Poincaré series and linking of Legendrian knots

Joint work with Nguyen Viet Dang (Université Lyon 1)

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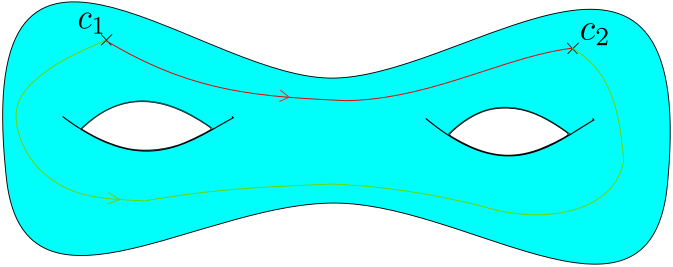
November, 12th 2020

Introduction

In all this talk, (X, G) will be a smooth ($\mathcal{C}^\infty$), compact, connected, oriented and Riemannian **surface** which has no boundary and which has **negative curvature** (a priori non constant).



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Theorem (Delsarte (1942), Huber (1959), Margulis (1969))

There exists $A_{c_1,c_2} > 0$ such that, as $T \to +\infty$,

$$\boxed{\mathcal{N}_T(c_1,c_2) := |\{\gamma \in \mathcal{P}_{c_1,c_2} :\ 0 < \ell(\gamma) \leq T\}| \sim A_{c_1,c_2} e^{Th_{top}},}$$

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- For more informations on the behaviour of $\mathcal{N}_T(c_1, c_2)$ and its applications, see the survey of Parkkonen and Paulin (LMS Lecture notes **425**, 2016).

Poincaré series as zeta renormalization of $\mathcal{N}_T(c_1, c_2)$

Let $s \in \mathbb{C}$ and set

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defines a holomorphic function in the half plane

$$\{w \in \mathbb{C} : \mathrm{Re}(w) > h_{\mathrm{top}}\}.$$

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- **Similar results for Ruelle zeta functions** which "count periodic orbits" (Selberg, Smale, Ruelle, Rugh, Fried, Kitaev, Baladi-Tsujii, Giulietti-Liverani-Pollicott, Dyatlov-Zworski, Faure-Tsujii, Dyatlov-Guillarmou, Jezequel, etc.)

Meromorphic continuation of Poincaré series

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For $\gamma \in \mathcal{P}_{c_1,c_2}$, one has

$$\varphi^{\ell(\gamma)}(S_{c_1}^*X) \cap S_{c_2}^*X \neq \emptyset.$$

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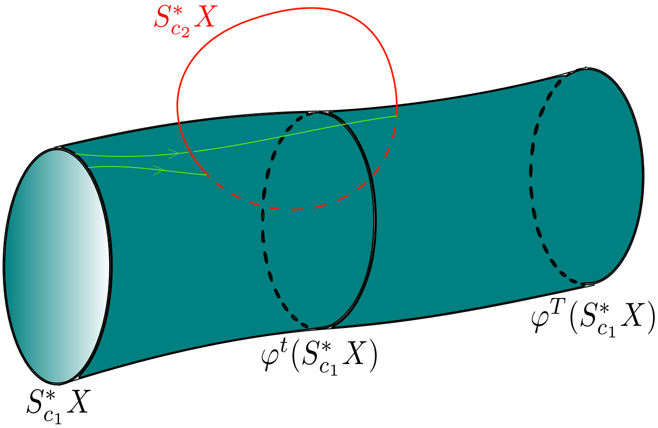
&= [S^*_c X] - d\left(\int_0^T \iota_V \varphi^{-t*}[S^*_c X] dt\right).
\end{aligned}$$

The current

$$R_T := -\int_0^T \iota_V \varphi^{-t*}[S_c^* X] dt$$

represents the integration on the surface

$$\left\{\varphi^t(x) : x \in S_c^* X \text{ and } 0 \le t \le T\right\} \subset S^* X.$$



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Let c_1 and c_2 be two points in X and let $T > 0$. Then, one has

$$\boxed{\mathcal{N}_T(c_1, c_2, s) = -\int_{S^*X} [S^*_{c_2}X] \wedge \int_0^T e^{-st} \iota_V \varphi^{-t*}[S^*_{c_1}X] dt.}$$

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$\rightsquigarrow$ Related to the sudy of transfer operators in dynamical systems.

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$\rightsquigarrow$ Non exhaustive list...

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Moreover, they describe $\mathrm{Ran}(\pi_0)$ in terms of generators of the De Rham cohomology.

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In summary, the value at 0 is given by :

$$\mathcal{N}_\infty(c_1, c_2, 0) = -\int_{S^*X} [S_{c_2}^* X] \wedge R_{c_1} =: -\mathbf{L}(c_1, c_2).$$

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The value at 0 as the linking of two knots

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We say that $S_c^* X$ is a **Legendrian knot** in S^*X.

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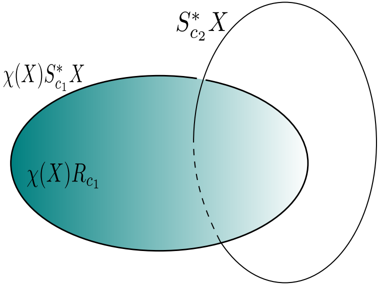

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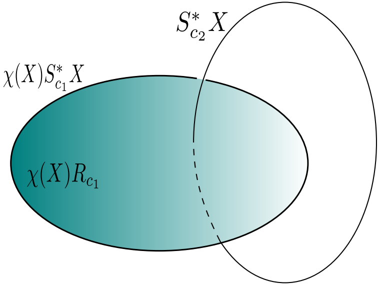

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&= -\chi(X) \mathbf{L}(c_1, c_2).
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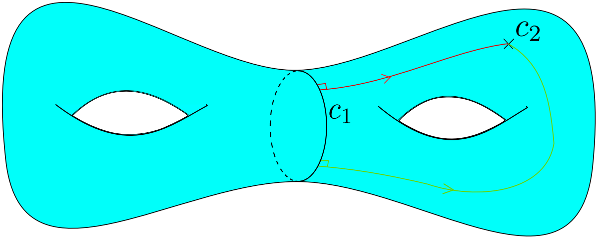
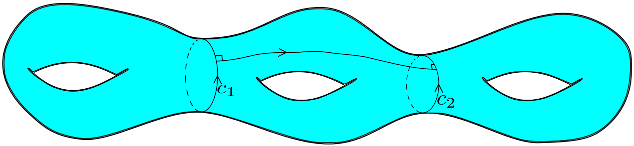
A more general picture

Let $\mathbf{c}_1$ and $\mathbf{c}_2$ in $\pi_1(X)$. If $\mathbf{c}_i$ is nontrivial, one can find an unique geodesic c_i in the conjugacy class of $\mathbf{c}_i \in \pi_1(X)$. We say that it is a **geodesic representative** of $\mathbf{c}_i$.





Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be two elements in $\pi_1(X)$ and let c_1 and c_2 be two of their geodesic representatives in X. Set

$$\mathcal{P}_{\mathbf{c}_1,\mathbf{c}_2} := \{\text{geodesic arcs joining } c_1 \text{ and } c_2 \text{ and directly} \perp \text{to } c_1 \text{ and } c_2\}.$$

Theorem (Dang-Rivière, 2020)

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$$s \mapsto \mathcal{N}_\infty(c_1, c_2, s) := \sum_{\gamma \in \mathcal{P}_{\mathbf{c_1},\mathbf{c_2}}:\ \ell(\gamma)>0} e^{-s\ell(\gamma)}$$

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- *in that case, $\mathcal{N}_\infty(c_1, c_2, 0)$ is the* ***linking number of the unit (direct) conormal bundles of c_1 and c_2.***

Comments.

- When c_1 and c_2 are both nontrivial in $\pi_1(X)$, the linking number we obtain is also the **linking number of the closed orbits** lifting c_1 and c_2 in S^*X.

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- There is an **explicit expression** for the linking number of c_1 and c_2 in terms of Euler characteristics :

  $$\pm\mathcal{N}_\infty(c_1, c_2, 0) = \frac{\chi(X(c_1))\chi(X(c_2))}{\chi(X)} - \chi(X(c_1)\cap X(c_2)) + \frac{1}{2}\chi(c_1\cap c_2).$$

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- What about higher dimensions ? Probably a pole when the dimension is odd.
- What can be extracted using the spectral decomposition of the Laplacian ?

Thank you for your attention.