WAVE DECAY ON CONVEX CO-COMPACT HYPERBOLIC MANIFOLDS

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ABSTRACT. For convex co-compact hyperbolic quotients $X = \Gamma\backslash\mathbb{H}^{n+1}$, we analyze the long-time asymptotic of the solution of the wave equation $u(t)$ with smooth compactly supported initial data $f = (f_0, f_1)$. We show that, if the Hausdorff dimension δ of the limit set is less than $n/2$, then $u(t) = C_\delta(f)e^{(\delta-\frac{n}{2})t}/\Gamma(\delta - n/2 + 1) + e^{(\delta-\frac{n}{2})t}R(t)$ where $C_\delta(f) \in C^\infty(X)$ and $||R(t)|| = \mathcal{O}(t^{-\infty})$. We explain, in terms of conformal theory of the conformal infinity of X, the special cases $\delta \in n/2 - \mathbb{N}$ where the leading asymptotic term vanishes. In a second part, we show for all $\epsilon > 0$ the existence of an infinite number of resonances (and thus zeros of Selberg zeta function) in the strip $\{-n\delta - \epsilon < \mathrm{Re}(\lambda) < \delta\}$. As a byproduct we obtain a lower bound on the remainder $R(t)$ for generic initial data f.

1. INTRODUCTION

It is well-known that on a compact Riemannian manifold (X, g), any solution $u(t, z)$ of the wave equation $(\partial_t^2 + \Delta_g)u(t, z) = 0$ expands as a sum of oscillating terms of the form $e^{i\lambda_j t}a_j(z)$ where λ_j^2 are the eigenvalues of the Laplacian Δ_g and a_j some associated eigenvectors. The eigenvalues then give the frequencies of oscillation in time. For non-compact manifolds, the situation is much more complicated and no general theory describes the behaviour of waves as time goes to infinity, at least in terms of spectral data. A first satisfactory description has been given by Lax-Phillips [22] and Vainberg [39] for the Laplacian Δ_X with Dirichlet condition on $X := \mathbb{R}^n \setminus \mathcal{O}$ where $\mathcal{O}$ is a compact obstacle and n odd; indeed if $u(t)$ is the solution of $(-\partial_t^2 - \Delta_X)u(t, z) = 0$ with compactly supported smooth initial data in X and under a *non-trapping* condition, they show an expansion as $t \to +\infty$ of the form

$$u(t, z) = \sum_{\substack{\lambda_j \in \mathcal{R} \\ \mathrm{Im}(\lambda_j) < N}} \sum_{k=1}^{m(\lambda_j)} e^{i\lambda_j t}t^{k-1}u_{j,k}(z) + \mathcal{O}(e^{-(N-\epsilon)t}), \quad \forall N > 0, \forall \epsilon > 0$$

uniformly on compacts, where $\mathcal{R} \subset \{\lambda \in \mathbb{C}, \mathrm{Im}(\lambda) \geq 0\}$ is a discrete set of complex numbers called *resonances* associated with a multiplicity function $m : \mathcal{R} \to \mathbb{N}$, and $u_{j,k}$ are smooth functions. The real part of λ_j is a frequency of oscillation while the imaginary part is an exponential decay rate of the solution. Resonances can in general be defined as poles of the meromorphic continuation of the Schwartz kernel of the resolvent of Δ_X through the continuous spectrum.

In [38], Tang and Zworski extended this result for *non-trapping* black-box perturbation of $\mathbb{R}^n$ and considered also a strongly trapped setting, namely when there exist resonances λ_j such that[1] $\mathrm{Im}(\lambda_j) < (1 + |\lambda_j|)^{-N}$ for all $N > 0$, satisfying in addition some separation and multiplicity conditions. The expansion of wave solutions then involved these resonances and the error is $\mathcal{O}(t^{-N})$ for all $N > 0$. This last result has also been generalized by Burq-Zworski [5] for semi-classical problems.

It is important to notice that such results are almost certainly not optimal when the trapping is hyperbolic since, at least for all known examples, resonances do not seem to

[1] This is typically the case when P has elliptic trapped orbits as shown in [31]

approach the real line faster than polynomially. Christiansen and Zworski [6] studied two examples in hyperbolic geometry, the modular surface and the infinite volume cylinder, they showed a full expansion of waves in terms of resonances with exponentially decaying error terms. The proof is based on a separation of variables computation in the cylinder case (here the trapping geometry is that of a single closed hyperbolic orbit) while it relies on well-known number theoretic estimates for the Eisenstein series in the modular case. The case of De Sitter-Schwarzchild metrics has recently been studied by Bony-Häfner [1] using also separation of variables and rotational symmetry of the space. This is another example of hyperbolic trapping. Clearly, the general hyperbolic trapping situation is an issue and the above results are always based on very explicit computations or the arithmetic nature of the manifold. It is therefore of interest to consider more general cases of hyperbolic trapping geometries, the most basic examples being the convex co-compact quotients of the hyperbolic space $\mathbb{H}^{n+1}$ that can be considered as the simplest non-trivial models of open quantum chaotic systems.

Hyperbolic quotients $\Gamma\backslash\mathbb{H}^{n+1}$ by a discrete group of isometries with only hyperbolic elements (those that do not fix points in $\mathbb{H}^{n+1}$ but fix two points on the sphere at infinity $S^n = \partial\mathbb{H}^{n+1}$) and admitting a finite sided fundamental domain with infinite volume are called *convex co-compact*. The Laplacian on such a quotient X has for continuous and essential spectrum the half-line $[n^2/4, \infty)$, the natural wave equation is

$$(\partial_t^2 + \Delta_X - n^2/4)u(t,z) = 0, \quad u(0,z) = f_0(z), \quad \partial_t u(0,z) = f_1(z), \tag{1.1}$$

its solution is

$$u(t) = \cos\Big(t\sqrt{\Delta_X - \frac{n^2}{4}}\Big)f_0 + \frac{\sin\Big(t\sqrt{\Delta_X - \frac{n^2}{4}}\Big)}{\sqrt{\Delta_X - \frac{n^2}{4}}}f_1. \tag{1.2}$$

For a convex co-compact quotient $X = \Gamma\backslash\mathbb{H}^{n+1}$, the group Γ acts on $\mathbb{H}^{n+1}$ as isometries but also on the sphere at infinity $S^n = \partial\mathbb{H}^{n+1}$ as conformal transformations. The limit set $\Lambda(\Gamma)$ of the group is the set of accumulation points on S^n of the orbit $\Gamma.m$ for the Euclidean topology on the closed unit ball $\{z \in \mathbb{R}^{n+1}; |z| \leq 1\}$ for a chosen $m \in \mathbb{H}^{n+1}$; it is well known that $\Lambda(\Gamma)$ does not depend on the choice of m. We denote by $\delta \in (0,n)$ the Hausdorff dimension of $\Lambda(\Gamma)$,

$$\delta := \dim_H(\Lambda(\Gamma)).$$

It is proved by Patterson [28] and Sullivan [37] that δ is also the exponent of convergence of Poincaré series

$$P_\lambda(m,m') := \sum_{\gamma\in\Gamma} e^{-\lambda d_h(m,\gamma m')}, \quad m,m' \in \mathbb{H}^{n+1}, \tag{1.3}$$

where d_h is the hyperbolic distance. Standard coordinates on the unit sphere bundle $SX = \{(z,\xi) \in TX; |\xi| = 1\}$ show that $2\delta + 1$ is the Hausdorff dimension of the trapped set of the geodesic flow on SX.

We denote by $\Omega := S^n \setminus \Lambda(\Gamma)$ the domain of discontinuity of Γ, this is the largest open subset of S^n on which Γ acts properly discontinuously. The quotient $\Gamma\backslash\Omega$ is a compact manifold and X can be compactified into a smooth manifold with boundary $\bar{X} = X \cup \partial\bar{X}$ with $\partial\bar{X} = \Gamma\backslash\Omega$. It turns out that $\partial\bar{X}$ inherits from the hyperbolic metric g on X a conformal class of metrics $[h_0]$, namely the conformal class of $h_0 = x^2 g|_{T\partial\bar{X}}$ where x is any smooth boundary defining function of $\partial\bar{X}$ in $\bar{X}$.

In this paper, we focus on the case when $\delta < n/2$ since if $\delta > n/2$, the Laplacian Δ_X has pure point spectrum in $(0, n^2/4)$ that gives the leading asymptotic behaviour of $u(t)$ by usual spectral theory. We prove the following result.

Theorem 1.1. *Let X be an $(n+1)$-dimensional convex co-compact hyperbolic manifold such that $\delta < n/2$, and let $f_0, f_1, \chi \in C_0^\infty(X)$. With $u(t)$ defined by (1.2), as $t \to +\infty$, we have the asymptotic*

$$\chi u(t) = \frac{A_X}{\Gamma(\delta - \frac{n}{2} + 1)} e^{-t(\frac{n}{2}-\delta)} \langle u_\delta, (\delta - \frac{n}{2}) f_0 + f_1 \rangle \chi u_\delta + \mathcal{O}_{L^2}(e^{(\delta - \frac{n}{2})t} t^{-\infty}) \tag{1.4}$$

where u_δ is the Patterson generalized eigenfunction defined in (2.10) *and $\langle , \rangle$ is the distribution pairing, $A_X \in \mathbb{C} \setminus \{0\}$ is a constant depending on X.*

Remark 1: when $\delta \notin n/2 - \mathbb{N}$, this shows that the "dynamical dimension" δ controls the exponential decay rate of waves, or *quantum decay rate*[2]. It seems to be the first rather general example of hyperbolic trapping for which we have an explicit asymptotic for the waves, in terms of geometric data. However, we point out that the recent work of Petkov-Stoyanov [34] should in principle imply an expansion in terms of a finite number of resonances for the exterior problem with strictly convex obstacles. We also believe that a result similar to Theorem 4.3 holds for general negatively curved asymptotically hyperbolic manifolds, this will be studied in a subsequent work.

Remark 2: In the special case $\delta \in n/2 - \mathbb{N}$ (note that it can happen only for $n \geq 3$ i.e. for four and higher dimensional manifolds) the leading term vanishes in view of the Euler Γ function in (1.4). Waves for this special case turn out to decrease faster. We explain this fact in the last section of the paper, and it is somehow related to the conformal theory of $\partial\bar{X}$: what happens is that when $\delta \notin n/2 - \mathbb{N}$, $\lambda = \delta$ is always the closest pole to the continuous spectrum of the meromorphic extension of the resolvent $R(\lambda) := (\Delta_X - \lambda(n-\lambda))^{-1}$ and u_δ is an associated non-L^2 eigenstate $(\Delta_X - \delta(n-\delta))u_\delta = 0$, while when $\delta = n/2 - k$ with $k \in \mathbb{N}$, the extended resolvent $R(\lambda)$ is holomorphic at $\lambda = \delta$ and u_δ has asymptotic behaviour near $\partial\bar{X}$

$$u_\delta(z) = x(z)^\delta f_\delta + \mathcal{O}(x(z)^{\delta+1})$$

where $f_\delta \in C^\infty(\partial\bar{X})$ is an element of $\ker(P_k)$, P_k being the k-th GJMS conformal Laplacian [8] of the conformal boundary $(\partial\bar{X}, [h_0])$; more precisely $P_j > 0$ for all $j = 1, \dots, k-1$ while $\ker P_k = \mathrm{Span}(f_\delta)$. The manifold has a special conformal geometry at infinity that makes the resonance δ disappear and transform into a 0-eigenvalue for the conformal Laplacian P_k.

The proof uses methods of Tang-Zworski [38] together with information on the closest resonance to the critical line, that is δ when $\delta \notin n/2 - \mathbb{N}$ (the physical sheet for the resolvent $R(\lambda) := (\Delta - \lambda(n-\lambda))^{-1}$ is $\{\mathrm{Re}(\lambda) > n/2\}$) this last fact has been proved by Patterson [29] using Poincaré series and Patterson-Sullivan measure. The powerful dynamical theory of Dolgopyat [7] has been used by the second author [26] (for surfaces) and Stoyanov [36] (in higher dimension) to prove the existence of a strip with no zero on the left of the first zero $\lambda = \delta$ for the Selberg zeta function. Using results of Patterson-Perry [30], this implies a strip $\{\delta - \epsilon < \mathrm{Re}(\lambda) < \delta)\}$ with no resonance. Then we can view $u(t)$ as a contour integral of the resolvent $R(\lambda)$ and move the contour up to δ and apply residue theorem. This involves obtaining rather sharp estimates on the truncated (on compact sets) resolvent near the line $\{\mathrm{Re}(\lambda) = \delta\}$. This is achieved by combining the non-vanishing result with an a priori bound that results from a precise parametrix of the truncated resolvent.

A second result of this article is the proof of the existence of an explicit strip with infinitely many resonances.

Theorem 1.2. *Let $X = \Gamma\backslash\mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold and let $\delta \in (0, n)$ be the Hausdorff dimension of its limit set. Then for all $\varepsilon > 0$, there exist*

[2] This kind of result was predicted in [27].

infinitely many resonances in the strip $\{-n\delta-\varepsilon < \mathrm{Re}(s) < \delta\}$*. If moreover* Γ *is a Schottky group, then there exist infinitely many resonances in the strip* $\{-\delta^2 - \varepsilon < \mathrm{Re}(s) < \delta\}$.

Note that the existence of infinitely many resonances in some strips was proved by Guillopé-Zworski [21] in dimension 2 and Perry [33] in higher dimension, but in both cases, they did not provide any geometric information on the width of these strips. Our proof is based on a Selberg like trace formula and uses all previously known counting estimates for resonances. An interesting consequence is an explicit Omega lower bound for the remainder in (1.4) for generic compactly supported initial data.

Corollary 1.3. *Let* $K \subset X$ *be a relatively compact open set, then there exists a generic set* $\Omega \subset L^2(K)$ *such that for all* $f_1 \in \Omega, f_0 = 0$ *and all* $\epsilon > 0$*, the remainder in* (1.4) *is not a* $\mathcal{O}_{L^2}(e^{-(\frac{n}{2}+n\delta+\epsilon)t})$ *as* $t \to \infty$*. If* X *is Schottky,* $\mathcal{O}_{L^2}(e^{-(\frac{n}{2}+n\delta+\epsilon)t})$ *can be improved to* $\mathcal{O}_{L^2}(e^{-(\frac{n}{2}+\delta^2+\epsilon)t})$.

The meaning of "generic" above is in the Baire category sense, i.e. it is a G_δ-dense subset. We point out that when $n = 1$, all convex-cocompact surfaces are Schottky i.e. are obtained as $\Gamma\backslash\mathbb{H}^2$, where Γ is a Schottky group. For a definition of Schottky groups in our setting we refer for example to the introduction of [17]. In higher dimensions, not all convex co-compact manifolds are obtained via Schottky groups. For more details and references around these questions we refer to [15].

The rest of the paper is organized as follows. In §2, we review and prove some necessary bounds on the resolvent in the continuation domain. In §3 we prove the estimate on the strip with finitely many resonances. In §4, we derive the asymptotics by using contour deformation and the key bounds of §2. We also show how to relate §3 to an Omega lower bound of the remainder. The section §5 is devoted to the analysis of the special cases $\delta \in \frac{n}{2} - \mathbb{N}$ in terms of the conformal theory of the infinity.

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2. Resolvent

We start in this section by analyzing the resolvent of the Laplacian for convex cocompact quotient of $\mathbb{H}^{n+1}$ and we give some estimates of its norms.

2.1. **Geometric setting.** We let Γ be a convex co-compact group of isometries of $\mathbb{H}^{n+1}$ with Hausdorff dimension of its limit set satisfying $0 < \delta < n/2$, we set $X = \Gamma\backslash\mathbb{H}^{n+1}$ its quotient equipped with the induced hyperbolic metric and we denote the natural projection by

$$\pi_\Gamma : \mathbb{H}^{n+1} \to X = \Gamma\backslash\mathbb{H}^{n+1}, \quad \bar{\pi}_\Gamma : \Omega \to \partial\bar{X} = \Gamma\backslash\Omega. \tag{2.1}$$

By assumption on the group Γ, for any element $h \in \Gamma$ there exists $\alpha \in \mathrm{Isom}(\mathbb{H}^{n+1})$ such that for all $(x, y) \in \mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$,

$$\alpha^{-1} \circ h \circ \alpha(x, y) = e^{l(\gamma)}(O_\gamma(x), y),$$

where $O_\gamma \in SO_n(\mathbb{R}), l(\gamma) > 0$. We will denote by $\alpha_1(\gamma), \dots, \alpha_n(\gamma)$ the eigenvalues of O_γ, and we set

$$G_\gamma(k) = \det\left(I - e^{-kl(\gamma)}O_\gamma^k\right) = \prod_{i=1}^{n}\left(1 - e^{-kl(\gamma)}\alpha_i(\gamma)^k\right). \tag{2.2}$$

The Selberg zeta function of the group is defined by

$$Z(\lambda) = \exp\left(-\sum_\gamma\sum_{m=1}^{\infty}\frac{1}{m}\frac{e^{-\lambda ml(\gamma)}}{G_\gamma(m)}\right),$$

the sum converges for $\mathrm{Re}(\lambda) > \delta$ and admits a meromorphic extension to $\lambda \in \mathbb{C}$ by results of Fried [9] and Patterson-Perry [30].

2.2. **Extension of resolvent, resonances and zeros of Zeta.** The spectrum of the Laplacian Δ_X on X is a half line of absolutely continuous spectrum $[n^2/4, \infty)$, and if we take for the resolvent of the Laplacian the spectral parameter $\lambda(n-\lambda)$

$$R(\lambda) := (\Delta_X - \lambda(n-\lambda))^{-1},$$

this is a bounded operator on $L^2(X)$ if $\mathrm{Re}(\lambda) > n/2$. It is shown by Mazzeo-Melrose [25] and Guillopé-Zworski [19] that $R(\lambda)$ extends meromorphically in $\mathbb{C}$ as continuous operators $R(\lambda) : L^2_{\mathrm{comp}}(X) \to L^2_{\mathrm{loc}}(X)$, with poles of finite multiplicity, i.e. the rank of the polar part in the Laurent expansion of $R(\lambda)$ at a pole is finite. The poles are called *resonances* of Δ_X, they form the discrete set $\mathcal{R}$ included in $\mathrm{Re}(\lambda) < n/2$, where each resonance $s \in \mathcal{R}$ is repeated with the mutiplicity

$$m_s := \mathrm{rank}(\mathrm{Res}_{\lambda=s} R(\lambda)).$$

A corollary of the analysis of divisors of $Z(\lambda)$ by Patterson-Perry [30] and Bunke-Olbrich [4] is the

Proposition 2.1 (Patterson-Perry, Bunke-Olbrich). *Let $s \in \mathbb{C} \setminus (-\mathbb{N}_0 \cup (n/2 - \mathbb{N}))$, then $Z(\lambda)$ is holomorphic at s, and s is a zero of $Z(\lambda)$ if and only if s is a resonance of Δ_X. Moreover its order as zero of $Z(\lambda)$ coincide with the multiplicity m_s of s as a resonance.*

We insist on the fact that the correspondance between zeros of $Z(\lambda)$ and poles of the resolvent $R(\lambda)$ will be a crucial argument in our estimates for the solutions of wave equation. This correspondance can be understood as a Selberg trace formula and comes from the fact that the logarithmic derivative of Selberg zeta function is given by

$$\frac{Z'(\lambda)}{Z(\lambda)} = (2\lambda - n)\mathrm{FP}_{\epsilon\to 0}\left(\int_{\mathcal{F}\cap\{\rho(m)>\epsilon\}} (R(\lambda; m, m') - R_{\mathbb{H}^{n+1}}(\lambda; m, m'))|_{m=m'} \mathrm{dvol}(m)\right)$$

where $R_{\mathbb{H}^{n+1}}(\lambda; m, m')$ is the resolvent kernel of the Laplacian $\Delta_{\mathbb{H}^{\mathbb{N}+1}}$ on $\mathbb{H}^{n+1}$, $\mathcal{F} \subset \mathbb{H}^{n+1}$ is a fundamental domain of the group Γ, ρ is a boundary defining function of $X = \Gamma\backslash\mathbb{H}^{n+1}$ and FP means finite part (i.e. the ϵ^0 coefficient of the asymptotic expansion as $\epsilon \to 0$). The core of the proof of Patterson-Perry is to use the meromorphic extension of $R(\lambda)$ to $\lambda \in \mathbb{C}$ to prove meromorphic extension of $s(\lambda) := Z'(\lambda)/Z(\lambda)$ to $\lambda \in \mathbb{C}$, and then to show that the poles of $s(\lambda)$ are first order, located at the resonances (except for the negative integer points) and with integer residues given by the multiplicity of the resonance. This analysis strongly uses the scattering operator $S(\lambda)$ defined in Section 5.

2.3. **Estimates on the resolvent $R(\lambda)$ in the non-physical sheet.** The series $P_\lambda(m, m')$ defined in (1.3) converges absolutely in $\mathrm{Re}(\lambda) > \delta$, is a holomorphic function of λ there, with local uniform bounds in m, m', which clearly gives

$$\forall \epsilon > 0, \exists C_\epsilon(m, m') > 0, \forall \lambda \text{ with } \mathrm{Re}(\lambda) \in [\delta + \epsilon, n], \quad |P_\lambda(m, m')| \leq C_{\epsilon,m,m'}$$

and $C_{\epsilon,m,m'}$ is locally uniform in m, m'. We show the

Proposition 2.2. *With previous assumptions, there exists $\epsilon > 0$ and a holomorphic family in $\{\mathrm{Re}(\lambda) > \delta - \epsilon\}$ of continuous operators $K(\lambda) : L^2_{\mathrm{comp}}(X) \to L^2_{\mathrm{loc}}(X)$ such that the resolvent satisfies in $\mathrm{Re}(\lambda) > \delta$*

$$R(\lambda) = \frac{(2\pi)^{-\frac{n}{2}}\Gamma(\lambda)}{\Gamma(\lambda - \frac{n}{2})} P(\lambda) + K(\lambda)$$

where $P(\lambda)$ is the operator with Schwartz kernel $P_\lambda(m, m')$. Moreover there exists $M > 0$ such that for any $\chi_1, \chi_2 \in C_0^\infty(X)$, there is a $C > 0$ such that

$$||\chi_1 K(\lambda)\chi_2||_{\mathcal{L}(L^2(X))} \leq C(|\lambda| + 1)^M, \quad \mathrm{Re}(\lambda) > \delta - \epsilon$$

Proof: we choose a fundamental domain $\mathcal{F}$ for Γ with a finite number of sides paired by elements of Γ. By standard arguments of automorphic functions, the resolvent kernel $R(\lambda; m, m')$ for $m, m' \in \mathcal{F}$ and $\mathrm{Re}(\lambda) > \delta$ is the average

$$R(\lambda; m, m') = \sum_{\gamma \in \Gamma} G(\lambda; m, \gamma m') = \sum_{\gamma \in \Gamma} \sigma(d_h(m, \gamma m'))^{\lambda} k_{\lambda}(\sigma(d_h(m, \gamma m')))$$

$$\sigma(d) := (\cosh d)^{-1} = 2e^{-d}(1 + e^{-2d})^{-1}$$

where $G(\lambda; m, m')$ is the Green kernel of the Laplacian on $\mathbb{H}^{n+1}$ and $k_\lambda \in C^\infty([0,1))$ is the hypergeometric function defined for $\mathrm{Re}(\lambda) > \frac{n-1}{2}$

$$k_\lambda(\sigma) := \frac{2^{\frac{3-n}{2}} \pi^{\frac{n+1}{2}} \Gamma(\lambda)}{\Gamma(\lambda - \frac{n+1}{2} + 1)} \int_0^1 (2t(1-t))^{\lambda - \frac{n+1}{2}} (1 + \sigma(1-2t))^{-\lambda} dt$$

which extends meromorphically to $\mathbb{C}$ and whose Taylor expansion at order $2N$ can be written

$$k_\lambda(\sigma) = 2^{-\lambda-1} \sum_{j=0}^{N} \alpha_j(\lambda) \Big(\frac{\sigma}{2}\Big)^{2j} + k_\lambda^N(\sigma), \quad \alpha_j(\lambda) := \frac{\pi^{-\frac{n}{2}} \Gamma(\lambda + 2j)}{\Gamma(\lambda - \frac{n}{2} + 1)\Gamma(j+1)}$$

with $k_\lambda^N \in C^\infty([0,1))$ and the estimate for any $\epsilon_0 > 0$

$$|k_\lambda^N(\sigma)| \leq \sigma^{2N+2} C^N (|\lambda| + 1)^{CN}, \quad \sigma \in [0, 1-\epsilon_0), \quad \mathrm{Re}(\lambda) > \frac{n}{2} - N \tag{2.3}$$

for some $C > 0$ depending only on ϵ_0, see for instance [13, Lem. B.1]. Extracting the first term with α_0 in k_λ, we can then decompose

$$R(\lambda; m, m') = \frac{\pi^{\frac{n}{2}} \Gamma(\lambda)}{2\Gamma(\lambda - \frac{n}{2} + 1)} \Big(\sum_{\gamma \in \Gamma} e^{-\lambda d_h} + \sum_{\gamma \in \Gamma} e^{-(\lambda+1) d_h} f_\lambda(e^{-d_h}) \Big) + \sum_{\gamma \in \Gamma} \sigma(d_h)^\lambda k_\lambda^0(\sigma(d_h))$$

$$f_\lambda(x) := \frac{(1 + x^2)^{-\lambda} - 1}{x},$$

and where d_h means $d_h(m, \gamma m')$ here. Thus to prove the Proposition, we have to analyze the term $K(\lambda) := 2^{-1}\alpha_0(\lambda) K_1(\lambda) + K_2(\lambda)$ with

$$K_1(\lambda) := \sum_{\gamma \in \Gamma} e^{-(\lambda+1) d_h} f_\lambda(e^{-d_h}), \quad K_2(\lambda) := \sum_{\gamma \in \Gamma} \sigma(d_h)^\lambda k_\lambda^0(\sigma(d_h))$$

The first term K_1 is easy to deal with since $|f_\lambda(x)| \leq C(|\lambda| + 1)$ for $x \in [0,1]$, thus we can use the fact that $P_{\lambda+1}(m, m')$ converges absolutely in $\mathrm{Re}(\lambda) > \delta - 1$, is holomorphic there, and is locally uniformly bounded in (m, m') thus

$$|\alpha_0(\lambda) \chi_1(m) \chi_2(m') K_1(\lambda)| \leq C(|\lambda| + 1)^{\frac{n}{2}+1}$$

the same bound holds for the operator in $\mathcal{L}(L^2(X))$ with Schwartz kernel $\chi_1(m)\chi_2(m) F_1(\lambda)$. Note that $\alpha_0(\lambda)$ has no pole in $\mathrm{Re}(\lambda) > 0$, thus no pole in $\mathrm{Re}(\lambda) > \delta/2 > 0$.

For $K_2(\lambda)$ we can decompose it as follows: for $m \in \mathrm{Supp}(\chi_1)$, $m' \in \mathrm{Supp}(\chi_2)$ (which are compact in $\mathcal{F}$), for $\epsilon_0 > 0$ fixed there is only a finite number of elements $\Gamma_0 = \{\gamma_0, \dots, \gamma_L \in \Gamma\}$ such that $d_h(m, \gamma m') > \epsilon_0$ for any $\gamma \notin \Gamma_0$ and any m, m' in any fixed compact set $\mathcal{K}$ of $\mathcal{F}$, this is because the group acts properly discontinuously on $\mathbb{H}^{n+1}$. Thus we split the sum in $K_2(\lambda)$ into

$$K_2(\lambda) = \sum_{\gamma \in \Gamma_0} \sigma(d_h)^\lambda k_\lambda^0(\sigma(d_h)) + \sum_{\gamma \notin \Gamma_0} \sigma(d_h)^\lambda k_\lambda^0(\sigma(d_h)). \tag{2.4}$$

We first observe that the second term is a convergent series, holomorphic in λ, for $\mathrm{Re}(\lambda) > \delta - 1$ and uniformly bounded in $m, m' \in \mathcal{K}$. Indeed it is easily seen to be bounded by

$$CN(|\lambda|+1)\sum_{j=1}^{N}|\alpha_j(\lambda)|P_{\mathrm{Re}(\lambda)+2j}(m,m') + C^N(|\lambda|+1)^{CN}P_{\mathrm{Re}(\lambda)+2N+1}(m,m') \tag{2.5}$$

by assumption on Γ_0 and using (2.3), C depending on ϵ_0 only. Moreover since $\alpha_j(\lambda)$ is polynomially bounded by $C(|\lambda|+1)^{2j}$ we have a polynomial bound for (2.5) of degree depending on N. The first term in (2.4) has a finite sum thus it suffices to estimate each term, but because of the usual conormal singularity of the resolvent at the diagonal, it explodes as $d_h(m,m') \to 0$. We want to use Schur's lemma for instance, so we have to bound

$$\sup_{m\in\mathcal{F}}\int_{\mathcal{F}}|\chi_1(m)\chi_2(m')K_2(\lambda;m,m')|dm'_{\mathbb{H}^{n+1}}, \quad \sup_{m'\in\mathcal{F}}\int_{\mathcal{F}}|\chi_1(m)\chi_2(m')K_2(\lambda;m,m')|dm_{\mathbb{H}^{n+1}}.$$

First we recall that $\mathbb{H}^{n+1} = (0,\infty)_x \times \mathbb{R}^n_y$ has a Lie group structure with product

$$(x,y).(x',y') = (xx', y+xy'), \quad (x,y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$$

and neutral element $e := (1,0)$. Then if $(u,v) := (x',y')^{-1}.(x,y) = (x/x', (y-y')/x')$ we get

$$(\cosh(d_h(x,y;x',y')))^{-1} = \frac{2xx'}{x^2+x'^2+|y-y'|^2} = \frac{2u}{1+u+|v|^2} = (\cosh(d_h(u,v;1,0)))^{-1}. \tag{2.6}$$

Moreover the diffeomorphism $(u,v) \to m' = m.(u,v)^{-1}$ on $\mathbb{H}^{n+1}$ pulls the hyperbolic measure $dm'_{\mathbb{H}^{n+1}} = x'^{-n-1}dx'dy'$ back into the right invariant measure $u^{-1}dudv$ for the group action. This is to say that we have to bound

$$\sup_{m\in\mathcal{F}}\int_{\mathcal{F}^{-1}.m}|\chi_1(m)\chi_2(m.(u,v)^{-1})K_2(\lambda;m,m.(u,v)^{-1})|\frac{dudv}{u} \tag{2.7}$$

where $\mathcal{F}^{-1}.m := \{m'^{-1}.m; m' \in \mathcal{F}\}$ and similarly

$$\sup_{m'\in\mathcal{F}}\int_{\mathcal{F}^{-1}.m'}|\chi_1(m'.(u,v)^{-1})\chi_2(m')K_2(\lambda;m'.(u,v)^{-1},m')|\frac{dudv}{u}. \tag{2.8}$$

Because m, m' are in compact sets, the estimate (2.5) with $N = n$ gives a polynomial bounds in λ in $\{\mathrm{Re}(\lambda) > \delta - \epsilon\}$ for the terms coming from $\gamma \notin \Gamma_0$. To deal with the term of (2.4) containing elements $\gamma \in \Gamma_0$, we use Lemma B.1 of [13] which proves that for any compact K of $\mathbb{H}^{n+1}$, there exists a constant C_K such that

$$\int_K |G(\lambda;(u,v),e)|\frac{dudv}{u} \leq \frac{C_K^N(|\lambda|+1)^{n-1}}{\mathrm{dist}(\lambda,-\mathbb{N}_0)}, \quad \mathrm{Re}(\lambda) > \frac{n}{2} - N. \tag{2.9}$$

Now to bound (2.7) with $K_2(\lambda,\bullet,\bullet)$ replaced by $\sigma(d_h(\bullet,\gamma\bullet))^\lambda k'_\lambda(\sigma(d_h(\bullet,\gamma\bullet)))$ we note that before we did our change of variable in (2.7), we can make the change of variable $m' \to \gamma^{-1}m'$ which amounts to bound

$$\sup_{m\in\mathcal{F}}\int_{(\gamma\mathcal{F})^{-1}.m}\Big|\chi_1(m)\chi_2(\gamma^{-1}m.(u,v)^{-1})\Big(G(\lambda;(u,v),e) - 2^{-\lambda-1}\alpha_0(\lambda)\sigma^\lambda(d_h((u,v),e))\Big)\Big|\frac{dudv}{u}$$

where we used (2.6). But again, since χ_1, χ_2 have compact support, we get a polynomial bound in λ using (2.9) and a trivial polynomial bound for $k_\lambda(0)$. The term (2.8) can be dealt with similarly and we finally deduce that for some M,

$$||\chi_1K_2(\lambda)\chi_2||_{\mathcal{L}(L^2(X))} \leq C(|\lambda|+1)^M, \quad \mathrm{Re}(\lambda) > \delta - \epsilon$$

and the Proposition is proved. $\square$

This clearly shows that the resolvent extends to $\{\mathrm{Re}(\lambda) > \delta\}$ analytically. Actually, Patterson [29] (see also [31, Prop 1.1]) showed the following.

Proposition 2.3 (**Patterson**). *The family of operators $\Gamma(\lambda - n/2+1)R(\lambda)$ is holomorphic in $\{\mathrm{Re}(\lambda) > \delta\}$, has no pole on $\{\mathrm{Re}(\lambda) = \delta, \lambda \neq \delta\}$ and has a pole of order 1 at $\lambda = \delta$ with rank 1 residue given by*

$$\mathrm{Res}_{\lambda=\delta}\Gamma(\lambda - n/2 + 1)R(\lambda) = A_X u_\delta \otimes u_\delta$$

where $A_X \neq 0$ is some constant depending on Γ and u_δ is the Patterson generalized eigenfunction defined by

$$\pi_\Gamma^* u_\delta(m) = \int_{\partial_\infty \mathbb{H}^{n+1}} \Big(\mathcal{P}(m, y)\Big)^\delta d\mu_\Gamma(y) \tag{2.10}$$

$\mathcal{P}$ being the Poison kernel of $\mathbb{H}^{n+1}$ and $d\mu_\Gamma$ the Patterson-Sullivan measure associated to Γ on the sphere $\partial_\infty \mathbb{H}^{n+1} = \mathbb{R}^n \cup \{\infty\} = S^n$.

We can but notice that $\delta \in n/2 - \mathbb{N}$ is a special case since the resolvent becomes holomorphic at $\lambda = \delta$. We postpone the analysis of this phenomenon to section §5.

A rough exponential estimate in the non-physical sheet also holds using determinant method (used for instance in [20]).

Lemma 2.4. *For $\chi_1, \chi_2 \in C_0^\infty(X)$, $j \in \mathbb{N}_0$, and $\eta > 0$ there is $C > 0$ such that for $|\lambda| \leq N/16$ and dist$(\lambda, \mathcal{R}\} > \eta$,*

$$||\partial_\lambda^j \chi_1 R(\lambda)\chi_2||_{\mathcal{L}(L^2(X))} \leq e^{C(N+1)^{n+3}},$$

Proof: we apply the idea of [20, Lem. 3.6] with the parametrix construction of $R(\lambda)$ written in [19]. Let x be a boundary defining function of $\partial\bar{X}$ in $\bar{X}$, which can be considered as a weight to define Hilbert spaces $x^\alpha L^2(X)$, for any $\alpha \in \mathbb{R}$. For any large $N > 0$ that we suppose in $2\mathbb{N}$ for convenience, Guillopé and Zworski [19] construct operators

$$P_N(\lambda, \lambda_0) : x^N L^2(X) \to x^{-N} L^2(X), \quad K_N(\lambda, \lambda_0) : x^N L^2(X) \to X^N L^2(X)$$

meromorphic with finite multiplicity in $O_N := \{\mathrm{Re}(\lambda) > (n - N)/2\}$, whose poles are situated at $-\mathbb{N}_0$, and such that

$$(\Delta_X - \lambda(n - \lambda))P_N(\lambda, \lambda_0) = 1 + K_N(\lambda, \lambda_0)$$

with λ_0 large depending on N, take for instance $\lambda_0 = n/2 + N/8$. Moreover $K_N(\lambda, \lambda_0)$ is compact with characteristic values satisfying in $O_{N,\eta} := O_N \cap \{\mathrm{dist}(\lambda, -\mathbb{N}_0) > \eta\}$

$$\mu_j(K_N(\lambda, \lambda_0)) \leq C(1 + |\lambda - \lambda_0|)j^{-\frac{1}{n}} + \begin{cases} e^{CN} & \text{if } j \leq CN^{n+1} \\ e^{-N/C}j^2 & \text{if } j \geq CN^{n+1} \end{cases} \tag{2.11}$$

for some $0 < \eta < 1/4$ and $C > 0$ independent of λ, N. They also have $||K_N(\lambda_0, \lambda_0)|| \leq 1/2$ in $\mathcal{L}(x^N L^2(X))$, thus by Fredholm theorem

$$R(\lambda) = P_N(\lambda, \lambda_0)(1 + K_N(\lambda, \lambda_0))^{-1} : x^N L^2(X) \to x^{-N} L^2(X)$$

is meromorphic with poles of finite multiplicity in O_N. By standard method as in [20, Lem. 3.6] we define

$$d_N(\lambda) := \det(1 + K_N(\lambda, \lambda_0)^{n+2})$$

which exists in view of (2.11), and we have the rough bound

$$||(1 + K_N(\lambda, \lambda_0))^{-1}||_{\mathcal{L}(x^N L^2(X))} \leq \frac{\det(1 + |K_N(\lambda, \lambda_0)|^{n+2})}{|d_N(\lambda)|} \tag{2.12}$$

in $O_{N,\eta}$ and where $|A| := (A^*A)^{\frac{1}{2}}$ for A compact. The term in the numerator is easily shown to be bounded by $\exp(C(N + 1)^{n+2})$ in $O_{N,\eta}$ from (2.11), actually this is written in [19, Lem. 5.2]. It remains to have a lower bound of $|d_N(\lambda)|$. In Lemma 3.6 of [20], they use the minimum modulus theorem to obtain lower bound of a function using its upper bound, but this means that the function has to be analytic in $\mathbb{C}$. Here there is a substitute which is Cartan's estimate [23, Th. I.11]. We first need to multiply $d_N(\lambda)$ by a holomorphic function $J_N(\lambda)$ with zeros of sufficient multiplicity at $\{-k; k = 0, \dots, N/2\}$ to make $J_N(\lambda)d_N(\lambda)$ holomorphic in O_N, for instance the polynomial

$$J_N(\lambda) := \prod_{k=0}^{N/2} (\lambda - k)^{CN^{n+2}}$$

for some large integer $C > 0$ suffices in view of the order ($\leq CN^{n+2}$) of each $-k$ as a pole of $d_N(\lambda)$ proved in [19, Lem. A.1]. Then clearly $f_N(\lambda) := J_N(\lambda + \lambda_0)d_N(\lambda + \lambda_0)/(J_N(\lambda_0)d_N(\lambda_0))$ is holomorphic in $\{|\lambda| \leq N/4\}$ and satisfies in this disk

$$|f_N(\lambda)| \leq e^{C(N+1)^{n+3}}, \quad f_N(0) = 1,$$

where we used the maximum principle in disks around each $-k$ to estimate the norm there. Thus we may apply Cartan's estimate for this function in $|\lambda| < N/4$: for all $\alpha > 0$ small enough there exists $C_\alpha > 0$ such that, outside a family of disks the sum of whose radii is bounded by αN

$$\log |f_N(\lambda)| > -C_\alpha \log \Big(\sup_{|\lambda| \leq N/4} |f_N(\lambda)| \Big)$$

and $|\lambda| \leq N/4$. Fixing α sufficiently small, there exists $\beta_N \in (3/4, 1)$ so that

$$|d_N(\lambda)| > e^{-C(N+1)^{n+3}} \text{ for } |\lambda - \lambda_0| = \beta_N \frac{N}{4}.$$

Note that we can also choose β_N so that $\mathrm{dist}(\beta_N N/4, \mathbb{N}) > \eta$ for some small η uniform with respect to N. Thus the same bound holds for $||(1 + K_N(\lambda, \lambda_0))^{-1}||_{\mathcal{L}(x^N L^2(X))}$ using (2.12). Now we need a bound for $P_N(\lambda, \lambda_0)$ and it suffices to get back to its definition in the proof of Proposition 3.2 of [19]: it involves operators of the form $\iota^* \varphi R_{\mathbb{H}^{n+1}}(\lambda)\psi \iota_*$ for some cut-off functions $\psi, \varphi \in C^\infty(\mathbb{H}^{n+1})$ and isometry

$$\iota : U \subset X \to \{(x, y) \in (0, \infty) \times \mathbb{R}^n; x^2 + |y|^2 < 1\} \subset \mathbb{H}^{n+1},$$

and operators whose norm is explicitely bounded in [19, Sect. 4] by $e^{C(N+1)}$ in $O_{N,\eta}$. The appendix B of [13] gives an estimate of the same form for $||\varphi R_{\mathbb{H}^{n+1}}(\lambda)\psi||$ as an operator in $\mathcal{L}(x^N L^2(X), x^{-N} L^2(X))$ for $\lambda \in O_{N,\eta}$ (this is actually a direct consequence of (2.9) and (2.3)) thus we have the bound

$$||R(\lambda)||_{\mathcal{L}(x^N L^2(X), x^{-N} L^2(X))} \leq e^{C(N+1)^3}$$

in $\{|\lambda - \lambda_0| = \beta_N N/4\}$. Let $\mathcal{R}_N$ be the set of poles of $R(\lambda)$ in O_N, each pole being repeated according to its order; $\mathcal{R}_N$ has at most CN^{n+2} elements so we may multiply $R(\lambda)$ by

$$F_N(\lambda) := \prod_{s \in \mathcal{R}_N} E(\lambda/s, n+2)$$

where $E(z, p) := (1 - z) \exp(z + \dots + p^{-1} z^p)$ is the Weierstrass elementary function. It is rather easy to check that for all $\epsilon > 0$ small, we have the bounds

$$e^{C_\epsilon (N+1)^{n+3}} \geq |F_N(\lambda)| \geq e^{-C_\epsilon (N+1)^{(n+3)}} \tag{2.13}$$

for some C_ϵ and for all $\lambda \in O_N$ such that $\mathrm{dist}(\lambda, \mathcal{R}) > \epsilon$. Thus $R(\lambda)F_N(\lambda)$ is holomorphic in $\{|\lambda - \lambda_0| \leq \beta_N N/4\}$ and we can use the maximum principle which gives a upper bound $||F_N(\lambda)R(\lambda)||_{\mathcal{L}(x^N L^2, x^{-N} L^2)} \leq \exp(C_\epsilon (N+1)^{n+3})$ in $\{|\lambda - \lambda_0| \leq \beta_N N/4\}$. We get our conclusion using (2.13), the fact that χ_i is bounded by e^{CN} as an operator from L^2 to $x^N L^2$, and the Cauchy formula for the case $j > 0$ (estimates of the derivatives with respect to λ). □

Remark: Notice that similar estimates are obtained independently by Borthwick [3].

In the case of surfaces the second author [26] used the powerful estimates developped by Dolgopyat [7] to prove that the Selberg zeta function $Z(\lambda)$ is analytic and non-vanishing in $\{\mathrm{Re}(\lambda) > \delta - \epsilon, \lambda \neq \delta\}$ for some $\epsilon > 0$. In higher dimension, the same result holds, as was shown recently by Stoyanov [36].

Theorem 2.5 (Naud, Stoyanov). *There exists $\epsilon > 0$ such that the Selberg zeta function $Z(\lambda)$ is holomorphic and non-vanishing in $\{\lambda \in \mathbb{C}; \mathrm{Re}(\lambda) > \delta - \epsilon, \lambda \neq \delta\}$.*

Using Proposition 2.1, this result about zeta function implies that the resolvent $R(\lambda)$ is holomorphic in a similar set (possibly by taking $\epsilon > 0$ smaller). Then an easy consequence of the maximum principle as in [38, 2] together with a rough exponential bound for the resolvent allows to get a polynomial bound for $||\chi_1 R(\lambda)\chi_2||$ on the $\{\mathrm{Re}(\lambda) = \delta; \lambda \neq \delta\}$.

Corollary 2.6. *There is $\epsilon > 0$ such that the resolvent $R(\lambda)$ is meromorphic in $\mathrm{Re}(\lambda) > \delta - \epsilon$ with only possible pole the simple pole $\lambda = \delta$, the residue of which is given by*

$$Res_{\lambda=\delta} R(\lambda) = \frac{A_X}{\Gamma(\frac{n}{2} - \delta + 1)} u_\delta \otimes u_\delta$$

where u_δ is the Patterson generalized eigenfunction of (2.10), $A_X \neq 0$ a constant. Moreover for all $\chi_1, \chi_2 \in C_0^\infty(X)$, there exists $L \in \mathbb{N}, C > 0$ such that for $|\lambda - \delta| > 1$ and all $j \in \mathbb{N}_0$

$$||\partial_\lambda^j \chi_1 R(\lambda)\chi_2||_{\mathcal{L}(L^2(X))} \leq C(|\lambda| + 1)^{L+(n+3)j} \text{ in } \{\mathrm{Re}(\lambda) \geq \delta\}$$

Proof: This is a consequence of Proposition 2.2, Proposition 2.3, Theorem 2.5 and the maximum principle as in [2, Prop. 1]. First we remark from Proposition 2.2 and Proposition 2.3 that P_λ has a first order pole with rank one residue at $\lambda = \delta$ and, since $|P_\lambda(m, m')| \leq |P_{\mathrm{Re}(\lambda)}(m, m')|$, we have the estimate

$$||\chi_1 R(\lambda)\chi_2||_{\mathcal{L}(L^2(X))} \leq |\mathrm{Re}(\lambda) - \delta|^{-1} C(|\lambda| + 1)^M$$

for $\mathrm{Re}(\lambda) \in (\delta, n/2)$. This implies by the Cauchy formula that

$$||\partial_\lambda^j \chi_1 R(\lambda)\chi_2||_{\mathcal{L}(L^2(X))} \leq |\mathrm{Re}(\lambda) - \delta|^{-1-j} C(|\lambda| + 1)^M.$$

Let $A > 0$, and $\varphi, \psi \in L^2(X)$, we can apply the maximum principle to the function

$$f(\lambda) = e^{iA(-i(\lambda-\delta))^{n+4}} \langle \partial_\lambda^j \chi_1 R(\lambda)\chi_2 \varphi, \psi\rangle$$

which is holomorphic in the domain Λ bounded by the curves

$$\Lambda_+ := \{\delta + u^{-n-3} + iu; u > 1\}, \quad \Lambda_- := \{\delta - \epsilon + iu; u > 1\}, \quad \Lambda_0 := \{i + u; \delta - \epsilon < u < \delta + 1\}.$$

Then it is easy to check as in [2, Prop. 1] that by choosing $A > 0$ large enough

$$|f(\lambda)| < C(|\lambda| + 1)^{L+(n+3)j} ||\varphi||_{L^2} ||\psi||_{L^2}$$

in Λ for some L depending only on M. In particular, applying the same method in the symmetric domain $\bar{\Lambda} := \{\bar{\lambda}; \lambda \in \Lambda\}$, we obtain the polynomial bound $||\partial_\lambda^j \chi_1 R(\lambda)\chi_2|| \leq C(|\lambda| + 1)^{L+(n+3)j}$ on $\{\mathrm{Re}(\lambda) = \delta, |\mathrm{Im}(\lambda)| > 1\}$. $\square$

3. Width of the strip with finitely many resonances

As stated in Theorem 2.5, we know that there exists a strip $\{\delta - \epsilon < \mathrm{Re}(\lambda) < \delta\}$ with no resonance for Δ_g, or equivalently no zero for Selberg zeta function. However the proof of this result does not provide any effective estimate on the width of this strip (i.e. on ϵ above). More generally it is of interest to know the following

$$\rho_\Gamma := \inf \Big\{ s \in \mathbb{R}; Z(\lambda) \text{ has at most finitely many zeros in } \{\mathrm{Re}(\lambda) > s\} \Big\}$$

or equivalently

$$\rho_\Gamma = \inf\Big\{s \in \mathbb{R}; R(\lambda) \text{ has at most finitely many poles in } \{\mathrm{Re}(\lambda) > s\}\Big\}.$$

In this work, we give a lower bound for ρ_Γ:

Theorem 3.1. *Let $X = \Gamma\backslash\mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold and let $\delta \in (0, n)$ be the Hausdorff dimension of its limit set. Then for all $\varepsilon > 0$, there exist infinitely many resonances in the strip $\{-n\delta-\varepsilon < \mathrm{Re}(s) < \delta\}$. If moreover Γ is a Schottky group, then there exist infinitely many resonances in the strip $\{-\delta^2 - \varepsilon < \mathrm{Re}(s) < \delta\}$.*

Remark: In particular, we have $\rho_\Gamma \geq -\delta n$ in general and $\rho_\Gamma \geq -\delta^2$ for Schottky manifolds. The limit case $\delta \to 0$ may be viewed as a cyclic elementary group Γ_0, and resonances of the Laplace operator on $\Gamma_0\backslash\mathbb{H}^2$ are given explicitely [18, Appendix], they form a lattice $\{-k + i\alpha\ell; k \in \mathbb{N}_0, \ell \in \mathbb{Z}\}$ for some $\alpha \in \mathbb{R}$, in particular there are infinitely many resonances on the vertical line $\{\mathrm{Re}(s) = 0\}$. This heuristic consideration suggests that for small values of δ, our result is rather sharp.

Proof: The proof is based on the trace formula of [15] and estimates on the distribution of resonances due to Patterson-Perry [30], Guillopé-Lin-Zworski [17] (see also Zworski [40] for dimension 2). To make some computations clearer (Fourier transforms), we will use the spectral parameter z with $\lambda = \frac{n}{2} + iz$ and $\mathrm{Im} z > 0$ in the non-physical half-plane. We set $\beta := \delta$ if X is Schottky, while $\beta := n$ if X is not Schottky. We proceed by contradiction and assume that there is $\rho = n/2 + \beta\delta + \varepsilon$ for some $\varepsilon > 0$ such that there are at most finitely many resonances in $\mathrm{Im}(z) < \rho$. Let us first recall the trace formula of [15]: as distributions of $t \in \mathbb{R} \setminus \{0\}$, we have the identity

$$\frac{1}{2}\Big(\sum_{\frac{n}{2}+iz\in\mathcal{R}} e^{iz|t|} + \sum_{k\in\mathbb{N}} d_k e^{-k|t|}\Big) = \sum_{\gamma\in\mathcal{P}}\sum_{m=1}^{\infty} \frac{\ell(\gamma)e^{-\frac{n}{2}m\ell(\gamma)}}{2G_\gamma(m)}\delta(|t| - m\ell(\gamma)) + \frac{\chi(\bar{X})\cosh\frac{t}{2}}{(2\sinh\frac{|t|}{2})^{n+1}}, \tag{3.1}$$

where $\mathcal{P}$ denotes the set of primitive closed geodesics on $X = \Gamma\backslash\mathbb{H}^{n+1}$, $\ell(\gamma)$ stands for the length of $\gamma \in \mathcal{P}$, $G_\gamma(m)$ is defined in (2.2), $d_k := \dim\ker P_k$ if P_k is the k-th GJMS conformal Laplacian on the conformal boundary $\partial\bar{X}$, $\mathcal{R}$ is the set of resonances of Δ_X counted with multiplicity and $\chi(\bar{X})$ denotes the Euler characteristic of $\bar{X}$. Next we choose $\varphi_0 \in C_0^\infty(\mathbb{R})$ a positive weight supported on $[-1, +1]$ with $\varphi_0(0) = 1$ and $0 \leq \varphi_0 \leq 1$. We set

$$\varphi_{\alpha,d}(t) = \varphi_0\left(\frac{t-d}{\alpha}\right),$$

where d will be a large positive number and $\alpha > 0$ will be small when compared to d (typically $\alpha = e^{-\mu d}$). Pluging it into the trace formula (3.1) and assuming that d coincides with a large length of a closed geodesic, we get that for d large enough,

$$\sum_{\gamma,m} \frac{\ell(\gamma)e^{-\frac{n}{2}m\ell(\gamma)}}{2G_\gamma(m)}\varphi_{\alpha,d}(ml(\gamma)) \geq Ce^{-\frac{n}{2}d},$$

with a constant $C > 0$, whereas the other term can be estimated by

$$\alpha\chi(\bar{X})\int_{-1}^{1}\varphi(t)\frac{\cosh((d+t\alpha)/2)}{(2\sinh(|d+t\alpha|/2))^{n+1}}dt = \mathcal{O}(\alpha)e^{-\frac{n}{2}d}.$$

The key part of the proof is to estimate carefully the spectral side of the formula, i.e. we must examinate

$$\sum_{\frac{n}{2}+iz\in\mathcal{R}} \widehat{\varphi}_{\alpha,d}(-z) + \sum_{\substack{\frac{n}{2}+iz=-k\\ k\in\mathbb{N}_0}} d_k\widehat{\varphi}_{\alpha,d}(-z),$$

where $\widehat{\varphi}$ is the usual Fourier transform. Standard formulas for Fourier transform on the Schwartz space show that for all integer $M > 0$, there exists a constant $C_M > 0$ such that

$$|\widehat{\varphi}_{\alpha,d}(-z)| \leq \alpha C_M \frac{e^{-d\mathrm{Im}(z)+\alpha|\mathrm{Im}(z)|}}{(1+\alpha|z|)^M}. \tag{3.2}$$

To simplify, we denote by $\widetilde{\mathcal{R}}$ the set $\{z \in \mathbb{C}; \frac{n}{2} + iz \in \mathcal{R} \cup i\mathbb{N}\}$ where each element z is repeated with the multiplicity

$$\begin{cases} m_{n/2+iz} \text{ if } z \notin i\mathbb{N} \\ m_{n/2-k} + d_k \text{ if } z = ik \text{ with } k \in \mathbb{N} \end{cases}.$$

Our assumption now is that

$$\{0 \leq \mathrm{Im}(z) \leq \rho\} \cap \widetilde{\mathcal{R}}$$

is finite for $\rho = \frac{n}{2} + \beta\delta + \varepsilon$. We set $\overline{\rho} > \rho \geq 0$. The idea is to split the sum over resonances as

$$\sum_{z \in \widetilde{\mathcal{R}_X}} \widehat{\varphi}_{\alpha,d}(-z) = \sum_{\frac{n}{2}-\delta \leq \mathrm{Im}(z) \leq \rho} \widehat{\varphi}_{\alpha,d}(-z) + \sum_{\rho \leq \mathrm{Im}(z) \leq \overline{\rho}} \widehat{\varphi}_{\alpha,d}(-z) + \sum_{\overline{\rho} \leq \mathrm{Im}(z)} \widehat{\varphi}_{\alpha,d}(-z),$$

and estimate their contributions using dimensional and fractal upper bounds. Using (3.2) we can bound the last term (for d large) by

$$\left| \sum_{\overline{\rho} \leq \mathrm{Im}(z)} \widehat{\varphi}_{\alpha,d}(-z) \right| \leq C_M \alpha e^{-\overline{\rho}(d-\alpha)} \int_{\overline{\rho}}^{+\infty} \frac{d\mathcal{N}(r)}{(1+\alpha r)^M},$$

where $\mathcal{N}(r) = \#\{z \in \widetilde{\mathcal{R}}; |z| \leq r\}$. By [30, Th. 1.10] (see also [15, Lemma 2.3] for a discussion about the d_k terms), we know that $\mathcal{N}(r) = \mathcal{O}(r^{n+1})$, thus we can choose $M = n+2$ and obtain, after a Stieltjes integration by parts, the following upper bound

$$\left| \sum_{\overline{\rho} \leq \mathrm{Im}(z)} \widehat{\varphi}_{\alpha,d}(-z) \right| = \mathcal{O}(\alpha^{-n} e^{-\overline{\rho} d}).$$

Similarly, we have the estimate (for d large and α small)

$$\left| \sum_{\rho \leq \mathrm{Im}(z) \leq \overline{\rho}} \widehat{\varphi}_{\alpha,d}(-z) \right| \leq C_M \alpha e^{-\rho(d-\alpha)} \int_{\rho}^{+\infty} \frac{d\widetilde{\mathcal{N}}(r)}{(1+\alpha r)^M},$$

where $\widetilde{\mathcal{N}}(r) = \#\{z \in \widetilde{\mathcal{R}} \ : \ \rho \leq \mathrm{Im}(z) \leq \overline{\rho}, \ |z| \leq r\}$. This counting function is known to enjoy the "fractal" upper bound $\widetilde{\mathcal{N}}(r) = \mathcal{O}(r^{1+\delta})$ when X is Schottky [17] (see also [40] when $n = 1$), thus we can write $\widetilde{\mathcal{N}}(r) = \mathcal{O}(r^{1+\beta})$ where β is defined above. In other words, one obtains by choosing $M = n+2$,

$$\left| \sum_{\rho \leq \mathrm{Im}(z) \leq \overline{\rho}} \widehat{\varphi}_{\alpha,d}(-z) \right| = \mathcal{O}(\alpha^{-\beta} e^{-\rho d}).$$

Since we have assumed that $\{0 \leq \mathrm{Im}(z) \leq \rho\} \cap \widetilde{\mathcal{R}}$ is finite, and using the fact that resonances (in the z plane) have all imaginary part greater than $\frac{n}{2} - \delta$, we also get

$$\left| \sum_{\frac{n}{2}-\delta \leq \mathrm{Im}(z) \leq \rho} \widehat{\varphi}_{\alpha,d}(-z) \right| = \mathcal{O}(\alpha e^{(\delta - \frac{n}{2})d}).$$

Gathering all estimates, we have obtained as $d \to +\infty$,

$$e^{-\frac{n}{2}d}(C + \mathcal{O}(\alpha)) = \mathcal{O}(\alpha e^{(\delta-\frac{n}{2})d}) + \mathcal{O}(\alpha^{-\beta} e^{-\rho d}) + \mathcal{O}(\alpha^{-n} e^{-\overline{\rho} d}),$$

where all the implied constants do not depend on d and α. If we now set $\alpha = e^{-\mu d}$, we get a *contradiction* as $d \to +\infty$, provided that

$$\begin{cases} n\mu - \overline{\rho} & < & -\frac{n}{2} \\ \delta & < & \mu \\ \rho - \beta\mu & > & \frac{n}{2}. \end{cases}$$

The last inequality is satisfied if we set $\mu := \delta + \varepsilon/(2\beta)$ since $\rho = n/2 + \beta\delta + \varepsilon$ by assumption, then we can then choose $\overline{\rho} := n\mu + n/2 + \varepsilon$ which is larger than ρ and we have our contradiction for all $\varepsilon > 0$. □

The proof reveals that any precise knowledge in the asymptotic distribution of resonances in strips has a direct impact on resonances with small imaginary part.

4. Wave asymptotic

4.1. **The leading term.** Let $f, \chi \in C_0^\infty(X)$, it is sufficient to describe the large time asymptotic of the function

$$u(t) := \chi \frac{\sin(t\sqrt{\Delta_X - \frac{n^2}{4}})}{\sqrt{\Delta_X - \frac{n^2}{4}}} f$$

and $\partial_t u(t)$. We proceed using same ideas than in [6]. We first recall that from Stone formula the spectral measure is

$$d\Pi(v^2) = \frac{i}{2\pi}\Big(R(\frac{n}{2} + iv) - R(\frac{n}{2} - iv)\Big)dv$$

in the sense that for $h \in C^\infty([0, \infty))$ we have

$$h\Big(\sqrt{\Delta_X - \frac{n^2}{4}}\Big) = \int_0^\infty h(v) d\Pi(v^2) 2v dv.$$

Since sin is odd, then it is clear that $u(t)$ can be expressed by the integral

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itv}\Big(\chi R(\frac{n}{2} + iv)f - \chi R(\frac{n}{2} - iv)f\Big)dv \tag{4.1}$$

which is actually convergent since $f \in C_0^\infty(X)$ (this is shown below). We want to move the contour of integration into the non-physical sheet $\{\mathrm{Im}(v) > 0\}$ (which correponds with $\lambda = n/2 + iv$ to $\{\mathrm{Re}(\lambda) < n/2\}$) for the part with e^{itv} and into the physical sheet $\{\mathrm{Im}(v) < 0\}$ for the part with e^{-itv}. Let us define the operator $L(v) : L^2(X) \to L^2(X)$ by

$$L(v)\varphi := \Big(\chi R(\frac{n}{2} + iv)\varphi - \chi R(\frac{n}{2} - iv)\varphi\Big)$$

and let $\eta > 0$ be small. We study the following integral for $\beta := n/2 - \delta$

$$I_1(R, \eta, t) := \int_{\substack{\mathrm{Im}(v)=\beta \\ \eta<|\mathrm{Re}(v)|<R}} e^{itv}L(v)f dv, \quad I_2(R, t) := \int_{\substack{|\mathrm{Re}(v)|=R \\ 0<\mathrm{Im}(v)<\beta}} e^{itv}L(v)f dv,$$

where these terms are considered as $L^2(X)$ functions. In particular let us first show that

Lemma 4.1. *Assume that for all $j \in \mathbb{N}_0$ and $\chi' \in C_0^\infty(X)$ with $\chi' f = f$, there exists $C > 0$ and $M \in \mathbb{N}$ such that*

$$||\partial_v^j \chi R(\frac{n}{2} + iv)\chi'||_{\mathcal{L}(L^2)} \leq C(|v| + 1)^M$$

in $\{|\mathrm{Im}(v)| \leq \beta\}$. Then for all $j \in \mathbb{N}_0$, there exists $M' \in \mathbb{N}$ such that for all $N > 0$ there exists $C' > 0$ with

$$||\partial_v^j L(v)f||_{L^2} \leq C'(1 + |v|)^{-N+M'}||f||_{H^{2N}} \tag{4.2}$$

where H^{2N} denotes the L^2-Sobolev space of order $2N$ on X. Consequently,

$$\lim_{R\to\infty} I_2(R,t) = \lim_{R\to\infty} \partial_t I_2(R,t) = 0 \text{ in } L^2(X).$$

Proof: First remark that for $f \in C_0^\infty(X)$, one has $L(v)\Delta^N f = (n^2/4+v^2)^N L(v)f$ for all N and so, choosing $\chi' \in C_0^\infty(X)$ with $\chi' f = f$, it is straightforward to see that for all $N, j \in \mathbb{N}_0$ there exists $C_{j,N} > 0$ such that

$$||\partial_v^j L(v)f||_{L^2} \leq C_{j,N}(1+|v|^2)^{-N}||f||_{H^{2N}}$$
$$\times \max_{\ell\leq j}\Big(||\partial_v^\ell \chi R(\frac{n}{2}+iv)\chi'||_{\mathcal{L}(L^2)} + ||\partial_v^\ell \chi R(\frac{n}{2}-iv)\chi'||_{\mathcal{L}(L^2)}\Big).$$

To prove the statement about the last limit, it suffices to take $j=0$ and $N \gg M$ large enough. □

Now we get estimates in t for $I_1(R,\eta,t)$.

Lemma 4.2. *If for all $j \in \mathbb{N}_0$, there exists $C > 0, M \in \mathbb{N}$ such that $||\partial_v^j L(v)||_{\mathcal{L}(L^2)} \leq C(|v|+1)^M$ in $|\mathrm{Im}(v)| \leq \beta$, then in L^2 sense, $I_1(R,\eta,t)$ and $\partial_t I_1(R,\eta,t)$ have a limit as $R\to\infty, \eta\to 0$ and*

$$\lim_{\eta\to 0}\lim_{R\to\infty} I_1(R,\eta,t) = \pi i e^{-\beta t}\mathrm{Res}_{v=i\beta}(L(v)f) + \mathcal{O}_{L^2}(e^{-\beta t}t^{-\infty}), \quad t\to\infty,$$

$$\lim_{\eta\to 0}\lim_{R\to\infty} \partial_t I_1(R,\eta,t) = -\pi\beta i e^{-\beta t}\mathrm{Res}_{v=i\beta}(L(v)f) + \mathcal{O}_{L^2}(e^{-\beta t}t^{-\infty}), \quad t\to\infty$$

Proof: Let us first consider $I_1(R,\eta,t)$, it can clearly be written as

$$e^{-t\beta}\int_{\eta<|u|<R} e^{itu}L(u+i\beta)f du.$$

Since $L(u+i\beta)$ has a pole at $u=0$, we can write

$$L(u+i\beta)f = \frac{a}{u} + h(u)$$

for some residue $a \in L^2(X)$ and $h(u)$ analytic on $\mathbb{R}$ with values in $L^2(X)$. Set $\psi \in C_0^\infty((-1,1))$ even and equal to 1 near 0, then by (4.2) and properties of Fourier transform, the integral

$$\int_{\eta<|u|<R} e^{itu}\Big((1-\psi(u))L(u+i\beta)f + \psi(u)h(u)\Big)du,$$

converges as $R\to\infty, \eta\to 0$ to a function that is a $\mathcal{O}_{L^2}(t^{-N})$ for all $N\in\mathbb{N}$ when $t\to\infty$. Now it remains to consider

$$a\int_{\eta<|u|<R} e^{itu}\psi(u)u^{-1}du = 2ia\int_\eta^R \frac{\sin(ut)}{u}\psi(u)du$$

which clearly has a limit as $R\to\infty, \eta\to 0$, we denote by $s(t)$ this limit. Then since $s(0)=0$ and $\psi(-u)=\psi(u)$, we have

$$\partial_t s(t) = 2ia\int_0^\infty \psi(u)\cos(tu)du = ia\hat{\psi}(t), \quad s(t) = ia\int_0^t \hat{\psi}(\xi)d\xi = \frac{1}{2}ia\int_{-t}^t \hat{\psi}(\xi)d\xi$$

and it is clear that

$$s(t) = \lim_{t\to\infty} s(t) + \mathcal{O}_{L^2}(t^{-\infty}) = \pi i a + \mathcal{O}_{L^2}(t^{-\infty}).$$

The same arguments show that

$$\partial_t s(t) = \mathcal{O}_{L^2}(t^{-\infty})$$

and this proves the result. □

Now we can conclude

Theorem 4.3. *Let $\chi \in C_0^\infty(X)$, then the solution $u(t)$ of we wave equation* (1.1) *with initial data $f_0, f_1 \in C_0^\infty(X)$ satisfies the asymptotic*

$$\chi u(t) = \frac{A_X}{\Gamma(\frac{n}{2} - \delta + 1)} e^{-t(\frac{n}{2}-\delta)} \langle u_\delta, (\delta - n/2)f_0 + f_1 \rangle \chi u_\delta + \mathcal{O}_{L^2}(e^{-t(\frac{n}{2}-\delta)} t^{-\infty})$$

as $t \to +\infty$, where u_δ is the Patterson generalized eigenfunction.

Proof: we apply the residue theorem after changing the contour in (4.1) as explained above. This gives for instance for $f = (0, f_1)$,

$$\int_{-R}^{R} e^{itv} L(v) f dv = I_1(R, \eta, t) + I_2(R, t) + \int_{\substack{v = i\beta + \eta \exp(i\theta) \\ -\pi < \theta < 0}} e^{itv} L(v) f dv$$

The limit of the last integral as $\eta \to 0$ is given $\pi i e^{-\beta t} \mathrm{Res}_{v = i\beta} L(v) f$. It suffices to conclude by taking the limits $R \to \infty, \eta \to 0$ and using Lemmas 4.1 and 4.2 with Corollary 2.6. Then the case $f = (f_0, 0)$ is dealt with similarly by differentiating in t the equation above and using Lemmas 4.2, 4.1. $\square$

We now show a lower bound in t for the remainder in $u(t)$ using Theorem 3.1.

Proposition 4.4. *Let $K \subset X$ be a relatively compact open set, then there exists a generic set $\Omega \subset L^2(K)$ (i.e. a countable intersection of open dense sets) such that for all $f_1 \in \Omega$ and all $\varepsilon > 0$, we have $r(t) \neq \mathcal{O}_{L^2}(e^{-(\frac{n}{2} + n\delta + \varepsilon)t})$ where*

$$r(t) := \chi u(t) - \frac{A_X}{\Gamma(\frac{n}{2} - \delta + 1)} e^{-t(\frac{n}{2}-\delta)} \langle u_\delta, f_1 \rangle \chi u_\delta$$

is the remainder in the expansion of the solution $u(t)$ of the wave equation (1.1) *with initial data $(0, f_1)$. The lower bound can be improved by $r(t) \neq \mathcal{O}_{L^2}(e^{-(\frac{n}{2} + \delta^2 + \varepsilon)t})$ if X is Schottky.*

Proof: Let us define Ω. If λ_0 is a resonance, we denote by Π_{λ_0} the polar part in the Laurent expansion of $R(\lambda)$ at λ_0. It is a finite rank operator of the form

$$\Pi_{\lambda_0} = \sum_{j=1}^{k} (\lambda - \lambda_0)^{-j} \sum_{m=1}^{m_j(\lambda_0)} \varphi_{jm} \otimes \psi_{jm}$$

where $m_j(\lambda_0), k \in \mathbb{N}$ and $\psi_{jm}, \varphi_{jm} \in C^\infty(X)$ and they can not vanish on an open set since they are solution of the elliptic equation $(\Delta_X - \lambda_0(n - \lambda_0))^{m_j(\lambda_0)} u = 0$. Thus $\chi \Pi_{\lambda_0}|_{L^2(K)}$ is a non-zero continuous operator from $L^2(K)$ to $L^2(X)$ and the kernel of $\chi \Pi_{\lambda_0}|_{L^2(K)}$ is a closed nowhere dense set of $L^2(K)$, we then define $\Omega = \cap_{s \in \mathcal{R}} (L^2(K) \setminus \ker \chi \Pi_s|_{L^2(K)})$ which is a generic set of $L^2(K)$. The idea now is to use the existence of a resonance, say λ_0, in the strip $\{-n\delta + \varepsilon > \mathrm{Re}(\lambda) > \delta\}$ proved in Theorem 3.1 and the formula (for $\mathrm{Re}(\lambda) > \delta$)

$$\chi R(\lambda) f = \int_0^\infty e^{t(\frac{n}{2} - \lambda)} \chi u(t) dt.$$

Indeed, if $r(t) = \mathcal{O}(e^{-t(\frac{n}{2} + n\delta + \varepsilon)})$, the integral $\int_0^\infty e^{t(\frac{n}{2} - \lambda)} r(t) dt$ converges for $\mathrm{Re}(\lambda) > -n\delta - \varepsilon$, and so it provides a holomorphic continuation of $\chi R(\lambda) f$ in λ there. Now a straightforward computation combined with Corollary 2.6 shows that for $\mathrm{Re}(\lambda) > \delta$

$$\int_0^\infty e^{(\frac{n}{2} - \lambda)t} r(t) dt = \chi R(\lambda) f - (\lambda - \delta)^{-1} \chi \mathrm{Res}_{\lambda = \delta} R(\lambda) f.$$

This leads to a contradiction when $f_1 \in \Omega$ since $\ker \chi \Pi_{\lambda_0}|_{L^2(K)} \cap \Omega = \emptyset$ and so $\chi R(\lambda) f$ has a singularity at $\lambda = \lambda_0$. We thus obtain our conclusion. The same method applies when X is Schottky and the finer estimates are valid. $\square$

Remark: we can clearly replace the space $L^2(K)$ above by the space of smooth functions in X with support in K and a similar result holds, so that we are in the setting of Theorem 4.3.

5. Conformal resonances

In this section, we give an explaination of the special cases $\delta \in n/2 - \mathbb{N}$ in term of conformal theory of the conformal infinity. As emphasized before, a convex co-compact hyperbolic manifold (X, g) compactifies into a smooth compact manifold with boundary $\bar{X} = X \cup \partial\bar{X}$, where $\partial\bar{X} = \Gamma\backslash\Omega$ if Ω is the domain of discontinuity of the group Γ defined in the introduction. If x is a smooth boundary defining function of $\partial\bar{X}$, $x^2 g$ extends smoothly to $\bar{X}$ as a metric, the restriction

$$h_0 = x^2 g|_{T\partial\bar{X}}$$

is a metric on $\partial\bar{X}$ inherited from g but depending on the choice of x, however its conformal class $[h_0]$ is clearly independent of x, it is then called the *conformal infinity* of X. By Graham-Lee [11, 10], there is an identification between a particular class of boundary defining functions and elements of the class $[h_0]$: indeed, for any $h_0 \in [h_0]$, there exists near $\partial\bar{X}$ a unique boundary defining function x such that $|dx|_{x^2 g} = 1$ and $x^2 g|_{T\partial\bar{X}} = h_0$, this function will be called a *geodesic boundary defining function*.

We now recall the definition of the scattering operator $S(\lambda)$ as in [12, 24]. Let $\lambda \in \mathbb{C} \setminus (n/2+\mathbb{Z})$ such that $R(z)$ is holomorphic at $z = \lambda$ and let x be a geodesic boundary defining function, then for all $f \in C^\infty(\partial\bar{X})$ there exists a unique function $F(\lambda, f) \in C^\infty(X)$ which satisfies the boundary value problem

$$\begin{cases} (\Delta_X - \lambda(n-\lambda))F(\lambda, f) = 0, \\ \exists F_1(\lambda, f), F_2(\lambda, f) \in C^\infty(\bar{X}) \text{ such that} \\ F(\lambda, f) = x^{n-\lambda}F_1(\lambda, f) + x^\lambda F_2(\lambda, f) \text{ and } F_1(\lambda, f)|_{\partial\bar{X}} = f. \end{cases}$$

Then the operator $S(\lambda) : C^\infty(\partial\bar{X}) \to C^\infty(\partial\bar{X})$ is defined by

$$S(\lambda)f = F_2(\lambda, f)|_{\partial\bar{X}}.$$

It is clear that $S(\lambda)$ depends on choice of x, but it is conformally covariant under change of boundary defining function: if $\hat{x} := xe^\omega$ is another such function, then the related scattering operator is

$$\hat{S}(\lambda) = e^{-\lambda\omega_0}S(\lambda)e^{(n-\lambda)\omega_0}, \quad \omega_0 := \omega|_{\partial\bar{X}}.$$

It is proved in [12] that $S(\lambda)$ has simple poles at $\lambda = n/2 + k$ for all $k \in \mathbb{N}$, and after renormalizing $S(\lambda)$ into

$$\mathcal{S}(\lambda) := 2^{2\lambda-n}\frac{\Gamma(\lambda - \frac{n}{2})}{\Gamma(\frac{n}{2} - \lambda)}S(\lambda)$$

we obtain by the main result of [12] that $\mathcal{S}(n/2 + k) = P_k$ is the k-th GJMS conformal Laplacian on $(\partial\bar{X}, h_0)$ defined previously in [8]. In general $\mathcal{S}(\lambda)$ is a pseudodifferential operator of order $2\lambda - n$ with principal symbol $|\xi|_{h_0}^{2\lambda-n}$ but for $\lambda = n/2 + k$, it becomes differential.

Proposition 5.1. *If $\delta = n/2 - k$ with $k \in \mathbb{N}$, then the j-th GJMS conformal Laplacian $P_j > 0$ for $j < k$ while P_k has a kernel of dimension 1 with eigenvector given by $f_{n/2-k}$ defined below in (5.3) in term of Patterson-Sullivan measure.*

Proof: Let us fix $\delta \in (0, n/2)$ not necessarily in $n/2 - \mathbb{N}$ for the moment. In [14], the first author studied the relation between poles of resolvent and poles of scattering operator. If $\lambda \in \mathbb{C}$, we define its resonance multiplicity by

$$m(\lambda) := \operatorname{rank}\Big(\operatorname{Res}_{s=\lambda}((2s - n)R(s))\Big)$$

while its scattering pole multiplicity is defined by

$$\nu(\lambda) := -\mathrm{Tr}\Big(\mathrm{Res}_{s=\lambda}(\partial_s \mathcal{S}(s)\mathcal{S}^{-1}(s))\Big).$$

We proved in [14] (see also [15] for point in pure point spectrum) that for $\mathrm{Re}(\lambda) < n/2$

$$\nu(\lambda) = m(\lambda) - m(n-\lambda) + 1\!\!1_{\frac{n}{2}-\mathbb{N}}(\lambda)\dim\ker \mathcal{S}(n-\lambda),$$

which in our case reduces to

$$\nu(\lambda) = m(\lambda) + 1\!\!1_{\frac{n}{2}-\mathbb{N}}(\lambda)\dim\ker \mathcal{S}(n-\lambda) \tag{5.1}$$

by the holomorphy of $R(\lambda)$ in $\{\mathrm{Re}(\lambda) \geq n/2\}$, stated in Proposition 2.3. We know from [24, 12] that the Schwartz kernel of $\mathcal{S}(\lambda)$ is related to that of $R(\lambda)$ by

$$\mathcal{S}(\lambda; y, y') = 2^{2\lambda-n+1}\frac{\Gamma(\lambda-\frac{n}{2}+1)}{\Gamma(\frac{n}{2}-\lambda)}[x^{-\lambda}x'^{-\lambda}R(\lambda; x, y, x'y')]|_{x=x'=0} \tag{5.2}$$

where $(x, y) \in [0, \epsilon) \times \partial\bar{X}$ are coordinates in a collar neighbourhood of $\partial\bar{X}$, x being the geodesic boundary defining function used to define $\mathcal{S}(\lambda)$. This implies with Proposition 2.3 that $\mathcal{S}(\lambda)$ is analytic in $\{\mathrm{Re}(\lambda) > \delta\}$ and has a simple pole at δ with residue

$$\mathrm{Res}_{\lambda=\delta}\mathcal{S}(\lambda) = A_X\frac{2^{-2k+1}}{(k-1)!}f_\delta \otimes f_\delta, \quad f_\delta := (x^{-\delta}u_\delta)|_{x=0}.$$

Note that Perry [32] proved that f_δ is well defined and in $C^\infty(\partial\bar{X})$. The functional equation $\mathcal{S}(\lambda)\mathcal{S}(n-\lambda) = \mathrm{Id}$ (see for instance Section 3 of [12]) and the fact that $\mathcal{S}(\lambda)$ is analytic in $\{\mathrm{Re}(\lambda) > \delta\}$ clearly imply that $\ker \mathcal{S}(\lambda) = 0$ for $\mathrm{Re}(\lambda) \in (\delta, n-\delta)$, thus in particular $\ker P_j = 0$ for any $j \in \mathbb{N}$ with $j < n/2 - \delta$. Moreover, using [30, Lemma 4.16] and the fact that $m_{n/2} = 0$ since $R(\lambda)$ is holomorphic in $\{\mathrm{Re}(\lambda) > \delta\}$, one obtains $S(n/2) = \mathrm{Id}$ thus $\mathcal{S}(\lambda) > 0$ for all $\lambda \in (\delta, n-\delta)$ by continuity of $\mathcal{S}(\lambda)$ with respect to λ. We also deduce from the functional equation and the holomorphy of $\mathcal{S}(s)$ at $n-\delta$ that

$$\mathcal{S}(n-\delta)f_\delta = 0.$$

We thus see from this discussion and Proposition 2.3 that, in (5.1), the relation $m(\delta) = \nu(\delta) = 1$ holds when $\delta \notin n/2 - \mathbb{N}$ while $\nu(\delta) = \dim\ker P_k$ when $\delta = n/2 - k$ with $k \in \mathbb{N}$ since $m(\delta) = 0$ in that case by holomorphy of $R(\lambda)$ at $\delta = n/2-k$. To compute $\dim\ker P_k$ when $\delta = n/2-k$, one can use for instance Selberg's zeta function. Indeed by Proposition 2.1 of [32], $Z(\lambda)$ has a simple zero at δ but it follows from Theorems 1.5-1.6 of Patterson-Perry [30] that $Z(\lambda)$ has a zero at $\lambda = n/2 - k$ of order $\nu(n/2-k)$ if $k \in \mathbb{N}, k < n/2$, therefore $\nu(n/2-k) = 1$ and thus

$$\dim\ker P_k = 1.$$

One can now describe a bit more precisely the function f_δ. The Poisson kernel of Proposition 2.3 in the half-space model $\mathbb{R}^n_y \times \mathbb{R}^+_{y_{n+1}}$ of $\mathbb{H}^{n+1}$ is

$$\mathcal{P}(\lambda; y, y_{n+1}, y') = \frac{y_{n+1}}{y_{n+1}^2 + |y-y'|^2}$$

thus if x is the boundary defining function used to define $\mathcal{S}(\lambda)$ and if $(\pi_\Gamma^* x/y_{n+1})|_{y_{n+1}=0} = k(y)$ (recall $\pi_\Gamma, \bar{\pi}_\Gamma$ are the projections of (2.1)) for some $k(y) \in C^\infty(\mathbb{R}^n)$, so we can describe rather explicitely f_δ, we have

$$\bar{\pi}_\Gamma^* f_\delta(y) = k(y)^{-\delta}\int_{\mathbb{R}^n}|y-y'|^{-2\delta}d\mu_\Gamma(y'), \quad y \in \Omega. \tag{5.3}$$

□

To summarize the discussion, if $\delta < n/2$, the Patterson function u_δ is an eigenfunction for Δ_X with eigenvalue $\delta(n-\delta)$, it is not an L^2 eigenfunction though and it has leading asymptotic behaviour $u_\delta \sim x^\delta f_\delta$ as $x \to 0$, where $f_\delta \in C^\infty(\partial\bar{X})$ is in the kernel of the boundary operator $\mathcal{S}(n-\lambda)$. When $\delta \notin n/2 - \mathbb{N}$, this is a resonant state for Δ_X with associated resonance δ while when $\delta \in n/2-\mathbb{N}$ it is still a generalized eigenfunction of Δ_X but not a resonant state anymore, and δ is not a resonance yet in that case: the resonance disappears when δ reaches $n/2-k$ and instead the k-th GJMS at $\partial\bar{X}$ gains an element in its kernel given by the leading coefficient of $u_{n/2-k}$ in the asymptotic at the boundary.

Remark: Notice that the positivity of P_j for $j < n/2-\delta$ has been proved by Qing-Raske [35] and assuming a positivity of Yamabe invariant of the boundary. Our proof allows to remove the assumption on the Yamabe invariant, which, as we showed, is automatically satisfied if $\delta < n/2$.

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