Fourier decay and nonlinearity of dynamical systems

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- $\mu =$ middle third Cantor measure (distribution of $\sum \varepsilon_k 3^{-k}$, $\varepsilon_k \sim \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$) is **not** Rajchman:

$$\widehat{\mu}(3^k) = \widehat{\mu}(1) \neq 0, \quad \forall k \in \mathbb{N}.$$

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- Equilibrium states for non-linear iterated function systems (Kaufman, Mosquera-Shmerkin, S.-Stevens)

Fourier decay and Lebesgue like properties

- μ Rajchman $\implies \operatorname{spt}(\mu)$ is set of **multiplicity** for trigonometric series: i.e. $\exists$ sequences $(a_n)_{n\in\mathbb{Z}} \neq (b_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that $\forall x \in \mathbb{R} \setminus \operatorname{spt}(\mu)$:

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$$\mu\Big(x : \Big|x - \frac{p}{q_n}\Big| \leq \frac{\psi(q_n)}{q_n} \text{ for } \infty \text{ many } (p,n)\Big) = \begin{cases} 1, & \sum_{n=1}^\infty \psi(q_n) = \infty; \\ 0, & \sum_{n=1}^\infty \psi(q_n) < \infty \end{cases}$$

for any lacunary sequence $(q_n) \subset \mathbb{N}$. **(Khintchine type property)**

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- $|\widehat{\mu}(n)| = O\Big(|n|^{-\alpha}\Big),\ |n| \to \infty \quad \implies$
  - μ is L^p **improving**: $\forall f \in L^p(\mathbb{R})$, $1 < p < \infty$: $f * \mu \in L^{p+\varepsilon}(\mathbb{R})$
  - **Hausdorff dimension** of μ satisfies $\dim_{\mathrm{H}} \mu \geq \min\{1, 2\alpha\}$

Avoiding lattices and Fourier decay

- **Bernoulli convolution** μ_λ: distribution of $\sum_{n\in\mathbb{N}} \varepsilon_n \lambda^n$ for $0 < \lambda < 1$ where $\varepsilon_n \sim \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ i.i.d. for all $n \in \mathbb{N}$.

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- Erdös-Salem: μ_λ is Rajchman if and only if $(d(\lambda^{-n}, \mathbb{Z}))_{n\in\mathbb{N}} \notin \ell^2$.

IFS fractals

- Let $\{f_a : a \in \mathcal{A}\}$ be an **iterated function system**, i.e. $\mathcal{A} \subset \mathbb{N}$, $I \subset \mathbb{R}$ is an interval and $f_a \in C^2(I)$ is a contraction: $\|f_a'\|_\infty < 1$.

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- **Example**: If $\mathcal{A} = \{1, 2\}$, $I = [-(1-\lambda)^{-1}, (1-\lambda)^{-1}]$, $f_1(x) = \lambda x + 1$, $f_2(x) = \lambda x - 1$, $\varphi(x) \equiv \log(1/2)$, then $\mu_\varphi = \mu_\lambda$, the Bernoulli convolution associated to $0 < \lambda < 1$.

IFSs from the Gauss map

- $I = [0, 1]$, $\mathcal{A} = \mathbb{N}$, $f_a(x) = (x + a)^{-1}$, $a \in \mathbb{N}$. Then f_a are the inverse branches of the **Gauss map** $T(x) = \frac{1}{x} \mod 1$.

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- Jordan-S. 2016: For all equilibrium states μ_φ with $\dim_{\mathrm{H}} \mu_\varphi > 1/2$ and $\log|T'|$ has a light tail at infinity w.r.t. μ_φ, we have

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Application:

- Salem (1943) conjectured the Minkowski question mark measure $\mu_?$ is Rajchman, where $\mu_?$ is the Stieltjes measure associated to the **Minkowski's question mark bijection** $? : [0, 1] \to [0, 1]$ mapping quadratic irrational numbers onto dyadic rational numbers.
- Indeed, $\mu_?$ is an equilibrium state for $\varphi(x) = \log(2^{-a_1(x)})$ where $a_1(x)$ is the first continued fraction digit of $x \in [0, 1]$.

IFSs from convex co-compact hyperbolic surfaces

- $X = \Gamma \setminus \mathbb{H}$: convex co-compact hyperbolic surface.
  Limit set: $\Lambda_X \subset \partial\mathbb{H}$ can be represented as a subset of an IFS fractal for some collection of maps $f_a(x) = \frac{r_a x + b_a}{\varrho_a x + c_a}$, $a \in \mathcal{A}$.

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- If μ is the **Patterson-Sullivan measure** on Λ_X, it is equilibrium state with potential defined by $\varphi(f_a(x)) = \log |f_a'(x)|^\delta$, for $\delta = \dim_{\mathrm{H}} \Lambda_X$.

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$$\|\varphi\|_{C^1} + \|g\|_{C^2} < \infty \quad \text{and} \quad \inf|\varphi'| > 0 :$$

$$\Rightarrow \quad \Big| \int g(x) e^{-2\pi i n \varphi(x)}\, d\mu(x) \Big| = O(|n|^{-\alpha(\delta)}), \quad n \to \infty.$$

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Application:

- **Selberg zeta function** on X:

$$\zeta_X(s) = \prod_{\substack{\gamma \text{ primitive} \\ \text{closed geodesic in } X}} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell(\gamma)}), \quad \mathrm{Re}(s) \gtrsim 1, s \in \mathbb{C}.$$

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- $\forall \delta > 0$, $\exists \alpha_0(\delta) > 0$ such that $\zeta_X(s) = 0$ for only finitely many $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \delta - \alpha_0(\delta)$.

Nonlinearity

- Let $\{f_a : I \to \mathbb{R} : a \in \mathcal{A}\}$ be C^2 IFS. Assume $I_a := f_a(I)$, $a \in \mathcal{A}$, are disjoint. Then there is an expanding map $T : I \to \mathbb{R}$ with

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- $\{f_a : a \in \mathcal{A}\}$ is **totally non-linear** if it is not conjugated to a self-similar IFS.

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and μ_φ equilibrium state with $\varphi(x) = \log p_{a(x)}$ for some $\sum_{a \in \mathcal{A}} p_a = 1$, $0 < p_a < 1$, and $a(x) \in \mathbb{N}$ determined by $x = T(f_{a(x)}(x))$.

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- Varjú-Yu 2020: if $g = 0$, $\psi(I) \subset c\mathbb{Z}$ with e^{-c} is not Pisot nor Salem number, then μ_φ is Rajchman with polylogarithmic decay.

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$$\log|T'| = g \circ T - g + \psi$$

and μ_φ equilibrium state with $\varphi(x) = \log p_{a(x)}$ for some $\sum_{a \in \mathcal{A}} p_a = 1$, $0 < p_a < 1$, and $a(x) \in \mathbb{N}$ determined by $x = T(f_{a(x)}(x))$.

- Mosquera-Shmerkin 2018: $\inf |g'| > 0$ and $\psi =$ constant, then μ_φ is Rajchman with power decay.
- Li-S. 2019: if $g = 0$ and ψ is not a **lattice**: $\psi(I) \not\subset c\mathbb{Z}$ for some $c \in \mathbb{R}$, then μ_φ is Rajchman.
- Brémont 2019: if $g = 0$, $\psi(I) \subset c\mathbb{Z}$ and μ_φ is **not** Rajchman, then e^{-c} is a Pisot number.
- Varjú-Yu 2020: if $g = 0$, $\psi(I) \subset c\mathbb{Z}$ with e^{-c} is not Pisot nor Salem number, then μ_φ is Rajchman with polylogarithmic decay.
- Solomyak 2019: if $g = 0$, for all ψ except zero Hausdorff dimensional parameter set of ψ, μ_φ is Rajchman with power decay.

Totally non-linear case

Theorem (S.-Stevens 2020)

Assume $\{f_a : I \to \mathbb{R}\}$ is totally non-linear and $\mathcal{A}$ is finite. Then every non-atomic equilibrium state μ_φ is Rajchman with power decay.

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Theorem (S.-Stevens 2020)

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A. Algom, F.-R. Hertz, Z. Wang (work in progress) can also prove Rajchman property but not power decay for $C^{1+\gamma}$ IFSs when $\{-\log|f_a'(x_a)| : a \in \mathcal{A}\}$ is not contained in an arithmetic progression, where x_a is the fixed point of f_a.

Large deviations

Write $I_{\mathbf{a}} := f_{\mathbf{a}}(I)$ for the composition $f_{\mathbf{a}} := f_{a_1} \circ \cdots \circ f_{a_n}$, $\mathbf{a} \in \mathcal{A}^n$.

Large deviations

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- **Large deviations** for light tailed observables: for any $\varepsilon > 0$ and $n \in \mathbb{N}$, we can find words $\mathcal{R}_n(\varepsilon) \subset \mathcal{A}^n$ such that for $\mu = \mu_\varphi$:

$$\mu = \sum_{\mathbf{a} \in \mathcal{R}_n(\varepsilon)} \mu|_{I_{\mathbf{a}}} + \sum_{\mathbf{a} \in \mathcal{A}^n \setminus \mathcal{R}_n(\varepsilon)} \mu|_{I_{\mathbf{a}}}$$

  where

  (1) for $\lambda = \int \log |T'| \, d\mu$ and $\delta = \dim_{\mathrm{H}} \mu$ we have

$$e^{-\varepsilon n} e^{-\lambda n} \lesssim |I_{\mathbf{a}}| \lesssim e^{\varepsilon n} e^{-\lambda n}, \quad \mathbf{a} \in \mathcal{R}_n(\varepsilon)$$

$$e^{-\varepsilon n} |I_{\mathbf{a}}|^{\delta} \lesssim \mu(I_{\mathbf{a}}) \lesssim e^{\varepsilon n} |I_{\mathbf{a}}|^{\delta} \quad \mathbf{a} \in \mathcal{R}_n(\varepsilon)$$

  (2) and the tail is exponentially small:

$$\sum_{\mathbf{a} \in \mathcal{A}^n \setminus \mathcal{R}_n(\varepsilon)} \mu(I_{\mathbf{a}}) = O(e^{-\delta(\varepsilon) n}),$$

Non-concentration and spectral gap

- The key to find $\varepsilon_0 > 0$ and $c_0 > 0$ such that the derivatives

$$f'_{\mathbf{a}}(x), \quad \mathbf{a} \in \mathcal{R}_n(\varepsilon), \quad x \in I,$$

non-concentrate in the scales $m \in \mathbb{N}$, $\frac{\varepsilon_0}{2} n \leq m \leq \varepsilon_0 n$ in the following sense: for any $x \in I$, $y \in \mathbb{R}$:

$$\frac{\sharp\{\mathbf{a} \in \mathcal{R}_n(\varepsilon) : e^{\lambda n} f'_{\mathbf{a}}(x) \in B(y, e^{-\varepsilon_0 m})\}}{\sharp \mathcal{R}_n(\varepsilon)} \lesssim e^{\varepsilon n} e^{-c_0 m}$$

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- For us ε_0 and c_0 and depends on the **spectral gap** for $\mathcal{L}_{\varphi - s \log |T'|}$ with $s = \delta - 2\pi i \xi$. Stoyanov (2011) has a proof for the spectral gap under a local non-integrability assumption for the roof functions of the symbolic Markov codings of C^2 Axiom A flows on C^2 complete Riemannian manifolds. This follows under total non-linearity of T with the roof $\log |T'|$.

Reduction to sum-product bounds

Cauchy-Schwartz and bounded distortions give us whenever

$$|\xi| \sim e^{(2k+1)n\lambda} e^{\varepsilon_0 n}$$

that

$$|\widehat{\mu}(\xi)|^2 \lesssim e^{\kappa\varepsilon n} e^{-\lambda(2k+1)\delta n} \sum_{\mathbf{a}_0 \dots \mathbf{a}_k \in \mathcal{R}_n(\varepsilon)^{k+1}} \sup_{e^{\varepsilon_0 n/2} \leq |\eta| \leq e^{\varepsilon n} e^{\varepsilon_0 n}} \Big| \sum_{\mathbf{b}_1 \dots \mathbf{b}_k} e^{-2\pi i \eta \zeta_1(\mathbf{b}_1) \dots \zeta_k(\mathbf{b}_k)} \Big|.$$

for the maps

$$\zeta_j(\mathbf{b}) := e^{2\lambda n} f'_{\mathbf{a}_{j-1}\mathbf{b}}(x_{\mathbf{a}_j})$$

and $x_{\mathbf{a}_j}$ is the center point of $f_{\mathbf{a}_j}(I)$ and $f_{\mathbf{a}_j}$ is the composition of the maps corresponding f_a to the word $\mathbf{a}_j = (a_1, \dots, a_n)$.

Sum-product bound

Lemma 8.43 (J. Bourgain: *The Discretized Sum-Product and Projection Theorems*, 2010)

For all $\kappa > 0$, there exists $\varepsilon_3 > 0$, $\varepsilon_4 > 0$ and $k \in \mathbb{N}$ such that the following holds.

Let ν be a probability measure on $[\frac{1}{2}, 1]$ and let N be a large integer. Assume for all $1/N < \varrho < 1/N^{\varepsilon_3}$ that

$$\max_a \nu(B(a, \varrho)) < \varrho^\kappa.$$

Then for all $\xi \in \mathbb{R}$, $|\xi| \sim N$:

$$\Big| \int \int \dots \int e^{-2\pi i \xi x_1 \dots x_k} \, d\nu(x_1) \dots d\nu(x_k) \Big| < N^{-\varepsilon_4}.$$

One can make this into a version involving multiple $\nu_1, \nu_2, \dots, \nu_k$ for ν_j a scaled version of $\mu_j = \frac{1}{\sharp \mathcal{R}_n(\varepsilon)} \sum_{\mathbf{b} \in \mathcal{R}_n(\varepsilon)} \delta_{\zeta_j(\mathbf{b})}$.

Representation theory and higher dimensions

- Li 2018: Renewal theoretic approach for Fourier decay of the Furstenberg measures on the projective spaces. This should help to get higher dimensional, totally non-linear case.
- Li-Naud-Pan 2019: $\mathrm{PSL}(2, \mathbb{C})$ version of Bourgain-Dyatlov proved
- Li-S. 2019: Self-affine measures, non-commuting matrices using Li 2018
- Fourier decay for self-similar measures in higher dimensions when assuming dense rotations is difficult, closely related to problem of finding spectral gap for non-lattice random walks on $SO(d)$. Currently known for algebraic parameters by Benoist-Saxcé 2014.