ABSENCE OF RESONANCE NEAR THE CRITICAL LINE ON ASYMPTOTICALLY HYPERBOLIC SPACES

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ABSTRACT. As a consequence of a result of Cardoso-Vodev, we show that the resolvent of the Laplacian on asymptotically hyperbolic manifolds is analytic in an exponential neighbourhood of the critical line $\{\Re(\lambda) = \frac{n}{2}\}$. The case of non-trapping metrics with constant curvature near infinity is also considered: there is a strip $\{\Re(\lambda) > \frac{n}{2} - \epsilon\}$ with a finite number of resonances.

1. Introduction

The purpose of this note is to give some 'free of resonance' regions near the critical line for the Laplacian on asymptotically hyperbolic manifolds.

An asymptotically hyperbolic manifold is a smooth non-compact Riemannian manifold (X, g) of dimension $n+1$ which is the interior of a smooth compact manifold with boundary $\bar{X} = X \cup \partial\bar{X}$ and such that for all boundary defining function x of $\bar{X}$ (i.e. $\partial\bar{X} = \{x = 0\}$ and $dx|_{\partial\bar{X}} \neq 0$), $x^2 g$ extends to a smooth metric on $\bar{X}$ and $|dx|_{x^2 g} = 1$ on $\partial\bar{X}$. The metric can then be expressed in a collar neighbourhood of the boundary $(0, \epsilon)_x \times \partial\bar{X}_y$ by

$$g = \frac{dx^2 + h(x, y, dy)}{x^2} \tag{1.1}$$

with $h(x, y, dy)$ a smooth tensor up to the boundary $\{x = 0\}$. It can be seen that (X, g) is a complete manifold with curvatures approaching -1 near the boundary $\partial\bar{X}$ (the boundary is the infinity of $\bar{X}$ with respect to the metric g) and that the hyperbolic convex co-compact quotients are contained in this class of manifolds.

It is well known that the spectrum of the Laplacian Δ_g acting on functions splits into absolutely continuous spectrum $[\frac{n^2}{4}, \infty)$ and a finite set of eigenvalues $\sigma_{pp}(\Delta_g) \subset (0, \frac{n^2}{4})$. The modified resolvent

$$R(\lambda) := (\Delta_g - \lambda(n - \lambda))^{-1}$$

is then meromorphic on $\{\Re(\lambda) > \frac{n}{2}\}$ with finite rank poles at each λ_e satisfying $\lambda_e(n - \lambda_e) \in \sigma_{pp}(\Delta_g)$. Mazzeo and Melrose [6] have constructed the meromorphic extension of $R(\lambda)$ to $\mathbb{C} \setminus \frac{1}{2}(n - \mathbb{N})$ with poles of finite multiplicity, which are called *resonances*. Physically, the most interresting resonances are those in a neighbourhood of the critical line $\{\Re(\lambda) = \frac{n}{2}\}$ (which corresponds to the essential spectrum). It turns out that their localization is closely related to the number of geodesics trapped in compact sets of (X, g). However, a general principle which seems to hold for geometric scattering on a large class of infinite volume manifolds is that there exists a free of resonance region of the form

$$\{\lambda \in \mathbb{C}; \Im(\lambda) \leq e^{-C_1|\lambda|}, |\Re(\lambda)| \geq C_2\}, \quad C_1, C_2 > 0$$

when the critical line is the axis $\{\Im(\lambda) = 0\}$. This was first proved by Burq [1] in Euclidean scattering and Vodev [13] on some surfaces with negative constant curvature near infinity. It is worth noting that these results are optimal in the sense that, in general, the existence of elliptic closed geodesics implies the existence of resonances which are exponentially close to the critical line (see [11, 8, 9]).

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Here, we deal with the case of asymptotically hyperbolic manifolds:

Theorem 1.1. *Let (X, g) be an asymptotically hyperbolic manifold and x a boundary defining function, then there exist $C_1, C_2 > 0$ such that the weighted resolvent $x^{\frac{1}{2}}(\Delta_g - \lambda(n-\lambda))^{-1}x^{\frac{1}{2}}$ extends analytically from $\{\lambda \in \mathbb{C}; |\Im(\lambda)| > C_2, \Re(\lambda) > \frac{n}{2}\}$ to*

$$\{\lambda \in \mathbb{C}; |\Im(\lambda)| > C_2, \Re(\lambda) > \frac{n}{2} - e^{-C_1|\lambda|}\}. \tag{1.2}$$

as bounded operators on $L^2(X) := L^2(X, dvol_g)$.

The essential ingredients are a uniform bound of the weighted resolvent norm on the critical line and a sharp parametrix of the meromorphically continued resolvent. In our case, the resolvent bound $||\rho R(\lambda)\rho|| \leq Ce^{C|\lambda|}$ on the critical line has been proved by Cardoso and Vodev [2], ρ being a weight function decreasing to 0 near infinity. To extend analytically $\rho R(\lambda)\rho$ to the region (1.2), the main point is to see it as a perturbation of the resolvent of the Laplacian on a model space X_0 which is sufficiently close to our manifold. A good candidate for X_0 is the warped product $(0, \epsilon)_x \times \partial\bar{X}_y$ equipped with the metric

$$g_0 := \frac{dx^2 + h(0, y, dy)}{x^2}$$

and take Dirichlet condition at $x = \epsilon$, but for technical reasons we will better use $(0, \infty) \times \partial\bar{X}$ with the same metric and localize the resolvent near $x = 0$ with cut-off functions (a similar approach is used by Vodev [13]). This model resolvent $R_0(\lambda)$ needs to have an analytic extension on weighted spaces in a neighbourhood of the form (1.2), with a norm bounded by $Ce^{C|\lambda|}$. The classical resolvent equation

$$R(\lambda) - R(z) = (\lambda(n-\lambda) - z(n-z))R(\lambda)R(z)$$

in the physical sheet $\{\Re(\lambda) > \frac{n}{2}\}$ and the approximation of $R(\lambda)$ by $R_0(\lambda)$ allow to write

$$\rho R(\lambda)\rho(1 + (\lambda - z)K(\lambda, z)) = K_1(\lambda, z)$$

where $K(\lambda, z), K_1(\lambda, z)$ are some operators which are expressed in terms of the model resolvents, $\rho R(z)\rho$ and some error terms. At last, the extension properties of $\rho R_0(\lambda)\rho$ through the critical line and those of $\rho R(z)\rho$ up to this critical line can be used to extend $K(\lambda, z)$ and $K_1(\lambda, z)$ to $z \in \{\Re(\lambda) = \frac{n}{2}\}$ and λ in the neighbourhood (1.2). The bound on the norm of $\rho R(z)\rho$ and $\rho R_0(\lambda)\rho$ can then be used to show that $|\lambda - z|.||K(\lambda, z)|| \leq \frac{1}{2}$ for $\Re(z) = \frac{n}{2}$ and $|\lambda - z| \leq C^{-1}e^{-C|z|}$ with $C > 0$ large and independent of (λ, z); this allows to invert holomorphically $1 + (\lambda - z)K(\lambda, z)$ and to define $\rho R(\lambda)\rho$ in (1.2).

In the case of non-trapping metrics, Vodev [14] proved that the norm of the resolvent on the critical line grows not faster than $C|\lambda|^{-1}$ when $|\Im(\lambda)| \to \infty$, which implies a larger extension of the resolvent through the essential spectrum. We especially consider the case of non-trapping manifolds with constant curvature near infinity and obtain:

Theorem 1.2. *Let (X, g) be a conformally compact manifold with constant curvature outside a compact subset and let x be a boundary defining function. If g is non trapping, there exist $C_1, C_2 > 0$ such that the weighted resolvent $x^{\frac{1}{2}}(\Delta_g - \lambda(n-\lambda))^{-1}x^{\frac{1}{2}}$ extends analytically from $\{\lambda \in \mathbb{C}; |\Im(\lambda)| > C_2, \Re(\lambda) > \frac{n}{2}\}$ to*

$$\{\lambda \in \mathbb{C}; |\Im(\lambda)| > C_2, \Re(\lambda) > \frac{n}{2} - C_1\}$$

as bounded operators on $L^2(X)$.

Note that this non-trapping condition is not satisfied for non-elementary convex co-compact quotients of $\mathbb{H}^{n+1}$. When $X = \Gamma\backslash\mathbb{H}^{n+1}$ with Γ a non-elementary convex co-compact group of isometries, better results are available with the help of Selberg's zeta function: it is well known that there exists a half plane $\Re(\lambda) > \delta$ with no resonance, where δ is the dimension of the limit set (see also a result of Naud [7] in dimension 2). This shows that Theorem 1.2 is weak in the sense that we need the non-trapping assumption but the compensation is that we do not have to confine ourselves to the rigid class of constant curvature manifolds.

The paper is organized as follows: in Section 1, we recall Cardoso-Vodev Theorem, then we study our models in Section 2 and finally we give the proof of the results in Section 3.

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2. Cardoso-Vodev result

In [2], Cardoso and Vodev consider some Riemannian manifolds X with controlled structure near infinity and they obtain exponential bounds for the weighted resolvent norm on the critical line. These manifolds have the following properties outside a compact set Z

$$X \setminus Z \cong \big([R,\infty) \times S, g := dr^2 + \sigma(r)\big)\,, \quad R \gg 1, \tag{2.1}$$

where S is a n-dimensional smooth compact manifold, $\cong$ means 'isometric' and $\sigma(r) = \sigma(r,y,dy)$ is a family of metrics on $S_r := \{r\} \times S$ which satisfy

$$|q(r,y)| \leq C, \quad \partial_r q(r,y) \leq Cr^{-1-\delta}, \quad \delta > 0, \quad r > R, \tag{2.2}$$

$$-\partial_r(\sigma^{-1})(r,y,\xi) \geq \frac{C}{r}\sigma^{-1}(r,y,\xi), \quad \forall (y,\xi) \in T^*S_r, \tag{2.3}$$

$\sigma^{-1}(r)$ being the principal symbol of the Laplacian on $(S_r, \sigma(r))$ and $q(r,y)$ is an effective potential defined by

$$q(r,y) := (2^{-1}\partial_r \log \nu)^2 + (2\nu)^{-2}\sum_{i,j} \sigma^{ij}\partial_{y_i}\nu\partial_{y_j}\nu + 2^{-1}\nu\Delta_g(\nu^{-1})$$

with $\nu := (\det(\sigma_{ij}))^{\frac{1}{2}}$. An asymptotically hyperbolic metric $g = x^{-2}(dx^2 + h(x,y,dy))$ can be decomposed as in (2.1) by putting $x = e^{-r}$ and we get

$$\sigma(r,y,dy) = e^{2r}h(e^{-r},y,dy)$$

with $h(x,y,dy)$ smooth up to $x = 0$. In x coordinate, we have $\nu \in x^{-n}C^\infty(\bar{X})$, $\partial_r = -x\partial_x$ and $h^{ij} \in C^\infty(\bar{X})$ thus

$$(x\partial_x \log \nu)^2 + \nu^{-2}\sum_{i,j} x^2 h^{ij}\partial_{y_i}\nu\partial_{y_j}\nu \in C^\infty(\bar{X})$$

$$\nu\Delta_g\nu^{-1} = \nu\Big(- (x\partial_x)^2 - (x\partial_x \log \nu)x\partial_x - \nu^{-1}\sum_{i,j}\partial_{y_i}(\nu x^2 h^{ij}\partial_{y_j})\Big)\nu^{-1} \in C^\infty(\bar{X})$$

and (2.2) is satisfied for all $\delta > 0$. Moreover we have for all $(y,\xi) \in T^*\partial\bar{X}$

$$\frac{x\partial_x(x^2h^{-1})(x,y,\xi)}{x^2h^{-1}(x,y,\xi)} = 2 + x\frac{\partial_x(h^{-1})(x,y,\xi)}{h^{-1}(x,y,\xi)} \geq 1$$

if $x \leq \epsilon$ with ϵ small, and we obtain that (2.3) is satisfied. As a conclusion, asymptotically hyperbolic manifolds are in the class of manifolds studied by Cardoso and Vodev [2] and Vodev [14], so their results can be summarized in that case in the

Theorem 2.1. *Let (X,g) be an asymptotically hyperbolic manifold and x a boundary defining function. There exists $C > 0$ such that the weighted resolvent $x^{\frac{1}{2}}(\Delta_g - \lambda(n-\lambda))^{-1}x^{\frac{1}{2}}$ extends continuously from $\{\Re(\lambda) > \frac{n}{2}, |\Im(\lambda)| \geq 1\}$ to $\{\Re(\lambda) \geq \frac{n}{2}, |\Im(\lambda)| \geq 1\}$ on $L^2(X)$ and the extension satisfies*

$$||x^{\frac{1}{2}}R(\lambda)x^{\frac{1}{2}}||_{\mathcal{L}(L^2,H^p)} \leq Ce^{C|\lambda|}, \quad C > 0$$

for $p = 0, 1$, $|\Im(\lambda)| \geq 1$ and $0 \leq \Re(\lambda) - \frac{n}{2} \leq 1$, where H^p means the p-Sobolev space on X with respect to the metric g. If in addition g is non-trapping we have for $p = 0, 1$, $|\Im(\lambda)| \geq 1$ and $0 \leq \Re(\lambda) - \frac{n}{2} \leq 1$

$$||x^{\frac{1}{2}} R(\lambda) x^{\frac{1}{2}}||_{\mathcal{L}(L^2, H^p)} \leq C |\Im(\lambda)|^{-1+p}, \quad C > 0.$$

3. Two models

Before giving the models, we recall a few properties of some differential operators on X. Let (X, g) be a conformally compact manifold and Δ_g the Riemannian Laplacian. If x is a boundary defining function and $(y_i)_{i=1,\ldots,n}$ some coordinates on $\partial\bar{X}$, the space $\mathcal{V}_0(\bar{X})$ of smooth vector fields on $\bar{X}$ which vanish on $\partial\bar{X}$ is locally generated by $x\partial_x, x\partial_{y_i}$ for $i = 1, \ldots, n$ near the boundary. We denote by $\mathrm{Diff}_0^k(\bar{X})$ the space of differential operators of order k generated by k products of elements of $\mathcal{V}_0(\bar{X})$

$$\mathrm{Diff}_0^k(\bar{X}) := \operatorname*{Vect}_{0 \leq i \leq k} \mathcal{V}_0(\bar{X})^k, \quad \mathcal{V}_0(\bar{X})^0 = C^\infty(\bar{X}).$$

For example it is straightforward to check that

$$\Delta_g \in \mathrm{Diff}_0^2(\bar{X}).$$

Δ_g is now considered as the self-adjoint operator obtained by Friedrichs extension from the Laplacian on $C_0^\infty(X) \subset L^2(X) = L^2(X, dvol_g)$. For $k \in \mathbb{R}$, we define the k-Sobolev space by

$$H^k(X) := \mathcal{D}om((1 + \Delta_g)^{\frac{k}{2}}),$$

where $\mathcal{D}om$ means the domain. The Sobolev spaces associated to two different conformally compact metrics are the same (for instance, it is done for $k = 1, 2$ by Froese-Hislop [3, appendix] in a more general framework) and

$$\forall D^k \in \mathrm{Diff}_0^k(\bar{X}), \quad D^k \in \mathcal{L}(H^s(X), H^{s-k}(X)), \quad s \in [0, 2]. \tag{3.1}$$

Moreover a useful property of these differential operators is the following

$$x^{-\alpha} D^k x^\alpha \in \mathrm{Diff}_0^k(\bar{X}), \quad \alpha \in \mathbb{R}, \quad D^k \in \mathrm{Diff}_0^k(\bar{X}) \tag{3.2}$$

which is easily seen from the commutator $[x\partial_x, x^\alpha]$ in local charts near $\partial\bar{X}$.

Let us now study two models which will be respectively used for the parametrix construction of the general case and for the case of constant curvature near infinity.

Let (M, h_0) be a Riemannian compact manifold of dimension n and

$$X_0 := (0, +\infty)_x \times M, \quad g_0 := x^{-2}(dx^2 + h_0). \tag{3.3}$$

Though (X_0, g_0) is not conformally compact (there is a cusp end when $x \to \infty$), it has a conformally compact structure near $x = 0$. We could take as model operator the Laplacian on $((0, 1] \times \partial\bar{X}, g_0)$ with Dirichlet condition at $x = 1$ to have a conformally compact structure (with boundary), but we prefer to use X_0 since it carries more symmetry and is therefore easier to study.

As for conformally compact manifolds, let us denote $\bar{X}_0 := [0, +\infty) \times M$ and $\mathrm{Diff}_0^k(\bar{X}_0)$ the space of smooth differential operators of order k on $\bar{X}_0$ with support in $[0, 1] \times M$ and which can be locally written

$$\sum_{i + |\alpha| \leq k} a_{i,\alpha}(x, y)(x\partial_x)^i x^{|\alpha|} \partial_{y_i}^\alpha, \quad a_{i,\alpha} \in C^\infty(\bar{X}_0),$$

where $(y_i)_{i=1,\ldots,n}$ are some local coordinates on M.

By taking the new variable $r = \log x$, it is easy to see that the Laplacian Δ_{g_0} is unitarily equivalent to

$$P_0 = -\partial_r^2 + e^{2r} \Delta_{h_0} + \frac{n^2}{4}$$

on $L^2(\mathbb{R}\times M, drdvol_{h_0})$. As before, we define the Sobolev spaces by

$$H^k(X_0) := \mathcal{D}om((1+\Delta_{g_0})^{\frac{k}{2}}) \cong \mathcal{D}om((1+P_0)^{\frac{k}{2}}).$$

We first remark that the arguments given by Froese and Hislop in [3, appendix] prove that for $k=0,1,2$ and $s\in[0,2]$

$$\forall D^k \in \mathrm{Diff}_0^k(\bar{X}_0), \quad D^k \in \mathcal{L}(H^s(X_0), H^{s-k}(X_0)) \tag{3.4}$$

though they do not consider the 'cuspidal' part $\{r\in\mathbb{R}^+\}$. Of course the case $k=1$ can be directly obtained from the identity

$$||(1+P_0)^{\frac{1}{2}}u||^2_{L^2} = ||\partial_r u||^2_{L^2} + ||e^r\Delta_{h_0}^{\frac{1}{2}}u||^2_{L^2} + \left(\frac{n^2}{4}+1\right)||u||^2_{L^2}. \tag{3.5}$$

We shall first see how the resolvent of Δ_{g_0} can be extended to the non-physical sheet and we will give an upper bound of its weighted norm.

Lemma 3.1. *Let $x_0<1$, (X_0,g_0) defined in (3.3) and $\rho=\rho(x)$ a smooth function on X_0 with support in $\{x<1\}$ such that $\rho(x)=x^{\frac{1}{2}}$ for $x\le x_0$. Then the weighted resolvent*

$$\rho R_0(\lambda)\rho := \rho(\Delta_{g_0}-\lambda(n-\lambda))^{-1}\rho$$

extends analytically from $\{\Re(\lambda)>\frac{n}{2}, |\Im(\lambda)|>1\}$ to $\{\Re(\lambda)>\frac{n}{2}-\frac{1}{4}, |\Im(\lambda)|>1\}$ and it satisfies

$$||\partial_\lambda^q \rho R_0(\lambda)\rho||_{\mathcal{L}(L^2,H^p)} \le e^{C|\lambda|}, \quad C>0 \tag{3.6}$$

in that region for $q=0,1$, $p=0,1,2$.

Proof: let us begin by $p=2$. Let $\chi\in C_0^\infty([0,1))$ which is equal to 1 on $\mathrm{Supp}(\rho)$ and $\chi_1\in C_0^\infty([0,1))$ such that $\chi_1=1$ on $\mathrm{Supp}(\chi)$. We then have for $\lambda_0\in\{\Re(\lambda)>\frac{n}{2}\}$

$$(\Delta_{g_0}+1)\rho R_0(\lambda)\rho = [\Delta_{g_0},\rho]x^{-\frac{1}{2}}\chi(x)x^{\frac{1}{2}}R_0(\lambda)\rho + \rho^2 + (\Lambda+1)\rho R_0(\lambda)\rho,$$

$$\chi(x)x^{\frac{1}{2}}R_0(\lambda)\rho = R_0(\lambda_0)\Big([\Delta_{g_0},\chi(x)x^{\frac{1}{2}}]x^{-\frac{1}{2}}\chi_1(x)x^{\frac{1}{2}}R_0(\lambda) + x^{\frac{1}{2}} + (\Lambda-\Lambda_0)\chi(x)x^{\frac{1}{2}}R_0(\lambda)\Big)\rho$$

where $\Lambda:=\lambda(n-\lambda), \Lambda_0:=\lambda_0(n-\lambda_0)$. Since $[\Delta_{g_0},\rho]x^{-\frac{1}{2}}\in\mathrm{Diff}_0^1(\bar{X}_0)$ and $[\Delta_{g_0},\chi(x)x^{\frac{1}{2}}]x^{-\frac{1}{2}}\in\mathrm{Diff}_0^1(\bar{X}_0)$ in view of (3.2), it suffices to prove (3.6) for $p=0$ and use (3.4) to obtain the other cases.

Let us now use P_0 instead of Δ_{g_0}. We have a decomposition induced by the spectral resolution of Δ_{h_0}

$$P_0 = \bigoplus_{j\in\mathbb{N}_0} P_0^{(j)}, \quad P_0^{(j)} := -\partial_r^2 + e^{2r}\mu_j^2 + \frac{n^2}{4}$$

where $(\mu_j^2)_{j\in\mathbb{N}_0}$ are the eigenvalues of Δ_{h_0} (counted with multiplicities) associated to an orthonormal basis of $L^2(M)$ of eigenvectors $(\psi_j)_{j\in\mathbb{N}_0}$. If we denote the resolvent of $P_0^{(j)}$ on $L^2(\mathbb{R},dr)$ by

$$R_0^{(j)}(\lambda) := (P_0^{(j)}-\lambda(n-\lambda))^{-1}$$

we clearly have for $f\in L^2(X)$ and $f_j:=\langle f,\psi_j\rangle_{L^2(M)}$

$$\rho(\Delta_{g_0}-\lambda(n-\lambda))^{-1}\rho f = \sum_{j\in\mathbb{N}_0}(\rho R_0^{(j)}(\lambda)\rho f_j)\psi_j. \tag{3.7}$$

Note that for $\mu_j\neq 0$ the translation

$$U_j : \left\{\begin{array}{ccc} L^2(\mathbb{R},dr) & \to & L^2(\mathbb{R},dr) \\ f & \to & f(\log(\mu_j)+\bullet)\end{array}\right.$$

is an isometry and that $U_j^{-1}P_0^{(j)}U_j = Q$ with $Q:=-\partial_r^2+e^{2r}+\frac{n^2}{4}$. Let us set $k=\lambda-\frac{n}{2}$ for simplicity. The Green kernel for Q is then easy to find for $\Re(k)>0$ (see [12, Ex. 4.15])

$$R_Q(\lambda;r,t) := (Q-\lambda(n-\lambda))^{-1}(r,t) = -K_{-k}(e^r)I_k(e^t)H(r-t) - I_k(e^r)K_{-k}(e^t)H(t-r)$$

with H the Heaviside function and I_k, K_k the modified Bessel functions whose integral representations (when they converge) are

$$I_k(z) = \frac{2^{1-k}z^k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1 (1-u^2)^{k-\frac{1}{2}}\cosh(zu)du \tag{3.8}$$

$$I_k(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos(u)}\cos(ku)du - \frac{\sin(k\pi)}{\pi}\int_0^\infty e^{-z\cosh(u)-ku}du \tag{3.9}$$

$$K_k(z) = K_{-k}(z) = \frac{\Gamma(-k+\frac{3}{2})2^{-k+1}}{\Gamma(\frac{1}{2})}z^k\int_0^\infty \frac{\sin(tz)}{tz}t^2(1+t^2)^{k-\frac{3}{2}}dt \tag{3.10}$$

$$K_{-k}(z) = \int_0^\infty \cosh(ku)e^{-z\cosh(u)}du. \tag{3.11}$$

Moreover for $\mu_j = 0$, the expression of the meromorphically extended euclidian resolvent kernel is well-known in $\mathbb{C}\setminus\{0\}$

$$R_0^{(j)}(\lambda; r, t) = (2k)^{-1}e^{-k|r-t|}$$

and we have for $\Re(\lambda) > \frac{n}{2} - \frac{1}{4}$, $|\Im(\lambda)| \geq 1$ and $p = 0, 1, 2$

$$||\partial_r^p \rho(e^\bullet)R_0^{(j)}(\lambda)\rho(e^\bullet)||_{\mathcal{L}(L^2(\mathbb{R}))} \leq C\left|\lambda - \frac{n}{2}\right|^{-1+p} \quad \text{if } \mu_j = 0. \tag{3.12}$$

From these expressions, one can remark that there is no resonance except $\frac{n}{2}$ for this problem. Without loss of generality, we can suppose that $\rho(e^r) = e^{\frac{r}{2}}\chi(r)$ where χ is a smooth function on $\mathbb{R}$ such that $\chi(r) = 1$ when $r \leq -1$ and $\chi(r) = 0$ when $r \geq 0$. Since from (3.8)-(3.11) we have

$$|I_k(e^t)\rho(e^{t-\log(\mu_j)})| \leq \begin{cases} e^{C|k|}e^{e^t} & \text{for } t > 0 \\ e^{C|k|}e^{\frac{t}{4}} & \text{for } t \leq 0 \end{cases}$$

and

$$|K_{-k}(e^r)\rho(e^{r-\log(\mu_j)})| \leq \begin{cases} e^{C|k|}e^{-e^r} & \text{for } r > 0 \\ e^{C|k|}e^{\frac{r}{4}} & \text{for } r \leq 0 \end{cases}$$

for $|\Re(k)| \leq \frac{1}{4}$ and $|\Im(k)| \geq 1$, we can easily deduce that

$$|K_{-k}(e^r)\rho(e^{r-\log(\mu_j)})|\int_{-\infty}^r |I_k(e^t)\rho(e^{t-\log(\mu_j)})|dt \leq e^{C|k|}$$

$$|I_k(e^r)\rho(e^{r-\log(\mu_j)})|\int_r^\infty |K_{-k}(e^t)\rho(e^{t-\log(\mu_j)})|dt \leq e^{C|k|}$$

and Schur's lemma implies that

$$||U_j^{-1}(\rho)R_Q(\lambda)U_j^{-1}(\rho)||_{\mathcal{L}(L^2)} \leq e^{C|k|}$$

for $|\Re(k)| \leq \frac{1}{4}$, $|\Im(k)| \geq 1$. But since $U_j^{-1}(\rho(e^\bullet)) = U_j^{-1}\rho(e^\bullet)U_j$ as operators and U_j is an isometry, we use

$$R_0^{(j)}(\lambda) = U_jR_Q(\lambda)U_j^{-1}$$

to conclude that for $|\Im(\lambda)| \geq 1$ and $|\Re(\lambda) - \frac{n}{2}| \leq \frac{1}{4}$

$$||\rho(e^\bullet)R_0^{(j)}(\lambda)\rho(e^\bullet)||_{\mathcal{L}(L^2)} \leq e^{C|\lambda|}. \tag{3.13}$$

Finally we combine (3.13) with (3.12) and (3.7). The bound for $q = 1$ (one derivative with respect to λ) is directly obtained from the case $q = 0$ and Cauchy's formula. □

Remark: a better estimate can be obtained but we do not need it for our purpose.

The second model is the hyperbolic space $(\mathbb{H}^{n+1}, g_h)$ with its usual metric. To see $\mathbb{H}^{n+1}$ as a conformally compact manifold we take the Beltrami model

$$\mathbb{H}^{n+1} = \{m \in \mathbb{R}^{n+1}; |m| < 1\}, \quad g_h = \frac{4dm^2}{(1-|m|^2)^2}$$

and we set for example $x = 2(1-|m|)(1+|m|)^{-1}$ as boundary defining function.

Proposition 3.2. *The weighted hyperbolic resolvent*

$$x^{\frac{1}{2}} R_h(\lambda) x^{\frac{1}{2}} := x^{\frac{1}{2}} (\Delta_{g_h} - \lambda(n-\lambda))^{-1} x^{\frac{1}{2}}$$

extends holomophically from $\{\Re(\lambda) > \frac{n}{2}\}$ to $\{\Re(\lambda) > \frac{n}{2} - \frac{1}{8}\}$ with values in $\mathcal{L}(L^2(\mathbb{H}^{n+1}), H^1(\mathbb{H}^{n+1}))$ and it satisfies

$$\left|\left|\partial_\lambda^q x^{\frac{1}{2}} R_h(\lambda) x^{\frac{1}{2}}\right|\right|_{\mathcal{L}(L^2, H^p)} \leq C \left|\lambda - \frac{n}{2}\right|^{-1+p}, \quad C > 0 \tag{3.14}$$

in the same region (with $\lambda \neq \frac{n}{2}$) for $p = 0, 1$, $q = 0, 1$.

Proof: to show these bounds, we consider the wave operators

$$U_0(t) := \cos\left(t\sqrt{\Delta_{g_h} - \frac{n^2}{4}}\right), \quad U_1(t) := \left(\Delta_{g_h} - \frac{n^2}{4}\right)^{-\frac{1}{2}} \sin\left(t\sqrt{\Delta_{g_h} - \frac{n^2}{4}}\right).$$

Let $\delta > 0$ be small, $\chi \in C_0^\infty(\mathbb{R}^+)$ with support in $[0, 1 - \frac{\delta}{2}[$ and which is equal to 1 on $[0, 1-\delta]$. Then set

$$\chi_t(m) := \chi\left(\frac{4\text{argtanh}|m|}{t}\right)$$

whose support is included in the hyperbolic ball of radius $\frac{t}{2}$ and which is equal to 1 in the hyperbolic ball of radius $\frac{t(1-\delta)}{2}$.

When $n+1$ is odd, we have by Huygens principle (see [5])

$$\chi_t U_0(t) \chi_t = 0, \quad t > 0. \tag{3.15}$$

Now we split $x^{\frac{1}{2}}$ in the following way

$$x^{\frac{1}{2}} U_0(t) x^{\frac{1}{2}} = (1-\chi_t) x^{\frac{1}{2}} U_0(t) (1-\chi_t) x^{\frac{1}{2}} + (1-\chi_t) x^{\frac{1}{2}} U_0(t) \chi_t x^{\frac{1}{2}} + \chi_t x^{\frac{1}{2}} U_0(t)(1-\chi_t) x^{\frac{1}{2}} \tag{3.16}$$

for $t > 0$. It is clear that

$$||\chi_t x^{\frac{1}{2}}|| \leq C, \quad ||U_0(t)|| \leq 1 \tag{3.17}$$

and remark that $m \in \text{Supp}(1-\chi_t)$ only if $|m| \geq \tanh\left(\frac{(1-\delta)t}{4}\right)$, that is

$$x = 2\frac{1-|m|}{1+|m|} \leq 2\frac{1-\tanh(\frac{t(1-\delta)}{4})}{1+\tanh(\frac{t(1-\delta)}{4})} = 2e^{-\frac{t(1-\delta)}{2}}.$$

So we find

$$\left|\left|(1-\chi_t) x^{\frac{1}{2}}\right|\right| \leq Ce^{-\frac{t(1-\delta)}{4}}. \tag{3.18}$$

From (3.16), (3.17) and (3.18) we deduce that

$$\left|\left|x^{\frac{1}{2}} U_0(t) x^{\frac{1}{2}}\right|\right| \leq Ce^{-\frac{t(1-\delta)}{4}}.$$

It remains to use the Laplace transform of U_0

$$\left(\lambda - \frac{n}{2}\right) x^{\frac{1}{2}} R_h(\lambda) x^{\frac{1}{2}} = \int_0^\infty e^{t(\frac{n}{2}-\lambda)} x^{\frac{1}{2}} U_0(t) x^{\frac{1}{2}} dt$$

and (3.14) is proved when $p=0$ and $n+1$ odd by taking $\delta<\frac{1}{2}$. To deal with the case $n+1$ even, we study $x^{\frac{1}{2}}\partial_t U_1(t)x^{\frac{1}{2}}$ and use $U_0(t)=\partial_t U_1(t)$. We have

$$\chi_t\partial_t U_1(t)\chi_t=\partial_t\left(\chi_t U_1(t)\chi_t\right)-(\partial_t\chi_t)U_1(t)\chi_t-\chi_t U_1(t)(\partial_t\chi_t) \tag{3.19}$$

and the Schwartz kernel of $U_1(t)$ is (see [5])

$$U_1(t;x,y)=C_n\left(\sinh^2\left(\frac{t}{2}\right)-\sinh^2\left(\frac{d_{\mathbb{H}^{n+1}}(x,y)}{2}\right)\right)_+^{-\frac{n}{2}} \tag{3.20}$$

with $C_n\in\mathbb{R}$. Hence, by construction of χ_t, the operators in (3.19) have a smooth kernel with support in $\{(x,y)\in\mathbb{H}^{n+1}\times\mathbb{H}^{n+1}, d_{\mathbb{H}^{n+1}}(x,y)<t(1-\frac{\delta}{2})\}$ and (3.20) implies that there exists $T>0$ such that

$$||\chi_t\partial_t U_1(t)\chi_t||\leq Ce^{-\frac{nt}{2}} \tag{3.21}$$

for $t\geq T$. At last, we proceed as in the odd case: we split $x^{\frac{1}{2}}$ on the support of χ_t and outside, which shows (3.14) for $p=0$.

Now for $p=1$, it suffices to show that $Dx^{\frac{1}{2}}R_h(\lambda)x^{\frac{1}{2}}$ extends to $\{\Re(\lambda)>\frac{n}{2}-\frac{1}{8}\}$ in $\mathcal{L}(L^2)$ for a finite number of $D\in\mathrm{Diff}_0^1(\overline{\mathbb{H}}^{n+1})$ which span $\mathrm{Diff}_0^1(\overline{\mathbb{H}}^{n+1})$ over $C^\infty(\overline{\mathbb{H}}^{n+1})$. In fact since $Dx^{\frac{1}{2}}\in x^{\frac{1}{2}}\mathrm{Diff}_0^1(\overline{\mathbb{H}}^{n+1})$, we will study $x^{\frac{1}{2}}DR_0(\lambda)x^{\frac{1}{2}}$. With D as before, we have for $\Re(\lambda)>\frac{n}{2}$

$$x^{\frac{1}{2}}DR_0(\lambda)x^{\frac{1}{2}}=\int_0^\infty e^{t(\frac{n}{2}-\lambda)}x^{\frac{1}{2}}DU_1(t)x^{\frac{1}{2}}dt \tag{3.22}$$

but $||DU_1(t)||\leq C$ in view of the continuity of D from $H^1(\mathbb{H}^{n+1})$ to $L^2(\mathbb{H}^{n+1})$. Moreover

$$\chi_t DU_1(t)\chi_t=D\chi_t U_1(t)\chi_t-[D,\chi_t]U_1(t)\chi_t$$

has a smooth kernel with compact support and it is straightforward to check (from (3.20)) that there exists $T>0$ such that for all $t\geq T$

$$||\chi_t DU_1(t)\chi_t||\leq Ce^{-\frac{nt}{2}} \tag{3.23}$$

Splitting $x^{\frac{1}{2}}DU_1(t)x^{\frac{1}{2}}$ in the same way as (3.16) and using (3.23), (3.22), one deduces the bound (3.14) for $p=1$. The case $q=1$ is obtained by the Cauchy formula from $q=0$. □

4. Parametrix construction and proof of the main result

In this section, we will give the construction of a parametrix for the resolvent $R(\lambda)$ of Δ_g on an arbitrary asymptotically hyperbolic manifold (X,g) whose metric is

$$g=x^{-2}(dx^2+h(x))$$

in a collar $(0,\delta)_x\times\partial\bar{X}$ near the boundary, $h(x)$ being a smooth family of metrics on $\partial\bar{X}$. Of course this parametrix will only be sufficient to approach the resolvent in $\{\Re(\lambda)>\frac{n}{2}-\delta\}$ with $\delta>0$ small. Roughly, the singularities at $x=0$ and $x'=0$ of the kernel of $R(\lambda)$ are controlled by those of the resolvent kernel of the Laplacian induced by the model metric $x^{-2}(dx^2+h(0))$ near the boundary, already studied in the previous section.

Proof of Theorem 1.1: consider (X,g) an asymptotically hyperbolic manifold and $R(\lambda):=(\Delta_g-\lambda(n-\lambda))^{-1}$ the resolvent for Δ_g in the physical sheet $\{\Re(\lambda)>\frac{n}{2}\}$. Let V be an open subset in (X,g) isometric to $U:=(0,\delta)_x\times\partial\bar{X}$ (with $\delta>0$) equipped with the metric $x^{-2}(dx^2+h(x))$ and $i:V\to U$ this isometry. Note that it is always possible to renormalize x (take $x'=x\delta^{-1}$) to have $\delta=1$ and it does not change the structure of the metric, so we suppose $\delta=1$. We now consider (X_0,g_0) the Riemannian manifold defined in (3.3) with

$$(M,h_0):=(\partial\bar{X},h(0)). \tag{4.1}$$

Let $\mathcal{I}_U$ and $\mathcal{R}_U$ be the following bounded operators

$$\mathcal{R}_U : \begin{cases} L^2(X_0, dvol_{g_0}) & \to \quad L^2(U, dvol_{g_0}) \\ \qquad\qquad f & \to \quad f(\iota_U(\bullet)) \end{cases}$$

$$\mathcal{I}_U : \begin{cases} L^2(U, dvol_{g_0}) & \to \quad L^2(X_0, dvol_{g_0}) \\ \qquad\qquad f & \to \quad 1\!\!1_U f \end{cases}$$

with ι_U the inclusion $U \subset X_0$ and $1\!\!1_U$ the characteristic function of U. Similarly one can define the operators $\mathcal{I}_V$ and $\mathcal{R}_V$ induced by the inclusion $V \subset X$. Since i^*g_0 and g are quasi-isometric on V, we obtain that $i^* : L^2(U, dvol_{g_0}) \to L^2(V, dvol_g)$ and $i_* : L^2(V, dvol_g) \to L^2(U, dvol_{g_0})$ are bounded. We then set

$$I^* := \mathcal{I}_V i^* \mathcal{R}_U \in \mathcal{L}(L^2(X_0, dvol_{g_0}), L^2(X, dvol_g)),$$

$$I_* := \mathcal{I}_U i_* \mathcal{R}_V \in \mathcal{L}(L^2(X, dvol_g), L^2(X_0, dvol_{g_0})).$$

For $j = 1, 2, 3, 4$, let ψ_j a smooth function on $\mathbb{R}^+$ which is equal to 1 in $[0, \frac{j}{5}]$ and to 0 in $[\frac{j+1}{5}, +\infty)$; ψ_j will also be considered as a function on U depending only on x. With $\widetilde{\psi}_j := i^*\psi_j$, we have as operators

$$I_* I^* \psi_j = \psi_j, \quad I^* I_* \widetilde{\psi}_j = \widetilde{\psi}_j.$$

It is easy to check that there exist $D_R, D_L \in \mathrm{Diff}_0^2(\bar{X})$ such that

$$\Delta_g - I^* \Delta_{g_0} \psi_3 I_* = x D_R, \quad \Delta_g - I^* \psi_3 \Delta_{g_0} I_* = x D_L.$$

Firstly, we take $\Re(\lambda) > \frac{n}{2}$ and $R_0(\lambda)$ is the resolvent considered in Lemma 3.1. Observe that

$$(\Delta_{g_0} \psi_3 - \lambda(n-\lambda)) \psi_2 R_0(\lambda) \psi_1 = \psi_1 + [\Delta_{g_0}, \psi_2] R_0(\lambda) \psi_1 \tag{4.2}$$

since $\psi_3 \psi_2 = \psi_2$ and $\psi_2 \psi_1 = \psi_1$. Let $\chi_1 := 1 - \widetilde{\psi}_1$ and let χ_0 be a smooth function with compact support on X which is equal to 1 on the support of χ_1. For $\lambda_0 \in \{\Re(\lambda) > \frac{n}{2}\}$ fixed, $\Lambda := \lambda(n-\lambda)$ and $\Lambda_0 := \lambda_0(n - \lambda_0)$ we set

$$R_{0R}(\lambda) := I^* \psi_2 R_0(\lambda) \psi_1 I_*, \quad E_R(\lambda) := R_{0R}(\lambda) + \chi_0 R(\lambda_0) \chi_1,$$

$$L_R(\lambda) := [\Delta_g, \chi_0] R(\lambda_0) \chi_1 + (\Lambda_0 - \Lambda) \chi_0 R(\lambda_0) \chi_1 + I^* [\Delta_{g_0}, \psi_2] R_0(\lambda) \psi_1 I_* + x D_R R_{0R}(\lambda),$$

and we deduce from (4.2)

$$(\Delta_g - \Lambda) E_R(\lambda) = 1 + L_R(\lambda). \tag{4.3}$$

Similarly, we give a left parametrix for $\Delta_g - \Lambda$

$$R_{0L}(\lambda) := I^* \psi_1 R_0(\lambda) \psi_2 I_*, \quad E_L(\lambda) = R_{0L}(\lambda) + \chi_1 R(\lambda_0) \chi_0,$$

$$L_L(\lambda) = R_{0L}(\lambda) x D_L + I^* \psi_1 R_0(\lambda) [\psi_2, \Delta_{g_0}] I_* + \chi_1 R(\lambda_0) [\Delta_g, \chi_0] + (\Lambda_0 - \Lambda) \chi_1 R(\lambda_0) \chi_0,$$

and we have

$$E_L(\lambda)(\Delta_g - \Lambda) = 1 + L_L(\lambda). \tag{4.4}$$

Let $z, \lambda \in \{s \in \mathbb{C}; \Re(s) > \frac{n}{2}\}$ and $Z := z(n-z)$, from (4.3) and (4.4) we then obtain

$$R(\lambda) = E_R(\lambda) - R(\lambda) L_R(\lambda), \quad R(z) = E_L(z) - L_L(z) R(z). \tag{4.5}$$

On the other hand, the resolvent equation

$$R(\lambda) - R(z) = (\Lambda - Z) R(\lambda) R(z)$$

combined with the first identity of (4.5) yield

$$x^{\frac{1}{2}} R(\lambda) x^{\frac{1}{2}} (1 + K(\lambda, z)) = K_1(\lambda, z), \tag{4.6}$$

$$K(\lambda, z) := (\Lambda - Z) x^{-\frac{1}{2}} L_R(\lambda) R(z) x^{\frac{1}{2}}, \quad K_1(\lambda, z) := x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} + (\Lambda - Z) x^{\frac{1}{2}} E_R(\lambda) R(z) x^{\frac{1}{2}}.$$

For simplicity, we will now denote by D^p (resp. D^p_0) all differential operator in $\mathrm{Diff}^p_0(\bar{X})$ with support in $\mathrm{Supp}(\widetilde{\psi}_3)$ (resp. $\mathrm{Diff}^p_0(\bar{X}_0)$ with support in $\mathrm{Supp}(\psi_3)$). With these notations, (3.2) is summarized by

$$D^p x^\alpha = x^\alpha D^p$$

and we also get

$$D^p_0 I_* = I_* D^p, \quad D^p I^* = I^* D^p_0.$$

Using that χ_0, χ_1 and $[\Delta_{g_0}, \psi_2]$ have compact support, we obtain

$$x^{-\frac{1}{2}} L_R(\lambda) = \left(D^1 + (\Lambda_0 - \Lambda)x^{-1}\chi_0\right) x^{\frac{1}{2}} R(\lambda_0)\chi_1 + x^{\frac{1}{2}} I^* D^2_0 R_0(\lambda)\psi_1 I_*, \tag{4.7}$$

$$x^{\frac{1}{2}} E_R(\lambda) = x^{\frac{1}{2}} R_{0R}(\lambda) + x^{\frac{1}{2}}\chi_0 R(\lambda_0)\chi_1. \tag{4.8}$$

Similarly, the second indentity of (4.5) and the definition of $E_L(\lambda), L_L(\lambda)$ imply that

$$\begin{aligned} R(z)x^{\frac{1}{2}} &= I^*\psi_1 R_0(z)(I_* x^{\frac{1}{2}} D^2 x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} + \psi_2 I_* x^{\frac{1}{2}}) \\ &\quad + \chi_1 R(\lambda_0) x^{\frac{1}{2}} \left(\chi_0 + D^1 x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} + (Z - \Lambda_0) x^{-\frac{1}{2}} \chi_0 R(z) x^{\frac{1}{2}}\right). \end{aligned} \tag{4.9}$$

Combining this last expression with (4.7) gives

$$\begin{aligned} \frac{K(\lambda, z)}{\Lambda - Z} &= \left(D^1 + (\Lambda_0 - \Lambda)x^{-1}\chi_0\right) x^{\frac{1}{2}} R(\lambda_0)\chi_1 R(z) x^{\frac{1}{2}} \\ &+ x^{\frac{1}{2}} I^* D^2_0 \psi_4 R_0(\lambda)\psi_1 I_* x^{\frac{1}{2}} x^{-\frac{1}{2}} \chi_1 R(\lambda_0) x^{\frac{1}{2}} \left(\chi_0 + D^1 x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} + (Z - \Lambda_0) x^{-\frac{1}{2}}\chi_0 R(z) x^{\frac{1}{2}}\right) \\ &+ x^{\frac{1}{2}} I^* D^2_0 \psi_4 R_0(\lambda)\psi_1^2 R_0(z)\psi_4 I_* x^{\frac{1}{2}} (D^2 x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} + \widetilde{\psi}_2). \end{aligned}$$

The first two lines of this expression can be extended to $\{\Re(\lambda) > \frac{n}{2} - \frac{1}{4}\} \cap \{|\Im(\lambda)| \geq 1\}$ and $\{\Re(z) \geq \frac{n}{2}\} \cap \{|\Im(z)| \geq C\}$ as bounded operators on $L^2(X)$ by using Lemma 3.1 and Theorem 2.1 (to be in the settings of Lemma 3.1, we take $\rho := i_*(x^{\frac{1}{2}})\psi_4$ and we remark that $x^{\frac{1}{2}} I^* D^2_0 \psi_4 = I^* D^2_0 \rho$ and $\psi_1 I_* x^{\frac{1}{2}} = \rho\psi_1 I_*$), and their $\mathcal{L}(L^2)$ norms is bounded by $Ce^{C(|\lambda|+|z|)}$.

It remains to consider the last line in the expression of $(\Lambda - Z)^{-1} K(\lambda, z)$. Using (3.1) and Theorem 2.1, observe that

$$D^2 x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} = D^2(\Delta_g + i)^{-1}(D^1 x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} + x + (i + Z) x^{\frac{1}{2}} R(z) x^{\frac{1}{2}})$$

can be extended to $\Re(z) = \frac{n}{2}$ and $|\Im(z)| \gg 0$ with $\mathcal{L}(L^2)$ norm bounded by $Ce^{C|z|}$. Note that

$$\begin{aligned} \rho R_0(\lambda)\psi_1^2 R_0(z)\rho &= \rho R_0(\lambda)\psi_1(\Delta_{g_0} - \Lambda) R_0(\lambda) R_0(z)(\Delta_{g_0} - Z)\psi_1 R_0(z)\rho \\ &= \rho(\psi_1 + R_0(\lambda)[\psi_1, \Delta_{g_0}]) R_0(\lambda) R_0(z)(\psi_1 - [\psi_1, \Delta_{g_0}] R_0(z))\rho. \end{aligned}$$

Now it is easy to see that it can be rewritten by

$$(\psi_1 + \rho R_0(\lambda)[\psi_1, \Delta_{g_0}] x^{-\frac{1}{2}}) \frac{\rho R_0(\lambda)\rho - \rho R_0(z)\rho}{\Lambda - Z}(\psi_1 - x^{-\frac{1}{2}}[\psi_1, \Delta_{g_0}] R_0(z)\rho).$$

Recall that $x^{-\frac{1}{2}}[\psi_1, \Delta_{g_0}] x^{-\frac{1}{2}} \in \mathrm{Diff}^1_0(\bar{X}_0)$ since $[\psi_1, \Delta_{g_0}]$ has compact support in X_0. Using this expression with Lemma 3.1, we obtain that

$$x^{\frac{1}{2}} I^* D^2_0 \psi_4 R_0(\lambda)\psi_1^2 R_0(z)\psi_4 I_* x^{\frac{1}{2}} (D^2 x^{\frac{1}{2}} R(z) x^{\frac{1}{2}} + \widetilde{\psi}_2)$$

extends to $\{\Re(\lambda) > \frac{n}{2} - \frac{1}{4}, |\Im(\lambda)| \geq 1\}$ and $\{\Re(z) \geq \frac{n}{2}, |\Im(z)| \geq C\}$ as continuous operators on $L^2(X)$ with norm bounded by $Ce^{C(|\lambda|+|z|)}$. Fix $z = \frac{n}{2} + is$ with $|s|$ large, then all these estimates prove that

$$||K(\lambda, z)||_{\mathcal{L}(L^2)} \leq \left|\Re(\lambda) - \frac{n}{2}\right| Ce^{C|s|}$$

when $\Im(\lambda) = s$ and $\Re(\lambda) > \frac{n}{2} - \frac{1}{4}$. Moreover this term is bounded by $\frac{1}{2}$ if

$$\left|\Re(\lambda) - \frac{n}{2}\right| \leq \frac{1}{2} C^{-1} e^{-C|\Im(\lambda)|}, \quad \Im(\lambda) = s \tag{4.10}$$

Since $K(\lambda, z)$ is holomorphic in λ in $\{\Re(\lambda) > \frac{n}{2} - \frac{1}{4}, |\Im(\lambda)| > 1\}$ and C does not depend on s, one can invert $(1 + K(\lambda, z))$ holomorphically if λ satisfies (4.10).

Finally the term $K_1(\lambda, z)$ can be treated in a similar way, using (4.8) and (4.9). So the proof is complete in the general case. □

When g is non-trapping we could apply the same method and prove that there exists a free of resonance region polynomially close to the critical line. We prefer to only write down the case of constant curvature near infinity since we obtain a slightly better result.

Proof of Theorem 1.2: the proof is essentially similar. When the metric g has constant curvature near infinity there exist (see [4]) some charts $(V_j)_{j=1,\dots,M}$ covering a neighbourhood of the boundary $\partial\bar{X}$ such that each V_j is isometric (note i_j this isometry) to the open set

$$B := \{m = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1}; y_{n+1} > 0, \sum_i y_i^2 < 1\} \tag{4.11}$$

equipped with the hyperbolic metric $y_{n+1}^{-2}(\sum_i dy_i^2)$. Actually, a parametrix sufficient for our problem is given in [4, Prop. 3.1]. Fix $\lambda_0 \in \{\Re(\lambda) > \frac{n}{2}\}$ and recall that Δ_{g_h} is the Laplacian on $\mathbb{H}^{n+1}$ and $R_h(\lambda)$ its resolvent studied in Lemma 3.2. Let χ_i^j be some functions with support in (4.11), such that $\chi_1^j = 1$ on the support of χ_2^j and

$$\chi_i^j(y_1, \dots, y_{n+1}) = \phi_i^j(y_1, \dots, y_n)\psi_i^j(y_{n+1})$$

with $\phi_i^j \in C_0^\infty(\mathbb{R}^n)$ satisfying $\sum_j i_j^*\phi_2^j = 1$ on $\partial\bar{X}$ and $\psi_i^j \in C_0^\infty([0,1))$ which is equal to 1 for $y_{n+1} \leq \delta$ with $\delta < 1$ small (see [4] or [10, Lem. 3.2] for details). This implies that $\chi := 1 - \sum_{j=1}^M i_j^*\chi_2^j$ has compact support in X. Let $\chi_0 \in C_0^\infty(X)$ such that $\chi_0 = 1$ on the support of χ. Finally I_j^* and I_{j*} are defined as in the proof of previous theorem with our new isometries $V_j \to B$ and the inclusions $V_j \subset X$, $B \subset \mathbb{H}^{n+1}$. We then apply the proof of previous theorem but we replace E_R, E_L by our new parametrix

$$E_R(\lambda) := \chi_0 R(\lambda_0)\chi + \sum_{j=1}^M I_j^*\chi_1^j R_h(\lambda)\chi_2^j I_{j*}, \quad E_L(\lambda) := \chi R(\lambda_0)\chi_0 + \sum_{j=1}^M I_j^*\chi_2^j R_h(\lambda)\chi_1^j I_{j*}$$

and the error terms are

$$L_R(\lambda) := [\Delta_g, \chi_0]R(\lambda_0)\chi + (\Lambda_0 - \Lambda)\chi_0 R(\lambda_0)\chi + \sum_{j=1}^M I_j^*[\Delta_{g_h}, \chi_1^j]R_h(\lambda)\chi_2^j I_{j*},$$

$$L_L(\lambda) := \chi R(\lambda_0)[\chi_0, \Delta_g] + (\Lambda_0 - \Lambda)\chi R(\lambda_0)\chi_0 + \sum_{j=1}^M I_j^*\chi_2^j R_h(\lambda)[\chi_1^j, \Delta_{g_h}]I_{j*}$$

where $\Lambda := \lambda(n - \lambda)$, $\Lambda_0 := \lambda_0(n - \lambda_0)$. As before, $x_j := i_{j*}x$ is a boundary defining function of $\{y_{n+1} = 0\}$ in $\bar{B}$. Moreover it is easy to check (see [4, Prop. 3.1]) that

$$x_j^{-\frac{1}{2}}[\Delta_{g_h}, \chi_i^j]x_j^{-\frac{1}{2}} \in \mathrm{Diff}_0^1(\overline{\mathbb{H}}^{n+1}), \quad i = 1, 2 \tag{4.12}$$

where we consider $B \subset \mathbb{H}^{n+1}$. We then have the same as (4.6) with

$$\begin{aligned}\frac{K(\lambda, z)}{\Lambda - Z} &= x^{-\frac{1}{2}}\left([\Delta_g, \chi_0]R(\lambda_0)\chi + (\Lambda_0 - \Lambda)\chi_0 R(\lambda_0)\chi + \sum_{j=1}^M I_j^*[\Delta_{g_h}, \chi_1^j]R_h(\lambda)\chi_2^j I_{j*}\right) \\ &\times\Big(\chi R(\lambda_0)\chi_0 + \sum_{j=1}^M I_j^*\chi_2^j R_h(z)(\chi_1^j I_{j*} + [\Delta_{g_h}, \chi_1^j]I_{j*}R(z)) \\ &\quad + \chi R(\lambda_0)[\Delta_g, \chi_0]R(z) - (\Lambda_0 - Z)\chi R(\lambda_0)\chi_0 R(z)\Big)x^{\frac{1}{2}}\end{aligned}$$

for λ, z in the physical sheet. Note that Theorem 2.1, Lemma 3.2 and (4.12) prove that all these products extend to $\Re(z) = \frac{n}{2}$ and $\Re(\lambda) > \frac{n}{2} - \frac{1}{8}$ except maybe

$$x^{-\frac{1}{2}}\left(\sum_{j=1}^{M} I_j^*[\Delta_{g_h}, \chi_1^j]R_h(\lambda)\chi_2^j I_{j*}\right)\left(\sum_{j=1}^{M} I_j^*\chi_2^j R_h(\lambda)(\chi_1^j I_{j*} + [\Delta_{g_h}, \chi_1^j]I_{j*}R(z))\right)x^{\frac{1}{2}}. \tag{4.13}$$

Fix $\lambda_0 = \frac{n}{2} + \frac{1}{8} + is_0$ with $s_0 > 0$ large and consider some (λ, z) in

$$\mathcal{O} := \left\{(\lambda, z) \in \mathbb{C}^2; |\lambda - \lambda_0| < \frac{1}{4}, |z - \lambda_0| < \frac{1}{4}, \Re(z) \geq \frac{n}{2}\right\}.$$

Theorem 2.1 implies that for $\alpha = -\frac{1}{2}, 0$

$$\left|\left|x^{-\frac{1}{2}}[\Delta_g, \chi_0]R(\lambda_0)\chi x^{\alpha} + (\Lambda - \Lambda_0)x^{-\frac{1}{2}}\chi_0 R(\lambda_0)\chi x^{\alpha}\right|\right|_{\mathcal{L}(L^2)} \leq C, \tag{4.14}$$

$$\left|\left|x^{\alpha}\chi R(\lambda_0)\chi_0 x^{\frac{1}{2}} + (Z - \Lambda_0)x^{\alpha}\chi R(\lambda_0)\chi_0 R(z)x^{\frac{1}{2}} + x^{\alpha}\chi R(\lambda_0)[\Delta_g, \chi_0]R(z)x^{\frac{1}{2}}\right|\right|_{\mathcal{L}(L^2)} \leq Cs_0^{-1} \tag{4.15}$$

and C does not depend on s_0. Recall that I_j^*, I_{j*} are some isometries $L^2(V_j) \leftrightarrow L^2(B)$ and $x_j = i_{j*}x$, then Lemma 3.2 and (4.12) lead to

$$\left|\left|\sum_{j=1}^{M} I_j^* x_j^{-\frac{1}{2}}[\Delta_{g_h}, \chi_1^j]R_h(\lambda)\chi_2^j x_j^{\frac{1}{2}} I_{j*}\right|\right|_{\mathcal{L}(L^2)} \leq C \tag{4.16}$$

$$\left|\left|\sum_{j=1}^{M} I_j^* x_j^{\frac{1}{2}}\chi_2^j R_h(z)x_j^{\frac{1}{2}}(\chi_1^j I_{j*} + x_j^{-\frac{1}{2}}[\Delta_{g_h}, \chi_1^j]x_j^{-\frac{1}{2}}I_{j*}x^{\frac{1}{2}}R(z)x^{\frac{1}{2}})\right|\right|_{\mathcal{L}(L^2)} \leq Cs_0^{-1} \tag{4.17}$$

and it remains to consider (4.13), which a priori does not exist in $\mathcal{O}$. When $V_j \cap V_i = \emptyset$ it is clear that $\chi_2^j I_{j*}I_i^*\chi_2^i = 0$, so suppose that $V_j \cap V_i \neq \emptyset$. Using that the isometry $i_i \circ i_j^{-1} : i_j(V_i \cap V_j) \to i_i(V_i \cap V_j)$ extends to an isometry I_{ij} on $\mathbb{H}^{n+1}$ and following the proof of previous theorem, we can see that

$$x_j^{\frac{1}{2}}R_h(\lambda)\chi_2^j I_{j*}I_i^*\chi_2^i R_h(z)x_i^{\frac{1}{2}}$$

can be expressed for λ, z in the physical sheet $\{\Re(\lambda) > \frac{n}{2}\}$ by

$$(\chi_2^j + x_j^{\frac{1}{2}}R_h(\lambda)[\chi_2^j, \Delta_{g_h}]x_j^{-\frac{1}{2}})\frac{x_j^{\frac{1}{2}}R_h(\lambda)\widetilde{x}_j^{\frac{1}{2}} - x_j^{\frac{1}{2}}R_h(z)\widetilde{x}_j^{\frac{1}{2}}}{\Lambda - Z}(\widetilde{x}_j^{-\frac{1}{2}}[\widetilde{\chi}_2^i, \Delta_{g_h}]R_h(z)\widetilde{x}_j^{\frac{1}{2}} + \widetilde{\chi}_2^i)I_{ij}^*$$

where $\widetilde{\chi}_2^i := I_{ij}^*\chi_2^i$, $\widetilde{x}_j := I_{ij}^* x_i$. Observe that this operator on $\mathbb{H}^{n+1}$ extends to $(\lambda, z) \in \mathcal{O}$ and satisfies in $\mathcal{O}$

$$||x_j^{\frac{1}{2}}R_h(\lambda)\chi_2^j I_{j*}I_i^*\chi_2^i R_h(z)x_i^{\frac{1}{2}}||_{\mathcal{L}(L^2,H^1)} \leq Cs_0^{-1} \tag{4.18}$$

in view of Lemma 3.2 and the following bound (implicitly used in the proof of previous theorem)

$$\left|\left|\frac{x_j^{\frac{1}{2}}R_h(\lambda)\widetilde{x}_j^{\frac{1}{2}} - x_j^{\frac{1}{2}}R_h(z)\widetilde{x}_j^{\frac{1}{2}}}{\Lambda - Z}\right|\right|_{\mathcal{L}(L^2,H^1)} \leq Cs_0^{-1}\sup_{|\lambda-\lambda_0|<\frac{1}{4}}||\partial_\lambda x_j^{\frac{1}{2}}R_h(\lambda)\widetilde{x}_j^{\frac{1}{2}}||_{\mathcal{L}(L^2,H^1)}.$$

Combining (4.14)-(4.18) with Theorem 2.1, one concludes that

$$||K(\lambda, z)|| \leq C|\lambda - z|$$

for $(\lambda, z) \in \mathcal{O}$, hence by taking $z = \frac{n}{2} + is_0$ we see that $1 + K(\lambda, z)$ is holomorphically invertible for

$$\lambda \in \{\lambda \in \mathbb{C}; |\lambda - z| < \max(C^{-1}, \frac{1}{8})\}.$$

Since C does not depend on s_0 (thus on z), there exists a strip $\{|\Re(\lambda) - \frac{n}{2}| < \epsilon\}$ where $K(\lambda, z)$ is invertible except maybe at a finite number of points. Finally the term $K_1(\lambda, z)$ defined as in (4.6) with our new parametrix can be studied similarly. $\square$

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