How Lagrangian states evolve into random waves

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joint work with Alejandro Rivera

Let (X, g) be a smooth connected compact d-dimensional Riemannian manifold.

Theorem

There exists a sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \longrightarrow \infty$ and an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of $L^2(X, g)$, such that

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Quantum chaos studies the properties of φ_n as $n \to +\infty$ when the geodesic flow is *chaotic*. For instance, this is the case when (X, g) has negative sectional curvature.

Quantum ergodicity

Theorem (Schnirelman '74, Zelditch '87, Colin de Verdière '85)
Let (X, g) be a compact manifold with negative sectional curvature. Then there exists a subsequence n_k of density one such that for all $a \in C(X)$, we have

$$\int_X a(x)|\varphi_{n_k}|^2(x)\mathrm{d}x \longrightarrow \frac{1}{\mathrm{Vol(X)}} \int_X a(x)\mathrm{d}x.$$

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The Quantum Unique Ergodicity Conjecture
We don't have to extract a sub-sequence in the previous statement.

Berry's conjecture

We suppose here that the φ_n are real-valued.

Berry's conjecture, Version 1: We denote by X_n the random variable given by $\varphi_n(\mathrm{x}_0)$, where x_0 is a point chosen uniformly at random in X. Then X_n converges in distribution to $\mathcal{N}\left(0, \frac{1}{\sqrt{\mathrm{Vol}(X)}}\right)$.

This would imply the Quantum Unique Ergodicity conjecture.

Towards a stronger version of Berry's conjecture

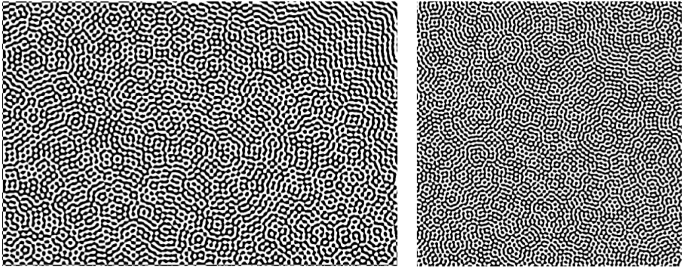

Figure 1. Left: Nodal domains of the eigenfunction of a quarter of the stadium with energy $E = 10092.029$. Right: Nodal domains of a random wave function (3) with $k = 100$.

Picture taken from E. Bogomolny, C. Schmit, *Random wave functions and percolation*, 2007.

A bit of probabilistic vocabulary

A *random field* is a probability measure on the set of (smooth) functions on $\mathbb{R}^d$.

In other words, a random field is a way of picking a smooth function at random.

Let (Ψ_n), Ψ be random fields. We say that Ψ_n converges in distribution to Ψ, written $\Psi_n \xrightarrow{d} \Psi$, if, for any bounded continuous functional $F : C^\infty(\mathbb{R}^d) \longrightarrow \mathbb{R}$, we have

$$\mathbb{E}_{\Psi_n}[F] \longrightarrow \mathbb{E}_{\Psi}[F].$$

Here, $C^\infty(\mathbb{R}^d)$ is equipped with the topology of convergence of all derivatives over all compact sets.

Gaussian random fields

Let μ be a measure on $(0, +\infty)$.

For each $k \in \mathbb{N}$, we give ourselves independent random variables

- φ_k uniform on $[0, 2\pi]$
- ξ_k uniform on $\mathbb{S}^{d-1}$
- λ_k following the law μ.

$$f_n(x) := \frac{\mu(0, +\infty)}{\sqrt{n}} \sum_{k=1}^{n} e^{i\lambda_k \xi_k \cdot x + \varphi_k}$$

induces a random field on $\mathbb{R}^d$. It converges in distribution to a random field, written $\Psi_{Berry,\mu}$, called a **isotropic**, **stationary**, **Gaussian** random field. It is **monochromatic** if μ is a multiple of a Dirac mass.

We write $\Psi_{Berry} := \Psi_{Berry,\delta_{\{1\}}}$.

Gaussian random fields (2)

In dimension 2, $\Re[\Psi_{Berry}]$ can alternatively be defined, in polar coordinates, as

$$\Re[\Psi_{Berry}(r,\theta)] = X_0 J_0(r) + \sqrt{2}\sum_{n\geq 1} J_n(r)\left[X_n\cos(n\theta) + Y_n\sin(n\theta)\right],$$

where J_n is the n-th Bessel function, and where the $(X_n)_{n\geq 0}, (Y_n)_{n\geq 1}$ are independent families of standard Gaussian variables.

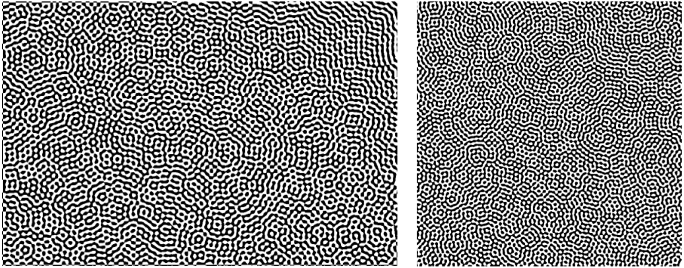

Figure 1. Left: Nodal domains of the eigenfunction of a quarter of the stadium with energy $E = 10092.029$. Right: Nodal domains of a random wave function (3) with $k = 100$. [1]

The right picture corresponds to a realization of $\Re[\Psi_{Berry}]$. Due to the ergodicity of translations for the measure Ψ_{Berry}, all realizations look the same.

[1] Picture taken from E. Bogomolny, C. Schmit, *Random wave functions and percolation*, 2007.

From deterministic functions to random fields

Let $\mathcal{U} \subset X$ be an open set, and let $(V_1, ..., V_d) : \mathcal{U} \longrightarrow (TX)^d$ be an orthonormal frame.
If (ψ_h) is a sequence of functions depending on a parameter $h > 0$, we define, for each $x \in \mathcal{U}$, a function $\psi_{h,x}$ on $\mathbb{R}^d$ by

$$\psi_{h,x}(y) = \psi_h \left[\exp_x(h(y_1 V_1(x) + \cdots + y_d V_d(x)))\right].$$

The function $\psi_{h,\mathrm{x}}$, with x taken uniformly at random in $\mathcal{U}$ induces a random field $\Psi_h^{\mathcal{U}}$ on $\mathbb{R}^d$.

A refined version of Berry's conjecture

Let X be a manifold of negative curvature. We fix an orthonormal frame defined on some open subset of X.
Let us denote by (ψ_{h_n}) an orthonormal basis of $L^2(X)$ such that $-h^2\Delta\psi_{h_n} = \psi_{h_n}$.

Berry's conjecture, Version 2

For any open set $\mathcal{U}$, we have

$$\Psi_{h_n}^{\mathcal{U}} \xrightarrow{d} \frac{1}{\sqrt{\mathrm{Vol}(X)}}\Psi_{Berry}.$$

Related works

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- Bourgain (2014), Buckley-Wigman (2016), I. (2017), Wigman-Yesha (2018), Sartori (2020) : Proof of Berry's conjecture on $\mathbb{T}^2$ for some generic families of eigenfunctions.

Extension to the Schrödinger propagator ?

Vague unformulated conjecture: Let (X, g) be a manifold/domain with chaotic classical dynamics.
Let $f_h \in C^\infty(X)$ be "generic" (and satisfy $-h^2\Delta f_h \approx f_h$).
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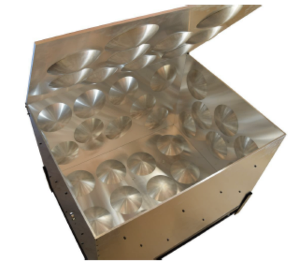
Application: Chaotic electromagnetic reverberation chambers



Lagrangian states

A *Lagrangian state* or *Lagrangian distribution* is a family of functions f_h of the form

$$f_h(x) = b(x)e^{\frac{i}{h}\varphi(x)},$$

where $b \in C^\infty(X), \varphi \in C^\infty(\mathrm{support}(b))$. We say it is *monochromatic* if $|\nabla\varphi| = 1$ for all $x \in X$. If this is the case, we have

$$(-h^2\Delta - 1)f_h = O(h).$$

If $\varphi \in C^\infty(\Omega; \mathbb{R})$, we define the Lagrangian manifold

$$\Lambda_\varphi := \{(x, \partial\varphi(x)); x \in \Omega\} \subset T^*X.$$

Let us write, if $0 < \lambda_1 < \lambda_2$,

$$Z_{\Omega}^{(\lambda_1,\lambda_2)} := \{\varphi \in C^{\infty}(\Omega;\mathbb{R}); \lambda_1 < |\nabla\varphi(x)| < \lambda_2 \ \forall x \in \Omega\},$$

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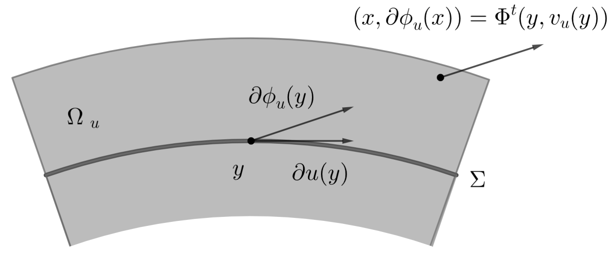

Transversality to the stable directions

For each $x \in X$, $\xi \in T_x^* X \backslash \{0\}$, let us denote by $E^-_{(x,\xi)} \subset T_{(x,\xi)} T^* X$ the *stable direction* at (x, ξ).

If $\varphi \in Z_\Omega^{(\lambda_1,\lambda_2)}$, we say that it is *transverse to the stable directions* (TSD) if

$$\forall x \in \Omega,\ T_{(x,\partial\varphi(x))}\Lambda_\varphi \cap E^-_{(x,\partial\varphi(x))} = \{0\}$$

Let us write, if $0 < \lambda_1 \leq \lambda_2$,

$$Z_{\Omega,TSD}^{(\lambda_1,\lambda_2)} := \{\varphi \in Z_\Omega^{(\lambda_1,\lambda_2)},\ \varphi \text{ is TSD}\}.$$

Quantum unique ergodicity for propagated Lagrangian states

Theorem (Schubert, 2005)
Let (X,g) be a compact manifold of negative curvature and let $\Omega \subset X$ be an open set. There exists $\gamma_X > 0$ such that for all $\varphi \in Z^{(\lambda_1,\lambda_2)}_{\Omega,TSD}$, the following holds.
If $b \in C_c^\infty(\Omega)$, write $f_h = b e^{\frac{i}{h}\varphi}$. For any $t_h \leq (\gamma_X - \varepsilon)|\log h|$ such that $t_h \underset{h\to 0}{\longrightarrow} +\infty$ and for any $a \in C(X)$, we have

$$\int_X |e^{iht_h\Delta} f_h|^2(x) a(x) \mathrm{d}x \underset{h\to 0}{\longrightarrow} \|b\|_{L^2} \int_X a(x)\mathrm{d}x.$$

Berry's conjecture for propagated Lagrangian states

Theorem (I.-Rivera, 2020)
Let (X, g) be a compact manifold of negative curvature, let $\Omega \subset X$ be an open set, and let $0 < \lambda_1 \leq \lambda_2$. There exists a G_δ-dense set $\widetilde{Z}_{\Omega,TSD}^{(\lambda_1,\lambda_2)} \subset Z_{\Omega,TSD}^{(\lambda_1,\lambda_2)}$ such that, for all $\varphi \in \widetilde{Z}_{\Omega,TSD}^{(\lambda_1,\lambda_2)}$, the following holds.
If $b \in C^\infty(\Omega)$, write $f_h = b e^{\frac{i}{h}\varphi}$, and $f_h(t) := e^{ith\Delta} f_h$.
Let $\mathcal{U} \subset X$ be an open set equipped with an orthonormal frame. $f_h(t)$ induces a random field $F_h^{\mathcal{U}}(t)$. Then we have

- For all t large enough, there exists a random field $F^{\mathcal{U}}(t)$ such that $F_h^{\mathcal{U}}(t) \xrightarrow{d} F^{\mathcal{U}}(t)$.
- $F^{\mathcal{U}}(t) \xrightarrow{d} \Psi_{Berry,\lambda}$. Here, λ is the push-forward of the measure $|b(x)|^2 \mathrm{d}x$ on X by the map $x \mapsto |\partial\varphi(x)| \in (0, \infty)$.

Ideas of the proof (1): the WKB method

If $(\tilde{X}, \tilde{g})$ is a *simply connected* manifold of negative curvature, and if $\tilde{f}_h = \tilde{b} e^{\frac{i}{h}\tilde{\varphi}}$ is a Lagrangian state with $\widetilde{\varphi}$ TSD, then for any t large enough,

$$\Phi^t(\{x, \partial\tilde{\varphi}(x)\}) = \{x, \partial\tilde{\varphi}_t(x)\} \subset T^*X,$$

and

$$\tilde{f}_h(t) = \tilde{b}_t e^{\frac{i}{h}\tilde{\varphi}_t} + O(h).$$

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On X, we have

$$f_h(t) = \sum_{j=1}^{N(t)} b_{j,t} e^{\frac{i}{h}\varphi_{j,t}} + O(h).$$

Ideas of the proof (2): Genericity implies irrationality

From now on, we work in a local chart around a point x_1, and pretend we are in $\mathbb{R}^d$. Let us write, for $x_1 \in X$, $\xi_{j,t}^{x_1} := \partial \varphi_{j,t}(x_1)$.

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Proof: Thom's transversality lemma

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Proof: Thom's transversality lemma and a bit of work...

Ideas of the proof (3): Working locally

Let $\alpha < \frac{1}{2}$. We will study the distribution of $f_h(\mathrm{x}_0)$ with x_0 chosen at random in $B(x_1, Rh^\alpha)$.

$$f_h(x_1 + h^\alpha x; t) = \sum_{j=1}^{N(t)} b_{j,t}(x_1 + h^\alpha x) e^{\frac{i}{h}\varphi_{j,t}(x_1 + h^\alpha x)} + O(h)$$

$$= \sum_{j=1}^{N(t)} b_{j,t}(x_1) e^{\frac{i}{h}\varphi_{j,t}(x_1) + ih^{\alpha-1} x \cdot \xi_{j,t}^{x_1}} + o(1).$$

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Here, $h^{\alpha-1}x$ is chosen at random in the big ball $B(0, h^{\alpha-1}R)$. By ergodicity, this will converge weakly, as $h \to 0$ to

$$\sum_{j=1}^{N(t)} |b_{j,t}(x_1)| e^{i\theta_j},$$

where the θ_j are independent, uniform in $[0, 2\pi)$.

End of the proof

- $\sum_{j=1}^{N(t)} |b_{j,t}(x_0)| e^{i\theta_j}$ behaves asymptotically as the Gaussian of variance $\sum_{j=1}^{N(t)} |b_{j,t}(x_0)|^2$.

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- $\sum_{j=1}^{N(t)} |b_{j,t}(x_0)| e^{i\theta_j}$ behaves asymptotically as the Gaussian of variance $\sum_{j=1}^{N(t)} |b_{j,t}(x_0)|^2$.
- Using ergodicity of the geodesic flow, one can show that $\sum_{j=1}^{N(t)} |b_{j,t}(x_0)|^2$ goes to $\|b\|_{L^2}$.

Thank you for your attention!