mathrm{tr}(P^3) + \frac{3}{8}h_0(h_0P, B) + \frac{3[E_1,\delta_0]}{80}d\mathrm{Scal} - \frac{3}{160}|P|^2\mathrm{Scal}\\
&\quad + \frac{\Delta^2\mathrm{Scal}}{160} + \frac{\Delta\mathrm{Scal}^2}{800} - \frac{\Delta|P|^2}{16} + \frac{3\mathrm{Scal}\Delta\mathrm{Scal}}{800} + \frac{3\mathrm{Scal}^3}{4000} - \frac{3\mathrm{Scal}|P|^2}{80}\\
&= \frac{153}{32000}\mathrm{Scal}^3 + \frac{3}{128}\mathrm{tr}\big((h_0^{-1}\mathrm{Ric})^3\big) - \frac{27}{1280}|\mathrm{Ric}|^2\mathrm{Scal} + \frac{3}{16}h_0(\mathrm{Ric}, B) + \frac{1}{160}\Delta\mathrm{Scal}^2\\
&\quad + \frac{\Delta^2\mathrm{Scal}}{160} - \frac{\Delta|\mathrm{Ric}|^2}{64} + \frac{3|d\mathrm{Scal}|^2}{320} + \frac{3h_0(\mathrm{Ric}, \mathrm{Hess\,Scal})}{160}
\end{aligned}$$

where, by the second Bianchi identity, we have that $\delta(J(P)d\mathrm{Scal}) = -\frac{|d\mathrm{Scal}|^2}{4} - \frac{h_0(\mathrm{Ric},\mathrm{Hess\,Scal})}{2} - \frac{\Delta\mathrm{Scal}^2}{40}$. Hence we get

$$-Q_0 = \frac{1}{1920}\Delta^2\mathrm{Scal} + \frac{h_0(\mathrm{Ric}, \mathrm{Hess\,Scal})}{320} - \frac{\Delta|\mathrm{Ric}|^2}{768} + \frac{29}{38400}\Delta\mathrm{Scal}^2 + \frac{|d\mathrm{Scal}|^2}{640}$$
$$+\frac{51}{64000}\mathrm{Scal}^3 + \frac{1}{256}\mathrm{tr}\big((h_0^{-1}\mathrm{Ric})^3\big) - \frac{9}{2560}|\mathrm{Ric}|^2\mathrm{Scal} + \frac{1}{32}h_0(\mathrm{Ric}, B)$$

The other computations are made similarly. For instance, for $k = n/2 - 1$,

$$\Delta\omega_{F_1} = x^2\delta_0 d\omega_0 + x^3(-1)^{\frac{n}{2}+1}\big(\frac{\delta_0 d\delta_0\omega_0}{2} - 2\delta_0 E_1\omega_0\big)\wedge dx + O(x^4),$$

and so

$$B_{\frac{n}{2}-1}\omega_0 = -\frac{\delta_0 d\delta_0\omega_0}{2} + 2\delta_0 E_1\omega_0.$$

We have $\delta\omega_{F_1} = \frac{x^4}{2}\delta_0 E_1\omega_0 + O(x^5)$, and so $C_{\frac{n}{2}-1} = \frac{\delta_0 E_1}{2}$. By Proposition 4.5,

$$G_{\frac{n}{2}-1} = (-1)^{\frac{n}{2}+1}\Big(\frac{\delta_0 d\delta_0}{4} - \frac{\delta_0 E_1}{2}\Big),$$

which implies the expression for $L_{\frac{n}{2}-2}$ by (4.11). For $k = \frac{n}{2} - 2$, we have

$$\begin{aligned}\omega_{F_1} &= \omega_0 + x^2\big(-\frac{d\delta\omega_0}{8} - \frac{\delta d\omega_0}{4} + \frac{(-1)^{\frac{n}{2}}\delta\omega_0}{4}\wedge\frac{dx}{x}\big)\\ &\quad + (-1)^{\frac{n}{2}}x^4\big(\frac{\delta E_1\omega_0}{4} - \frac{\delta d\delta\omega_0}{16} - \frac{E_1\delta\omega_0}{8}\big)\wedge\frac{dx}{x}\\ \delta &= \begin{pmatrix}0 & (-1)^{\frac{n}{2}}(x\partial_x - 6)\\ 0 & 0\end{pmatrix} + x^2\begin{pmatrix}\delta_0 & (-1)^{\frac{n}{2}}E_1\\ 0 & \delta_0\end{pmatrix} + x^4\begin{pmatrix}\frac{[\delta_0,E_1]}{2} & (-1)^{\frac{n}{2}}A_2\\ 0 & \frac{[\delta_0,E_1]}{2}\end{pmatrix}\\ &\quad + x^6\begin{pmatrix}\frac{2[\delta_0,A_2]+\big[[\delta_0,E_1],E_1\big]}{8} & *\\ 0 & *\end{pmatrix}\end{aligned}$$

and so

$$\begin{aligned}C_{\frac{n}{2}-2} &= \frac{\delta_0 A_2}{4} + \frac{\delta_0 E_1^2}{8} - \frac{\delta_0 E_1 d\delta_0}{16} - \frac{\delta_0 E_1\delta_0 d}{8},\\ B_{\frac{n}{2}-2} &= \frac{3\delta_0 A_2}{2} - \frac{\delta E_1\delta d}{2} - \frac{3\delta E_1 d\delta}{8} - \frac{\delta d\delta E_1}{4} + \frac{(\delta d)^2\delta}{16} + \frac{\delta dE_1\delta}{8} + \frac{3\delta E_1^2}{4},\\ G_{\frac{n}{2}-2} &= (-1)^{\frac{n}{2}+1}\Big(\frac{(\delta d)^2\delta}{64} + \frac{\delta dE_1\delta}{32} - \frac{\delta E_1\delta d}{16} - \frac{\delta E_1 d\delta}{16} - \frac{\delta d\delta E_1}{16} + \frac{\delta E_2}{16} + \frac{\delta E_1^2}{8}\Big).\end{aligned}$$

In the case $k = \frac{n}{2} - 1$, we have $\omega'_{F_1} = \omega_0\wedge\frac{dx}{x} + x^2\big(-\frac{d\delta}{4} + \frac{E_1}{2}\big)\wedge\frac{dx}{x}$, $D'_{\frac{n}{2}-2} = (-1)^{\frac{n}{2}+1}\frac{dE_1}{2}$ and $B'_{\frac{n}{2}-2} = (-1)^{\frac{n}{2}}d\delta E_1 - \frac{dE_1\delta}{2} - E_2 - \frac{(d\delta)^2}{4} + \delta dE_1 + E_1 d\delta - 2E_1^2 + \frac{\delta dE_1}{2}$. So we get

$$Q_{\frac{n}{2}-2} = -\frac{\Delta^2}{16} + \frac{d\delta E_1}{4} + \frac{\delta dE_1}{4} - \frac{dE_1\delta}{8} + \frac{E_1 d\delta}{4} - \frac{2E_1^2 + E_2}{4}.$$

□

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