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ON THE EQUATIONS FOR UNIVERSAL TORSORS OVER DEL PEZZO SURFACES

VERA V. SERGANOVA[1] AND ALEXEI N. SKOROBOGATOV[2]

[1]*Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA* (serganov@math.berkeley.edu)

[2]*Department of Mathematics, South Kensington Campus, Imperial College London, London SW7 2BZ, UK* (a.skorobogatov@imperial.ac.uk) *and Institute for Information Transmission Problems, Russian Academy of Sciences, 19 Bolshoi Karetnyi, Moscow 127994, Russia*

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à Jean-Louis Colliot-Thélène

Abstract We describe equations of the universal torsors over del Pezzo surfaces of degrees from 2 to 5 over an algebraically closed field in terms of the equations of the corresponding homogeneous space G/P. We also give a generalization for fields that are not algebraically closed.

Keywords: del Pezzo surfaces; universal torsors; semisimple Lie groups; homogeneous spaces

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Introduction

Universal torsors were invented by Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc; for smooth projective varieties X with $\mathrm{H}^1(X, \mathcal{O}) = 0$ they play the role similar to that of n-coverings of elliptic curves. The foundations of the theory of descent on torsors were laid in a series of notes in *Comptes Rendus de l'Académie des Sciences de Paris* in the second half of the 1970s, and a detailed account was published in [**4**]. The theory has strong number theoretic applications if the torsors can be described by explicit equations, and if the resulting system of equations can be treated using some other methods, whether algebraic or analytic. Such is the case for surfaces fibred into conics over $\mathbb{P}^1_k$: the universal torsors over these surfaces are closely related to complete intersections of quadrics of a rather special kind. To describe them we use the following terminology. If $Z \subset \mathbb{A}^m_k$ is a closed subset of an affine space with a coordinate system over a field k, then the variety obtained from Z by multiplying coordinates by non-zero numbers will be called a *dilatation* of Z. If exactly n geometric fibres of the conic bundle $X \to \mathbb{P}^1_k$ are singular, then there is a non-singular quadric $Q \subset \mathbb{A}^{2n}_k$ such that the universal torsors over X are stably birationally equivalent to the product of a complete intersection of $n-2$ dilatations of Q, and a Severi–Brauer variety (see [**4**, Theorem 2.6.1]). This description was key to a plethora of applications to the Hasse principle, weak approximation, zero-cycles, R-equivalence and rationality problems (see, for example, [**2**, **6**]). A similar approach to cubic and more general smooth del Pezzo surfaces without a pencil of rational curves requires better understanding of their universal torsors. Known descriptions of universal torsors over diagonal cubic surfaces (see [**4**, 2.5] and [**5**, 10]) lack simplicity and symmetry of the conic bundle case.

A non-singular quadric in $\mathbb{A}_k^{2n}$ can be regarded as a homogeneous space of the simple Lie group G associated with the root system D_n which naturally appears in connection with conic bundles with n singular fibres (see, for example, [**10**]). Indeed, over an algebraically closed field we can identify Q with the orbit of the highest weight vector of the fundamental $2n$-dimensional representation V of G. Then the 'essential' part of the torsor is the intersection of $n-2$ dilatations of this homogeneous space by the elements of a maximal torus in $\mathrm{GL}(V)$.

Our first aim in this paper is to obtain a similar description in the case of del Pezzo surfaces. (Recall that these two families exhaust all minimal smooth projective rational surfaces, according to the classification of Enriques, Manin and Iskovskih.) We build on the results of our previous paper [**16**], where we studied split del Pezzo surfaces, i.e. the case when the Galois action on the set of exceptional curves is trivial. The main result of [**16**] is a construction of an embedding of a universal torsor over a split del Pezzo surface X of degree 5, 4, 3 or 2 into the orbit of the highest weight vector of a fundamental representation of the simple simply connected Lie group G which has the same root system as X, i.e. A_4, D_5, E_6 or E_7, respectively. This orbit is the punctured affine cone over G/P, where $P \subset G$ is a maximal parabolic subgroup. The embedding is equivariant with respect to the action of the Néron–Severi torus T of X, identified with a split maximal torus of G extended by $\boldsymbol{G}_m$. In Theorem 2.5 we describe universal torsors over split del Pezzo surfaces of degree d as intersections of $6-d$ dilatations of the affine cone over G/P by k-points of the maximal torus of $\mathrm{GL}(V)$ which is the centralizer of T in $\mathrm{GL}(V)$. This gives a more conceptual approach to the equations appeared previously in the work of Popov [**14**] and Derenthal [**7**] (see also [**8**, **11**, **20**–**22**]). This approach can be called a *global description* of torsors compared to their *local description* obtained by Colliot-Thélène and Sansuc in [**4**, 2.3].

For a general del Pezzo surface X of degree 4, 3 or 2 with a *rational point* we construct an embedding of a universal torsor over X into the same homogeneous space as in the split case, but this time equivariantly with respect to the action of a (possibly, non-split) maximal torus of G (see Theorem 4.4). The case of del Pezzo surfaces of degree 5, where a rational point comes for free by a theorem of Enriques and Swinnerton-Dyer, was already known [**18**, Theorem 3.1.4]. The proof of Theorem 4.4 uses a recent result of Gille [**9**] and Raghunathan [**15**] which classifies maximal tori in quasi-split algebraic groups. This result implies that the Néron–Severi torus T of X embeds into the same split group G extended by $\boldsymbol{G}_m$, exactly as in the case of a split del Pezzo surface.

The condition on the existence of a rational point on X is not a restriction in the case of degree 5, but is clearly a restriction for smaller degrees, limiting the scope of possible applications. However, if X is a del Pezzo surface of degree 4, this condition is necessary as well as sufficient for our construction: if X can be realized inside a twisted form of the quotient of G/P by a maximal torus, then X has a rational point (see Corollary 4.5 (i)). Finally, in Corollary 4.6 we show that any universal torsor over a del Pezzo surface of degree 4 with a k-point is a dense open subset of the intersection of the affine cone over a twisted form of G/P with its dilatation by a k-point of the centralizer of T in $\mathrm{GL}(V)$.

We recall the construction of [**16**] in § 1 alongside with all necessary notation. In § 2 we describe torsors over split del Pezzo surfaces as intersections of dilatations of the affine cone over G/P. In § 3 we prove a uniqueness property used in the proof of the main results in the non-split case in § 4.

1. Review of the split case

Preliminary remarks

Let k be a field of characteristic 0 with an algebraic closure $\bar{k}$.

Let V be a vector space over k, and let $T \subset \mathrm{GL}(V)$ be a split torus, i.e. $T \simeq \boldsymbol{G}_m^n$ for some n. Let $\Lambda \subset \hat{T}$ be the set of weights of T in V, and let $V_\lambda \subset V$ be the subspace of weight λ. We have $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$. Let S be the centralizer of T in $\mathrm{GL}(V)$, i.e.

$$S = \prod_{\lambda\in\Lambda} \mathrm{GL}(V_\lambda) \subset \mathrm{GL}(V).$$

In what follows we always assume that $\dim V_\lambda = 1$ for all $\lambda \in \Lambda$; then S is a maximal torus in $\mathrm{GL}(V)$. Let $\pi_\lambda : V \to V_\lambda$ be the natural projection. For $A \subset V$ we write $A^\times$ for the set of points of A outside $\bigcup \pi_\lambda^{-1}(0)$.

Let $r = 4$, 5, 6 or 7. A split del Pezzo surface X of degree $d = 9 - r$ is the blowing-up of $\mathbb{P}^2$ in r k-points in general position (i.e. no three points are on a line and no six are on a conic). The Picard group $\mathrm{Pic}\,X$ is a free abelian group of rank $r+1$, generated by the classes of exceptional curves on X. Let $T = \boldsymbol{G}_m^{r+1}$. Once an isomorphism $\hat{T} \xrightarrow{\sim} \mathrm{Pic}\,X$ is fixed, T is called the Néron–Severi torus of X. A universal torsor $f : \mathcal{T} \to X$ is an X-torsor with structure group T, whose type is the isomorphism $\hat{T} \xrightarrow{\sim} \mathrm{Pic}\,X$ (see [**18**, p. 25]). We call a divisor in $\mathcal{T}$ an *exceptional divisor* if it is the inverse image of an exceptional curve in X.

Now suppose that $\dim V$ equals the number of exceptional curves on X. We can make an obvious but useful observation.

Lemma 1.1. *Let $\mathcal{T} \to X$ be a universal torsor over a split del Pezzo surface X. Let ϕ and ψ be T-equivariant embeddings $\mathcal{T} \to V$ such that for each weight $\lambda \in \Lambda$ the divisors of functions $\pi_\lambda\phi$ and $\pi_\lambda\psi$ are equal to the same exceptional divisor with multiplicity* 1. *Then $\psi = s \circ \phi$ for some $s \in S(k)$.*

Proof. Since $\mathcal{T}$ is a universal torsor we have $k[\mathcal{T}]^* = k^*$, hence two regular functions with equal divisors differ by a non-zero multiplicative constant. □

Construction in the split case

Let the pair consisting of a root system R of rank r and a simple root α be one of the pairs in the list

$$(\mathrm{A}_4, \alpha_3), \qquad (\mathrm{D}_5, \alpha_5), \qquad (\mathrm{E}_6, \alpha_6), \qquad (\mathrm{E}_7, \alpha_7). \tag{1.1}$$

Here and elsewhere in this paper we enumerate roots as in [**3**]. Let G be the split simply connected simple group with split maximal torus H and root system R. Let ω be the fundamental weight dual to α, and let $V = V(\omega)$ be the irreducible G-module with the highest weight ω. It is known that V is faithful and minuscule (see [**3**]). Let $P \subset G$ be the maximal parabolic subgroup such that $G/P \subset \mathbb{P}(V)$ is the orbit of the highest weight vector. The affine cone over G/P is denoted by $(G/P)_{\mathrm{a}}$.

It is easy to check that the G-module $S^2(V)$ is the direct sum of two irreducible submodules $V(\omega_1) \oplus V(2\omega)$. For $r \leqslant 6$, $V(\omega_1)$ is a non-trivial irreducible G-module of least dimension; it is a minuscule representation of G. If $r = 7$, then $V(\omega_1)$ is the adjoint representation; it is quasi-minuscule, that is, all the non-zero weights have multiplicity 1 and form one orbit of the Weyl group W of R. If pr is the natural projection $S^2(V) \to V(\omega_1)$, and $\mathrm{Ver} : V \to S^2(V)$ is the Veronese map $x \mapsto x^2$, then it is well known that $(G/P)_{\mathrm{a}}$ is the fibre $(pr \circ \mathrm{Ver})^{-1}(0)$ (as a scheme, see [**1**, Proposition 4.2], and references therein).

Let S be the centralizer of H in $\mathrm{GL}(V)$. Since the eigenspaces of H in V are one dimensional, V has a coordinate system with respect to which S is the diagonal torus. Let the torus $T \subset S$ be the extension of H by the scalar matrices $\boldsymbol{G}_m \subset \mathrm{GL}(V)$. Note that an eigenspace of H in V is also an eigenspace of T, so that there is a natural bijection between the corresponding sets of characters.

As in [**16**] we denote by V^{sf} the dense open subset of V consisting of the points whose H-orbits are closed and whose stabilizers in T are trivial; the first of these conditions is stability in the sense of Mumford [**13**]. Let $(G/P)_{\mathrm{a}}^{\mathrm{sf}} = (G/P)_{\mathrm{a}} \cap V^{\mathrm{sf}}$. In [**16**] we constructed a T-equivariant closed embedding of $\mathcal{T}$ into $(G/P)_{\mathrm{a}}^{\mathrm{sf}}$ such that each weight hyperplane section $\mathcal{T} \cap \pi_\lambda^{-1}(0)$ is an exceptional divisor with multiplicity 1. Then $X^\times = f(\mathcal{T}^\times)$ is the complement to the union of exceptional curves on X.

We need to recall the details of this construction. It starts with the case $(\mathrm{R}, \alpha) = (\mathrm{A}_4, \alpha_3)$ where the torsor $\mathcal{T}$ is the set of stable points of $(G/P)_{\mathrm{a}}$ which is the affine cone over the Grassmannian $\mathrm{Gr}(2,5)$ (see [**18**, 3.1]). Thus $\mathcal{T}$ is open and dense in $(G/P)_{\mathrm{a}}$ in this case. As in [**16**] we use dashes to denote the previous pair in (1.1); the previous pair of (A_4, α_3) is $(\mathrm{A}_1 \times \mathrm{A}_2, \alpha_1^{(1)} + \alpha_2^{(2)})$, though it will not be used. For $r \geqslant 5$ we assume that a torsor $\mathcal{T}' \subset (G'/P')_{\mathrm{a}}^{\mathrm{sf}}$ over a split del Pezzo surface of degree $10 - r$ is already constructed, and proceed to construct $\mathcal{T}$ as follows.

Let $\Lambda_n \subset \Lambda$ be the set of weights λ such that n is the coefficient of α in the decomposition of $\omega - \lambda$ into a linear combination of simple roots. Let $V_n = \bigoplus_{\lambda \in \Lambda_n} V_\lambda$, then

$$V = \bigoplus_{n \geqslant 0} V_n. \tag{1.2}$$

The subspaces V_n are G'-invariant. In fact, $V_n = 0$ for $n > 3$ so that

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3,$$

and $V_3 = 0$ unless $r = 7$. The degree 0 component $V_0 \simeq k$ is the highest weight subspace, and the degree 1 component V_1 is isomorphic to V' as a G'-module. The G'-module V_2 is irreducible with highest weight ω_1. For $r = 7$ we have $V_3 \simeq k$.

As in [**16**], let g_t be the element of T such that $g_t|_{V_i} = t^{1-i}\,\mathrm{id}$, where $t \in \bar{k}^*$. Let $U \subset (G/P)_{\mathrm{a}}$ be the set of points of $(G/P)_{\mathrm{a}}$ outside $(V_0 \oplus V_1) \cup (V_2 \oplus V_3)$. The natural projection $\pi : V \to V_1$ defines a morphism $U \to V_1 \setminus \{0\}$ which is the composition of a torsor under $\boldsymbol{G}_m = \{g_t \mid t \in \bar{k}^*\}$ and the morphism inverse to the blowing-up of $(G'/P')_{\mathrm{a}} \setminus \{0\}$ in $V_1 \setminus \{0\}$ [**16**, Corollary 4.2]. There is a G'-equivariant affine morphism $\exp : V_1 \to (G/P)_{\mathrm{a}}$ such that $\pi \circ \exp = \mathrm{id}$, and the affine cone over $\exp(V_1)$ is dense in $(G/P)_{\mathrm{a}}$. As in [**16**] we write $\exp(x) = (1, x, p(x), q(x))$. We mentioned above that the G'-module V_2 is a direct summand of $S^2(V_1)$. The map p can be identified, up to a non-zero constant, with the composition of the natural map $V_1 \to S^2(V_1)$ with the projection to V_2. Is this notation the scheme $(G'/P')_{\mathrm{a}}$ is the fibre $p^{-1}(0)$.

Since V_2 is the direct sum of one-dimensional weight spaces, it has a natural coordinate system. The weight coordinates of $p(x)$ will be written as $p_\mu(x)$, where $\mu \in W'\omega_1$.

The choice of a point in $V^\times$ defines an isomorphism $V^\times \simeq S$ compatible with the action of S. Using this isomorphism we define a multiplication on $V^\times$, and then extend it to V. However, none of our formulae will depend of this isomorphism.

Suppose that $\mathcal{T}' \subset (G'/P')_{\mathrm{a}} \subset V' = V_1$ is such that $f' : \mathcal{T}' \to X' = \mathcal{T}'/T'$ is a universal torsor over a del Pezzo surface X', moreover, the T'-invariant hyperplane sections of $\mathcal{T}'$ are the exceptional divisors. In [**16**] we proved that for any $\bar{k}$-point x_0 in $\mathcal{T}'^\times$ there exists a non-empty open subset $\Omega(x_0) \subset (G'/P')_{\mathrm{a}}^\times$, whose definition is recalled in the beginning of the next section, such that for any y_0 in $\Omega(x_0)$ the orbit $T'y_0$ is the scheme-theoretic intersection $x_0^{-1}y_0\mathcal{T}' \cap (G'/P')_{\mathrm{a}}$ (see [**16**, Corollary 6.4]). Therefore, if $\mathcal{T}$ is the proper transform of $x_0^{-1}y_0\mathcal{T}'$ in U, then $X = \mathcal{T}/T$ is the blowing-up of X' at $f'(x_0)$. Consequently, one proves that $\mathcal{T} \subset (G/P)_{\mathrm{a}}^{\mathrm{sf}}$. Equivalently, $\mathcal{T}$ can be defined as the affine cone (without zero) over the Zariski closure of $\exp(x_0^{-1}y_0\mathcal{T}' \setminus T'y_0)$ in $(G/P)_{\mathrm{a}}^{\mathrm{sf}}$.

The construction of an embedding of a universal torsor over X into $(G/P)_{\mathrm{a}}^{\mathrm{sf}}$ is the main result of [**16**] (Theorem 6.1). The following corollary to this theorem complements it by showing that our embedding is in a sense unique.

Corollary 1.2. *Let $\mathcal{T} \subset V^{\mathrm{sf}}$ be a closed T-invariant subvariety such that $\mathcal{T}/T$ is a split del Pezzo surface and the weight hyperplane sections of $\mathcal{T}$ are the exceptional divisors with multiplicity* 1. *Then for some $s \in S(k)$ the torsor $s\mathcal{T}$ is a subset of $(G/P)_{\mathrm{a}}$ obtained by our construction (for some choice of a basis of simple roots of our root system* R*).*

Proof. The construction of [**16**] recalled above produces a universal torsor $\tilde{\mathcal{T}}$ over the same split del Pezzo surface X inside $(G/P)_{\mathrm{a}}$, satisfying the condition that the weight hyperplane sections are the exceptional divisors with multiplicity 1. The identifications of the exceptional curves on X with the weights of V coming from $\mathcal{T}$ and $\tilde{\mathcal{T}}$ may be different, however the permutation that links them is an automorphism of the incidence graph of the exceptional curves on X. It is well known (see [**12**, Chapter 4]) that the automorphism group of this graph is the Weyl group W of R. Thus replacing $\tilde{\mathcal{T}}$ by its image under the action of an appropriate element of W (that is, a representative of this element in the normalizer of H in G), we ensure that the identification of the weights with the exceptional curves is the same for both embeddings. (The choice of this element in W is equivalent to the choice of a basis of simple roots in our construction.) The multiplicity 1 condition in the construction of [**16**] is easily checked by induction from the case $r = 4$ where we consider the Plücker coordinate hyperplane sections of $\mathrm{Gr}(2, 5)$. It remains to apply Lemma 1.1. □

Let us recall some more notation. For $\mu \in W\omega_1 \subset \hat{H}$ we write $S^2_\mu(V)$ for the H-eigenspace of $S^2(V)$ of weight μ, and $S^2_\mu(V)^*$ for the dual space. Let Ver_μ be the Veronese map $V \to S^2(V)$ followed by the projection to $S^2_\mu(V)$. For $r = 6$ we write $S^3_0(V)$ for the zero weight H-eigenspace in $S^3(V)$, and $\mathrm{Ver}_0 : V \to S^3_0(V)$ for the corresponding natural map.

As in the previous corollary, we denote by $\mathcal{T} \subset V^{\mathrm{sf}}$ a closed T-invariant subvariety such that $\mathcal{T}/T$ is a split del Pezzo surface and the weight hyperplane sections of $\mathcal{T}$ are exceptional divisors with multiplicity 1. Let $I \subset k[V^*]$ be the ideal of $\mathcal{T}$, $I_\mu = I \cap S^2_\mu(V)^*$, and, for $r = 6$, let $I_0 = I \cap S^3_0(V)^*$.

Let $\tilde{\mu}$ be the character by which T acts on $S^2_\mu(V)$. The T-invariant hypersurface in $\mathcal{T}$ cut by the zeros of a form from $S^2_\mu(V)^* \setminus I_\mu$ is mapped by $f : \mathcal{T} \to X$ to a conic on X. The class of this conic in $\mathrm{Pic}\, X$, up to sign, is $\tilde{\mu} \in \hat{T}$ under the isomorphism $\hat{T} \simeq \mathrm{Pic}\, X$ given by the type of the torsor $f : \mathcal{T} \to X$ (see the comments before Proposition 6.2 in [**16**]). The conics on X in a given class form a two-dimensional linear system, hence the codimension of I_μ in $S^2_\mu(V)^*$ is 2 (see [**16**, Formula (15)]). Let $I^\perp_\mu \subset S^2_\mu(V)$ be the two-dimensional zero set of I_μ. The corresponding projective system defines a morphism $f_\mu : X \to \mathbb{P}^1_k = \mathbb{P}(I^\perp_\mu)$ whose fibres are the conics of the class $\tilde{\mu}$. The link between Ver_μ and f_μ is described in the following commutative diagram:

$$\begin{array}{ccccccc} V & \supset & (G/P)_{\mathrm{a}} & \supset & \mathcal{T} & \longrightarrow & X \\ \downarrow{\scriptstyle \mathrm{Ver}_\mu} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_\mu} \\ S^2_\mu(V) & \supset & p^\perp_\mu & \supset & I^\perp_\mu \setminus \{0\} & \longrightarrow & \mathbb{P}^1_k \end{array} \qquad (1.3)$$

Here $p^\perp_\mu$ is the zero set of $p_\mu \in S^2_\mu(V)^*$.

Lemma 1.3. *For $\mu \in W\omega_1$ the vertical maps in (1.3) are surjective. Moreover, we have $\dim S^2_\mu(V) = r - 1$.*

Proof. For the two right-hand maps the statement is clear. The map $V \to S^2_\mu(V)$ is surjective because all eigenspaces of T in V are one dimensional. Since $\dim S^2_\mu(V)$ does not change if we replace μ by $w\mu$ for any $w \in W$, to calculate $\dim S^2_\mu(V)$ we can assume that $\mu = \omega_1$. But ω_1 is a weight of H' in V_2, so we have $S^2_{\omega_1}(V) = S^2_{\omega_1}(V_1) \oplus (V_2)_{\omega_1}$, where $\dim(V_2)_{\omega_1} = 1$. Starting with the case of the Plücker coordinates for $r = 4$, one shows by induction that $\dim S^2_{\omega_1}(V) = r-1$. Hence $\dim S^2_\mu(V) = r-1$ and so $\dim p_\mu^\perp = r-2$ for any μ.

To compute $\mathrm{Ver}_\mu((G/P)_{\mathrm{a}})$ we can continue to assume that $\mu = \omega_1$. If x is in V_1, then Ver_{ω_1} sends $\exp(x) \in (G/P)_{\mathrm{a}}$ to $\mathrm{Ver}_{\omega_1}(x) + p_{\omega_1}(x)$. Thus the projection of $\mathrm{Ver}_{\omega_1}((G/P)_{\mathrm{a}})$ to $S^2_{\omega_1}(V_1) = \mathrm{Ver}_{\omega_1}(V_1)$, which is a vector space of dimension $r-2$, is surjective. Hence Ver_{ω_1} maps $(G/P)_{\mathrm{a}}$ surjectively onto $p_{\omega_1}^\perp$. $\square$

Let $\{x^\nu \mid \nu \in \Lambda_1\}$ be a set of T-homogeneous coordinates in V_1. Arguing by induction as in this proof, it is easy to show starting with the case of the Plücker coordinates on $\mathrm{Gr}(2,5)$ that $p_\mu(x)$ is the sum of all the monomials of weight μ with non-zero coefficients:

$$p_\mu(x) = \sum_{\{\nu,\eta \in \Lambda_1 \mid \nu+\eta=\mu\}} c_{\nu\eta} x^\nu x^\eta, \quad c_{\nu\eta} \neq 0.$$

We finish this section with some remarks on the equations of G/P in the case E_7. In this case V is the minimal 56-dimensional representation of G. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{h}$ be the Lie algebra of $H \subset G$, so that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. We have the decomposition $S^2(V) \simeq V(2\omega) \oplus \mathfrak{g}$, so $p : V \to \mathfrak{g}$ is the composition of the natural map $V \to S^2(V)$ and the projection $S^2(V) \to \mathfrak{g}$. If α is a root of $\mathfrak{g}$, we define $p_\alpha(x)$ as the composition of p with the projection to $\mathfrak{g}_\alpha$. We have $(G/P)_{\mathrm{a}} = p^{-1}(0)$. Moreover, the ideal $I((G/P)_{\mathrm{a}})$ is generated by the quadratic forms $p_\alpha(x)$, and by seven quadratic forms of weight zero which are the compositions of p with the projection $\mathfrak{g} \to \mathfrak{h}$ followed by $\alpha : \mathfrak{h} \to k$, where $\alpha \in \mathfrak{h}^*$ is a simple root.

If μ is a weight of V we denote by H_μ the image of μ in $\mathfrak{h}$ under the natural isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ defined by the Killing form.

Lemma 1.4. *The set $p(V) \cap \mathfrak{h}$ is the union of lines spanned by the vectors H_μ, where $\mu \in \Lambda$.*

Proof. Let $h \in p(V) \cap \mathfrak{h}$. Since p is a G-invariant map, we have $Gh \subset p(V)$. Note that

$$\dim Gh = \dim \mathfrak{g} - \dim Z(h),$$

where $Z(h)$ is the centralizer of h in $\mathfrak{g}$. On the other hand, we have $\dim p(V) \leqslant 56$, hence

$$\dim Z(h) \geqslant 77. \tag{1.4}$$

Note that $Z(h) \subset \mathfrak{g}$ is a reductive Lie algebra containing $\mathfrak{h}$. Thus the Dynkin diagram of the semisimple part of $Z(h)$ is obtained from the Dynkin diagram E_7 by removing some of the nodes; the dimension of the centre of $Z(h)$ is the number of removed nodes. By direct inspection $Z(h)$ satisfies (1.4) only if its semisimple part is of type E_6. Thus $Z(h)$ is conjugate to the centralizer of H_ω, since the latter is defined by all the simple roots except α. Therefore, for some $w \in W$ the vector $w(h)$ is proportional to H_ω, thus h is proportional to H_μ for some weight $\mu \in W\omega = \Lambda$. $\square$

Lemma 1.5. *Let N be the zero set of the quadratic forms $p_\alpha(x)$ for all the roots α of $\mathfrak{g}$. Then the irreducible components of N are $(G/P)_{\mathrm{a}}$ and the two-dimensional spaces $V_\mu \oplus V_{-\mu}$ for all $\mu \in \Lambda$.*

Proof. Let $x \in N$. If $p(x) = 0$, then $x \in (G/P)_{\mathrm{a}}$. So assume $p(x) \neq 0$. Then by the previous lemma $p(x)$ is a multiple of H_μ for some $\mu \in \Lambda$, and we may assume without loss of generality that $\mu = \omega$. Since p is G-equivariant, the Lie algebra $\mathfrak{g}_x$ of the stabilizer of x in G is a subalgebra of $Z(H_\omega)$. But $Z(H_\omega)$ is isomorphic to the direct sum of the simple Lie algebra $\mathfrak{g}'$ of type E_6, $\dim \mathfrak{g}' = 78$, and the one-dimensional centre. It is clear that $\dim \mathfrak{g}_x \geqslant \dim \mathfrak{g} - 56 = 77$, therefore we must have $\mathfrak{g}' \subset \mathfrak{g}_x$. Consider the decomposition $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$, and let x_i be the component of x in V_i. Then $\mathfrak{g}_x$ contains $\mathfrak{g}'$ if and only if $x_1 = x_2 = 0$. Thus $x \in V_\omega \oplus V_{-\omega}$. (Note that $p(x) \neq 0$ implies $\mathfrak{g}_x = \mathfrak{g}'$.) □

Corollary 1.6. *The closed set $(G/P)^{\mathrm{sf}}_{\mathrm{a}} \subset V^{\mathrm{sf}}$ is given by the equations $p_\alpha(x) = 0$ for all the roots α of $\mathfrak{g}$.*

Proof. The stabilizer of any vector $x \in V_\mu \oplus V_{-\mu}$ in T has positive dimension, so x is not stable. Thus $N^{\mathrm{sf}} = (G/P)^{\mathrm{sf}}_{\mathrm{a}}$. □

2. Torsors over split del Pezzo surfaces

Unless stated otherwise we assume that $r \geqslant 5$, so that G' is of type A_4, D_5 or E_6. Recall that we use dashes to denote objects related to the 'previous' root system. If $r(u)$ is a symmetric n-form on V_1, then for any $x \in V_1$, $r(xu)$ is another symmetric form (we multiply $x, u \in V_1$ coordinate-wise). In what follows u is always a mute variable.

Let x_0 be a k-point of $\mathcal{T}'^{\times}$. We define the dense open subset $\Omega(x_0) \subset (G'/P')^{\times}_{\mathrm{a}}$ as the set of $\bar{k}$-points x such that $\exp(x_0^{-1} x \mathcal{T}'^{\times})$ is not contained in $V \setminus V^{\times}$, that is, in the union of weight hyperplanes of V. For $r = 5$ or 6 the set $\Omega(x_0)$ is the complement to the union of the closed subsets

$$Z_\mu(x_0) = \{x \in (G'/P')_{\mathrm{a}} \mid p_\mu(x_0^{-1} x u) \in I'_\mu\}$$

for all weights μ of V_2; for $r = 7$ one also removes the closed subset

$$Z_0(x_0) = \{x \in (G'/P')_{\mathrm{a}} \mid q(x_0^{-1} x u) \in I'_0\}.$$

The condition $y_0 \in \Omega(x_0)$ implies that for all μ the vectors $\mathrm{Ver}_\mu(x_0)$ and $\mathrm{Ver}_\mu(y_0)$ are not proportional. Since $\dim S^2_\mu(V_1)^* = 2 + \dim I'_\mu$, we see that for any $y_0 \in \mathcal{T}' \cap \Omega(x_0)$ the subspace $I'_\mu \subset S^2_\mu(V_1)^*$ consists of the forms vanishing at x_0 and y_0.

Recall that $\pi : V \to V_1$ is the natural projection (cf. (1.2)).

Lemma 2.1. *Let x_0 be a k-point of $\mathcal{T}'^{\times}$, and let y_0 be a k-point of $\Omega(x_0) \cap \mathcal{T}'$. Let $\mathcal{T}$ be the torsor defined as the affine cone over the Zariski closure of $\exp(x_0^{-1} y_0 \mathcal{T}' \setminus T' y_0)$ in $(G/P)^{\mathrm{sf}}_{\mathrm{a}}$. Then we have the following statements.*

(i) *The closed set $Z_\mu(x_0) \subset (G'/P')_{\mathrm{a}}$ consists of the $\bar{k}$-points x in $(G'/P')_{\mathrm{a}}$ such that $p_\mu(x_0^{-1} y_0 x) = 0$. For $r = 7$ the closed set $Z_0(x_0)$ consists of the $\bar{k}$-points x in $(G'/P')_{\mathrm{a}}$ such that $q(x_0^{-1} y_0 x) = 0$.*

(ii) *The open set $\Omega(x_0)\cap\mathcal{T}'$ is the inverse image of the complement to all exceptional curves on X' and to all conics on X' passing through $f'(x_0)$. For $r=7$ one also removes from the cubic surface $X'\subset\mathbb{P}^3_k$ the nodal curve cut by the tangent plane to X' at $f'(x_0)$. We have $\mathcal{T}^\times=\pi^{-1}(\Omega(x_0)\cap\mathcal{T}')$.*

(iii) *We have $t=\exp(x_0^{-1}y_0^2)\in\mathcal{T}^\times$.*

Proof. (i) The inclusion of $Z_\mu(x_0)$ into the hypersurface given by $p_\mu(x_0^{-1}y_0x)=0$ is clear: assigning the variable u the value $y_0\in\mathcal{T}'$ we see that $p_\mu(x_0^{-1}xu)\in I'_\mu$ implies that $p_\mu(x_0^{-1}xy_0)=0$. Conversely, let us prove that every point x of $(G'/P')_{\mathrm{a}}$ satisfying the condition $p_\mu(x_0^{-1}y_0x)=0$, is in $Z_\mu(x_0)$. Using Lemma 1.3 we see that the set of quadratic forms $p_\mu(x_0^{-1}yu)$ on V_1 for a fixed x_0 and arbitrary $y\in(G'/P')_{\mathrm{a}}$ is a vector subspace $L\subset S^2_\mu(V_1)^*$ of codimension 1, in fact this is the space of forms vanishing at x_0. As was pointed out before the statement of the lemma, I'_μ is the subspace of L of codimension 1 consisting of the forms vanishing at y_0. This proves the desired inclusion.

Now let $r=7$. The inclusion of $Z_0(x_0)$ into the hypersurface $q(x_0^{-1}y_0x)=0$ is clear for the same reason as above. Conversely, let $x\in(G'/P')_{\mathrm{a}}(\bar k)$ be such that $q(x_0^{-1}y_0x)=0$. We need to prove that $q(x_0^{-1}xu)$ vanishes for any $\bar k$-point u of $\mathcal{T}'$. In the end of the proof of Proposition 6.2 of [**16**] we showed that the dual space $\mathrm{H}^0(X',\mathcal{O}(-K_{X'}))^*$ is a four-dimensional vector subspace of $S^3_0(V_1)$, so that we have a commutative diagram similar to (1.3):

$$\begin{array}{ccccc}
V_1 & \supset & \mathcal{T}' & \longrightarrow & X' \\
\downarrow{\scriptstyle \mathrm{Ver}_0} & & \downarrow & & \downarrow{\scriptstyle \varphi} \\
S^3_0(V_1) & \supset & \mathrm{H}^0(X',\mathcal{O}(-K_{X'}))^*\setminus\{0\} & \longrightarrow & \mathbb{P}(\mathrm{H}^0(X',\mathcal{O}(-K_{X'}))^*)
\end{array}$$

where φ is the anticanonical embedding $X'\hookrightarrow\mathbb{P}^3_k$. In [**16**] we also showed that for any $x\in(G'/P')_{\mathrm{a}}(\bar k)$ the cubic form $q(x_0^{-1}xu)$, considered as a linear form on $S^3_0(V_1)$, vanishes on the tangent space $T_{x_0}\simeq\mathbb{P}^2_k$ to $\varphi(X')\subset\mathbb{P}^3_k$ at $\varphi f'(x_0)$. It is thus obvious that if $q(x_0^{-1}xu)$ vanishes at any point of $\varphi(X')$ outside of T_{x_0}, then $q(x_0^{-1}xu)$ vanishes at any $\bar k$-point u of $\mathcal{T}'$. But $\varphi f'(y_0)\notin T_{x_0}$, otherwise $q(x_0^{-1}y_0z)=0$ for any $\bar k$-point z of $(G'/P')_{\mathrm{a}}$ contradicting the assumption that y_0 is in $\Omega(x_0)$. Thus $q(x_0^{-1}xy_0)=0$ implies that $q(x_0^{-1}xu)\in I'_0$.

(ii) The geometric description of $\Omega(x_0)\cap\mathcal{T}'$ follows from [**16**, Corollary 6.3]. Hence $\Omega(x_0)\cap\mathcal{T}'$ is obtained from $\mathcal{T}'$ by removing the images $\pi(E)$ of all exceptional divisors $E\subset\mathcal{T}$, so that $\pi(\mathcal{T}^\times)=\Omega(x_0)\cap\mathcal{T}'$.

(iii) Recall that $\exp(x)$ gives a section of the natural morphism $\pi:\mathcal{T}\to x_0^{-1}y_0\mathcal{T}'$ over the complement to the fibre $T'y_0$. Thus $t\in\mathcal{T}$. Since y_0 is in $\Omega(x_0)\cap\mathcal{T}'$ we see from (ii) that t is in $\mathcal{T}^\times$. □

Let $\mathcal{T}\subset V^{\mathrm{sf}}$ be a closed T-invariant subvariety such that $\mathcal{T}/T$ is a split del Pezzo surface and the weight hyperplane sections of $\mathcal{T}$ are exceptional divisors with multiplicity 1. The torsor $\mathcal{T}$ defines an important subset of the torus S. Namely, let $\mathcal{Z}$ be the closed subset of S consisting of the points s such that $s\mathcal{T} \subset (G/P)_{\mathrm{a}}$ (which implies $s\mathcal{T} \subset (G/P)_{\mathrm{a}}^{\mathrm{sf}}$). Equivalently, $\mathcal{Z} = \bigcap_{x\in\mathcal{T}^\times(\bar{k})} x^{-1}(G/P)_{\mathrm{a}}^\times$. The set $\mathcal{Z}$ is T-invariant, since such are $(G/P)_{\mathrm{a}}$ and $\mathcal{T}$. In the case when $\mathcal{T} \subset (G/P)_{\mathrm{a}}^{\mathrm{sf}}$, the variety $\mathcal{Z}$ contains the identity element $1 \in S(k)$.

Lemma 2.2. *Under the assumptions of Lemma 2.1 for $r = 4$ we have $\mathcal{Z} = T$, and for $r \geqslant 5$ we have $\pi(\mathcal{Z}) = y_0^{-1}\Omega(x_0)$ which is dense and open in $y_0^{-1}(G'/P')_{\mathrm{a}}^\times$. The closed subvariety $\mathcal{Z} \subset S$ is the affine cone (without zero) over $t^{-1}\exp(x_0^{-1}y_0\Omega(x_0))$; in particular, $\mathcal{Z}$ is geometrically integral, and $t^{-1}\mathcal{T}^\times \subset \mathcal{Z}$. For $r = 5$ this inclusion is an equality.*

Proof. The statement in the case $r = 4$ is clear since $\mathcal{T}$ is dense in $(G/P)_{\mathrm{a}}$, and the only elements of S that leave $G/P = \mathrm{Gr}(2, 5)$ invariant are the elements of T. Now assume that $r \geqslant 5$. For a fixed x_0, in order to construct an embedding $\mathcal{T} \subset (G/P)_{\mathrm{a}}$ we can choose any y in the dense open subset $\Omega(x_0) \subset (G'/P')_{\mathrm{a}}$. The embeddings defined by (x_0, y_0) and (x_0, y) satisfy the conditions of Lemma 1.1. We obtain an element $s \in \mathcal{Z}$ such that $\pi(s) = y_0^{-1}y$. Thus $\pi(\mathcal{Z})$ contains $y_0^{-1}\Omega(x_0)$.

Let us prove that $\pi(\mathcal{Z}) \subset y_0^{-1}(G'/P')_{\mathrm{a}}^\times$. Let $\pi_0 : V \to V_0 \simeq k$ be the natural projection. Choose $y \in \mathcal{T} \subset (G/P)_{\mathrm{a}}$ such that $\pi(y) = y_0 \in \Omega(x_0) \subset (G'/P')_{\mathrm{a}}^\times$. By Lemma 4.1 of [**16**] we have $\pi_0(y) = 0$. Thus $\pi_0(sy) = 0$ for any $s \in \mathcal{Z}$. But since $sy \in (G/P)_{\mathrm{a}}$, an inspection of cases in Lemma 4.1 of [**16**] shows that $\pi(sy) = \pi(s)y_0 \in (G'/P')_{\mathrm{a}}^\times$. Therefore, $\pi(\mathcal{Z}) \subset y_0^{-1}(G'/P')_{\mathrm{a}}^\times$. Next, we note that $st \in (G/P)_{\mathrm{a}}^\times$ (since $t \in \mathcal{T}^\times$ by Lemma 2.1). The coordinates of the projection of st to V_2 equal $p_\mu(\pi(s)x_0^{-1}y_0^2)$, up to a non-zero constant, hence $p_\mu(\pi(s)x_0^{-1}y_0^2) \neq 0$ for all μ. But for $r \leqslant 6$ the open set $y_0^{-1}\Omega(x_0) \subset y_0^{-1}(G'/P')_{\mathrm{a}}^\times$ is given by $p_\mu(x_0^{-1}y_0^2u) \neq 0$, by Lemma 2.1 (i). For $r = 7$ a similar argument shows that $q(\pi(s)x_0^{-1}y_0^2) \neq 0$. Thus we obtain the equality $\pi(\mathcal{Z}) = y_0^{-1}\Omega(x_0)$.

By Lemma 2.1 (iii), $t = \exp(x_0^{-1}y_0^2)$ is in $\mathcal{T}^\times$ so we have $t\mathcal{Z} \subset (G/P)_{\mathrm{a}}^\times$. Since $\mathcal{Z}$ is invariant under the action of $\boldsymbol{G}_m = \{g_t \mid t \in \bar{k}^*\}$, we see from Lemma 4.1 of [**16**] that $\mathcal{Z}$ is a $\boldsymbol{G}_m$-torsor over $\pi(\mathcal{Z}) = y_0^{-1}\Omega(x_0)$. Moreover, $t^{-1}\exp(x_0^{-1}y_0^2x)$ is a section of this torsor. This proves that $\mathcal{Z}$ is the affine cone over $t^{-1}\exp(x_0^{-1}y_0\Omega(x_0))$.

If $r = 5$, then $\Omega(x_0)$ is a dense open subset of $\mathcal{T}'$ as both sets are Zariski open in $\mathrm{Gr}(2, 5)$. Thus the last statement follows from Lemma 2.1 (ii). $\square$

This lemma implies that $\dim \mathcal{Z} = 2 + \dim G'/P'$ which equals $8, 12, 18$ for $r = 5, 6, 7$, respectively.

Definition 2.3. We say that $r - 3$ points $z_0, \ldots, z_{r-4}$ in $\mathcal{Z}(\bar{k})$ are in general position if for any weight $\mu \in W\omega_1$ the vectors $\mathrm{Ver}_\mu(z_i)$, $i = 0, \ldots, r - 4$, are linearly independent.

Lemma 2.4. *Let $\mathcal{T} \subset V^{\mathrm{sf}}$ be a closed T-invariant subvariety such that $\mathcal{T}/T$ is a split del Pezzo surface and the weight hyperplane sections of $\mathcal{T}$ are exceptional divisors with multiplicity 1. Then $\mathcal{Z}$ contains $r - 3$ k-points in general position. More precisely, for any k-point z_0 of $\mathcal{Z}$ the points $(z_1, \ldots, z_{r-4}) \in \mathcal{Z}(k)^{r-4}$ such that $z_0, z_1, \ldots, z_{r-4}$ are in general position, form a dense open subset of $\mathcal{Z}^{r-4}$.*

Proof. We first note that $\mathrm{Ver}_\mu(\mathcal{Z})$ is dense in a vector subspace of $S^2_\mu(V)$ of dimension $r-3$. Indeed, assume without loss of generality that $\mu=\omega_1$. Then, as in the proof of Lemma 1.3, we have $S^2_{\omega_1}(V)=S^2_{\omega_1}(V_1)\oplus(V_2)_{\omega_1}$. The image of $t\mathcal{Z}$ consists of the points $\mathrm{Ver}_{\omega_1}(x_0^{-1}y_0u)+p_{\omega_1}(x_0^{-1}y_0u)$, where u is in $\Omega(x_0)$, by Lemma 2.2. Since Ver_{ω_1} sends $(G'/P')_{\mathrm{a}}$ to a vector space of dimension $r-3$, by Lemma 1.3, we see that $\mathrm{Ver}_{\omega_1}(t\mathcal{Z})$ is a dense subset of a vector space of this dimension. Hence the same is true for $\mathcal{Z}$.

We can choose the points $z_1,\ldots,z_{r-4}$ in $\mathcal{Z}(k)$ one by one, in such a way that z_n is in the complement to the union of the inverse images under Ver_μ of the linear span of $\mathrm{Ver}_\mu(z_i)$, $i=0,\ldots,n-1$. This complement is non-empty since $\mathrm{Ver}_\mu(\mathcal{Z})$ is a Zariski dense subset of a vector space of dimension $r-3$. □

Equations for $\mathcal{T}$ have been given by Popov [**14**] and Derenthal [**7**, **8**]. The following result gives a concise natural description of these equations, in terms of the well-known equations of $(G/P)_{\mathrm{a}}\subset V$.

Theorem 2.5. *Let $r=4$, 5, 6 or 7. Every split del Pezzo surface X of degree $9-r$ has a universal torsor $\mathcal{T}$ which is an open subset of the intersection of $r-3$ dilatations of $(G/P)_{\mathrm{a}}$ by k-points of the diagonal torus S:*

$$\mathcal{T}=\bigcap_{z\in\mathcal{Z}(\bar{k})} z^{-1}(G/P)^{\mathrm{sf}}_{\mathrm{a}}=\bigcap_{i=0}^{r-4} z_i^{-1}(G/P)^{\mathrm{sf}}_{\mathrm{a}}, \tag{2.1}$$

where $z_0=1,z_1,\ldots,z_{r-4}$ are k-points of $\mathcal{Z}\subset S$ in general position.

For $r=4,5,6,7$ the number of quadratic equations defining $\mathcal{T}\subset V^{\mathrm{sf}}$ is 5, 20, 81, 504, respectively (recall that E_7 has 126 roots). Our equations appear to be the same as those obtained in [**14**] and [**7**] except for $r=7$. This is probably because we describe $\mathcal{T}$ in an open subset V^{sf} of V, so we obtain a subset of the 529 equations for the Zariski closure of $\mathcal{T}$ in V (cf. Corollary 1.6). See [**8**], [**11**], [**20**], [**21**] and [**22**] for more details and explicit computations.

Proof of Theorem 2.5. We proceed by induction on r. For $r=4$ our statement is [**18**, Theorem 3.1]. Recall that the ideal of $(G'/P')_{\mathrm{a}}$ is generated by the quadratic forms $p_\mu(x)$, $\mu\in W'\omega_1$. Thus the induction assumption implies that I', the ideal of $\mathcal{T}'$, is the radical of the ideal generated by the weight μ components $I'_\mu=I'\cap S^2_\mu(V_1)^*$, for all $\mu\in W'\omega_1$.

By Corollary 1.2 it is enough to prove the theorem for $\mathcal{T}$ which satisfies the assumptions of Lemma 2.1. The torsor $\mathcal{T}$ is clearly contained in the closed set

$$\mathcal{S}=\bigcap_{s\in\mathcal{Z}(\bar{k})} s^{-1}(G/P)^{\mathrm{sf}}_{\mathrm{a}}\subset V^{\mathrm{sf}}.$$

Since $\mathcal{T}$ is closed in V^{sf}, for the first equality in (2.1) it is enough to prove that $\mathcal{T}$ is dense in $\mathcal{S}$. For this it is enough to show that $x_0^{-1}y_0\mathcal{T}'$ is dense in $\pi(\mathcal{S})$.

For $v \in V \otimes \bar{k}$ we write $v = (v_0, v_1, v_2, v_3)$, where $v_i \in V_i \otimes \bar{k}$. Similarly, we write $s \in S(\bar{k})$ as (s_0, s_1, s_2, s_3), where $s_i \in \mathrm{GL}(V_i \otimes \bar{k})$. In this notation the set

$$\bigcap_{s \in \mathcal{Z}(\bar{k})} \{(s_0^{-1}t, s_1^{-1}tx, s_2^{-1}tp(x), s_3^{-1}tq(x)) \mid x \in V_1 \otimes \bar{k},\ t \in \bar{k}^*\}$$

is dense in $\mathcal{S}$. This set can also be written as

$$\bigcap_{s \in \mathcal{Z}(\bar{k})} \{(t, x, (ts_0)^{-1}s_2^{-1}p(s_1x), (ts_0)^{-2}s_3^{-1}q(s_1x)) \mid x \in V_1 \otimes \bar{k},\ t \in \bar{k}^*\}.$$

Since $(1,1,1,1) \in \mathcal{Z}$, we see that $\pi(\mathcal{S})$ is contained in the set of $x \in V_1 \otimes \bar{k}$ such that for all $s \in \mathcal{Z}(\bar{k})$ we have

$$s_0^{-1}s_2^{-1}p(s_1x) = p(x).$$

Let $J \subset k[V_1^*]$ be the ideal of $x_0^{-1}y_0\mathcal{T}'$, and $J_\mu = J \cap S^2_\mu(V_1)^*$. Then $\dim I'_\mu = \dim J_\mu$, so that J_μ has codimension 2 in $S^2_\mu(V_1)^*$. Lemma 1.3 implies that the linear span L of the quadratic forms $p_\mu(yy_0^{-1}x)$ on V_1 for a fixed $y_0 \in (G'/P')^\times_{\mathrm{a}}$ and arbitrary $y \in (G'/P')^\times_{\mathrm{a}}$ has codimension 1 in $S^2_\mu(V_1)^*$ (in fact, L is the space of forms vanishing at y_0). Lemma 2.2 implies that L coincides with the linear span of the quadratic forms $p_\mu(s_1x)$, for all $s \in \mathcal{Z}(\bar{k})$. Hence the linear span of the forms $s_0^{-1}s_{2,\mu}^{-1}p_\mu(s_1x) - p_\mu(x)$, for all $s \in \mathcal{Z}(\bar{k})$, has codimension at most 2 in $S^2_\mu(V_1)^*$. However, the inclusion $x_0^{-1}y_0\mathcal{T}' \subset \pi(\mathcal{S})$ implies that this space is in J_μ, and thus coincides with J_μ. This holds for every μ, but the induction assumption implies that J is the radical of the ideal generated by the components J_μ for all $\mu \in W'\omega_1$. Hence $x_0^{-1}y_0\mathcal{T}' = \pi(\mathcal{S})$.

Let us prove the second equality in (2.1). For $r < 7$ the ideal of $(G/P)_{\mathrm{a}}$ is generated by the forms $p_\mu(x)$, $\mu \in W\omega_1$, so that the closed set $(G/P)^{\mathrm{sf}}_{\mathrm{a}} \subset V^{\mathrm{sf}}$ is given by the equations $p_\mu(x) = 0$. The last statement also holds for $r = 7$ by Corollary 1.6: here $(G/P)^{\mathrm{sf}}_{\mathrm{a}} \subset V^{\mathrm{sf}}$ is given by the equations $p_\mu(x) = 0$ for all the roots μ of $\mathfrak{g}$. If $z_0, \ldots, z_{r-4}$ are k-points of $\mathcal{Z}$ in general position, then the quadratic forms $p_\mu(z_iu)$ span a subspace of dimension $r-3$ in $I_\mu = I \cap S^2_\mu(V^*)$, where I is the ideal of $\mathcal{T}$. But $\dim I_\mu = r-3$, hence I_μ is generated by the $p_\mu(z_iu)$, $i = 0, \ldots, r-4$. The already proved first equality in (2.1) implies that I is the radical of the ideal generated by the I_μ. This proves the second equality in (2.1), and so completes the proof of the theorem. □

Remark. In the case $r = 5$ the general position condition has a simple geometric meaning. By the last claim of Lemma 2.2 we have $\mathcal{T}^\times = s\mathcal{Z}$ for some $s \in S(k)$ well defined up to an element of $T(k)$. If $\mathcal{T} \subset (G/P)^{\mathrm{sf}}_{\mathrm{a}}$, then $\mathcal{Z}$ contains 1, so that s is a k-point of $\mathcal{T}^\times$. Then the previous theorem implies

$$\mathcal{T}^\times = s\mathcal{Z} = (G/P)^\times_{\mathrm{a}} \cap r^{-1}s(G/P)^\times_{\mathrm{a}}, \quad \text{so that } \mathcal{T} = (G/P)^{\mathrm{sf}}_{\mathrm{a}} \cap r^{-1}s(G/P)^{\mathrm{sf}}_{\mathrm{a}}, \tag{2.2}$$

where r is a k-point in $\mathcal{T}^\times$ such that $f(s)$ and $f(r)$ are points in $X^\times$ not contained in a conic on X (cf. diagram (1.3)). Here $f(s)$ is uniquely determined by $\mathcal{T}$, whereas $f(r)$ can be any point of $X^\times$ outside the 10 conics through $f(s)$.

This remark can be seen as a particular case of the following description of $\mathcal{Z}$. For any g and h in $\mathcal{T}^\times(k)$ such that $\mathrm{Ver}_\mu(h)$ and $\mathrm{Ver}_\mu(g)$ are not proportional for any $\mu \in W\omega_1$, we have $\mathcal{Z} = g^{-1}(G/P)^\times_{\mathrm{a}} \cap h^{-1}(G/P)^\times_{\mathrm{a}}$. The proof is similar to that of Theorem 2.5; we omit it here since we shall not need this fact.

To construct $r-3$ points in $\mathcal{Z}$ in general position is not hard, because the points of $\mathcal{Z}$ are parametrized by polynomials. Indeed, decompose $V_1 = V_{1,0} \oplus V_{1,1} \oplus V_{1,2}$ similarly to (1.2), and consider the points $t^{-1}\exp(x_0^{-1}y_0\exp(v_i))$, where $v_1,\ldots,v_{r-3}$ in $V_{1,1}$ satisfy certain open conditions which are easy to write down using Lemma 2.2.

3. A uniqueness result

Before we proceed to the main result of this section (Proposition 3.4) we prove an auxiliary statement concerning the divisors on del Pezzo surfaces.

We continue to assume that X is a split del Pezzo surface of degree $9-r$. Let $X^\times$ be the complement to the union of exceptional curves on X. The group $\mathrm{Div}_{X\setminus X^\times}(X)$ of divisors supported in $X \setminus X^\times$ is freely generated by the exceptional curves. The kernel of the natural surjective map $\mathrm{Div}_{X\setminus X^\times}(X) \to \mathrm{Pic}\, X$ is the subgroup of principal divisors supported in $X\setminus X^\times$, and so is identified with $k[X^\times]^*/k^*$. Thus we have an exact sequence

$$0 \to k[X^\times]^*/k^* \to \mathrm{Div}_{X\setminus X^\times}(X) \to \mathrm{Pic}\, X \to 0. \tag{3.1}$$

The elements of the set Λ of weights of T in V are in a canonical bijection with the set of exceptional curves on X. For $\lambda \in \Lambda$ we denote by ℓ_λ the exceptional curve corresponding to λ. Let X_λ be the del Pezzo surface of degree $10-r$ obtained from X by contracting ℓ_λ. The morphism $X \to X_\lambda$ defines the injective maps

$$\mathrm{Div}_{X_\lambda\setminus X_\lambda^\times}(X_\lambda) \to \mathrm{Div}_{X\setminus X^\times}(X), \qquad \mathrm{Pic}\, X_\lambda \to \mathrm{Pic}\, X,$$

and identifies $X^\times$ with an open subset of $X_\lambda^\times$. We obtain the following commutative diagram of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \prod_{\lambda\in\Lambda} k[X_\lambda^\times]^*/k^* \arrow[r] \arrow[d] & \prod_{\lambda\in\Lambda} \mathrm{Div}_{X_\lambda\setminus X_\lambda^\times}(X_\lambda) \arrow[r] \arrow[d] & \prod_{\lambda\in\Lambda} \mathrm{Pic}\, X_\lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k[X^\times]^*/k^* \arrow[r] & \mathrm{Div}_{X\setminus X^\times}(X) \arrow[r] & \mathrm{Pic}\, X \arrow[r] & 0
\end{tikzcd} \tag{3.2}$$

Lemma 3.1. *When $4 \leqslant r \leqslant 7$ the left-hand vertical map in (3.2) is surjective.*

Proof. The intersection index defines an integral bilinear form on $\mathrm{Div}_{X\setminus X^\times}(X)$ whose kernel is $k[X^\times]^*/k^*$, so that the induced form on $\mathrm{Pic}\, X$ is non-degenerate. We note that any exceptional curve ℓ_λ can be included into the support of a principal divisor as follows: if $\ell_{\lambda'}$ is an exceptional curve such that $(\ell_{\lambda'}.\ell_\lambda) = 1$, then $\ell_\lambda + \ell_{\lambda'}$ is a degenerate fibre of a pencil of conics on X. Any such pencil has $r-1 \geqslant 3$ degenerate fibres each consisting of a pair of exceptional curves intersecting transversally at one point. Let $\ell_1 + \ell_2$ and

$\ell_\nu+\ell_{\nu'}$ be two more degenerate fibres of the same pencil. Then $\ell_\lambda+\ell_{\lambda'}-\ell_1-\ell_2$ is in $k[X_\nu^\times]^*/k^*$. Moreover, ℓ_1 and ℓ_2 are in $\mathrm{Div}_{X_\lambda\setminus X_\lambda^\times}(X_\lambda)$.

Choose any weight λ. If $r\leqslant 6$, then any two intersecting exceptional curves intersect transversally at one point. Thus any element $x\in k[X^\times]^*/k^*$ can be written as $x=m\ell_\lambda+\sum m_i\ell_i+D$, where $m,m_i\in\boldsymbol{Z}$, the divisor D is in $\mathrm{Div}_{X_\lambda\setminus X_\lambda^\times}(X_\lambda)$, and $(\ell_\lambda.\ell_i)=1$ for all i. Since $(\ell_\lambda^2)=-1$ we have $m=\sum m_i$. The argument of the previous paragraph shows that x is a sum of elements from $k[X_\nu^\times]^*/k^*$, for some $\nu\in\Lambda$, and a principal divisor in $\mathrm{Div}_{X_\lambda\setminus X_\lambda^\times}(X_\lambda)$. By injectivity of the map $\mathrm{Pic}\,X_\lambda\to\mathrm{Pic}\,X$ this principal divisor comes from $k[X_\lambda^\times]^*/k^*$.

If $r=7$ there is exactly one exceptional curve $\ell_{\lambda'}$ such that $(\ell_\lambda.\ell_{\lambda'})=2$. Modifying our x by an element from some $k[X_\nu^\times]^*/k^*$ we can arrange that $\ell_{\lambda'}$ is not in the support of x. Then we conclude as above. □

We denote by R the torus S/T, so that we have an exact sequence of split k-tori:

$$1\to T\to S\to R\to 1. \tag{3.3}$$

Our construction identifies $\hat{S}$ with the free abelian group $\mathrm{Div}_{X\setminus X^\times}(X)$. The type of the universal torsor $\mathcal{T}\to X$ is an isomorphism of free abelian groups $\hat{T}\xrightarrow{\sim}\mathrm{Pic}\,X$. Thus (3.3) is the dual sequence of (3.1). Let R^λ, S^λ, T^λ be tori whose groups of characters are $k[X_\lambda^\times]^*/k^*$, $\mathrm{Div}_{X_\lambda\setminus X_\lambda^\times}(X_\lambda)$, $\mathrm{Pic}\,X_\lambda$, respectively. Dualizing diagram (3.2) we obtain the following commutative diagram of tori:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & S & \longrightarrow & R & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \prod_{\lambda\in\Lambda}T^\lambda & \longrightarrow & \prod_{\lambda\in\Lambda}S^\lambda & \longrightarrow & \prod_{\lambda\in\Lambda}R^\lambda & \longrightarrow & 1 \end{array} \tag{3.4}$$

Corollary 3.2. *If $s\in S(\bar{k})$ is such that for every $\lambda\in\Lambda$ the image of s in S^λ belongs to T^λ, then $s\in T(\bar{k})$.*

Proof. By Lemma 3.1 the right-hand map in (3.4) is injective. □

After this digression we return to universal torsors over del Pezzo surfaces and their embeddings into homogeneous spaces.

The choice of y_0 plays the role of a 'normalization' for the embedding of the torsor $\mathcal{T}$ into $(G/P)_{\mathrm{a}}$. It is convenient to choose these normalizations in a coherent way. Let $M_1,\dots,M_r$ be k-points in general position in $\mathbb{P}^2_k$, and let X_r be the blowing-up of $\mathbb{P}^2_k$ in $M_1,\dots,M_r$. We can identify $X_r^\times$ with an open subset $\mathcal{U}\subset\mathbb{P}^2_k$. Choose $u_0\in\mathcal{U}(k)$. At every step of our inductive process we can choose the points y_0 in the fibre of $\mathcal{T}'\to X'$ over u_0. Thus we get a compatible family of the y_0 (more precisely, of torus orbits) that are mapped to each other by the surjective maps $\mathcal{T}\to\mathcal{T}'$. In our previous notation, the point $t=\exp(x_0^{-1}y_0^2)$ must be taken for the point y_0 of the next step.

If A is a subset of the torus S, then we denote by $\mathrm{P}^n(A)\subset S$ the set of products of n elements of A in S. We define $\mathrm{P}^0(A)=T$.

Proposition 3.3. *Let r and n be integers satisfying $4 \leqslant r \leqslant 7$, $0 \leqslant n \leqslant r-4$. Under the assumptions of Lemma 2.1, if at every step of our construction we choose the points y_0 over a fixed point of $\mathcal{U}$, then we have the following statements:*

(i) $\mathrm{P}^{n+1}(t^{-1}\mathcal{T}^\times) \subset t^{-1}(G/P)_{\mathrm{a}}^\times$;

(ii) $\mathrm{P}^{n}(t^{-1}\mathcal{T}^\times) \subset \mathcal{Z}$.

Proof. Parts (i) and (ii) of the proposition are clearly equivalent. For $n=0$ the inclusion (i) is the main theorem of [**16**], and this finishes the case $r=4$. Let $n \geqslant 1$. Recall that the projection π maps $t^{-1}\mathcal{T}$ onto $y_0^{-1}\mathcal{T}'$. Assume that we have the desired inclusions for $n-1$ and for both torsors $\mathcal{T}'$ and $\mathcal{T}$, namely

$$\mathrm{P}^{n}(t^{-1}\mathcal{T}^\times) \subset t^{-1}(G/P)_{\mathrm{a}}^\times, \qquad \mathrm{P}^{n}(y_0^{-1}\mathcal{T}'^\times) \subset y_0^{-1}(G'/P')_{\mathrm{a}}^\times.$$

By Lemma 4.1 of [**16**] every $\bar{k}$-point of $(G/P)_{\mathrm{a}}^\times$ can be written as $g_x \cdot \exp(v)$, where $x \in \bar{k}^*$, and $v \in V_1 \otimes \bar{k}$. By the first inclusion in induction assumption this is also true for elements of $t\,\mathrm{P}^{n}(t^{-1}\mathcal{T}^\times)$. Since $\mathcal{T}$ is g_x-invariant, we have $\exp(v) \in t\,\mathrm{P}^{n}(t^{-1}\mathcal{T}^\times)$. On applying π to both sides we deduce $v \in x_0^{-1}y_0^2\,\mathrm{P}^{n}(y_0^{-1}\mathcal{T}'^\times)$. Applying the second inclusion in induction assumption we obtain $v \in x_0^{-1}y_0(G'/P')_{\mathrm{a}}^\times$. Therefore, $\mathrm{P}^{n}(t^{-1}\mathcal{T}^\times)$ is contained in the affine cone over $t^{-1}\exp(x_0^{-1}y_0(G'/P')_{\mathrm{a}}^\times)$. But this implies $\mathrm{P}^{n}(t^{-1}\mathcal{T}^\times) \subset \mathcal{Z}$, since $t\mathcal{Z}$ is the intersection of the affine cone over $\exp(x_0^{-1}y_0(G'/P')_{\mathrm{a}}^\times)$ with $V^\times$, by Lemma 2.2. This proves (ii), and hence also (i). □

Proposition 3.4. *Let $r = 4, 5, 6$ or 7, and let $\mathcal{T} \subset V^{\mathrm{sf}}$ be a closed T-invariant subvariety such that $X = \mathcal{T}/T$ is a split del Pezzo surface of degree $9-r$, and the weight hyperplane sections of $\mathcal{T}$ are exceptional divisors with multiplicity 1. Let $\mathcal{Z} \subset S$ be the closed subset of points z such that $z\mathcal{T} \subset (G/P)_{\mathrm{a}}$. Then there is a unique $s \in S(k)$ defined up to an element of $T(k)$, such that $\mathrm{P}^{r-4}(\mathcal{T}^\times) \subset s\mathcal{Z}$.*

Proof. By Corollary 1.2, up to replacing $\mathcal{T}$ by its dilatation by an element of $S(k)$, we can assume that $\mathcal{T} \subset (G/P)_{\mathrm{a}}$ is constructed as in Lemma 2.1. Thus the existence of s follows from Proposition 3.3. We prove the uniqueness by induction on r. For $r=4$ the only elements of S that leave $\mathrm{Gr}(2,5)$ invariant are the elements of T, so that $\mathcal{Z} = T$ in this case, hence our statement is clear.

Now let us assume that $r \geqslant 5$. By Lemma 2.2 the inclusion $\mathrm{P}^{r-4}(t^{-1}\mathcal{T}^\times) \subset s\mathcal{Z}$ implies that $\mathrm{P}^{r-4}(y_0^{-1}\mathcal{T}'^\times)$ is contained in $\pi(s)y_0^{-1}(G'/P')_{\mathrm{a}}^\times$, from which it follows that $\mathcal{T}'^\times \cdot \mathrm{P}^{r-5}(y_0^{-1}\mathcal{T}'^\times) \subset \pi(s)(G'/P')_{\mathrm{a}}^\times$. By the definition of $\mathcal{Z}'$ this implies $\mathrm{P}^{r-5}(y_0^{-1}\mathcal{T}'^\times) \subset \pi(s)\mathcal{Z}'$. By induction assumption $\pi(s)$ is unique up to an element of $T'(k)$.

Recall that $\mathcal{T}'$ is a universal torsor over the surface X' obtained from X by blowing down an exceptional curve, say ℓ_λ in the notation of the beginning of this section. In this notation we have $X' = X_\lambda$ and $T' = T^\lambda$. The argument in the previous paragraph works for any weight $\lambda \in \Lambda$. Thus Corollary 3.2 implies that our s is unique up to an element of $T(k)$. □

Remark. For $r=5$ the inclusion $\mathrm{P}^{r-4}(\mathcal{T}^\times) \subset s\mathcal{Z}$ is an equality by the last claim of Lemma 2.2, but this is no longer so for $r=6$ or 7, for dimension reasons.

4. Non-split del Pezzo surfaces

Let $\varGamma = \mathrm{Gal}(\bar{k}/k)$. Let G be a split simply connected semisimple group over k with a split maximal k-torus H and the root system R. Let N be the normalizer of H in G, and let $W = N/H$ be the Weyl group. The action of N by conjugation gives rise to an action of W on the torus H. Since H is split, the Galois group $\varGamma$ acts trivially on W. Thus the continuous 1-cocycles of $\varGamma$ with values in W are homomorphisms $\varGamma \to W$, and the elements of $\mathrm{H}^1(k, W)$ are homomorphisms $\varGamma \to W$ considered up to conjugation in W.

Theorem 4.1 (Gille–Raghunathan). *For any $\sigma \in \mathrm{Hom}(\varGamma, W)$ the twisted torus H_σ is isomorphic to a maximal torus of G.*

Proof. See [**9**, Theorem 5.1 (b)] or [**15**, Theorem 1.1]. □

Recall from [**17**, I.5.4] that we have an exact sequence of pointed sets

$$1 \to N(k) \to G(k) \to (G/N)(k) \xrightarrow{\varphi} \mathrm{H}^1(k, N) \to \mathrm{H}^1(k, G).$$

(Note by the way that the last map here is known to be surjective.) The homogeneous space G/N is the variety of maximal tori of G, so that an equivalent form of the Gille–Raghunathan theorem is the surjectivity of the composite map

$$(G/N)(k) \to \mathrm{H}^1(k, N) \to \mathrm{H}^1(k, W) = \mathrm{Hom}(\varGamma, W)/\mathrm{conj}\,.$$

We fix an embedding of H_σ as a maximal torus of G, this produces a k-point $[H_\sigma]$ in G/N. The choice of a $\bar{k}$-point g_0 in G such that $g_0 H g_0^{-1} = H_\sigma$ defines a 1-cocycle $\rho : \varGamma \to N(\bar{k})$, $\rho(\gamma) = g_0^{-1} \cdot {}^\gamma g_0$, which is a lifting of $\sigma \in Z^1(k, W) = \mathrm{Hom}(\varGamma, W)$. We have $[\rho] = \varphi[H_\sigma]$, see [**17**, I.5.4], moreover, the image of $[\rho]$ in $\mathrm{H}^1(k, G)$ is trivial.

Let $G \to \mathrm{GL}(V)$ be an irreducible representation of G. Define $T \subset \mathrm{GL}(V)$ as the torus generated by H and the scalar matrices $\boldsymbol{G}_m$. The group N acts by conjugation on T. The twisted torus T_σ is an extension of H_σ by $\boldsymbol{G}_m$.

Let $(G/P)_{\mathrm{a}} \subset V$ be the orbit of the highest weight vector (with zero added to it); $P \subset G$ is a parabolic subgroup, and $(G/P)_{\mathrm{a}}$ is the affine cone over G/P. The maximal torus $H_\sigma \subset G$ acts on $(G/P)_{\mathrm{a}}$, and so does T_σ. Define $(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma}$ to be the dense open subset of $(G/P)_{\mathrm{a}}$ consisting of the points with closed H_σ-orbits and trivial stabilizers in T_σ.

The group $N \subset G$ acts on V preserving V^{sf} and $V^\times$, thus giving rise to the action of W on V^{sf}/T and on $V^\times/T$ by automorphisms of algebraic varieties (not necessarily preserving some group structure on $V^\times/T$). The action of N preserves $(G/P)_{\mathrm{a}}^{\mathrm{sf}} \subset V$, thus W acts on $Y = (G/P)_{\mathrm{a}}^{\mathrm{sf}}/T$. Hence we define the twisted forms $(V^{\mathrm{sf}}/T)_\sigma$, $(V^\times/T)_\sigma$ and Y_σ. The variety $(V^\times/T)_\sigma$ is an open subset of the quasi-projective toric variety $(V^{\mathrm{sf}}/T)_\sigma$ which contains Y_σ as a closed subset.

Lemma 4.2. *The k-varieties Y_σ and $(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma}/T_\sigma$ are isomorphic.*

Proof. Recall that $g_0 \in G(\bar{k})$ is a point such that $\rho(\gamma) = g_0^{-1} \cdot {}^\gamma g_0 \in Z^1(k, N)$ is a cocycle that lifts $\sigma \in Z^1(k, W) = \mathrm{Hom}(\varGamma, W)$. The image of ρ in $Z^1(k, G)$ is a coboundary, so that the inner form G_ρ is isomorphic to G, and the twisted space $(G/P)_{a,\rho}$ is isomorphic to $(G/P)_a$. The map $x \mapsto g_0 x$ on $\bar{k}$-points of $(G/P)_a$ gives rise to an isomorphism of k-varieties $(G/P)_{a,\rho} \xrightarrow{\sim} (G/P)_a$ compatible with the isomorphism $G_\rho \xrightarrow{\sim} G$ that sends g to $g_0 g g_0^{-1}$. The embedding $H \hookrightarrow G$ gives rise to an embedding $H_\sigma = H_\rho \hookrightarrow G_\rho$, so that T_σ acts on $(G/P)_{a,\rho}$ on the left. We obtain a T_σ-equivariant isomorphism $(G/P)_{a,\rho} \xrightarrow{\sim} (G/P)_a$.

Let $(G/P)_{a,\rho}^{\mathrm{sf}}$ be the subset of $(G/P)_{a,\rho}$ consisting of the points with closed H_σ-orbits with trivial stabilizers in T_σ. The closedness of orbits and the triviality of stabilizers are conditions on $\bar{k}$-points, hence we obtain a T_σ-equivariant k-isomorphism $(G/P)_{a,\rho}^{\mathrm{sf}} \xrightarrow{\sim} (G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma}$. It descends to an isomorphism $Y_\sigma \xrightarrow{\sim} (G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma}/T_\sigma$. □

Corollary 4.3. *For any homomorphism $\sigma : \Gamma \to W$ the twisted variety Y_σ has a k-point, and so does $(V^\times/T)_\sigma$, so that $(V^\times/T)_\sigma \simeq R_\sigma$.*

Proof. Since k is an infinite field, any dense open subset of $(G/P)_{\mathrm{a}}$ contains k-points. Thus $Y_\sigma^\times(k) \neq \emptyset$, but this is a subset of $(V^\times/T)_\sigma$, so that this variety also has a k-point. □

Remark. This approach via the Gille–Raghunathan theorem generalizes a key ingredient in the second author's proof of the Enriques–Swinnerton-Dyer theorem that every del Pezzo surface of degree 5 has a k-point, from quotients of Grassmannians by the action of a maximal torus to quotients of homogeneous spaces of quasi-split semisimple groups [**19**].

We now assume that R is the root systems of rank r in (1.1), and that the highest weight of the G-module V is the fundamental weight dual to the root indicated in (1.1). Then V is minuscule, so that the centralizer S of H in $\mathrm{GL}(V)$ is a torus. Let $R = S/T$. The group N acts by conjugation on T and hence also on S and R. The connected component of 1 acts trivially, so we obtain an action of W on these tori (preserving the group structure). On twisting T, S and R by σ we obtain an exact sequence of k-tori:

$$1 \to T_\sigma \to S_\sigma \to R_\sigma \to 1. \tag{4.1}$$

Note in passing that the character group $\hat{S}$ has an obvious W-invariant basis, which gives rise to a Galois invariant basis of $\hat{S}_\sigma$. In other words, S_σ is a quasi-trivial torus; in particular, $\mathrm{H}^1(k, S_\sigma) = \{1\}$ as follows from Hilbert's Theorem 90. Note also that $V^\times/T$ is a torsor under R, so that $(V^\times/T)_\sigma$ is a torsor under R_σ. By Corollary 4.3 this torsor is trivial, that is, there is a (non-canonical) isomorphism $(V^\times/T)_\sigma \simeq R_\sigma$.

Let X be a del Pezzo surface over k, not necessarily split, of degree $9-r$, where r is the rank of the root system R. Let $\bar{X}$ be the surface obtained from X by extending the ground field from k to $\bar{k}$. Our construction identifies $\hat{S}$ with the free abelian group $\mathrm{Div}_{\bar{X}\setminus\bar{X}^\times}(\bar{X})$ generated by the exceptional curves on $\bar{X}$, and $\hat{T}$ with $\mathrm{Pic}\,\bar{X}$ (via the type of the universal torsor $\mathcal{T} \to X$). The Galois group permutes the exceptional curves on $\bar{X}$, thus defining a homomorphism $\sigma_X : \Gamma \to W$, where W is the Weyl group of R. This homomorphism is well defined up to conjugation in W, so we have a well defined class $[\sigma_X] \in \mathrm{H}^1(\Gamma, W)$, where Γ acts trivially on W.

We now assume $\sigma = \sigma_X$. Then we get isomorphisms of Γ-modules

$$\hat{S}_\sigma = \mathrm{Div}_{\bar{X}\setminus\bar{X}^\times}(\bar{X}), \qquad \hat{T}_\sigma = \mathrm{Pic}\,\bar{X},$$

thus T_σ is the Néron–Severi torus of X. The dual sequence of (4.1) coincides with the natural exact sequence of Γ-modules

$$0 \to \bar{k}[X^\times]^*/\bar{k}^* \to \mathrm{Div}_{\bar{X}\setminus\bar{X}^\times}(\bar{X}) \to \mathrm{Pic}\,\bar{X} \to 0.$$

There is a natural bijection between the morphisms $X^\times \to R_\sigma$ and the homomorphisms of Γ-modules $\hat{R}_\sigma \to \bar{k}[X^\times]^*$. Universal torsors on X exist if and only if the exact sequence of Γ-modules

$$1 \to \bar{k}^* \to \bar{k}[X^\times]^* \to \bar{k}[X^\times]^*/\bar{k}^* \to 1 \tag{4.2}$$

is split [**18**, Corollary 2.3.10]. Any splitting of this sequence gives a map

$$\hat{R}_\sigma = \bar{k}[R_\sigma]^*/\bar{k}^* = \bar{k}[X^\times]^*/\bar{k}^* \to \bar{k}[X^\times]^*,$$

and hence defines a morphism $\phi : X^\times \to R_\sigma$. By the 'local description of torsors' (see [**4**, 2.3] or [**18**, Theorem 4.3.1]) the restriction of a universal X-torsor to $X^\times$ is the pull-back of the torsor $S_\sigma \to R_\sigma$ to $X^\times$ via ϕ. Moreover, this gives a bijection between the splittings of (4.2) and the universal X-torsors. In our case it is easy to see that ϕ is an embedding. The isomorphism $\hat{R}_\sigma = \bar{k}[X^\times]^*/\bar{k}^*$ comes from our construction, thus after extending the ground field to $\bar{k}$, the morphism ϕ coincides, up to translation by a $\bar{k}$-point of R, with the embedding of $\bar{X}^\times$ into $(V \otimes_k \bar{k})^\times/\bar{T}$ obtained from the embedding $\bar{\mathcal{T}}^\times \subset \bar{V}^\times$.

Theorem 4.4. *Let $r = 4$, 5, 6 or 7. Let X be a del Pezzo surface of degree $9-r$ with a k-point, and let $\sigma \in \mathrm{H}^1(\Gamma, W)$ be the class defined by the action of the Galois group on the exceptional curves of X. There exists an embedding $X \hookrightarrow Y_\sigma$ such that the divisors in $Y_\sigma \setminus Y_\sigma^\times$ cut the exceptional curves on X with multiplicity* 1. *The restriction of $(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma} \to Y_\sigma$ to $X \subset Y_\sigma$ is a universal X-torsor whose type is the isomorphism* $\hat{T}_\sigma = \mathrm{Pic}\,\bar{X}$.

Proof. A del Pezzo surface X of degree 4 with a k-point is unirational, i.e. X is dominated by a k-rational variety (see Chapter IV, Theorem 29.4 and Theorem 30.1 in [**12**]). Therefore, k-points are Zariski dense in X, in particular, $X^\times(k) \neq \emptyset$.

From Corollary 4.3 we get an embedding $Y_\sigma^\times \hookrightarrow R_\sigma$, which becomes unique if we further assume that a given k-point of $Y_\sigma^\times$ goes to the identity element of R_σ.

Since $X(k) \neq \emptyset$, there is a unique embedding $\phi : X^\times \to R_\sigma$ such that the induced map $\phi^* : \hat{R}_\sigma \to \bar{k}[X^\times]^*$ is a lifting of the isomorphism $\hat{R}_\sigma = \bar{k}[R_\sigma]^*/\bar{k} \xrightarrow{\sim} \bar{k}[X^\times]^*/\bar{k}^*$, and ϕ sends a given k-point of $X^\times$ to 1.

Let $\mathcal{L}$ be the k-subvariety of the torus R_σ whose points are $r \in R_\sigma(\bar{k})$ such that $rX^\times \subset Y_\sigma^\times$, where the multiplication is the group law of R_σ. To prove the first statement we need to show that $\mathcal{L}(k) \neq \emptyset$. Let $\mathrm{P}^n(X^\times)$ be the k-subvariety of R_σ whose $\bar{k}$-points are products of n elements of $X^\times(\bar{k})$ in $R_\sigma(\bar{k})$. The surface $\bar{X}$ is split, hence it follows from Proposition 3.4 that there exists a unique $c \in R_\sigma(\bar{k})$ such that $\mathrm{P}^{r-4}(X^\times)(\bar{k}) \subset c\mathcal{L}(\bar{k})$.

But since $\mathrm{P}^{r-4}(X^\times)$ and $\mathcal{L}$ are subvarieties of R_σ defined over k we conclude that c is a k-point. If m is a k-point of $X^\times$, then $c^{-1}m^{r-4}$ is a k-point of $\mathcal{L}$, as required.

To check that the restriction of $(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma} \to Y_\sigma$ to $X \subset Y_\sigma$ is a universal torsor we can go over to $\bar{k}$ where it follows from our main theorem in the split case. □

Remark. Let X be a del Pezzo surface with a k-point, of degree 5, 4, 3 or 2. Although $(G/P)_{\mathrm{a}}$ contains some universal X-torsor, other universal X-torsors of the same type are naturally embedded into certain twists of $(G/P)_{\mathrm{a}}$. Indeed, all torsors of the same type are obtained from any of them by twisting by the cocycles in $Z^1(k, T_\sigma)$. The natural map $H_\sigma \to T_\sigma$ gives a surjection $\mathrm{H}^1(k, H_\sigma) \to \mathrm{H}^1(k, T_\sigma)$ since $\mathrm{H}^1(k, \boldsymbol{G}_m) = \{1\}$ by Hilbert's Theorem 90. Therefore, it is enough to consider the twists of $\mathcal{T} \subset (G/P)_{\mathrm{a}}$ by cocycles $\theta \in Z^1(k, H_\sigma)$. The twisted torsor $\mathcal{T}_\theta$ is contained in the twist of $(G/P)_{\mathrm{a}}$ by the 1-cocycle in $Z^1(k, G)$ coming from $\theta \in Z^1(k, H_\sigma)$. By general theory [**17**, I.5] the twisted variety ${}_\theta(G/P)_{\mathrm{a}}$ is a left homogeneous space of the inner form of G defined by θ. (We note for the sake of completeness that by Steinberg's theorem every class in $\mathrm{H}^1(k, G)$ comes from $\mathrm{H}^1(k, H_\sigma)$ for some maximal torus $H_\sigma \subset G$. Note also that since $\mathrm{H}^1(k, \mathrm{SL}(n)) = \{1\}$ all twisted forms of $\mathrm{Gr}(2,5)$ that appear in out context are isomorphic to $\mathrm{Gr}(2,5)$.) The twisted form ${}_\theta(G/P)_{\mathrm{a}}$ naturally embeds into the twist of V by the image of the cocycle θ in $Z^1(k, S_\sigma)$. Since $\mathrm{H}^1(k, S_\sigma) = \{1\}$ this is a vector space (non-canonically) isomorphic to V and acted on by S_σ.

Corollary 4.5. *Let X be a del Pezzo surface of degree 4 such that universal X-torsors exist. Let $\sigma \in \mathrm{H}^1(\Gamma, W)$ be the class defined by the action of the Galois group on the exceptional curves of X.*

(i) *$X^\times$ and $Y_\sigma^\times$ are k-subvarieties of R_σ. Moreover, Y_σ contains cX for some $c \in R_\sigma(k)$ if and only if X has a k-point.*

(ii) *If $X \subset Y_\sigma$, then $X = Y_\sigma \cap cd^{-1}Y_\sigma$ for some $c, d \in X^\times(k)$. More precisely, c is the same as in the proof of Theorem 4.4, and d is any k-point of $X^\times$ which is not on a conic through c.*

Proof. (i) In view of Theorem 4.4 it remains to prove the 'only if' part. We note that $Y_\sigma^\times$ embeds into R_σ by Corollary 4.3. The existence of universal X-torsors implies that $X^\times$ embeds into R_σ, as was discussed before Theorem 4.4. For $r = 5$ the inclusion $X^\times \subset c\mathcal{L}$ from the proof of Theorem 4.4 is an equality by the remark in the end of § 3. Hence if $\mathcal{L}$ has a k-point, then so does $X^\times$.

(ii) By unirationality of X the set $X(k)$ is Zariski dense in X, and hence $\mathcal{L}(k) = c^{-1}X^\times(k)$ is Zariski dense in $\mathcal{L}$. Since $X \subset Y_\sigma$, the variety X is contained in $Y_\sigma \cap cd^{-1}Y_\sigma$ for any $d \in X^\times(k)$, and this inclusion is an equality for any d not on a conic in X that passes through c, see the remark after the proof of Theorem 2.5. □

Corollary 4.6. *Let X be a del Pezzo surface of degree 4 with a k-point.*

(i) *X has a universal torsor which is the intersection*

$$(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma} \cap s(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma},$$

where s is a k-point of S_σ whose image in R_σ is c/d for some $c, d \in X^\times(k)$.

(ii) *Let $\mathcal{T}$ be any universal X-torsor. Then there exists a cocycle $\theta \in Z^1(k, H_\sigma)$ such that*

$$\mathcal{T} = {}_\theta(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma} \cap s \cdot {}_\theta(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma},$$

where s is as in part (i).

Proof. By Theorem 4.4 the inverse image of $X \subset Y_\sigma$ in $(G/P)_{\mathrm{a}}^{\mathrm{sf},\sigma}$ is a universal X-torsor. By part (ii) of Corollary 4.5 it remains to show that we can choose $d \in X^\times(k)$ not on a conic through c with the additional condition that c/d is in the image of $S_\sigma(k)$ in $R_\sigma(k)$. Let $\tilde{X} \subset \mathbb{P}_k^3$ be the cubic surface obtained by blowing-up c in X. The exceptional divisor $E \subset \tilde{X}$ is a line in $\mathbb{P}_k^3$, and the proper transforms of the 10 conics through c on X are the 10 lines in $\tilde{X}$ that meet E. Let $x \in E(k)$ be a point that does not belong to these 10 lines. The intersection of the tangent plane $T_{\tilde{X},x}$ with $\tilde{X}$ is the union of E and a geometrically integral conic C. Since C contains the k-point x we have $C \simeq \mathbb{P}_k^1$. Taking the projection to X we construct a morphism $\psi : \mathbb{A}_k^1 \to X$ such that $\psi(0) = c$ and $\psi(\mathbb{A}_k^1 \setminus \{0\})$ does not meet the 10 conics through c on X. It is well known and easy to prove that $\mathrm{H}^1(\mathbb{A}_k^1, T_\sigma) = \mathrm{H}^1(k, T_\sigma)$, thus for any k-point $d \in \psi(\mathbb{A}_k^1(k)) \cap X^\times$ the class $[\mathcal{T}_d] \in \mathrm{H}^1(k, T_\sigma)$ of the fibre of $\mathcal{T} \to X$ at d equals $[\mathcal{T}_c]$. Thus the map $R_\sigma(k) \to \mathrm{H}^1(k, T_\sigma)$ defined by the exact sequence (4.1) sends c/d to zero, so that c/d is in the image of $S_\sigma(k)$ in $R_\sigma(k)$.

Since every universal X-torsor can be obtained from any other torsor of the same type by twisting by some cocycle $\theta \in Z^1(k, H_\sigma)$ (cf. the remark after Theorem 4.4), part (ii) is a consequence of part (i). □

Thus any universal torsor over X is an open subset of the intersection of two k-dilatations of ${}_\theta(G/P)_{\mathrm{a}}$ for some cocycle $\theta \in Z^1(k, H_\sigma)$.

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