INVERSE PROBLEMS FOR EINSTEIN MANIFOLDS

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ABSTRACT. We show that the Dirichlet-to-Neumann operator of the Laplacian on an open subset of the boundary of a connected compact Einstein manifold with boundary determines the manifold up to isometries. Similarly, for connected conformally compact Einstein manifolds of even dimension $n + 1$, we prove that the scattering matrix at energy n on an open subset of its boundary determines the manifold up to isometries.

1. INTRODUCTION

The purpose of this note is to prove two results: first that compact connected Einstein manifolds with boundary are determined modulo isometries from the Dirichlet-to-Neumann map on an open subset of its boundary. Secondly, that a conformally compact connected Einstein manifolds of even dimension $n+1$ is determined, modulo isometries, by the scattering matrix on an open subset of the boundary.

The Dirichlet-to-Neumann (DN in short) map $\mathcal{N} : C^\infty(\partial \bar{X}) \to C^\infty(\partial \bar{X})$ for the Laplacian on a Riemannian manifold with boundary $(\bar{X}, g)$ is defined by solving the Dirichlet problem

$$(1.1) \qquad \Delta_g u = 0, \quad u|_{\partial \bar{X}} = f$$

where $f \in C^\infty(\partial \bar{X})$ is given, then $\mathcal{N}f := -\partial_n u|_M$ where ∂_n is the interior pointing normal vector field to the boundary for the metric g. It is an elliptic pseudo-differential operator of order 1 on the boundary, see for example [16]. Mathematically, it is of interest to know what this map determines about the geometry of the manifold, but $\mathcal{N}$ can also be interpreted as a boundary measurement of current flux in terms of voltage in electrical impedance tomography. We refer to [24] for a survey in the field, and to [14, 15, 16, 22, 23] for significant results about that problem.

Our first result answers a conjecture of Lassas and Uhlmann [15]

Theorem 1.1. *Let $(\bar{X}_1, g_1)$ and $(\bar{X}_2, g_2)$ be two smooth connected compact manifolds with respective boundaries $\partial \bar{X}_1$ and $\partial \bar{X}_2$. We suppose that g_1 and g_2 are Einstein with the same constant $\lambda \in \mathbb{R}$, i.e. $\mathrm{Ric}(g_i) = \lambda g_i$ for $i = 1, 2$. Assume that $\partial \bar{X}_1$ and $\partial \bar{X}_2$ contain a common open set Γ such that the identity map $\mathrm{Id} : \Gamma \subset \bar{X}_1 \to \Gamma \subset \bar{X}_2$ is a smooth diffeomorphism. If the Dirichlet-to-Neumann map $\mathcal{N}_i$ of Δ_{g_i} on $\bar{X}_i$ for $i = 1, 2$ satisfy $(\mathcal{N}_1 f)|_\Gamma = (\mathcal{N}_2 f)|_\Gamma$ for any $f \in C_0^\infty(\Gamma)$, then there exists a diffeomorphism $J : \bar{X}_1 \to \bar{X}_2$, such that $J^* g_2 = g_1$.*

Then we consider a class of non-compact complete Einstein manifolds, but conformal to a compact manifold. In this case we say that (X, g) is Einstein, with $\dim X = n + 1$, if

$$\mathrm{Ric}(g) = -ng.$$

We say that a Riemannian manifold (X, g) is conformally compact if X compactifies into a smooth manifold with boundary $\bar{X}$ and for any smooth boundary defining function ρ of $\bar{X}$, $\bar{g} := \rho^2 g$ extends to $\bar{X}$ as a smooth metric. Such a metric g is necessarily complete on X and its sectional curvatures are pinched negatively

outside a compact set of X. If in addition the sectional curvatures of g tends to -1 at the boundary, we say that (X, g) is *asymptotically hyperbolic*.

It has been shown in [9, 10] that if (X, g) is asymptotically hyperbolic, or in particular if (X, g) is Einstein, then there exists a family of boundary defining functions ρ (i.e. $\partial\bar{X} = \{\rho = 0\}$ and $d\rho|_{\partial\bar{X}}$ does not vanish) such that $|d\rho|_{\rho^2 g} = 1$ near the boundary. These will be called *geodesic boundary defining functions*. Note that, in this case, a DN map can not be defined as in (1.1) since Δ_g is not an elliptic operator at the boundary. The natural analogue of the DN map on a conformally compact Einstein manifold (X, g) is related to scattering theory, at least in the point of view of Melrose [21]. We consider an $n + 1$-dimensional conformally compact Einstein manifold (X, g) with $n + 1$ even. Following [11, 13], the *scattering matrix or scattering map* in this case, and more generally for asymptotically hyperbolic manifolds, is an operator $\mathcal{S} : C^\infty(\partial\bar{X}) \to C^\infty(\partial\bar{X})$, constructed by solving a Dirichlet problem in a way similar to (1.1). This will be discussed in details in section 4. We show that for all $f \in C^\infty(\partial\bar{X})$, there exists a unique function $u \in C^\infty(\bar{X})$ such that

$$(1.2) \qquad\qquad \Delta_g u = 0 \text{ and } u|_{\partial X} = f.$$

Since there is no canonical normal vector field at the boundary defined from g (recall that g blows-up at the boundary), we can consider $\bar{g} := \rho^2 g$ for some geodesic boundary defining function and take the unit normal vector field for $\bar{g}$, that is $\nabla^{\bar{g}}\rho$, which we denote by ∂_ρ. It turns out that $(\partial_\rho^k u|_{\partial\bar{X}})_{k=1,\dots,n-1}$ are locally determined by $u|_{\partial\bar{X}} = f$ and the first term in the Taylor expansion of u which is global is the n-th $\partial_\rho^n u|_{\partial\bar{X}}$. We thus define $\mathcal{S}f \in C^\infty(\partial\bar{X})$ by

$$(1.3) \qquad\qquad \mathcal{S}f := \frac{1}{n!}\partial_\rho^n u|_{\partial\bar{X}}.$$

Notice that $\mathcal{S}$ a priori depends on the choice of ρ, we shall say that it is associated to ρ. It can be checked that if $\hat{\rho} = e^\omega \rho$ is another geodesic boundary defining function with $\omega \in C^\infty(\bar{X})$, then the scattering map $\hat{\mathcal{S}}$ associated to $\hat{\rho}$ satisfy $\hat{\mathcal{S}} = e^{-n\omega_0}\mathcal{S}$ where $\omega_0 = \omega|_{\partial\bar{X}}$, see [11] and Subsection 4.1 below.

We also remark that the fact that $u \in C^\infty(\bar{X})$ strongly depends on the fact that the manifold under consideration is Einstein and has even dimensions. For more general asymptotically hyperbolic manifolds, the solution u to (1.2) possibly has a logarithmic singularity as shown in [11]. Our second result is the following

Theorem 1.2. *Let (X_1, g_1) and (X_2, g_2) be connected, C^∞, $(n + 1)$-dimensional conformally compact manifolds, with $n+1$ even. Suppose that g_1 and g_2 are Einstein and that $\partial\bar{X}_1$ and $\partial\bar{X}_2$ contain a common open set Γ such that the identity map $\mathrm{Id} : \Gamma \subset \bar{X}_1 \to \Gamma \subset \bar{X}_2$ is a smooth diffeomorphism. If for $i = 1, 2$, there exist boundary defining functions ρ_i of $\partial\bar{X}_i$ such that the scattering maps $\mathcal{S}_i$ of Δ_{g_i} associated to ρ_i satisfy $(\mathcal{S}_1 f)|_\Gamma = (\mathcal{S}_2 f)|_\Gamma$ for all $f \in C_0^\infty(\Gamma)$. Then there is a diffeomorphism $J : \bar{X}_1 \to \bar{X}_2$, such that $J^* g_2 = g_1$ in X_1.*

The proofs are based on the results of Lassas and Uhlmann [15], and Lassas, Taylor and Uhlmann [14], and suitable unique continuation theorems for Einstein equation.

It is shown in [15] that a connected compact manifold with boundary ($\bar{X} = X \cup \partial\bar{X}, g$), is determined by the Dirichlet-to-Neumann if the interior (X, g) is real analytic, and if there exists an open set Γ of the boundary $\partial\bar{X}$ which is real analytic with g real analytic up to Γ. In [14] Lassas, Taylor and Uhlmann prove the analogue of this result for complete manifolds.

A theorem of De Turck and Kazdan, Theorem 5.2 of [6], says that if $(\bar{X}, g)$ is a connected Einstein manifold with boundary then the collection of harmonic

coordinates give X, the interior of $\bar{X}$, a real analytic structure which is compatible with its C^∞ structure, and moreover g is real analytic in those coordinates. The principle is that Einstein's equation becomes a non-linear elliptic system with real analytic coefficient in these coordinates, thus the real analyticity of the metric. But since the harmonic coordinates satisfy the Laplace equation, they are analytic as well.

However this construction is not necessarily valid at the boundary. Therefore one cannot guarantee that $(\bar{X}, g)$ is real analytic at the boundary, and hence one cannot directly apply the results of [14].

To prove Theorems 1.1 and 1.2, we first show that the DN map (or the scattering map) determines the metric in a small neighbourhood U of a point $p \in \Gamma \subset \partial\bar{X}$, then we shall prove that this determines the Green's function in $U \times U$. However one of the results of [14] says that this determines the whole Riemannian manifold, provided it is real-analytic, but as mentioned above, this is the case of the interior of an Einstein manifold.

The essential part in this paper is the reconstruction near the boundary. This will be done using the ellipticity of Einstein equation in harmonic coordinates, and by applying a unique continuation theorem for the Cauchy problem for elliptic systems with diagonal principal part. The unique continuation result we need in the compact case was essentially proved by Calderón [4, 5]. The conformally compact case is more involved since the system is only elliptic in the uniformly degenerate sense of [19, 18, 20, 17], see also [1]. When the first version of this paper was completed we learned that O. Biquard [3] proved a unique continuation result for Einstein manifolds without using the DN map for functions, which was a problem that was part of the program of M. Anderson [2]. Under our assumptions, it seems somehow natural to use harmonic coordinates for Einstein equation, and we notice that our approach is self-contained and does not require the result of [3].

Throughout this paper when we refer to the real analyticity of the metric, we mean that it is real analytic with respect to the real analytic structure defined from harmonic coordinates corresponding to the Einstein metric g.

2. Acknowledgments

3. Inverse problem for Einstein manifolds with boundary

The result of De Turck and Kazdan concerning the analyticity of the metric does not apply to Einstein manifolds with boundary $(\bar{X} = X \cup \partial\bar{X}, g)$. Their argument breaks down since the boundary can have low regularity even though g has constant Ricci curvature. This means that the open incomplete manifold (X, g) is real-analytic with respect to the analytic structure defined by harmonic coordinates, but a priori $(\bar{X}, g)$ does not satisfy this property. We will use the Dirichlet-to-Neumann map to overcome this difficulty.

3.1. **The Dirichlet-to-Neumann map.** As in section 1,

$$\mathcal{N} : C^\infty(\partial\bar{X}) \to C^\infty(\partial\bar{X})$$

is defined by solving the Dirichlet problem (1.1) with $f \in C^\infty(\partial \bar{X})$, and setting $\mathcal{N}f := -\partial_n u|_M$ where ∂_n is the interior pointing normal vector to the boundary for the metric g. Its Schwartz kernel is related to the Green function $G(z, z')$ of the Laplacian Δ_g with Dirichlet condition on $\partial \bar{X}$ by the following identity

Lemma 3.1. *The Schwartz kernel* $\mathcal{N}(y, y')$ *of* $\mathcal{N}$ *is given for* $y, y' \in \partial \bar{X}$, $y \neq y'$, *by*

$$\mathcal{N}(y, y') = \partial_n \partial_{n'} G(z, z')|_{z=y, z'=y'}$$

where $\partial_n, \partial_{n'}$ *are respectively the inward pointing normal vector fields to the boundary in variable z and z'.*

Proof: Let x be the distance function to the boundary in $\bar{X}$, it is smooth in a neighbourhood of $\partial \bar{X}$ and the normal vector field to the boundary is the gradient $\partial_n = \nabla^g x$ of x. The flow $e^{t\partial_n}$ of $\nabla^g x$ induces a diffeomorphism $\phi : [0, \epsilon)_t \times \partial \bar{X} \to \phi([0, \epsilon) \times \partial \bar{X})$ defined by $\phi(t, y) := e^{t\partial_n}(y)$ and we have $x(\phi(t, y)) = t$. This induces natural coordinates $z = (x, y)$ near the boundary, these are normal geodesic coordinates. The function u in (1.1) can be obtained by taking

$$u(z) := \chi(z) - \int_{\bar{X}} G(z, z')(\Delta_g \chi)(z') dz'$$

where χ is any smooth function on $\bar{X}$ such that $\chi = f + O(x^2)$. Now using Green's formula and $\Delta_g(z) G(z, z') = \delta(z - z') = \Delta_g(z') G(z, z')$ where $\delta(z - z')$ is the Dirac mass on the diagonal, we obtain for $z \in X$

$$u(z) = \int_{\partial \bar{X}} \left(\partial_{n'} G(z, z') \chi(z') \right)|_{z'=y'} dy' - \int_{\partial \bar{X}} \left(G(z, z')(\partial_n \chi)(z') \right)|_{z'=y'} dy'$$

$$u(z) = \int_{\partial \bar{X}} \left(\partial_{n'} G(z, z') \right)|_{z'=y'} f(y') dy'$$

We have a Taylor expansion $u(x, y) = f(y) + x \mathcal{N} f(y) + O(x^2)$ near the boundary. Let $y \in \partial X$ and take $f \in C^\infty(X)$ supported near y. Thus pairing with $\phi \in C^\infty(\partial \bar{X})$ gives

$$(3.1) \qquad \int_{\partial \bar{X}} u(x, y) \phi(y) dy = \int_{\partial \bar{X}} f(y) \phi(y) dy - x \int_{\partial \bar{X}} \phi(y) \mathcal{N} f(y) dy + O(x^2).$$

Now taking ϕ with support disjoint to the support of f, thus $\phi f = 0$, and differentiating (3.1) in x, we see, using the fact that Green's function $G(z, z')$ is smooth outside the diagonal, that

$$\int_{\partial \bar{X}} \phi(y) \mathcal{N} f(y) dy = \int_{\partial \bar{X}} \int_{\partial \bar{X}} \left(\partial_n \partial_{n'} G(z, z') \right)|_{z=y, z'=y'} f(y') \phi(y) dy dy',$$

which proves the claim. $\square$

3.2. **The Ricci tensor in harmonic coordinates and unique continuation.** Let us take coordinates $x = (x_0, x_1, \ldots, x_n)$ near a point $p \in \partial \bar{X}$, with x_0 a boundary defining function of $\partial \bar{X}$, then $\mathrm{Ric}(g)$ is given by definition by

$$(3.2) \qquad \mathrm{Ric}(g)_{ij} = \sum_k \left(\partial_{x_k} \Gamma_{ji}^k - \partial_{x_j} \Gamma_{ki}^k + \sum_l \Gamma_{kl}^k \Gamma_{ji}^l - \sum_l \Gamma_{jl}^k \Gamma_{ki}^l \right)$$

with

$$(3.3) \qquad \Gamma_{ji}^k = \frac{1}{2} \sum_m g^{km} \left(\partial_{x_i} g_{mj} + \partial_{x_j} g_{mi} - \partial_{x_m} g_{ij} \right).$$

Lemma 1.1 of [6] shows that $\Delta_g x_k = \sum_{i,j} g^{ij} \Gamma_{ij}^k$, so Einstein equation $\mathrm{Ric}(g) = \lambda g$ for some $\lambda \in \mathbb{R}$ can be written as the system (see also Lemma 4.1 in [6])

$$(3.4) \quad -\frac{1}{2} \sum_{\mu, \nu} g^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} g_{ij} + \frac{1}{2} \sum_r (g_{ri} \partial_{x_j} (\Delta_g x_r) + g_{rj} \partial_{x_i} (\Delta_g x_r)) + Q_{ij}(g, \partial g) = 0$$

where $Q_{ij}(A,B)$ is smooth and polynomial of degree two in B, where A,B denote vectors $(g_{kl})_{k,l} \in \mathbb{R}^{(n+1)^2}$ and $(\partial_{x_m} g_{kl})_{k,l,m} \in \mathbb{R}^{(n+1)^3}$. From this discussion we deduce the following

Proposition 3.2. *Let $(x_0, x_1, \ldots, x_n)$ be harmonic coordinates for Δ_g near a point $p \in \{x_0 = 0\}$, then there exist $Q_{ij}(A,B)$ smooth, polynomial of degree 2 in $B \in \mathbb{R}^{(n+1)^3}$, such that $Ric(g) = \lambda g$ is equivalent near p to the system*

$$(3.5) \qquad \sum_{\mu,\nu} g^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} g_{ij} + Q_{ij}(g, \partial g) = 0, \quad i,j = 0, \ldots, n$$

with $\partial g := (\partial_{x_m} \bar{g}_{kl})_{k,l,m} \in \mathbb{R}^{(n+1)^3}$.

Now we may use a uniqueness theorem for the Cauchy problem of such elliptic systems.

Proposition 3.3. *If $C := (c_{ij})_{i,j=0,\ldots,n}, D := (d_{ij})_{i,j=0,\ldots,n}$ are smooth symmetric 2-tensors near $p \in \{x_0 = 0\}$, with C positive definite, the system (3.5) near p with boundary conditions $g_{ij}|_{x_0=0} = c_{ij}$ and $\partial_{x_0} g_{ij}|_{x_0=0} = d_{ij}$, $i,j = 0, \ldots, n$, has at most a unique smooth solution.*

Proof: The system is elliptic and the leading symbol is a scalar times the identity, the result could then be proved using Carleman estimates. For instance, uniqueness properties are proved by Calderon [4, 5] for elliptic systems when the characteristics of the system are non-multiple, but in our case they are multiple. However, since the leading symbol is scalar and this scalar symbol has only non-multiple characteristics, the technics used in Calderon could be applied like in the case of a single equation with non-multiple characteristics. Since we did not find references that we can cite directly, we prefer to use Proposition 4.3 which is a consequence of a uniqueness result of Mazzeo [19]. Indeed, first it is straightforward to notice, by using boundary normal coordinates, that two solutions of (3.5) with same Cauchy data agree to infinite order at the boundary, therefore we may multiply (3.5) by x_0^2 and (3.5) becomes of the form (4.7) thus Proposition 4.3 below proves uniqueness. $\square$

3.3. **Reconstruction near the boundary.** Throughout this section we assume that $(\bar{X}_1, g_1), (\bar{X}_2, g_2)$ are C^∞ connected Einstein manifolds with boundary $M_j = \partial \bar{X}_j$, $j = 1, 2$, such that M_1 and M_2 contain a common open set Γ, and that the identity map $\mathrm{Id} : \Gamma \subset \partial X_1 \longrightarrow \Gamma \subset \partial X_2$ is a C^∞ diffeomorphism. Moreover we assume that for every $f \in C_0^\infty(\Gamma)$, the Dirichlet-to-Neumann maps satisfy

$$\mathcal{N}_1 f|_\Gamma = \mathcal{N}_2 f|_\Gamma.$$

We first prove

Lemma 3.4. *For $i = 1, 2$, there exists $p \in \Gamma$, some neighbouroods U_i of p in $\bar{X}_i$ and a diffeomorphism $F : U_1 \to U_2$, $F|_{U_1 \cap X_1}$ analytic, such that $F^* g_2 = g_1$ and $F|_{U_1 \cap \Gamma} = \mathrm{Id}$.*

Proof: For $i = 1, 2$, let $t_i = \mathrm{dist}(., \partial \bar{X}_i)$ be the distance to the boundary in $\bar{X}_i$, then the flow $e^{t \nabla^{g_i} t_i}$ of the gradient $\nabla^{g_i} t_i$ induces a diffeomorphism

$$\phi^i : [0, \epsilon) \times \partial \bar{X}_i \to \phi^i([0, \epsilon) \times \partial \bar{X}_i)$$
$$\phi^i(t, y) := e^{t \nabla^{g_i} t_i}(y),$$

and we have the decomposition near the boundary $(\phi^i)^* g_i = dt^2 + h_i(t)$ for some one-parameter family of metrics $h_i(t)$ on $\partial \bar{X}_i$. Lee-Uhlmann [15] proved that $\mathcal{N}_1|_\Gamma = \mathcal{N}_2|_\Gamma$ implies that

$$(3.6) \qquad \partial_t^k h_1|_\Gamma = \partial_t^k h_2|_\Gamma, \quad \forall \ k \in \mathbb{N}_0.$$

Let us now consider $H_i := {\phi^i}^* g_i$ on the collar $[0, \epsilon)_t \times \Gamma$. Let $p \in \Gamma$ be a point of the boundary and $(y_1, \ldots, y_n)$ be a set of local coordinates in a neighbourhood of p in Γ, and extend each y_j to $[0, \epsilon) \times \Gamma$ by the function $(t, m) \to y_j(m)$. Notice that $\phi^2 \circ (\phi^1)^{-1}$ is a smooth diffeomorphism from a neighbourhood of p in $\bar{X}_1$ to a neighbourhood of p in $\bar{X}_2$, this is a consequence of the fact that $\mathrm{Id} : \Gamma \subset \bar{X}_1 \to \Gamma \subset \bar{X}_2$ is a diffeomorphism. Using $z := (t, y_1, \ldots, y_n)$ as coordinates on $[0, \epsilon) \times \Gamma$ near p, then (3.6) shows that there is an open neighbourhood U of p in $[0, \epsilon)_t \times \Gamma$ such that $H_2 = H_1 + O(t^\infty)$ and we can always assume $U \cap \{t = 0\} \neq \Gamma$. Let $y_0 \in C_0^\infty(\Gamma)$ with $y_0 = 0$ on $U \cap \{t = 0\}$ but y_0 not identically 0, and by cutting off far from p me may assume that $y_j \in C_0^\infty(\Gamma)$ for $j = 1, \ldots, n$. Now let $(x_0^1, x_1^1, \ldots, x_n^1)$ and $(x_0^2, x_1^2, \ldots, x_n^2)$ be harmonic functions near p in $[0, \epsilon) \times \Gamma$ for respectively H_1 and H_2 such that $x_j^1 = x_j^2 = y_j$ on $\{t = 0\}$. These functions are constructed by solving the Dirichlet problem $\Delta_{g_i} w_j^i = 0$ on $\bar{X}_i$ with boundary data $w_j^i|_{M_i} = y_j$, $i = 1, 2$, and $j = 0, \ldots, n$, and by setting $x_j^i = {\phi^i}^* w_j^i$. Note that $\{m \in U \cap \Gamma; x_0^i(m) = 0, dx_0^i(m) = 0\}$ is a closed set with empty interior in $U \cap \{t = 0\}$, since otherwise x_0^i would vanish to order 2 on an open set of $\{t = 0\}$, thus by unique continuation it would be identically 0 since it is harmonic. Then $(x_0^1, \ldots, x_n^1)$ and $(x_0^2, \ldots, x_n^2)$ form smooth coordinate systems near at least a common point of $U \cap \{t = 0\}$; for convenience let us denote again p this point and $U \subset [0, \epsilon) \times \Gamma$ an open set containing p where they both form smooth coordinates.

We have $\Delta_{H_1}(x_j^1 - x_j^2) = O(t^\infty)$ and $\partial_t x_j^1|_{t=0} = \partial_t x_j^2|_{t=0}$ for all j since $\mathcal{N}_1|_\Gamma = \mathcal{N}_2|_\Gamma$. Since $u = x_j^1 - x_j^2$ is solution of $\Delta_{H_1} u = O(t^\infty)$ in U with u vanishing at order 2 at the boundary $t = 0$, a standard Taylor expansion argument shows that $x_j^1 = x_j^2 + O(t^\infty)$ in U for all j. Now define $\psi : U \to \psi(U) \subset U$ so that $(x_0^1, \ldots, x_n^1) = (\psi^* x_0^2, \ldots, \psi^* x_n^2)$. Then $\psi = \mathrm{Id} + O(t^\infty)$ in U, and consequently we obtain in U

$$(3.7) \qquad \psi^* H_2 = H_1 + O(t^\infty).$$

The metrics $g = H_1$ and $g = \psi^* H_2$ both satisfy Einstein equation $\mathrm{Ric}(g) = \lambda g$ in U. Moreover in coordinates $(x_0^1, \ldots, x_n^1)$ this correspond to the system (3.5) and since the coordinates are harmonic with respect to g, the system is elliptic and diagonal to leading order. From the unique continuation result in Proposition 3.3, we conclude that there exists a unique solution to this system in U_1 with given initial data $g|_{U \cap \{t=0\}}$ and $\partial_{x_0^1} g|_{U \cap \{t=0\}}$. In view of (3.7), this proves that $H_1 = \psi^* H_2$ in U. Although it is not relevant for the proof, we remark that ψ is actually the Identity on U since $\psi|_{U \cap \Gamma} = \mathrm{Id}$ and it pulls back one metric in geodesic normal coordinates to the other. Now it suffices to go back to $\bar{X}_1$ and $\bar{X}_2$ through ϕ^1, ϕ^2 and we have proved the Lemma by setting $U_i := \phi^i(U)$ and

$$(3.8) \qquad F := \phi^2 \circ \psi \circ (\phi^1)^{-1}.$$

Remark that F is analytic from $U_1 \cap \{t_1 \neq 0\}$ to $U_2 \cap \{t_2 \neq 0\}$ since the harmonic functions w_j^i define the analytic structure in $U_i \cap \{t_i \neq 0\}$ for all $j = 0, \ldots, n$ and F is the map that identify w_j^1 to w_j^2 for all j. $\qquad \square$

Next we prove

Corollary 3.5. *For $i = 1, 2$, let $G_i(z, z')$ be the Green function of Δ_{g_i} in $\bar{X}_i$ with Dirichlet condition at M, then $\mathcal{N}_1|_\Gamma = \mathcal{N}_2|_\Gamma$ implies that there exists an open set $U_1 \subset X_1$ with*

$$G_2(F(z), F(z')) = G_1(z, z'), \quad (z, z') \in (U_1 \times U_1) \setminus \{z = z'\},$$

where F was defined in (3.8)

Proof: First we remark that g_1 is Einstein and thus real analytic in $U_1 \setminus (U_1 \cap M)$, so is any harmonic function in this open set. Let $\partial_n, \partial_{n'}$ be the normal vector fields to the boundary in the first and second variables in $U_1 \times U_1$ respectively, as defined in Lemma 3.1. We see from the proof of Lemma 3.4 that $F_* \partial_n$ and $F_* \partial_{n'}$ are the normal vector fields to the boundary in the first and second variable in $U_2 \times U_2$ (since $\psi = \mathrm{Id} + O(t^\infty)$ in that Lemma). So we get $\partial_{n'} G_2(F(z), F(z')) = (F_* \partial_{n'}) G_2(F(z), z')$ for $z' \in M$ since $F|_{U_1 \cap \Gamma} = \mathrm{Id}$.

We first show that $\partial_{n'} G_2(F(z), F(z')) = \partial_{n'} G_1(z, z')$ for any $(z, z') \in U_1 \times (U_1 \cap \Gamma) \setminus \{z = z'\}$. Now fix $z' \in U_1 \cap \Gamma$, then the function $T_1(z) := \partial_{n'} G_1(z, z')$ solves $\Delta_{g_1} T_1 = 0$ in $U_1 \setminus \{z'\}$ and, using Lemma 3.1, it has boundary values $T_1|_{U_1 \cap \Gamma \setminus \{z'\}} = 0$ and $\partial_n T_1|_{U_1 \cap \Gamma \setminus \{z'\}} = \mathcal{N}_1(., z')$ where $\mathcal{N}_i(.,.)$ denote the Schwartz kernel of $\mathcal{N}_i$, $i = 1, 2$. The function $T_2(z) := \partial_{n'} G_2(F(z), F(z'))$ solves $\Delta_{F^* g_2} T_2(z) = \Delta_{g_1} T_2(z) = 0$ in $U_1 \setminus \{z'\}$. We also have $\partial_n T_2|_{U_1 \cap \Gamma \setminus \{z'\}} = F^*[(F_* \partial_n)(F_* \partial_{n'}) G_2(., z')|_{U_1 \cap \Gamma \setminus \{z'\}}]$ and $T_2|_{U_1 \cap \Gamma \setminus \{z'\}} = 0$. But from Lemma 3.1, $(F_* \partial_n)(F_* \partial_{n'}) G_2(., z')|_{U_1 \cap \Gamma \setminus \{z'\}} = \mathcal{N}_2(., z')$ where $\mathcal{N}_2(.,.)$ is the Schwartz kernel of $\mathcal{N}_2$. Using again that $F|_{U_1 \cap \Gamma} = \mathrm{Id}$, we deduce that $\partial_n T_2|_{U_1 \cap \Gamma \setminus \{z'\}} = \mathcal{N}_2(., z')$. By our assumption $\mathcal{N}_1|_\Gamma = \mathcal{N}_2|_\Gamma$, we conclude that T_1 and T_2 solve the same Cauchy problem near $U_1 \cap \Gamma \setminus \{z'\}$, so first by unique continuation near the boundary and then real analyticity in $U_1 \setminus (U_1 \cap \Gamma)$, we obtain $T_1 = T_2$ there.

Now we can use again similar arguments to prove that $G_1(z, z') = G_2(F(z), F(z'))$ in $(U_1 \times U_1) \setminus \{z = z'\}$. Indeed, fix $z' \in U_1$, then $T_1(z) := G_1(z', z)$ and $T_2(z) := G_2(F(z'), F(z))$ solve $\Delta_{g_1} T_i = 0$ in $U_1 \setminus \{z'\}$ and with boundary values $T_i|_\Gamma = 0$ and $\partial_n T_1|_{U_1 \cap \Gamma} = \partial_n T_2|_{U_1 \cap \Gamma}$ by what we proved above. Thus unique continuation for Cauchy problem and real analyticity allow us to conclude that $T_1 = T_2$. $\qquad\square$

3.4. **Proof of Theorem 1.1.** To conclude the proof of 1.1, we use the following Proposition which is implicitly proved by Lassas-Taylor-Uhlmann [14]

Proposition 3.6. *For $i = 1, 2$, let $(\bar{X}_i, g_i)$ be C^∞ connected Riemannian manifolds with boundary, assume that its interior X_i has a real-analytic structure compatible with the smooth structure and such that the metric g_i is real analytic on X_i. Let $G_i(z, z')$ be the Green function of the Laplacian Δ_{g_i} with Dirichlet condition at $\partial \bar{X}_i$, and assume there exists an open set $U_1 \subset X_1$ and an analytic diffeomorphism $F : U_1 \to F(U_1) \subset X_2$ such that $G_1(z, z') = G_2(F(z), F(z'))$ for $(z, z') \in (U_1 \times U_1) \setminus \{z = z'\}$. Then there exists a diffeomorphism $J : X_1 \to X_2$ such that $J^* g_2 = g_1$ and $J|_{U_1} = F$.*

The proof is entirely done in section 3 of [14], although not explicitly written under that form. The principle is to define maps

$$\mathcal{G}_j : X_j \to H^s(U_1), \quad \mathcal{G}_1(z) := G_1(z, .), \quad \mathcal{G}_2(z) := G_2(z, F(.))$$

where $H^s(U_1)$ is the s-Sobolev space of U_1 for some $s < 1 - (n+1)/2$, then prove that $\mathcal{G}_j$ are embeddings with $\mathcal{G}_1(X_1) = \mathcal{G}_2(X_2)$, and finally show that $J := \mathcal{G}_2^{-1} \circ \mathcal{G}_1 : X_1 \to X_2$ is an isometry. Note that J restricts to F on U_1 since $G_1(z, z') = G_2(F(z), F(z'))$.

Proposition 3.6 and Corollary 3.5 imply Theorem 1.1, after noticing that an isometry $\psi : (X_1, g_1) \to (X_2, g_2)$ extends smoothly to the manifold with boundary $(\bar{X}_1, g_1)$ by smoothness of the metrics g_i up to the boundaries $\partial \bar{X}_i$.

4. INVERSE SCATTERING FOR CONFORMALLY COMPACT EINSTEIN MANIFOLDS

Consider an $n+1$ dimensional connected conformally compact Einstein manifold $(\bar{X}, g)$ with $n + 1$ even, and let ρ be a geodesic boundary defining function and

$\bar{g} := \rho^2 g$. Using the flow $\phi_t(y)$ of the gradient $\nabla^{\rho^2 g}\rho$, one has a diffeomorphism $\phi : [0, \epsilon)_t \times \partial\bar{X} \to \phi([0, \epsilon) \times \partial\bar{X}) \subset \bar{X}$ defined by $\phi(t, y) := \phi_t(y)$, and the metric pulls back to

$$(4.1) \qquad \phi^* g = \frac{dt^2 + h(t)}{t^2}$$

for some smooth one-parameter family of metrics $h(t)$ on the boundary $\partial\bar{X}$. Note that here $\phi^*\rho = t$.

4.1. **The scattering map.** The scattering map $\mathcal{S}$, or scattering matrix, defined in the introduction is really $\mathcal{S} = S(n)$, where $S(\lambda)$ for $\lambda \in \mathbb{C}$ is defined in [13, 11]. Let us construct $\mathcal{S}$ by solving the boundary value problem $\Delta_g u = 0$ with $u \in C^\infty(\bar{X})$ and $u|_{\partial\bar{X}} = f$ where $f \in C^\infty(\partial\bar{X})$ is given. This follows the construction in section 4.1 of [11]. Writing Δ_g in the collar $[0, \epsilon)_t \times \partial\bar{X}$ through the diffeomorphism ϕ, we have

$$\Delta_g = -(t\partial_t)^2 + (n - \frac{t}{2}\text{Tr}_{h(t)}(\partial_t h(t)))t\partial_t + t^2 \Delta_{h(t)}$$

and for any $f_j \in C^\infty(\partial\bar{X})$ and $j \in \mathbb{N}_0$

$$(4.2) \qquad \Delta_g(f_j(y)t^j) = j(n-j)f_j(y)t^j + t^j(H(n-j)f_j)(t, y),$$

$$(H(z)f_j)(t, y) := t^2 \Delta_{h(t)} f_j(y) - \frac{(n-z)t}{2}\text{Tr}_{h(t)}(\partial_t h(t))f_j(y).$$

Now recall that since g is Einstein and even dimensional, we have $\partial_t^{2j+1} h(0) = 0$ for $j \in \mathbb{N}_0$ such that $2j + 1 < n$, see for instance Section 2 of [9]. Consequently, $H(n-j)f_j$ is an even function of t modulo $O(t^n)$ for $j \neq 0$, and modulo $O(t^{n+2})$ when $j = 0$. Since $H(n-j)f_j$ also vanishes at $t = 0$, we can construct by induction a Taylor series using (4.2)

$$(4.3) \qquad F_j = \sum_{k=0}^{j} t^k f_k(y), \quad F_0 = f_0 = f, \quad F_j = F_{j-1} + t^j \frac{[t^{-j}(\Delta_g F_{j-1})]|_{t=0}}{j(j-n)}$$

for $j < n$ such that $\Delta_g F_j = O(t^{j+1})$. Note that, since $H(n-j)f_j$ has even powers of t modulo $O(t^n)$, we get $f_{2j+1} = 0$ for $2j + 1 < n$. For $j = n$, the construction of F_n seems to fail but actually we can remark that $\Delta_g F_{n-1} = O(t^{n+1})$ instead of $O(t^n)$ thanks to the fact that $t^{2j}H(n-2j)f_{2j}$ has even Taylor expansion at $t = 0$ modulo $O(t^{2j+n+2})$ by the discussion above. So we can set $F_n := F_{n-1}$ and then continue to define F_j for $j > n$ using (4.3). Using Borel's Lemma, one can construct $F_\infty \in C^\infty(\bar{X})$ such that $\phi^* F_\infty - F_j = O(t^{j+1})$ for all $j \in \mathbb{N}$ and $\Delta_g F_\infty = O(\rho^\infty)$. Now we finally set $u = F_\infty - G\Delta_g F_\infty$ where $G : L^2(X, \text{dvol}_g) \to L^2(X, \text{dvol}_g)$ is the Green operator, i.e. such that $\Delta_g G = \text{Id}$, recalling that $\ker_{L^2} \Delta_g = 0$ by [17]. From the analysis of G in [17], one has that G maps $\dot{C}^\infty(\bar{X}) := \{v \in C^\infty(\bar{X}), v = O(\rho^\infty)\}$ to $\rho^n C^\infty(\bar{X})$. This proves that $u \in C^\infty(\bar{X})$ and has an asymptotic expansion

$$(4.4) \qquad \phi^* u(t, y) = f(y) + \sum_{0 < 2j < n} t^{2j} f_{2j}(y) - \phi^*(G\Delta_g F_\infty) + O(t^{n+1}).$$

In particular the first odd power is of order t^n and its coefficient is given by the smooth function $[t^{-n}\phi^*(G\Delta_g F_\infty)]|_{t=0}$ of $C^\infty(\partial\bar{X})$. Notice that the f_{2j} in the construction are local with respect to f, more precisely $f_{2j} = p_{2j}f$ for some differential operator p_{2j} on the boundary. Note that we used strongly that the Taylor expansion of the metric $t^2\phi^* g$ at $t = 0$ is even to order t^n, which comes from the fact that X is Einstein and has even dimensions. Indeed for a general asymptotically hyperbolic manifold, u has logarithmic singularities, see [11, 12].

Since $\phi^* \nabla^{\rho^2 g} \rho = \partial_t$, the definition of $\mathcal{S}f$ in the Introduction is equivalent to $\mathcal{S}f = \frac{1}{n!}\partial_t^n \phi^* u|_{t=0}$, i.e. the n-th Taylor coefficient of the expansion of $\phi^* u$ at $t = 0$, in other words

$$\mathcal{S}f = -[t^{-n}\phi^*(G\Delta_g F_\infty)]_{t=0} = -[\rho^{-n}G\Delta_g F_\infty]|_{\partial\bar{X}}.$$

From the analysis of Mazzeo-Melrose [17], one can describe the behaviour of the Green kernel $G(z, z')$ near the boundary and outside the diagonal $\text{diag}_{\bar{X} \times \bar{X}}$:

(4.5) $$\rho(z)^{-n}\rho(z')^{-n}G(z, z') \in C^\infty(\bar{X} \times \bar{X} \setminus \text{diag}_{\bar{X} \times \bar{X}}).$$

We can show easily that the kernel of $\mathcal{S}$ is the boundary value of (4.5) at the corner $\partial\bar{X} \times \partial\bar{X}$:

Lemma 4.1. *The Schwartz kernel $\mathcal{S}(y, y')$ of the scattering map $\mathcal{S}$ is, for $y \neq y'$,*

$$\mathcal{S}(y, y') = n[\rho(z)^{-n}\rho(z')^{-n}G(z, z')]_{z=y, z'=y'}$$

where $G(z, z')$ is the Green kernel for Δ_g.

Proof: Consider $(G\Delta_g F_\infty)(z)$ for $z \in X$ fixed and use Green formula on the compact $U_\epsilon := \{z' \in X; \rho(z) \geq \epsilon, \text{dist}(z', z) \geq \epsilon\}$

$$\int_{U_\epsilon} G(z, z')\Delta_g F_\infty(z')\text{dv}_g(z') = \int_{\partial U_\epsilon} (G(z, z')\partial_{n'}F_\infty(z') - \partial_{n'}G(z, z')F_\infty(z'))d\nu_\epsilon(z')$$

where $\partial_{n'}$ is the unit normal interior pointing vector field of ∂U_ϵ (in the right variable z') and $d\nu_\epsilon$ the measure induced by g there. Consider the part $\rho(z') = \epsilon$ in the variables as in (4.1) using the diffeomorphism ϕ, i.e. $\phi(t', y') = z'$, then $\phi^*\partial_{n'} = t'\partial_t'$ and $\phi^*(d\nu_{t'}) = t'^{-n}\text{dvol}_{h(t')}$. Using (4.5) and $F_\infty = f + O(\rho^2)$ by the construction of F_∞ above the Lemma, we see that the integral on $\rho' = \epsilon$ converges to

$$n\int_{\partial\bar{X}}[\rho(z')^{-n}G(z, z')]_{z'=y'}f(y')\text{dv}_{h(0)}(y').$$

as $\epsilon \to 0$. As for the part on $\text{dist}(z', z) = \epsilon$, by another application of Green formula and $\Delta_g(z')G(z, z') = \delta(z - z')$, this converges to $F_\infty(z)$ as $\epsilon \to 0$. We deduce that the solution u of $\Delta_g u$ with $u|_{\partial\bar{X}} = f$ is given by

(4.6) $$u(z) = n\int_{\partial\bar{X}}[\rho(z')^{-n}G(z, z')]_{z'=y'}f(y')\text{dv}_{h(0)}(y').$$

Let us write dy for $\text{dv}_{h_0}(y)$. So given $y \in \partial X$, let f be supported in a neighborhood of y and take $\psi \in C^\infty(\partial\bar{X})$ with $\psi f = 0$ and consider the pairing

$$\int_{\partial\bar{X}} \phi^* u(t, y)\psi(y)dy.$$

The Taylor expansion of u at $t = 0$ and the structure of $G(z, z')$ given by (4.5) show that

$$\int_{\partial\bar{X}} \psi(y)\mathcal{S}f(y)dy = n\int_{\partial\bar{X}}[\rho(z)^{-n}\rho(z')^{-n}G(z, z')]|_{z=y, z'=y'}\psi(y)f(y')dy'dy,$$

which proves the claim. $\qquad\square$

Remark: A more general relation between the kernel of the resolvent of Δ_g, $(\Delta_g - \lambda(n - \lambda))^{-1}$, and the kernel of the scattering operator $S(\lambda)$ holds, as proved in [13]. But since the proof of Lemma 4.1 is rather elementary, we included it to make the paper essentially self-contained.

4.2. **Einstein equation for g.** We shall analyze Einstein equation in a good system of coordinates, actually constructed from harmonic coordinates for Δ_g. First choose coordinates $(y_1, \ldots, y_n)$ in a neighbourhood $V \subset \partial \bar{X}$ of $p \in \partial \bar{X}$. Take an open set $W \subset \partial \bar{X}$ which contains V, we may assume that $y_i \in C_0^\infty(W)$. Let ϕ be the diffeomorpism as in (4.1). From the properties of the solution of the equation $\Delta_g u = 0$, as in subsection 4.1 (which follows Graham-Zworski [11]), there exists n smooth functions $(x_1, \ldots, x_n)$ on $\bar{X}$ such that

$$\Delta_g x_i = 0, \quad \phi^* x_i = y_i + \sum_{0 < 2k < n} t^{2k} p_{2k} y_i + t^n \mathcal{S} y_i + O(t^{n+1})$$

where p_k are differential operators on $\partial \bar{X}$ determined by the $(\partial_t^k h(0))_{k=0,\ldots,n-1}$ at the boundary (using the form (4.1)). Similarly let $y_0 \in C_0^\infty(W)$ be a non zero smooth function such that $y_0 = 0$ in V, then by Subsection 4.1 there exists $v \in C^\infty(\bar{X})$ such that

$$\Delta_g v = 0, \quad \phi^* v = y_0 + \sum_{0 < 2k < n} t^{2k} p_{2k} y_0 + t^n \mathcal{S} y_0 + O(t^{n+1}).$$

Thus in particular v vanishes near p to order ρ^n since $p_k y_0 = 0$ in V for $k = 1, \ldots, n$, thus one can write

$$v = \rho^n(w + O(\rho)),$$

where w is a smooth function on $\partial \bar{X}$ near p. The set $\{m \in V; w(m) \neq 0\}$ is an open dense set of V. Indeed, otherwise w would vanish in an open set of V but an easy computation shows that if $U \in \rho^j C^\infty(\bar{X})$ then $\Delta_g U = -j(j-n)U + O(\rho^{j+1})$ so v would vanish to infinite order at an open set of V and by Mazzeo's unique continuation result [19], it would vanish identically in $\bar{X}$. Thus, possibly by changing p to another point (still denoted p for convenience), there exists $v \in C^\infty(\bar{X})$ such that v is harmonic for Δ_g and $v = \rho^n(w + O(\rho))$ with $w > 0$ near p, the function $x_0 := v^{1/n}$ then defines a boundary defining function of $\partial \bar{X}$ near p, it can be written as ρe^f for some smooth f. Then $(x_0, x_1, \ldots, x_n)$ defines a system of coordinates near p.

Let us now consider Einstein equations in these coordinates. Again like (3.4), the principal part of $\text{Ric}(g)$ is given by

$$-\frac{1}{2} \sum_{\mu,\nu} g^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} g_{ij} + \frac{1}{2} \sum_r (g_{ri} \partial_{x_j}(\Delta_g x_r) + g_{rj} \partial_{x_i}(\Delta_g x_r)).$$

But all functions x_r are harmonic, except x_0, and the latter satisfies

$$0 = \Delta_g x_0^n = -n \text{div}_g(x_0^{n-1} \nabla^g x_0) = -n x_0^{n-1} \Delta_g x_0 - n(n-1) x_0^{n-2} |dx_0|_g^2$$

or equivalently $\Delta_g x_0 = (1-n) x_0 |dx_0|_{x_0^2 g}$. But this involves only terms of order 0 in the metric g or $\bar{g} := x_0^2 g$ so the principal part of $\text{Ric}(g)$ in these coordinates is

$$-\frac{1}{2} \sum_{\mu,\nu} g^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} g_{ij}$$

which is elliptic in the interior X. We multiply the equation $\text{Ric}(g) = -ng$ by x_0^2 near p and using (3.2) and (3.3), with the commutations relations $[x_0 \partial_{x_0}, x_0^\alpha] = \alpha x_0^\alpha$ for all $\alpha \in \mathbb{C}$, it is straightforward to obtain

Lemma 4.2. *Let $x = (x_0, x_1, \ldots, x_n)$ be the coordinates defined above near a point $p \in \{x_0 = 0\}$, then Einstein equation for g can be written under the system*

$$(4.7) \qquad \sum_{\mu,\nu} x_0^2 \bar{g}^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} \bar{g}_{ij} + Q_{ij}(x_0, \bar{g}, x_0 \partial \bar{g}) = 0, \quad i,j = 0, \ldots, n$$

where $\bar{g} = x_0^2 g$ near p, $Q_{ij}(x_0, A, B)$ are smooth and polynomial of order 2 in B, and $x_0 \partial \bar{g} := (x_0 \partial_{x_m} \bar{g}_{ij})_{m,i,j} \in \mathbb{R}^{(n+1)^3}$.

This is a non-linear system of order 2, elliptic in the uniformly degenerate sense of [18, 20, 17] and diagonal at leading order. We state the following unique continuation result for this system:

Proposition 4.3. *Assume $\bar{g}_1$ and $\bar{g}_2$ are two smooth solutions of the system (4.7) with $\bar{g}_1 = \bar{g}_2 + O(x_0^\infty)$ near p. Then $\bar{g}_1 = \bar{g}_2$ near p.*

Proof: This is an application of Mazzeo's unique continuation result [19]. We work in a small neighbourhood U of p and set $w = (\bar{g}_1 - \bar{g}_2)$ near p. For h metric near p and ℓ symmetric tensor near p, let

$$G(x_0, h, \ell) := -\sum_{\mu, \nu} x_0^2 h^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} \bar{g}_2 - Q(x_0, h, \ell)$$

where $Q := (Q_{ij})_{i,j=0,\ldots,n}$. Note that G is smooth in all its components. We have from (4.7)

$$(4.8) \qquad \sum_{\mu, \nu} x_0^2 \bar{g}_1^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} w = G(x_0, \bar{g}_1, x_0 \partial \bar{g}_1) - G(x_0, \bar{g}_2, x_0 \partial \bar{g}_2).$$

Let $g_1 := x_0^{-2} \bar{g}_1$ and let ∇ be the connection on symmetric 2 tensors on U induced by g_1, then $\nabla^* \nabla w = \sum_{\mu, \nu} g_1^{\mu\nu} \nabla_{\partial_{x_\nu}} \nabla_{\partial_{x_\mu}} w$ and in coordinates it is easy to check that $x_0(\nabla_{\partial_{x_\mu}} - \partial_{x_\mu})$ is a zeroth order operator with smooth coefficients up to the boundary, using (3.3) for instance. Therefore one obtains, using (4.8),

$$|\nabla^* \nabla w|_{g_1} \leq C(|w|_{g_1} + |\nabla w|_{g_1})$$

for some C depending on $\bar{g}_1, \bar{g}_2$. It then suffices to apply Corollary 11 of [19], this proves that $w = 0$ and we are done. $\square$

4.3. **Reconstruction near the boundary and proof of Theorem 1.2.** The proof of Theorem 1.2 is fairly close to that of Theorem 1.1. Let $(\bar{X}_1, g_1)$ and $(\bar{X}_2, g_2)$ be conformally compact Einstein manifolds with geodesic boundary defining functions ρ_1 and ρ_2. Let $\mathcal{S}_i$ be the scattering map for g_i defined by (1.3) using the boundary defining function ρ_i, assume that $\partial \bar{X}_1$ and $\partial \bar{X}_2$ contain a common open set Γ such that the identity map which identifies the copies of Γ is a diffeomorphism, and that $\mathcal{S}_1 f|_\Gamma = \mathcal{S}_2 f|_\Gamma$ for all $f \in C_0^\infty(\Gamma)$. Using the geodesic boundary defining function ρ_i for $g_i, i = 1, 2$, there is a diffeomorphism $\phi^i : [0, \epsilon)_t \times \partial \bar{X}_i \to \phi^i([0, \epsilon) \times \partial \bar{X}_i) \subset \bar{X}_i$ as in (4.1) so that

$$(4.9) \qquad (\phi^i)^* g_i = \frac{dt^2 + h_i(t)}{t^2}$$

where $h_i(t)$ is a family of metric on $\partial \bar{X}_i$. We first show the

Lemma 4.4. *The metrics $h_1(t)$ and $h_2(t)$ satisfy $\partial_t^j h_1(0)|_\Gamma = \partial_t^j h_2(0)|_\Gamma$ for all $j \in \mathbb{N}_0$.*

Proof: For a compact manifold M, let us denote $\Psi^z(M)$ the set of classical pseudo-differential operators of order $z \in \mathbb{R}$ on M. Since $\mathcal{S}_i$ is the scattering operator $S_i(\lambda)$ at energy $\lambda = n$ for Δ_{g_i} as defined in [13], we can use [13, Th.1.1], then we have that $\mathcal{S}_i \in \Psi^n(\partial \bar{X}_i)$ for $i = 1, 2$, with principal symbol $\sigma_n^i(y, \xi) = 2^{-n} \Gamma(-\frac{n}{2}) / \Gamma(\frac{n}{2}) |\xi|_{h_i(0)}$, thus $h_1(0) = h_2(0)$ on Γ and $\chi(\mathcal{S}_1 - \mathcal{S}_2)\chi \in \Psi^{n+1}(\Gamma)$ for all $\chi \in C_0^\infty(\Gamma)$. Now we use Einstein equation, for instance the results of [7, 8] (see also [9, Sec. 2]) show, using only Taylor expansion of $\text{Ric}(g) = -ng$ at the boundary, that

$$\partial_t^j h_1(0)|_\Gamma = \partial_t^j h_2(0)|_\Gamma, \quad j = 0, \ldots, n-1.$$

Then we use Theorem 1.2 of [13] which computes the principal symbol of $\mathcal{S}_1 - \mathcal{S}_2$. Since this result is entirely local, we can rephrase it on the piece Γ of the boundary: if there exists a symmetric 2-tensor L on Γ such that $h_1(t) = h_2(t) + t^k L + O(t^{k+1})$ on $[0, \epsilon)_t \times \Gamma$ for some $k \in \mathbb{N}$, then for any $\chi \in C_0^\infty(\Gamma)$ we have $\chi(\mathcal{S}_1 - \mathcal{S}_2)\chi \in \Psi^{n-k}(\Gamma)$ and the principal symbol of this operator at $(y, \xi) \in T^*\Gamma$ is [1]

$$(4.10) \qquad A_1(k, n) L(\xi^*, \xi^*) |\xi|_{h_0}^{n-k-2} + A_2(k, n) \frac{k(k-n) \mathrm{Tr}_{h_0}(L)}{4} |\xi|_{h_0}^{n-k}$$

where $h_0 := h_1(0)|_\Gamma = h_2(0)|_\Gamma$, $\xi^* := h_0^{-1}\xi \in T_y\Gamma$ is the dual of ξ through h_0, and $A_i(k, \lambda)$ are the meromorphic functions of $\lambda \in \mathbb{C}$ defined by

$$A_1(k, \lambda) := -\pi^{-\frac{n}{2}} 2^{k-2\lambda+n} \frac{\Gamma(\frac{n}{2} - \lambda + \frac{k}{2} + 1)}{\Gamma(\lambda - \frac{k}{2} - 1)} \frac{\Gamma(\lambda)^2}{\Gamma(\lambda - \frac{n}{2} + 1)^2} \frac{T_1(k, \lambda)}{M(\lambda)}$$

$$A_2(k, \lambda) := \pi^{-\frac{n}{2}} 2^{k-2\lambda+n-2} \frac{\Gamma(\frac{n}{2} - \lambda + \frac{k}{2})}{\Gamma(\lambda - \frac{k}{2})} \frac{\Gamma(\lambda)^2}{\Gamma(\lambda - \frac{n}{2} + 1)^2} \frac{T_2(k, \lambda)}{M(\lambda)}$$

where $T_l(k, \lambda)$ is defined, when the integral converges, by

$$T_l(k, \lambda) := \int_0^\infty \int_{\mathbb{R}^n} \frac{u^{2\lambda-n+k+3-2l}}{(u^2 + |v|^2)^\lambda (u^2 + |e_1 - v|^2)^\lambda} dv_{\mathbb{R}^n} du, \quad e_1 = (1, 0, \ldots, 0),$$

and $M(\lambda) \in \mathbb{C}$ is a constant not explicitly computed in [13]. However at $\lambda = n$ the constant $M(n)$ is defined in [13, Sec. 4] such that $u(z)/n = M(n)f + O(\rho(z))$ where $u(z)$ is the function of (4.6), so $M(n) = n$ by (4.4). Since we are interested in the case $k = n$, only the term with $A_1(n, n)$ appears, and setting $\lambda = n$ in $A_1(n, \lambda)$, with the explicit formulae above and the fact that $T_1(n, n) > 0$ converges by Lemma 5.2 of [13], we see easily that $A_1(n, n) \neq 0$ if $n > 2$. Since we assumed $\chi\mathcal{S}_1\chi = \chi\mathcal{S}_2\chi$, this implies that $L = 0$ and $h_1 - h_2 = O(t^{n+1})$ near Γ. We finally use again [7] (see [8, Sect. 4] for full proofs), where it is proved that if $g_1 = g_2 + O(\rho^{n-1})$ with g_1, g_2 conformally compact Einstein and n odd, then $g_1 = g_2 + O(\rho^\infty)$. Notice that their arguments are entirely local near any point of the boundary, so we can apply it near the piece Γ of the boundary. $\qquad \square$

Lemma 4.5. *For $i = 1, 2$, there exist $p \in \Gamma$, neighbourhoods U_i of p in $\bar{X}_i$ and a diffeomorphism $F : U_1 \to U_2$, $F|_{U_1 \cap X_1}$ analytic, such that $F^*g_2 = g_1$ and $F|_{U_1 \cap \Gamma} = \mathrm{Id}$.*

Proof: We work in the collar $[0, \epsilon)_t \times \Gamma$ through the diffeomorphisms ϕ^i as in (4.9). In a neighbourhood $U \subset [0, \epsilon) \times \Gamma$ of $p \in \Gamma$, we use coordinates $\bar{x}^i := (\bar{x}_0^i, \ldots, \bar{x}_n^i)$ where $\bar{x}_j^i := {\phi^i}^* x_j^i$ and x_j^i is the function defined in Subsection 4.2 for g_i with boundary values $x_j^1|_{\rho_1=0} = x_j^2|_{\rho_2=0} \in C_0^\infty(\Gamma)$ for all j, Now set $\psi : U \to \psi(U) \subset [0, \epsilon) \times \Gamma$ such that $\bar{x}_j^1 = \psi^* \bar{x}_j^2$. This is a diffeomorphism near p and moreover ${\phi^1}^* g_1$ and ${\phi^2}^* g_2$ coincide to infinite order at $t = 0$ by Lemma 4.4, so the coordinates $\bar{x}^1$ and $\bar{x}^2$ satisfy $\Delta_{{\phi^1}^* g_1}(\bar{x}_j^1 - \bar{x}_j^2) = O(t^\infty)$ for all j. Since $\bar{x}^1, \bar{x}^2$ have the same boundary values, they agree to order $O(t^n)$ using the construction of F_{n-1} in (4.3). But since $\mathcal{S}_1(x_j^1|_\Gamma) = \mathcal{S}_2(x_j^2|_\Gamma)$ on Γ, one has $\bar{x}_j^1 = \bar{x}_j^2 + O(t^{n+1})$ near p, which again

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[1]It is important to notice that the coefficient of $|\xi|^{n-k}$ in (4.10) is not exactly that of Theorem 1.2 of [13], indeed there is a typo in equation (3.5) in [13, Prop 3.1]: the coefficient in front of $T = \mathrm{Tr}_{h_0}(L)$ there should be $k(k-n)/4$ instead of $k(k+1)/4$, this comes from the fact that, in the proof of [13, Prop 3.1], the term

$$\frac{1}{16} x^2 f \partial_x \log(\delta_2/\delta_1) \partial_x \log(\delta_2 \delta_1) = -\frac{k(n+1)}{4} f x^k T + O(x^{k+1})$$

while it has been considered as a $O(x^{k+1})$ there.

by induction and (4.2) shows that $\bar{x}_j^1 = \bar{x}_j^2 + O(t^\infty)$ near p. In particular, setting $\bar{g}_1 := (\bar{x}_0^1)^2 \phi^{1*} g_1$, $\bar{g}_2 := \psi^*((\bar{x}_0^2)^2 \phi^{2*} g_2) = (\bar{x}_0^1)^2 \psi^*(\phi^{2*} g_2)$, one obtains that $\bar{g}_1 = \bar{g}_2 + O((\bar{x}_0^1)^\infty)$, i.e. the metrics agree to infinite order in the coordinates $\bar{x}^1$. Thus from Lemma 4.2, $\bar{g}_1$ and $\bar{g}_2$ both satisfy the same system (4.7) and agree to infinite order at the boundary $\{t = 0\}$ near p in the coordinate system $\bar{x}^1$, so by Proposition 4.3, we deduce that $\phi^{1*} g_1 = \psi^* \phi^{2*} g_2$ and this ends the proof by setting $F := \phi^2 \circ \psi \circ (\phi^1)^{-1}$. $\qquad\square$

We finish by the following Corollary, similar to Corollary 3.5.

Corollary 4.6. *Let $G_i(z, z')$ be the Green kernel for g_i, $i = 1, 2$. Then $\mathcal{S}_1|_\Gamma = \mathcal{S}_2|_\Gamma$ implies that there exists an open set U_1 such that $G_1(z, z') = G_2(F(z), F(z'))$ for all $(z, z') \in (U_1 \times U_1) \setminus \{z = z'\}$.*

Proof: We first take $y' \in U_1 \cap \Gamma$, and consider $T_1(z) := [\rho_1(z')^{-n} G_1(z, z')]_{z'=y'}$ and $T_2(z) := [F^* \rho_2(z')^{-n} G_2(F(z), F(z'))]|_{z'=y'}$. They both satisfy $\Delta_{g_1} T_i(z) = 0$ for $z \in U_1$ and by Lemma 4.1 and the assumption $\mathcal{S}_1|_\Gamma = \mathcal{S}_2|_\Gamma$, we have that $T_1 - T_2 = O(\rho_1^{n+1})$ near $\Gamma \setminus \{y'\}$, so by induction on (4.2), $T_1 = T_2 + O(\rho_\infty^1)$ in $U_1 \setminus \{y'\}$, and then by the unique continuation result of Mazzeo [19], $T_1 = T_2$ in the same set. Now this means that for $z' \in U_1$, $z \to G_1(z', z)$ and $z \to G_2(F(z'), F(z))$ are harmonic for Δ_{g_1} in $U_1 \setminus \{z'\}$, and they coincide to order ρ_1^{n+1} at Γ, so again by unique continuation they are equal. $\qquad\square$

4.4. **Proof of Theorem 1.2.** Using Corollary 4.6 and the fact that (X_1, g_1) and (X_2, g_2) Einstein, and by Theorem 5.2 of [6] are analytic in harmonic coordinates, it suffices to apply Theorem 4.1 of [14], which is essentially the same as Proposition 3.6 but for a complete manifold.

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