LOCAL-GLOBAL PRINCIPLE FOR RATIONAL POINTS AND ZERO-CYCLES
ARIZONA WINTER SCHOOL 2015

JEAN-LOUIS COLLIOT-THÉLÈNE

ABSTRACT. At the AWS 1999, I discussed work on the Hasse principle achieved during the previous decade. The present lectures are meant as an introduction to significant progress achieved during the last 5 years, in particular to work of Browning, Harpaz, Liang, Matthiesen, Skorobogatov, Wittenberg.

The text, which in part uses some earlier notes of mine, is not in final form. It is mainly a survey, but an attempt has been made at giving proofs in suitably chosen simple cases, for example in section 7.4.

CONTENTS

Date: 22nd March 2015; some misprints corrected, subsection 7.4 reorganized, 30th May 2015.

Some introductions to the topic

J.-L. Colliot-Thélène, L'arithmétique des variétés rationnelles, Annales scientifiques de Toulouse (1992).

J.-L. Colliot-Thélène, The Hasse principle in a pencil of algebraic varieties, Contemp. Math. **210** (1998), 19–39.

J.-L. Colliot-Thélène, Points rationnels sur les fibrations (Chapter I), Budapest conference, Springer, Berlin 2003.

J.-L. Colliot-Thélène, The Brauer-Manin obstruction and the fibration method, Lecture at Bremen 2005.
http://www.math.u-psud.fr/~colliot/CTBremenBrauerplusJuly2012.pdf

J.-L. Colliot-Thélène, Notes sur le groupe de Brauer.
http://www.math.u-psud.fr/~colliot/CTnotesBrauer.pdf

D. Harari, Zéro-cycles et points rationnels sur les fibrations en variétés rationnellement connexes (d'après Harpaz et Wittenberg) , séminaire Bourbaki, janvier 2015 (version 28/1/2015).
Vidéo : http://www.youtube.com/watch?v=6VFdk9UD54Y

E. Peyre, Obstructions au principe de Hasse et à l'approximation faible, dans Séminaire Bourbaki 56-ème année, exposé 931, Astérisque 299 (2005), 165–193.

A. N. Skorobogatov, Torsors and rational points, Cambridge tracts in mathematics **144** (2001).

A. N. Skorobogatov, Arithmetic geometry : rational points, TCC lecture notes, 2013.
https://wwwf.imperial.ac.uk/~anskor/arith_geom_files/TCCnotes.pdf

A video :

https://video.ias.edu/jointiasnts/2014/0410-AlexeiSkorobogatov
T. Browning, A survey of applications of the circle method to rational points, https://www.maths.bris.ac.uk/~matdb/preprints/him.pdf

Various lectures and slides on my home page http://www.math.u-psud.fr/~colliot/liste-cours-exposes.html

For related material you may also want to visit the webpages of Misha Borovoi, Timothy Browning, Cyril Demarche, Ulrich Derenthal, David Harari, Yongqi Liang, Bjorn Poonen, Alexei Skorobogatov, Michael Stoll, Bianca Viray, Tony Várilly-Alvarado, Dasheng Wei, Olivier Wittenberg.

I thank Alexei Skorobogatov and Olivier Wittenberg for discussions on several points of this report.

1. Introduction

Let k be a number field, Ω the set of its places. For each place v, let k_v denote the completion.

Given an algebraic variety X over k, one would like to decide whether the set $X(k)$ of k-rational points is not empty.

One necessary condition is that each set $X(k_v)$ of local points be nonempty.

Deciding whether the condition $\prod_v X(k_v) \neq \emptyset$ is fulfilled is a relatively easy matter.

When X is a smooth projective quadric, weak approximation holds : the image of the diagonal map

$$X(k) \to \prod_v X(k_v)$$

has dense image in the topological produce. In particular, the Hasse principle holds : $\prod_v X(k_v) \neq \emptyset$ implies $X(k) \neq \emptyset$.

It has been known for a long time that such a nice principle holds very rarely.

In 1970, Yu. I. Manin noticed that the Brauer group of scheme, as developed by Grothendieck, produces a common explanation for many of the counterexamples to the Hasse principle hitherto exhibited. In short, for X/k smooth, projective, the Brauer group $\mathrm{Br}(X)$ cuts out a subset $X(A_k)^{\mathrm{Br}} \subset X(A_k) = \prod_v X(k_v)$ and there are inclusions

$$X(k) \subset X(A_k)^{\mathrm{Br}} \subset X(A_k).$$

The set $X(A_k)^{\mathrm{Br}}$ is referred to as the Brauer-Manin set of X. Most countexamples known at the time have since been explained in this simple fashion : $X(A_k) \neq \emptyset$ but $X(A_k)^{\mathrm{Br}} = \emptyset$.

This raised the question whether there are interesting classes of varieties X for which $X(A_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$.

The set $X(A_k)^{\mathrm{Br}}$ is closed in $X(A_k)$. One may ask for general classes of varieties for which the following property holds :

(WABM) *The topological closure of* $X(k)$ *in* $X(A_k)$ *coincides with the Brauer-Manin set* $X(A_k)^{\mathrm{Br}}$:

$$\overline{X(k)}^{top} = X(A_k)^{\mathrm{Br}}$$

WABM stands for : weak approximation for the Brauer-Manin set.

Work of Sansuc and of Borovoi, building upon earlier work on the Hasse principle for principal homogeneous spaces of semisimple simply connected groups (Eichler, Kneser, Harder, Chernousov) showed that this is indeed so for varieties which are k-birational to homogeneous spaces of connected linear algebraic groups with connected geometric stabilizers.

In work done in 1984-1987, Sansuc, Swinnerton-Dyer and I [41] proved

$$\overline{X(k)}^{top} = X(A_k)^{\mathrm{Br}}$$

for smooth projective models of affine surfaces given by an equation in three variables (x, y, t)

$$y^2 - az^2 = P(t)$$

with $a \in k^{\times}$ and $P(x)$ a separable polynomial of degree 3 or 4.

Such surfaces are in general NOT k-birational to a homogenous space of the type considered by Sansuc and Borovoi, as mentioned above (question : can you prove this ?).

Since then, one kept on asking for more general classes of varieties for which $X(k)^{top} = X(A_k)^{\mathrm{Br}}$ holds.

From a naive point of view, one asks whether $\overline{X(k)}^{top} = X(A_k)^{\mathrm{Br}}$ holds for smooth projective models of varieties defined by an equation

$$Norm_{K/k}(\Xi) = P(t)$$

where $P(t) \in k[t]$ is a nonzero polynomial and K/k is a finite field extension.

The study of such equations leads to the study of systems of equations

$$Norm_{K_i/k}(\Xi_i) = P_i(t), \;\; i = 1, \ldots, n,$$

where each $P_i(t)$ is a nonzero polynomial and each K_i is an étale extension of k, i.e. a finite product of finite field extensions of k.

For the smooth k-variety V defined by such a system, projection to the t-coordinate defines a morphism $V \to \mathbf{A}^1_k$. Over the complement U of the closed set defined by $\prod_i P_i(t) = 0$, projection $V \to U$ is a principal homogeneous space under the k-torus T defined by the equations

$$1 = Norm_{K_1/k}(\Xi_1) = \cdots = Norm_{K_n/k}(\Xi_n),$$

which is the product over all i of the tori

$$R^1_{K_i/k}\mathbf{G}_m = \mathrm{Ker}[Norm_{K_i/k} : R_{K_i/k}\mathbf{G}_m \to \mathbf{G}_{m,k}].$$

We shall be interested in smooth compactifications X of V with a map $X \to \mathbf{P}^1_k$ extending $V \to U \subset \mathbf{P}^1_k$. From the geometric point, such varieties are rather simple. Indeed, over $\overline{k}$, they are $\overline{k}$-rational varieties.

Around 1990, I put forward the conjecture – extending one done for surfaces by Sansuc and me in 1979 :

Conjecture **For any smooth, projective, geometrically rational connected variety over a number field k, the closure of $X(k)$ in $X(A_k)$ coincides with the Brauer-Manin set $X(A_k)^{\mathrm{Br}}$.**

That is, WABM should hold for such varieties.

It seems natural to try an induction process on dimension. Let X/k be a smooth, projective, geometrically connected variety, equipped with a dominant fibration $X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. If WABM holds for the smooth fibres, does WABM hold for X ?

[The analogous question for the Hasse principle and weak approximation already has a negative answer for one parameter families of conics.]

This question is hard. One could for instance dream of understanding cubic surfaces by fibering them into curves of genus one, for which some version of (WABM) is conjectured. Except in very special cases, we do not know how to do this.

In trying to prove an inductive result, one should therefore put some restriction on the fibres of $X \to \mathbf{P}^1_k$.

Geometrically rational varieties are special cases of (geometrically) rational connected varieties. In the geometric classification of higher dimensional varieties, rationally connected varieties appeared as the good analogue of rational curves and rational surfaces (work of Kollár, Miyaoka, Mori, of Campana). A fundamental theorem due to Graber, Harris and Starr [60] asserts that for any fibration $X \to Y$ whose base and general fibre is rationally connected, then the total space X is also rationally connected.

The purpose of these lectures is to give an introduction to recent progress on the arithmetic of rationally connected fibres under inductive procedures.

It will turn out that in the process of handling general fibrations $X \to \mathbf{P}^1_k$ with rationally connected generic fibre, concrete auxiliary varieties will be forced upon us which are very close to the special systems

$$Norm_{K_i/k}(\Xi_i) = P_i(t), \;\; i = 1, \ldots, n,$$

considered above.

There will be general results on rational points conditional on some conjectures for these auxiliary varieties.

Over $k = \mathbf{Q}$, and under specific assumptions on the singular fibres of $X \to \mathbf{P}^1_k$, unconditional results on rational points have been achieved thanks to great recent advances in additive combinatorics, due to Green, Tao, Ziegler, with further work by T. Browning and L. Mathiesen. We shall mention work of the last two authors, Harpaz, Skorobogatov, Wittenberg.

Over an arbitrary number field, Harpaz and Wittenberg have obtained a completely satisfying inductive result for analogous questions on zero-cycles (rather than rational points). A typical question here is : rather than asking for existence of a rational point, one ask whether the degrees of field extensions K/k over which a given k-variety aquires a K-rational point is equal to 1. In other words, given any prime p, is there a field extension of degree prime to p over which X aquires a rational point ?

2. Basic properties of number fields; Hasse principle and weak approximation (definitions)

Let k be a number field. We let Ω_k, or sometimes simply Ω, denote the set of its places. The completion of k at a place v is denoted k_v. For a finite (=nonarchimedean) place v, the notation v is also used for the normalized valuation attached to the place v.

2.1. Hasse principle, weak approximation, strong approximation.

Definition 2.1. (Hasse principle) Let k be a number field. The Hasse principle fails for a k-variety if $\prod_{v\in\Omega} X(k_v) \neq \emptyset$ and $X(k) = \emptyset$.

A class of algebraic varieties over k satisfies the Hasse principle if any k-variety in that class satisfies the Hasse principle.

Definition 2.2. (Weak approximation) Let k be a number field. Weak approximation holds for a k-variety X if the image of the diagonal map

$$X(k) \to \prod_{v\in\Omega} X(k_v)$$

is dense in the right hand side equipped with the product topology.

Assuming $\prod_{v\in\Omega} X(k_v) \neq \emptyset$, this amounts to the statement that $X(k)$ is not empty and that for any finite S of place, the image of $X(k)$ under the diagonal embedding

$$X(k) \to \prod_{v\in S} X(k_v)$$

is dense.

According to this definition, if weak approximation holds for a k-variety X, then the Hasse principle holds for X. One should however be aware that for some classes of varieties it may be easy to prove weak approximation for varieties in the class which already have a k-rational point, and hard to prove the Hasse principle. The simplest example here is that of quadrics.

Let X be an integral k-variety. A subset $H \subset X(k)$ is a Hilbert set if there exists an integral k-variety Z and a dominant quasi-finite k-morphism $Z \to X$ such that H is the set of k-points P whose fibre Z_P is integral.

The intersection of two Hilbert sets in $X(k)$ contains a Hilbert set.

Definition 2.3. (Hilbertian weak approximation) Let k be a number field. A geometrically integral k-variety X satisfies hilbertian weak approximation if for any Hilbert set $H \subset X(k)$ the image of H under the diagonal map

$$X(k) \to \prod_{v\in\Omega} X(k_v)$$

is dense in the RHS equipped with the product topology.

Assume $X_{smooth}(k) \neq \emptyset$. The definition then implies that any Hilbert set in $X(k)$ is Zariski dense in X, and in particular is not empty.

Definition 2.4. (Weak weak approximation) Let k be a number field. A k-variety X with a k-point satisfies weak weak approximation if there exists a finite set $S_0 \subset \Omega_k$ such that the image of the diagonal map

$$X(k) \to \prod_{v \in \Omega, v \notin S_0} X(k_v)$$

is dense in the RHS equipped with the product topology. This amounts to requiring that for any finite set $S \subset \Omega$ with $S \cap S_0 = \emptyset$, the image of the diagonal map

$$X(k) \to \prod_{v \in S} X(k_v)$$

is dense.

Proposition 2.5. *(Kneser) Let k be a number field. Let X and Y be two smooth, geometrically integral k-varieties. Assume that X and Y are k-birationally equivalent, and assume*

$$\prod_v X(k_v) \neq \emptyset.$$

Then weak approximation holds for X if and only if it holds for Y.

Definition 2.6. (Strong approximation) Let k be a number field. Let X be a k-variety with a k-point. We say that X satisfies strong approximation with respect to a finite set $S \subset \Omega$ if the image of the diagonal map

$$X(k) \to X(\mathbb{A}_k^S)$$

is dense in the space of S-adèles of X, i.e. of adèles where the components at places in S have been omitted.

Definition 2.7. (Hilbertian strong approximation) Let k be a number field. Let X be a k-variety with a k-point. We say that X satisfies strong approximation with respect to a finite set $S \subset \Omega$ if for any Hilbert set $H \subset X(k)$ the image of H under the diagonal map

$$X(k) \to X(\mathbb{A}_k^S)$$

is dense in the space of S-adèles of X.

If weak approximation holds for a *proper* k-variety X with a k-point, then strong approximation holds for X with respect to any finite set $S \subset \Omega$, in particular for $S = \emptyset$. Similarly in the Hilbertian case.

2.2. **Basic properties of number fields.**

Theorem 2.8. *(Ekedahl, [57]) Let k be a number field, $S \subset \Omega$ a finite set, and for each $v \in S$, let $\lambda_v \in k_v$. Let $H \subset \mathbf{A}^1(k) = k$ be a Hilbert set. For each $\varepsilon > 0$, there exists $\lambda \in H$ such that $|\lambda - \lambda_v|_v < \varepsilon$ for each $v \in S$.*

Thus Hilbertian weak approximation holds for any nonempty Zariski open set of the projective line.

The following theorem may be viewed as an extension of the Chinese remainder theorem.

Theorem 2.9. *(Ekedahl, [57]) Let k be a number field, $S \subset \Omega$ a finite set, and for each $v \in S$, let $\lambda_v \in k_v$. Let $\varepsilon > 0$. Let $v_0 \notin S$ be a place of k. Let $H \subset k$ be a Hilbert set. There then exists $\lambda \in H$ such that*
(i) $|\lambda - \lambda_v|_v < \varepsilon$ for each $v \in S$,
(ii) $v(\lambda) \geq 0$ at each finite place $v \notin S \cup v_0$.

Thus Hilbertian strong approximation holds for the affine line $\mathbf{A}^1_k$ and any *nonempty* finite set $S \subset \Omega$.

Note that one may choose any $v_0 \notin S$.

Ekedahl's theorem is actually more general.

Theorem 2.10. *(Ekedahl, [57]) Let R be the ring of integers of a number field k, let $\pi : X \to \mathrm{Spec} R$ be a morphism of finite type, and let $\rho : Y \to X$ be an étale covering. Assume that the generic fibre of $\pi.\rho$ is geometrically irreducible. Assume moreover that weak approximation, resp. strong approximation away from a finite $S \subset \Omega$, holds for $X \times_R k$. Then weak approximation, resp. strong approximation away from S, holds for the set of $x \in X(k)$ with $\rho^{-1}(x)$ connected.*

A useful result in this context is given by A. Smeets [130, Prop. 6.1].

The next theorem is the extension to number fields of Dirichlet's theorem on primes in an arithmetic progression.

Theorem 2.11. *(Dirichlet, Hasse) Let k be a number field, $S \subset \Omega$ a finite set of finite places, and for each $v \in S$, let $\lambda_v \in k_v^{\times}$. Let $\varepsilon > 0$. There exist $\lambda \in k^*$ and a finite place $v_0 \notin S$, of absolute degree 1, such that*
(i) $|\lambda - \lambda_v|_v < \varepsilon$ for each place $v \in S$,
(ii) $\lambda > 0$ in each real completion of k,
(iii) λ is a unit at any place $v \notin S \cup v_0$ and $v_0(\lambda) = 1$.

Here v_0 may not be chosen at the outset.

The next statement (easy for $k = \mathbf{Q}$) enables one to approximate also at the archimedean places, if one accepts to loose control over an infinite set of places of k, which one may choose at the outset. Typically, this will be the set of places split in a fixed, finite extension of k.

Theorem 2.12. *(Dirichlet, Hasse, Waldschmidt, Sansuc)[112] Let k be a number field, $S \subset \Omega$ a finite set, and for each $v \in S$, let $\lambda_v \in k_v^{\times}$. Let $\varepsilon > 0$. Let V be an* *infinite set of places of k. There exists $\lambda \in k^*$ and a finite place $v_0 \notin S$ of absolute degree 1 such that*

(i) $|\lambda - \lambda_v|_v < \varepsilon$ for each $v \in S$,

(iii) λ is a unit at each finite place $v \notin S \cup v_0 \cup V$ and $v_0(\lambda) = 1$.

Here again v_0 may not be chosen at the outset.

This is to be compared with the following proposition, coined by Harpaz and Wittenberg [80, Lemma 5.2]. Though relatively easy to prove, it plays an important rôle in their work.

Proposition 2.13. *Let K/k be a finite extension of number fields. Let S be a finite set of places of k. For each place $v \in S$, let $\xi_v \in k_v^\times \cap Norm_{K/k}(K_v^\times)$. Then there exists $\xi \in k^\times$ arbitrarily close to ξ_v for $v \in S$ and such that ξ is a unit outside S except possibly at places v above which there exists a place w of K of degree one. In addition, if v_0 is a place of k, not in S, over which K possesses a place of degree 1, one may ensure that ξ is integral outside of $S \cup \{v_0\}$.*

Tchebotarev's theorem is used to prove the existence of such a place v_0, but the proof otherwise only uses the strong approximation theorem.

Theorem 2.14. *(Tchebotarev) Let K/k be a finite extension of number fields. There exists an infinite set of places v of k which are split in K, i.e. the k_v-algebra $K \otimes_k k_v$ is a product of copies of k_v.*

This special case of Tchebotarev's theorem admits of an elementary proof (reference given in [80, Lemma 5.2]).

We shall also use :

Theorem 2.15. *Let K/k be a finite nontrivial extension of number fields. There exist infinitely many places v of k such that $K \otimes_k k_v$ has no k_v-factor. That is, given an irreducible polynomial $P(t)$ of degree at least two, there exist infinitely many places v such that $P(t)$ has no root in k_v.*

As is well known, the second statement does not hold for reducible polynomials. A classical example is $P(t) = (t^2 - 13)(t^2 - 17)(t^2 - 221) \in \mathbf{Q}[t]$.

We may also use the following statement (see [62, Prop. 2.2.1]).

Theorem 2.16. *Let $L/K/k$ be finite extensions of number fields, with L/K cyclic. There exist infinitely many places w of K of degree 1 over k which are inert in the extension L/K.*

Let us now give a few reminders of class field theory.

Theorem 2.17. *Let k be a number field, Ω the set of its places.*

(i) There are embeddings

$$i_v : \mathrm{Br}(k_v) \hookrightarrow \mathbf{Q}/\mathbf{Z}.$$

For nonarchimedan v, the map i_v is an isomorphism. For a real place v, the map i_v induces $\mathrm{Br}(k_v) = \mathbf{Z}/2$. For a complex place v, $\mathrm{Br}(k_v) = 0$.

(ii) The image of natural map $\mathrm{Br}(k) \to \prod_{v\in\Omega} \mathrm{Br}(k_v)$ *lies in the direct sum* $\oplus_{v\in\Omega}\mathrm{Br}(k_v)$
(iii) The maps i_v *fit into an exact sequence*

$$0 \to \mathrm{Br}(k) \to \oplus_{v\in\Omega}\mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z} \to 0.$$

That the sequence (iii) is a complex is a generalisation of Gauss's quadratic reciprocity laws.

Theorem 2.18. *(Poitou-Tate) Let* k *be a number field,* Ω *the set of its places. Let* T *be an algebraic* k*-torus. Let* $\hat{T}$ *be its character group. This is a torsion free finitely generated Galois module. There is a natural exact sequence of abelian groups*

$$H^1(k,T) \to \oplus_{v\in\Omega}H^1(k_v,T) \to \mathrm{Hom}(H^1(k,\hat{T}),\mathbf{Q}/\mathbf{Z}) \to H^2(k,T) \to \oplus_{v\in\Omega}H^2(k_v,T)$$

and a perfect duality of finite abelian groups $Ш^1(k,T) \times Ш^2(k,\hat{T}) \to \mathbf{Q}/\mathbf{Z}$.

Remark 2.1. Here is an amusing consequence. Let K/k be a finite extension of number fields. The quotient $k^\times/Norm_{K/k}(K^\times)$ is finite if and only if the kernel of the restriction map $Br(k) \to Br(K)$ is finite.

As a matter of fact, for $K \neq k$, these groups are infinite. The only known proof (Fein, Kantor, Schacher) uses the classification of finite simple groups.

Proposition 2.19. *Let* K/k *be a cyclic extension of number fields, and* $G = Gal(K/k)$. *For each place* $v \in \Omega$*, there are embeddings*

$$k_v^*/N_{K/k}((K \otimes_k k_v)^*) \to G$$

and there is an exact sequence

$$1 \to k^*/N_{K/k}(K^*) \to \oplus_{v\in\Omega}k_v^*/N_{K/k}((K \otimes_k k_v)^*) \to G \to 1.$$

In particular, for a cyclic extension K/k, an element a in $k^\times$ is a global norm if and only if it is a local norm everywhere locally. This is a celebrated result of Hasse. Moreover, it is enough that a be a local norm at all places except possibly one.

As pointed out in [80], one also has :

Proposition 2.20. *Let* K/k *be an abelian extension of number fields. If* $c \in k^\times$ *is local norm for* K/k *at all places of* k *except possibly one place* v_0*, then it is also a norm at* v_0.

3. The Brauer-Manin obstruction

3.1. **Counterexamples to the Hasse principle and to weak approximation.** In the literature one finds many counterexamples to the local-global principle and weak approximation.

Singular varieties

There are examples of singular varieties X/k with points in all completions k_v but with the property that there exists at least one place v such that $X_{sing}(k_v) = X(k_v)$ and $X_{sing}(k) = \emptyset$. One may give examples with X_{sing} finite.

Reducible varieties

Proposition 3.1. *Let $X = Y \cup Z$ be the disjoint union of two k-varieties Y and Z. Assume X has points in all k_v's, but there exists a place v_1 with $Y(k_{v_1}) = \emptyset$, hence $Z(k_{v_1}) \neq \emptyset$, and a place $v_2 \neq v_1$ with $Z(k_{v_2}) = \emptyset$, hence $Y(k_{v_2}) \neq \emptyset$. Then*
(i) $X(k) = \emptyset$.
(ii) $X(A_k)^{\mathrm{Br}} = \emptyset$.

Proof. Statement (i) is obvious. Let us prove (ii). We have the natural map :

$$\mathrm{Br}(k) \oplus \mathrm{Br}(k) \to \mathrm{Br}(Y) \oplus \mathrm{Br}(Z) = \mathrm{Br}(X).$$

Choose $\alpha \in \mathrm{Br}(k)$ with $\alpha_v = 1/2 \in \mathrm{Br}(k_v)$ for $v = v_1$ and $v = v_2$, and $\alpha_v = 0$ for any other place of v. Let $A \in \mathrm{Br}(X)$ be the image of $(\alpha, 0) \in \mathrm{Br}(k) \times \mathrm{Br}(k)$. Then for any $\{M_v\} \in \prod_{v \in \Omega} X(k_v)$, we have $M_{v_1} \in Z(k_{v_1})$ and $M_{v_2} \in Y(k_{v_2})$, hence

$$\sum_v A(M_v) = A(M_{v_1}) + A(M_{v_2}) = \alpha_{v_2} = 1/2 \neq 0.$$

□

Let us discuss the famous counterexample $(x^2 - 13)(x^2 - 17)(x^2 - 221) = 0$ over $\mathbf{Q}$ from another point of view.

This defines a closed set $Z \subset \mathbf{G}_{m,\mathbf{Q}}$. Let $A = (x, 13) \in \mathrm{Br}(\mathbf{G}_{m,\mathbf{Q}})$. For any point $P_v \in Z(Q_v)$ we have $A(P_v) = 0$ if $v \neq 2, 13, 17$. This obvious for $\mathbb{R}$. This is also clear for any finite place $v = p$ distinct from $2, 13, 17$.

Indeed, x_p is then a unit in $\mathbf{Z}_p$.

Let $v = 13$. Then $x_{13}^2 = 17$ in $\mathbf{Q}_{13}$ and $x_{13} = \pm 2$ up to a square in $\mathbf{Q}_{13}$, hence $A(P_{13}) = (\pm 2, 13)_{13} = 1 \in \mathbf{Z}/2$.

Let $v = 17$. Then $x_{17}^2 = 13$ in $\mathbf{Q}_{17}$, hence $x_{17} = \pm 8$ up to a square in $\mathbf{Q}_{17}$. Then $A(P_{17}) = (\pm 8, 13)_{17} = 0$.

Let $v = 2$. Then $x_2^2 = 17$ in $\mathbf{Q}_2$, hence $x_2 = \pm 5$ up to a square in $\mathbf{Q}_2$. Then $A(P_2) = (\pm 5, 13)_2 = 0$, as may be seen by writing this as $(\pm 5, 13)_5 + (\pm 5, 13)_{13} = 1 + 1 = 0$ in $\mathbf{Z}/2$.

This produces a Brauer-Manin obstruction attached to the algebra induced on Z by the class $A \in Br(\mathbf{G}_{m,\mathbf{Q}})$.

Compare with Stoll [132], Liu–Xu [96], Jahnel–Loughran [87].

For smooth, projective, geometrically integral varieties, many counterexamples have been produced.

Genus one curves

There is a famous equation considered independently by Reichardt and by Lind in the 40's. An affine equation is :

$$2y^2 = x^4 - 17$$

over the rationals.

Let us give the "elementary " proof. If there is a solution over $\mathbf{Q}$, then there is a solution of

$$2u^2 = v^4 - 17w^4 \neq 0$$

with $u, v, w \in \mathbf{Z}$ and $(v, w) = 1$. Reducing modulo 17, we find $u \neq 0 \ (17)$. Now 2 is not a fourth power modulo 17 (indeed $2^4 = -1$ modulo 17). The equation then gives that u is not a square modulo 17. If an odd prime p divides u, then the equation yields $(17/p) = 1$. By the law of quadratic reciprocity for the Legendre symbol, we then have $(p/17) = 1$. Thus p is a square modulo 17. Both 2 and -1 are squares modulo 17. Thus u is a square modulo 17. Contradiction.

A challenge was to understand such examples so as to systematically study other equations.

The more famous Selmer example

$$3x^3 + 4y^3 + 5z^3 = 0$$

is harder to handle.

Principal homogeneous spaces of connected linear algebraic groups

The first counterexamples to the Hasse principle were here given by Hasse and by Witt around 1934.

They are given by an equation

$$N_{K/k}(\Xi) = c$$

where K/k is a Galois extension with group $\mathbf{Z}/2 \times \mathbf{Z}/2$, $c \in k^*$, and Ξ is a "variable" in K (which corresponds to 4 variables). This defines a principal homogeneous space under the 3-dimensional torus given by the equation

$$N_{K/k}(\Xi) = 1.$$

In his book Cohomologie galoisienne [117], Serre constructs a principal homogeneous space of a (not simply connected) semisimple group which is a counterexample to the Hasse principle.

Geometrically rational surfaces

Counterexamples to the Hasse principle for any smooth projective model were given for :

- Smooth cubic surfaces (Swinnerton-Dyer 1962)
- Diagonal cubic surfaces (Cassels–Guy, 1966)

$$5x^3 + 9y^3 + 10z^3 + 12t^3 = 0.$$

A further example was given by A. Bremner, then all diagonal cubic surfaces with a, b, c, d integer smaller than 100 were studied by CT-Kanevsky-Sansuc (1987) [32].

- Singular intersection of two quadrics in P^4, as well as conic bundles over $\mathbf{P}^1$ (Iskovskikh 1970)

$$y^2 + z^2 = (3 - x^2)(x^2 - 3).$$

CT-Coray-Sansuc [30] later proved that any

$$y^2 + z^2 = (c - x^2)(x^2 - c + 1)$$

with $c > 0$ integer congruent 3 modulo 4 gives a counterexample. Here is the elementary argument. One checks that there are solutions in all completions $\mathbf{Q}_p$. A rational solution lead to an integral solution of

$$y^2 + z^2 = (3v^2 - u^2)(u^2 - 2v^2) \neq 0$$

with $(u, v) = 1$. In the LHS, the primes congruent 3 mod 4 appear with an even exponent. As $3v^2 - u^2$ and $u^2 - 2v^2$ are coprime, the same property hods for each of them. Over the reals, the equation gives $3v^2 - u^2 > 0$ et $u^2 - 2v^2 > 0$. Thus the pair $(3v^2 - u^2, u^2 - 2v^2)$ may take the following values modulo 4 :

$$(1, 1); (2, 1); (0, 1); (2, 1); (0, 1).$$

But the possible values of the pair (u^2, v^2) modulo 4 are

$$(0, 1); (1, 0); (1, 1)$$

which for $(3v^2 - u^2, u^2 - 2v^2)$ modulo 4 gives $(3, 2); (3, 1); (2, 3)$, hence none of the previous values. Contradiction, there is no rational solution for

$$y^2 + z^2 = (c - x^2)(x^2 - c + 1)$$

with c as above.

• Smooth complete intersections of two quadrics in $\mathbf{P}^4_k$: Birch et Swinnerton-Dyer, Crelle 1975.

$$uv = x^2 - 5y^2,$$

$$(u + v)(u + 2v) = x^2 - 5z^2.$$

Counterexamples to weak approximation for any smooth projective model were given for

• Singular cubic surfaces with two conjugate singular points (Swinnerton-Dyer 1962)

$$y^2 + z^2 = (4x - 7)(x^2 - 2).$$

The $\mathbf{Q}$-rational points are not dense in the set of real points, for any $\mathbf{Q}$-rational point one has $x \geq 7/4$.

• Smooth intersection of two quadrics in $\mathbf{P}^4_k$: CT-Sansuc , Note CRAS 1977.

See also Coray-Tsfasman [49] for various concrete birational models.

3.2. **The Brauer–Manin set.**

Lemma 3.2. *Let k be a local field, X a smooth quasi-projective k-variety, $A \in \mathrm{Br}(X)$. The evaluation map $ev_A : X(k) \to \mathrm{Br}(k) \subset \mathbf{Q}/\mathbf{Z}$ into the discrete group $\mathbf{Q}/\mathbf{Z}$ is a continous map and its image is finite.*

Proof. Because X/k is smooth, $Br(X)$ is torsion hence A , by a theorem of Gabber, represented by an Azumaya algebra $\mathcal{A}/X$. Let $Y \to X$ be the Severi-Brauer scheme associated to an Azumaya algebra $\mathcal{A}$. The class $A \in Br(X)$ is annihilated by some integer $n > 0$. The image of ev_A is thus contained in the finite subgroup $\mathbf{Z}/n \subset \mathbf{Q}/\mathbf{Z}$. The inverse image of 0 in $X(k)$ coincides with the image of $Y(k)$ in $X(k)$. As $Y \to X$ is smooth, this image is open in $X(k)$ (implicit function theorem). For any $\beta \in \mathrm{Br}(k)$, $A - \beta$ belongs to $\mathrm{Br}_{Az}(X)$. The inverse image of any element in $\mathbf{Z}/n$ is thus open in $X(k)$. This also proves that each of these open sets is closed.

A known theorem of Gabber (see also J. de Jong's webpage) ensures that the statement holds for any torsion element in $\mathrm{Br}(X)$, which for X/k smooth implies the result for any element of $\mathrm{Br}(X)$. □

Lemma 3.3. *Let k be a number field, X a smooth quasi-projective k-variety and $A \in \mathrm{Br}(X)$.*

(i) There exists a nonempty open set $T = \mathrm{Spec}(O_S)$ of the ring of integers of k, a model $\mathcal{X}/T$ of X/k and an element $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$ with image $A \in \mathrm{Br}(X)$. For such a model, for any place $v \in T$ and any point $M_v \in \mathcal{X}(O_v) \subset X(k_v)$, $A(M_v) = 0 \in \mathrm{Br}(k_v)$.

(ii) If X is a proper k-variety, there exists a finite set S of places of k such that for all $v \notin S$, for any $M_v \in X(k_v)$, $A(M_v) = 0$.

Given $A \in \mathrm{Br}(X)$, these lemmas show that there is a well defined, continuous map

$$ev_A : X(\mathbb{A}_k) \to \mathbf{Q}/\mathbf{Z}$$

which sends an adèle $\{M_v\}$ to the finite sum $\sum_v A(M_v) \in \mathbf{Q}/\mathbf{Z}$.

The Brauer–Manin pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbf{Q}/\mathbf{Z})$$

is given by

$$(\{M_v\}, A) \mapsto \sum_v A(M_v) \in \mathbf{Q}/\mathbf{Z}.$$

One checks this is well defined, i.e. each sum only involves finitely many nonzero terms.

For X/k proper, $X(\mathbb{A}_k) = \prod_v X(k_v)$, one may rewrite the map as

$$\prod_v X(k_v) \times \mathrm{Br}(X) \to \mathbf{Q}/\mathbf{Z}.$$

For any subset $B \subset \mathrm{Br}(X)$, we let $X(\mathbb{A}_k)^B \subset X(\mathbb{A}_k)$ denote the set of adèles orthogonal to B with respect to the above pairing. It is a closed subset of $X(\mathbb{A}_k)$.

For $B = \mathrm{Br}(X)$, the set $X(\mathbb{A}_k)^B$ is referred to as the Brauer-Manin set of X.

If X/k is moreover projective, $X(\mathbb{A}_k)$ is compact. If $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ is empty, there exists a *finite* subset $B \subset \mathrm{Br}(X)$ such that $X(\mathbf{A}_k)^B = \emptyset$.

Theorem 3.4. *(Manin) Let k be a number field and X a smooth k-variety. The closure of the image of the diagonal map*

$$X(k) \to X(\mathbb{A}_k)$$

lies in the Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$.

Proof. This follows immediately from the class field theory exact sequence

$$0 \to \mathrm{Br}(k) \to \oplus_v \mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z} \to 0$$

and the fact that $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ is closed in $X(\mathbb{A}_k)$. □

There may thus exist a k-variety X with $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^{\mathrm{Br}(X)} = \emptyset$. In that case one says that there is a Brauer-Manin obstruction to the Hasse principle for X.

When the inclusion $X(\mathbb{A}_k)^{\mathrm{Br}(X)} \subset X(\mathbb{A}_k)$ is a proper inclusion, one says that there is a Brauer-Manin obstruction to strong approximation for X. If X/k is proper, then this is a Brauer-Manin obstruction to weak approximation.

Let $S \subset \Omega_k$ be a finite set containing all archimedean places and let $T = \mathrm{Spec}(O_S)$ be the ring of S-integers of a number field k. Let $\mathcal{X}/T$ be a separated T-scheme of finite type and let $X = \mathcal{X} \times_T \mathrm{Spec}(k)$. For each subset $B \subset \mathrm{Br}(X)$, there is an inclusion

$$\mathcal{X}(O_S) \subset [\prod_{v \in S} X(k_v) \times \prod_{v \in T} \mathcal{X}(O_v)]^B.$$

Using the Brauer group of X one may thus sometimes show that there is a lack of strong approximation outside a finite set S of places. For $A \in \mathrm{Br}(\mathcal{X})$ computing the image of ev_A on the set

$$\prod_{v \in S} X(k_v) \times \prod_{v \in T} \mathcal{X}(O_v)$$

may be done in a finite number of steps, because for any $v \in T$, A vanishes on $\mathcal{X}(O_v)$. On the whole of $X(\mathbb{A}_k)$, unless X is proper over k, there is no such finite process.

Remark 3.5. Quite generally, as soon as one has a contravariant functor F from the category of k-schemes to the category of sets, for any k-variety X there is a commutative diagram

$$\begin{array}{ccc} X(k) & \to & F(k) \\ \downarrow & & \downarrow \\ X(\mathbb{A}_k) & \to & \prod_v F(k_v) \end{array}$$

This imposes restrictions on the image of $X(k)$ in $X(\mathbb{A}_k)$. We shall later see functors which are useful from this point of view. One such is $H^1_{\text{ét}}(\bullet, G)$ for G an algebraic group over k.

Let $\mathrm{Br}_0(X) \subset \mathrm{Br}(X)$ be the image of $\mathrm{Br}(k) \to \mathrm{Br}(X)$ under the structure morphism.

If $X(\mathbb{A}_k) \neq \emptyset$, then $\mathrm{Br}(k) \stackrel{\simeq}{\to} \mathrm{Br}_0(X)$.

(1) For $B \subset \mathrm{Br}(X)$, the set $X(\mathbb{A}_k)^B$ only depends on the image of B in $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

(2) Let $Б(X) \subset \mathrm{Br}(X)$ be the subgroup consisting of elements $A \in \mathrm{Br}(X)$ such that for each place v, there exists $\alpha_v \in \mathrm{Br}(k_v)$ with image equal to $A \otimes_k k_v \in \mathrm{Br}(X_{k_v})$. Let us assume $X(\mathbb{A}_k) \neq \emptyset$. This implies that $\mathrm{Br}(k_v) \to \mathrm{Br}(X_{k_v})$ is injective for each v. Then for each v, α_v is well defined, it is equal to the evaluation of A at any k_v-point of X. By lemma 3.3, $\alpha_v = 0$ for almost all v. For each adèle $\{M_v\} \in X(\mathbb{A}_k)$, one then has

$$\sum_v A(M_v) = \sum_v \alpha_v \in \mathbf{Q}/\mathbf{Z}.$$

The value of this sum does not depend on the adèle $\{M_v\}$. The Brauer-Manin obstruction attached to the "small" subgroup $Б(X) \subset \mathrm{Br}(X)$ plays a great rôle in the study of the Hasse principle for homogeneous spaces of connected linear algebraic groups – but it is too small to control weak approximation.

The group $Б(X)/Br_0(X)$ is conjecturally finite. Indeed :

Proposition 3.6. *Let X be a smooth, projective, geometrically connected k-variety. Assume $X(\mathbb{A}_k) \neq \emptyset$. If $Ш(\mathrm{Pic}^0_{X/k})$ is finite, then $Б(X)/Br_0(X)$ is finite.*

Here are some references on the group $Б(X)$: Sansuc [111], Borovoi, Borovoi-CT-Skorobogatov [6], Wittenberg [150], Harari-Szamuely [77].

3.3. **The obstruction on a few examples.**

3.3.1. *The Reichardt and Lind counterexample to the Hasse principle.* Let X be the smooth compactification of the smooth $\mathbf{Q}$-curve U defined by the affine equation

$$2y^2 = x^4 - 17 \neq 0.$$

One easily checks $X(\mathbf{A_Q}) \neq \emptyset$.

One checks that the Azumaya algebra on U defined by the quaternion algebra $A = (y, 17)$ defines an element of $\mathrm{Br}(X) \subset \mathrm{Br}(U)$.

One then checks that A on each $U(\mathbf{Q}_p)$ takes the following values. For $p \neq 17$ and $p = \infty$, A vanishes on $U(\mathbf{Q}_p)$. For $p = 17$, A takes on $U(\mathbf{Q}_{17})$ the constant value $1/2 \in \mathbf{Q}/Z$. Since evaluation is continous, this still holds on all of $X(\mathbf{Q}_p)$. Thus

$$\sum_p A(M_p) = 1/2$$

for any adèle $\{M_p\} \in X(\mathbf{A_Q})$, hence $X(\mathbf{Q}) = \emptyset$. [One may check $A \in Б(X \subset Br(X)$.]

3.3.2. *The Swinnerton-Dyer counterexample to weak approximation.* Let $U/\mathbf{Q}$ be the affine surface defined by the equation

$$y^2 + z^2 = (4x - 7)(x^2 - 2) \neq 0$$

sur $\mathbf{Q}$. Its set $U(\mathbb{R})$ of real points decomposes into two connected components, the first one with $-\sqrt{2} < x < \sqrt{2}$, lthe second one with $7/4 < x$. Consider the Azumaya algebra $A = (-1, 4x - 7)$ on U. One shows that on any smooth compactification X of U, the class of A belongs to $\mathrm{Br}(X) \subset \mathrm{Br}(U)$.

For p prime, $p \neq 2$, using the equality

$$(4x - 7)(4x + 7) - 16(x^2 - 2) = 1$$

one checks that A vanished on $U(\mathbf{Q}_p)$ hence also on $X(\mathbf{Q}_p)$. For $p = 2$, one checks that A also vanishes on $U(\mathbf{Q}_2)$. On $U(\mathbb{R})$, A takes the two values $(0, 1/2) \in \mathbf{Q}/\mathbf{Z}$. The reciprocity law then shows that on any $\mathbf{Q}$-rational point M, $A(M) = 0 \in \mathrm{Br}(\mathbb{R})$, that is $x > 7/4$.

3.3.3. *The Iskovskikh counterexample to the Hasse principle.* In that example, and more generally in the example [30, Exemple 5.4]

$$y^2 + z^2 = (c - x^2)(x^2 - c + 1) \neq 0$$

with $c \in \mathbb{N}$, c congruent 3 modulo 4, one uses the Azumaya algebra on U defined by the quaternion algebra $A = (c - x^2, -1)$. Let X be a smooth compactification of U. One checks that A defines a class in $\mathrm{Br}(X) \subset \mathrm{Br}(U)$.

For $p \neq 2$ (also for $p = \infty$), one checks $A(M_p) = 0$ for any $M_p \in U(\mathbf{Q}_p)$, For $p = 2$, one checks $A(M_2) = 1/2 \in \mathbf{Q}/\mathbf{Z}$ for any $M_2 \in U(\mathbf{Q}_2)$, Thus

$$\sum_p A(M_p) = 1/2$$

for any adèle $\{M_p\} \in X(\mathbf{A}_{\mathbf{Q}})$, hence $X(\mathbf{Q}) = \emptyset$.

3.3.4. *Principal homogeneous spaces under a specific torus.* The following example is discussed in more detail in [29].

Let k be a number field, $a, b, c \in k^\times$, and let U be the k-variety defined by the equation

$$(x^2 - ay^2)(z^2 - bt^2)(u^2 - abw^2) = c.$$

Let X/k be a smooth compactification of U. Computing residues, one easily checks that the class of the quaternion algebra $A = (x^2 - ay^2, b) \in \mathrm{Br}(U)$ lies in the subgroup $\mathrm{Br}(X)$.

Proposition 3.7. *With notation as above, assume that for each place v of k the fields extension $k_v(\sqrt{a}, \sqrt{b})$ is cyclic, hence of degree at most 2. Then*

(i) The class A belongs to Б(X).

(ii) For each adèle $\{M_v\}_{v \in \Omega}$ of X, one has :

$$\sum_{v \in \Omega} A(M_v) = \sum_{v,\ a \notin k_v^{*2}} (c, b)_v \in \mathbf{Z}/2.$$

Proof. For a field F containing k, if one of a, b or ab is a square, then on the equation of U one sees that U is F-rational, hence $\mathrm{Br}(F) \xrightarrow{\simeq} \mathrm{Br}(X_F)$. This proves (i).

Let $v \in \Omega$. Let $M_v \in U(k_v)$ be a point with coordinates $x_v, y_v, z_v, t_v, u_v, w_v$. Let us compute $(x_v^2 - aby_v^2, b) \in \mathrm{Br}(k_v)$. If a is not a square in k_v, then either b or ab is a square in k_v. In the first case, $(x_v^2 - ay_v^2, b) = 0$, in the second case, $(x_v^2 - ay_v^2, b) = (x_v^2 - ay_v^2, a) = 0$. We have $(z_v^2 - bt_v^2, b) = 0$. Assume a is a square. Then $(u_v^2 - abw_v^2, b) = (u_v^2 - abw_v^2, ab) = 0$. Using the equation of U we then conclude $(x_v^2 - ay_v^2, b) = (c, b)_v$. By continuity of the evaluation map of A, the same result holds on any point of $X(k_v)$. □

Starting from this explicit formula, one easily produces counterexamples to the Hasse principle. Here is an example. Take $k = \mathbf{Q}$, $a = 17$, $b = 13$ and take for c a prime number which is not a square mod 17 and not a square mod 13, for instance $c = 5$.

Many more examples have been constructed.

3.4. **Beyond the Brauer-Manin obstruction.** Papers :

Skorobogatov [122], then Harari [65], Harari–Skorobogatov [71], Demarche [52], Skorobogatov [124] : definition of the étale Brauer-Manin set, comparison with the descent obstruction.

Examples of Enriques surfaces (Harari–Skorobogatov [73], Várilly-Alvarado–Viray [142]).

Poonen [107] : the étale Brauer-Manin obstruction is not enough (for a threefold). Poonen uses a threefold with a fibration to a curve with finitely many points, the generic fibre being a Châtelet surface.

Harpaz and Skorobogatov [78] : the étale Brauer–Manin obstruction is not enough (for a surface). Again a fibration over a curve with finitely many points, but with a tricky argument with a singular fibre consisting of a union of curves of genus zero.

CT–Pál–Skorobogatov [35] : the étale Brauer-Manin obstruction is not enough for the total space of families of quadrics over a curve with finitely many points – already for families of conics.

In the last three sets of examples, the varieties have a nonconstant map to a curve of genus at least one, hence have a nontrivial Albanese variety. A. Smeets [131] has now given examples with trivial Albanese varieties.

4. Algebra : Calculating the Brauer group

This chapter is extracted from my IU Bremen notes, 2005.

`www.math.u-psud.fr/~colliot/CTBremenBrauerplusJuly2012.pdf`

I only added brief mentions of later developments. There is more in my "Notes sur le groupe de Brauer" :

`http://www.math.u-psud.fr/~colliot/CTnotesBrauer.pdf`

One would like to be able to compute $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$. For this, a prerequisite is to compute the Brauer group $\mathrm{Br}(X)$, or a least a system of representives of $\mathrm{Br}(X)/\mathrm{Br}(k)$.

Suppose $\mathrm{char}\, k$ is zero, and X/k is smooth, projective and geometrically connected. We write $\overline{X} := X \times_k \overline{k}$, where $\overline{k}$ is an algebraic closure of k.

4.1. **The "geometric" Brauer group.** For computing $\mathrm{Br}\,\overline{X}$, we have an exact sequence

$$0 \to (\mathbf{Q}/\mathbf{Z})^{b_2-\rho} \to \mathrm{Br}\,\overline{X} \to \mathrm{H}^3(\overline{X}, \mathbf{Z})_{tors} \to 0.$$

Here b_2 is the second Betti number, which one computes by using either l-adic cohomology $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbf{Q}_l)$ for an arbitrary prime l or by using classical cohomology $\mathrm{H}^2(X \times_k \mathbb{C}, \mathbf{Q})$ if an embedding $k \subset \mathbb{C}$ is given. The integer $\rho = \mathrm{rk}\,\mathrm{NS}\,\overline{X}$ is the rank of the geometric Néron-Severi group. The vanishing of $b_2 - \rho$ is equivalent to the vanishing of the coherent cohomology group $\mathrm{H}^2(X, O_X)$. The group $\mathrm{H}^3(\overline{X}, \mathbf{Z})_{tors}$ is a finite group, which one computes either as the direct sum over all primes l of the torsion in integral l-adic cohomology $\mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbf{Z}_l)$ or as the torsion in classical cohomology $\mathrm{H}^3(X \times_k \mathbb{C}, \mathbf{Z})$ if an embedding $k \subset \mathbb{C}$ is given. If X is a curve, or if $\overline{X}$ is birational to a projective space, then $\mathrm{Br}\,\overline{X} = 0$.

Remarks

1. It is in general quite difficult to exhibit the Azumaya algebras on $\overline{X}$ corresponding to the divisible subgroup $(\mathbf{Q}/\mathbf{Z})^{b_2-\rho}$.
2. When k is a number field, it is an open question whether the group of fixed points $(\mathrm{Br}\,\overline{X})^{\mathrm{Gal}(\overline{k}/k)}$ is finite.

Further works on this topic

Skorobogatov and Zarhin [127, 128, 129]
Ieronymou, Skorobogatov, Zarhin [85] [86]
Colliot-Thélène et Skorobogatov [44]
Hassett and Várilly-Alvarado [82]

4.2. **The "algebraic" Brauer group.** Define $\mathrm{Br}_1(X) := \mathrm{Ker}[\mathrm{Br}\,X \to \mathrm{Br}\,\overline{X}]$. For computing this group, we have the exact sequence

$$0 \to \mathrm{Pic}\,X \to (\mathrm{Pic}\,\overline{X})^{\mathrm{Gal}(\overline{k}/k)} \to^* \mathrm{Br}\,k \to \mathrm{Br}_1\,X \to \mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) \to^* \mathrm{H}^3(k, \overline{k}^\times)$$

where the maps marked with a $*$ are zero when $X(k)$ is nonempty. The group $\mathrm{H}^3(k, \overline{k}^\times)$ is trivial when k is a number field (this is a nontrivial result from class field theory).

There are cases where it is easy to explicitly compute the group $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ but where it is difficult to lift a given element of that group to an explicit element of $\mathrm{Br}_1(X)$: Even if one knows a 3-cocycle is a 3-coboundary, it is not easy to write it down as an explicit 3-coboundary. This may create difficulties for deciding whether a given $X(\mathbb{A}_k)^{\mathrm{Br}\,X}$ is empty or not. Such a delicate situation arises in the study of diagonal cubic surfaces ([32], [88]).

To get a hold on $\mathrm{H}^1(k, \operatorname{Pic} \overline{X})$ one uses the exact sequence of Galois-modules

$$0 \to \operatorname{Pic}^0_{X/k}(\overline{k}) \to \operatorname{Pic} \overline{X} \to \operatorname{NS} \overline{X} \to 0\,.$$

Here $\operatorname{NS} \overline{X}$ is of finite type. If $\operatorname{NS} \overline{X}_{tors} = 0$, then $\mathrm{H}^1(k, \operatorname{NS} \overline{X})$ is a finite group.

4.3. **Curves.** If $X = C$ is a curve, then $\operatorname{Br}_1 C = \operatorname{Br} C$ (as noted above), and the above exact sequence reads

$$0 \to \operatorname{Jac} C(\overline{k}) \to \operatorname{Pic} \overline{C} \to \mathbf{Z} \to 0.$$

Since $\mathrm{H}^1(k, \mathbf{Z}) = 0$, we thus have the exact sequence

$$(\operatorname{Pic} \overline{C})^{\operatorname{Gal}(\overline{k}/k)} \to \mathbf{Z} \to \mathrm{H}^1(k, \operatorname{Jac} C(\overline{k})) \to \mathrm{H}^1(k, \operatorname{Pic} \overline{C}) \to 0$$

which one may combine with the above long exact sequence. The group $\mathrm{H}^1(k, \operatorname{Jac} C(\overline{k}))$ classifies principal homogeneous spaces under $\operatorname{Jac} C = \operatorname{Pic}^0_{C/k}$. The map $\mathbf{Z} \to \mathrm{H}^1(k, \operatorname{Jac} C(\overline{k}))$ sends 1 to the class of the principal homogeneous space $\operatorname{Pic}^1_{C/k}$. If k is a number field, we thus have a surjective map from $\operatorname{Br} C$ to a quotient of $\mathrm{H}^1(k, \operatorname{Jac} C(\overline{k}))$. In practice, it is quite hard to lift an element of this quotient to an explicit element of $\operatorname{Br} C$.

Examples

1. If $C = \mathbb{P}^1_k$, then the natural map $\operatorname{Br} k \to \operatorname{Br} \mathbb{P}^1_k$ is an isomorphism.
2. If C is a smooth projective conic with no rational point, we have an exact sequence

$$\begin{gathered} 0 \to \mathbf{Z}/2 \to \operatorname{Br} k \to \operatorname{Br} C \to 0 \\ 1 \mapsto [A_C] \end{gathered}$$

where $[A_C] \in {}_2\operatorname{Br} k$ is the class corresponding to C.

4.4. **Residues.** Let A be a discrete valuation ring with field of fractions F and with residue field κ of characteristic zero. There is a natural "residue map" $\operatorname{Br} F \to \mathrm{H}^1(\kappa, \mathbf{Q}/\mathbf{Z})$ and an exact sequence

$$0 \to \operatorname{Br} A \to \operatorname{Br} F \to \mathrm{H}^1(\kappa, \mathbf{Q}/\mathbf{Z})\,.$$

Let k be a field of characteristic zero. Let X be a smooth, integral, k-variety with function field $k(X)$. Given a closed integral subvariety $Y \subset X$ of codimension 1, with function field $k(Y)$, we may consider the residue map $\operatorname{Br} k(X) \to \mathrm{H}^1(k(Y), \mathbf{Q}/\mathbf{Z})$. One then has (Grothendieck) the exact sequence

$$0 \to \operatorname{Br} X \to \operatorname{Br} k(X) \to \bigoplus_Y \mathrm{H}^1(k(Y), \mathbf{Q}/\mathbf{Z}),$$

where Y runs through all codimension 1 subvarieties of X as above.

From the exactness of this sequence one deduces that $\operatorname{Br} X$ is a birational invariant for smooth, projective, integral k-varieties.

4.5. **The projective line.** Let us consider the special case $X = \mathbb{P}^1_k$. As noted above, $\operatorname{Br}\mathbb{P}^1_k = \operatorname{Br} k$. The short exact sequence above thus reads

$$0 \to \operatorname{Br} k \to \operatorname{Br} k(\mathbb{P}^1) \to \bigoplus_{P\in\mathbb{P}^1_k} \mathrm{H}^1(k_P, \mathbf{Q}/\mathbf{Z}),$$

where P runs through the closed points of $\mathbb{P}^1_k$ and k_P is the residue field at such a point P.

One may compute the cokernel of the last map : there is an exact sequence

$$0 \longrightarrow \operatorname{Br} k \longrightarrow \operatorname{Br} k(\mathbb{P}^1) \longrightarrow \bigoplus_{P\in\mathbb{P}^1_k} \mathrm{H}^1(k_P, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sum_P N_{k_P/k}} \mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z}) \longrightarrow 0,$$

where $N_{k_{P/k}}$ is the corestriction map.

4.6. **Conic bundles over the projective line.** Let X/k be a smooth, projective, geometrically connected surface equipped with a morphism $X \to \mathbb{P}^1_k$ whose generic fibre X_η is a smooth conic over $K = k(\mathbb{P}^1_k) = k(t)$. After performing k-birational transformations one may assume that for each closed point $P \in \mathbb{P}^1_k$, the fibre X_P is a conic over the residue field k_P, and that $X \to \mathbb{P}^1_k$ is relatively minimal. There are finitely many points $P \in \mathbb{P}^1_k$ for which X_P is not smooth. At such a point P, there is a quadratic extension F_P/k_P over which X_P splits into a pair of transversal lines. Write $F_P = k_P(\sqrt{a_P})$.

Let $A \in \operatorname{Br} K$ be the class of a quaternion algebra over K associated to the conic X_η/K, as in example 2 of section 2.3.

We shall assume that A does not come from $\operatorname{Br} k$. In the long exact sequence associated to $\mathbb{P}^1$ in section 2.5, for each closed point $P \in \mathbb{P}^1_k$, the residue $\delta_P(A) \in \mathrm{H}^1(k_P, \mathbf{Q}/\mathbf{Z})$ lies in $\mathrm{H}^1(F_P/k_P, \mathbf{Z}/2) = \mathbf{Z}/2$.

Using the last three subsections, one shows that there is an exact sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br} X \to (\oplus_P(\mathbf{Z}/2)_P)/(\{\delta_P(A)\}) \to k^\times/k^{\times 2}.$$

The last map sends the class of the element $1 \in (\mathbf{Z}/2)_P = \mathrm{H}^1(F_P/k_P, \mathbf{Z}/2) \subset \mathrm{H}^1(k_P, \mathbf{Q}/\mathbf{Z})$ to the class of $N_{k_P/k}(a_P) \in k^\times/k^{\times 2}$.

In this situation one may give explicit generators for $\operatorname{Br} X/\operatorname{Br} k$. They are given as the images under $\operatorname{Br} k(t) \to \operatorname{Br} k(X)$ of suitable linear combinations of elements of the shape $\mathrm{Cores}_{k_P/k}(t-\alpha_P, \beta_P) \in \operatorname{Br} k(t)$, where $k_P = k(\alpha_P)$, $\beta_P \in k_P^\times$, and $(t-\alpha_P, \beta_P)$ is a quaternion algebra over the field $k_P(t)$.

Since a conic bundle $X/\mathbb{P}^1_k$ contains a smooth conic $Y \subset X$, functoriality of the exact sequence

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\mathrm{Gal}(\overline{k}/k)} \to^* \operatorname{Br} k \to \operatorname{Br}_1 X \to \mathrm{H}^1(k, \operatorname{Pic}\overline{X}) \to^* \mathrm{H}^3(k, \overline{k}^\times)$$

implies that the map $\mathrm{H}^1(k, \operatorname{Pic}\overline{X}) \to^* \mathrm{H}^3(k, \overline{k}^\times)$ is zero.

In these notes, we shall often consider a smooth, projective, geometrically connected k-variety X equipped with a dominant k-morphism $X \to \mathbf{P}^1_k$ with geometrically integral generic fibre. We shall be concerned with the so-called **vertical**

Brauer group *of X with respect ot the fibration $f : X \to \mathbf{P}^1_k$. This is the group*

$$Br_{vert}(X) = \{\alpha \in \mathrm{Br}(X) \cap f^*(\mathrm{Br}(k(\mathbf{P}^1))) \subset \mathrm{Br}(X)\}.$$

For conic bundles, and more generally for one-parameter families of Severi-Brauer varieties, one has $Br_{vert}(X) = Br(X)$.

4.7. **Computing when no smooth projective model is available.** One is often confronted with the following problem : given a smooth, affine, geometrically connected variety U over a field k, compute the Brauer group of a smooth compactification X of U without knowing a single such smooth compactification. The point as far as local to global problems are concerned is that it is only the Brauer group of smooth compactifications which naturally produces obstructions to the existence of rational points. A preliminary question is to compute $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ (also a birational invariant of smooth, projective varieties).

Assume $U = T$ is a k-torus, i.e. an algebraic group which over $\overline{k}$ becomes isomorphic to a product of multiplicative groups. To such a k-torus there is associated its character group $\hat{T}$ (over $\overline{k}$). This is a g-lattice (g being the Galois group of $\overline{k}$ over k). For X any smooth k-compactification of T, one has

$$\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) = \mathrm{Ker}[\mathrm{H}^2(g, \hat{T}) \to \prod_h \mathrm{H}^2(h, \hat{T})],$$

where $h \subset g$ runs through all closed pro-cyclic subgroups of g – as a matter of fact the computation of this kernel may be done after going over only to a suitable finite Galois extension of k.

It seems hard to lift the elements of $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ to explicit elements in $\mathrm{Br}\,X$. The situation gets worse if X is a smooth compactification of a principal homogeneous space U under T. We have the same formula for $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ as above, but in this case for k arbitrary there is no reason why the map $\mathrm{Br}_1\,X \to \mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ should be surjective. If k is a number field, the map is surjective but lifting seems nevertheless very hard. Hence it seems difficult to test the condition $X(\mathbb{A}_k)^{\mathrm{Br}\,X_c} \neq \emptyset$.

Probably the simplest nontrivial example is the norm 1 torus $T = R^1_{K/k}\mathbb{G}_m$ associated to a biquadratic extension $K = k(\sqrt{a}, \sqrt{b})/k$. In this case one finds $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) = \mathbf{Z}/2$. The same result holds if $X = X_c$ is a smooth compactification of a principal homogeneous space of $R^1_{K/k}\mathbb{G}_m$, that is a variety $U = U_c$ given by an equation $\mathrm{Norm}_{K/k}(\mathbf{z}) = c$ for some $c \in k^\times$. If k is a number field, U_c has points in all completions of k, and X_c is a smooth compactification of U, then there exists some $A \in \mathrm{Br}\,X_c$ such that $X(\mathbb{A}_k)^{\mathrm{Br}\,X_c} = X(\mathbb{A}_k)^A$. But how to compute such an A in a systematic fashion ?

The question is important, since in this case it is known that $X_c(\mathbb{A}_k)^{\mathrm{Br}\,X_c} \neq \emptyset$ implies $X_c(k) \neq \emptyset$. The latter statement is a general fact for principal homogeneous spaces of connected linear algebraic groups (Sansuc [111]), and it holds more generally for smooth compactifications of homogeneous spaces under connected linear algebraic groups, at least when the geometric stabilizer group is connected (Borovoi).

Coming back to the case of equations $\mathrm{Norm}_{K/k}(\mathbf{z}) = c$, in the case $K = k(\sqrt{a}, \sqrt{b})$, Sansuc [113] gives an algorithm to decide whether $X_c(\mathbb{A}_k)^{\mathrm{Br}\, X_c} \neq \emptyset$. It would be interesting to understand this algorithm better.

It is natural to study varieties which are given as the total space of a one-parameter family of principal homogeneous spaces. A special but already difficult case is that of varieties given by an affine equation

$$\mathrm{Norm}_{K/k}(\mathbf{z}) = P(t)$$

where K/k is a finite field extension and $P(t)$ a polynomial in one variable. In [31] the group $\mathrm{H}^1(k, \mathrm{Pic}\, \overline{X})$ for smooth projective models X of varieties defined by such an equation was computed for many cases, but it could not computed in all cases. See the questions raised at the end of section 2 of [31].

Further work on this section, various papers of Bogomolov, Borovoi, Demarche, Harari, Kunyavskiĭ, Skorobogatov, D. Wei, Wittenberg, myself : [33] [8] [148] [31] [9] [34] [26] [53] [54] [5] [7] [28] [29] [147] [48].

5. Harari's formal lemma and variants

5.1. **The formal lemma for the Brauer group.** This subsection is a mere translation from [24].

Theorem 5.1. *(Harari,* [62, Thm. 2.1.1 p. 226]*) Let k be a number field and X be a smooth connected k-variety. Let $\mathcal{X}/O$ be an integral model over an open set* $\mathrm{Spec}\,(O)$ *of the spectrum of the ring of integers of k.*

Let $U \subset X$ be a nonempty open subet of X. For any element $\alpha \in \mathrm{Br}\,(U)$ which does not belong to $\mathrm{Br}\,(X) \subset \mathrm{Br}\,(U)$, there exist infinitely many places v of k for which there exists $M_v \in U(k_v) \cap \mathcal{X}(O_v)$ with $\alpha(M_v) \neq 0$.

Proof. There exists a codimension 1, integral closed subvariety $Z \subset X$, with generic point ζ, such that the residue

$$\partial_\zeta(\alpha) \in H^1(k(Z), \mathbf{Q}/\mathbf{Z})$$

at ζ does not vanish. After replacing X by a suitable open set, one may assume that Z is smooth over k, that α has only one nontrivial residue on X, to wit the residue at ζ, and that the residue $\partial_\zeta(\alpha)$ belongs to $H^1(Z, \mathbf{Q}/\mathbf{Z}) \subset H^1(k(Z), \mathbf{Q}/\mathbf{Z})$ (here we use étale cohomology). Replacing X by a smaller open set, we may assume that Z is a finite cover of affine space $\mathbf{A}^d_k$. Let Z_1/Z be the finite (cyclic) cover defined by $\partial_\zeta(\alpha)$. Hilbert's irreducibility theorem, when applied to the composite cover $Z_1/\mathbf{A}^d_k$, produce k-point of $\mathbf{A}^d_k$ whose fibre is integral. Let us pick such a k-point. It inverse image under the map $Z \to \mathbf{A}^d_k$ is a closed point $P \in Z$ such that $\partial_\zeta(\alpha)(P) \neq 0 \in H^1(k(P), \mathbf{Q}/\mathbf{Z})$.

A local equation of $Z \subset X$ en P may be written as part of a regular system of parameters of the regular local ring $O_{X,P}$. One thus finds a closed integral curve

$C \subset X$ containing P as a smooth closed point, transversal to Z at P. Shrinking X some more, one may assume that C/k is smooth and $Z \cap C = P$. We have $\alpha \in \mathrm{Br}(X \setminus Z)$. Thus α induces an element $\alpha_C \in \mathrm{Br}(C \setminus P)$. Because C and Z are transveral at P one has

$$\partial_P(\alpha_C) = \partial_\zeta(\alpha)(P) \in H^1(k(P), \mathbf{Q}/\mathbf{Z}).$$

Thus $\partial_P(\alpha_C) \neq 0$. The embedding $C \subset X$ extends to an embedding of integral models over a suitable open set of the spectrum of integers of k. It is enough to prove the statement of the theorem for the smooth connected curve C.

Let now C be a connected, smooth (hence integral) k-curve, $P \in C$ a closed point, $U = C \backslash P$. Let $\alpha \in \mathrm{Br}(U)$ with nonzero residue $\chi = \partial_P(\alpha) \in H^1(k(P), \mathbf{Q}/\mathbf{Z})$. Let $r > 1$ be the order of $\partial_P(\alpha)$. We thus have

$$\partial_P(\alpha) \in H^1(k(P), \mathbf{Z}/r) \subset H^1(k(P), \mathbf{Q}/\mathbf{Z}).$$

Replacing C by a suitable open set, we may assume that C is affine, $C = \mathrm{Spec}(A)$, and that P is defined be the vanishing of $f \in A$. Let A_{hs} be the henselisation of A in P. Restriction $H^1(A_{hs}, \mathbf{Z}/r) \to H^1(k(P), \mathbf{Z}/r)$ is an isomorphism. There thus exists a connected étale open set $q : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, say $q : D \to C$, such that $q : Q = q^{-1}(P) \to P$ is an isomorphism, and such that χ is the restriction of some element $\xi \in H^1(D, \mathbf{Z}/r)$.

Let $V = D \setminus Q$. Consider the cup-product $(f, \xi) \in \mathrm{Br}(V)$ of the class of $f \in k[V]^*/k[V]^{*r} \subset H^1(V, \mu_r)$ with $\xi \in H^1(D, \mathbf{Z}/r)$. The difference $\beta = \alpha_D - (f, \xi) \in \mathrm{Br}(V)$ has a trivial residue at Q, it thus belongs to $\mathrm{Br}(D)$.

There exists an open set $\mathrm{Spec}(O)$ of the spectrum of the ring of integers of k, a flat affine, finite type curve $\mathcal{C}/\mathrm{Spec}(O)$, a morphism $\mathcal{D} \to \mathcal{C}$, extending $D/C/\mathrm{Spec}(k)$, with the following properties : the element f comes from an element f in the ring of $\mathcal{C}$, the element $\xi \in H^1(D, \mathbf{Z}/r)$ is the restriction of an element $\xi \in H^1(\mathcal{D}, \mathbf{Z}/r)$, and $\beta \in \mathrm{Br}(D)$ is the restriction of an element $\beta \in \mathrm{Br}(\mathcal{D})$. We may moreover assume that the closed set in $\mathcal{D}$ and $\mathcal{C}$ defined by $f = 0$ are integral and isomorphic though the map $\mathcal{D} \to \mathcal{C}$ and that the projection of these closed sets to $\mathrm{Spec}(O)$ is finite and étale. Let R be the ring whose spectrum is the closed set defined by $f = 0$ in $\mathcal{C}$ (and in $\mathcal{D}$). The fraction field of R is the field $k(P)$.

Tchebotarev's theorem ensures that there exist infinitely many places v of k such that there exists a place w of $k(P)$ with $k_v \xrightarrow{\simeq} k(P)_w$ (i.e. w is of degree 1 over v) and w is inert in the cyclic extension $k(P)(\chi)/k(P)$ defined by $\chi \in H^1(k(P), \mathbf{Z}/r)$ [62, Prop. 2.2.1, p. 226].

For such a place v, there exists an O_v-point N_v^0 of the closed set $f = 0$ of $\mathcal{D}$ which maps isomorphically to an O_v-point M_v^0 of $\mathcal{C}$.

Let $N_v \in \mathcal{D}(O_v)$ be such that $f(N_v) \neq 0$ and let $M_v \in \mathcal{C}(O_v) \subset C(k_v)$ be its image. One then has

$$\alpha(M_v) = \alpha(N_v) = \beta(N_v) + (f(N_v), \chi(N_v)) \in \mathrm{Br}(k_v) \xrightarrow{\simeq} \mathbf{Q}/\mathbf{Z}.$$

We have $\beta(N_v) \in \mathrm{Br}(O_v) = 0$. As w is inert in the cyclic extension $k(P)(\chi)/k(P)$, if N_v is close enough to N_v^0 for the v-adic topology on $\mathcal{D}(O_v)$, the class $\chi(N_v) \in H^1(k(P)_w, \mathbf{Z}/r) = H^1(k_v, \mathbf{Z}/r)$ has order r. From the standard formula for the tame symbol we then get that $\alpha(M_v) \in \mathbf{Z}/r \subset \mathbf{Q}/\mathbf{Z}$ is equal to the class of the valuation $v(f(N_v))$ modulo r. As the closed set $f = 0$ of $\mathcal{D} \times_O O_v$ contains the O_v-section of $\mathcal{D} \times_O O_v \to \mathrm{Spec}(O_v)$ defined by N_v^0, and is finite and étale over O_v, there exists $N_v \in \mathcal{D}(O_v)$, close to N_v^0, such that $v(f(N_v)) \equiv 1 \mod r$, hence with image $M_v \in \mathcal{C}(O_v) \subset C(k_v)$ satisfying $\alpha(M_v) \neq 0$. □

Remark 5.1. (i) Here is the simplest case, which the reader should have first handled as an exercise before reading the proof just given. Let $a \in k^\times$, a not a square in k, let $X = \mathbf{A}^1_k = \mathrm{Spec}(k[t])$ then $U \subset X$ the open set given by $t \neq 0$, then $\alpha \in \mathrm{Br}(k(t))$ the class of the quaternion algebra (a,t). There exist infinitely many places v for which there exists $t_v \in k_v^\times$ with $(a,t_v) \neq 0 \in \mathrm{Br}(k_v)$.

(ii) In this example, there also exist infinitely many places v such that α identically vanishes on $U(k_v)$. The analogous property holds more generally for any smooth connected curve X.

This does not extend to higher dimension. Suppose X/k is smooth, geometrically integral of dimension at least 2, $U \subset X$ is an open set, $\alpha \in \mathrm{Br}(U)$ and there exists a codimension 1 subvariety $Z \subset X$ such that $\partial(\alpha) \in H^1(k(Z), \mathbf{Q}/\mathbf{Z})$ defines a cyclic extension $L/k(Z)$ with the property that k is algebraically closed in L. Then for almost all places v of k, the class α takes on $U(k_v)$ at least one value other than 0.

Starting from Theorem 5.1, a combinatorial argument leads to the following extremely useful result. The present version is a variation, first stated in [42], of D. Harari's "formal lemma" [62, Corollaire 2.6.1, p. 233].

Theorem 5.2. *Let k be a number field and X be smooth, geometrically connected k-variety. Let $U \subset X$ be a nonempty open set and $B \subset \mathrm{Br}(U)$ a finite subgroup. Let $\{P_v\} \in U(\mathbf{A}_k)$. Assume that for any α in the finite group $B \cap \mathrm{Br}(X)$,*

$$\sum_{v \in \Omega} \alpha(P_v) = 0.$$

Then for any finite set S of places of k there exists an adèle $\{M_v\} \in U(\mathbf{A}_k)$ such that $M_v = P_v$ for $v \in S$ and a finite set S_1 of places, $S \subset S_1$ such that for any $\beta \in B$

$$\sum_{v \in S_1} \beta(M_v) = 0$$

and

$$\sum_{v \in \Omega} \beta(M_v) = 0.$$

Proof. As the group B is finite, upon replacing S by a bigger finite set, we may assume that the embedding $U \subset X$ extends to an embedding of integral models $\mathcal{U} \subset \mathcal{X}$ over the ring O of S-integers of k, that $P_v \in \mathcal{U}(O_v) \subset \mathcal{X}(O_v)$ for $v \notin S$, that $B \subset \mathrm{Br}(\mathcal{U})$ and that $B \cap \mathrm{Br}(X) \subset \mathrm{Br}(\mathcal{X})$. For each $\beta \in B$, and for $v \notin S$, we have $\beta(P_v) = 0$. For each $\beta \in B \cap \mathrm{Br}(X)$ each $v \notin S$, each $M_v \in \mathcal{X}(O_v) \subset X(k_v)$, we have $\beta(M_v) = 0$.

Let $\alpha \in B, \alpha \notin \mathrm{Br}(X)$. According to theorem 5.1, there exists an infinite set T_α of places $v \notin S$, and a family $\{N_v\} \in \prod_{v \in T_\alpha} U(k_v) \cap \mathcal{X}(O_v)$ such that $\alpha(N_v) \neq 0$ for each $v \in T_\alpha$. The group $B/(B \cap \mathrm{Br}(X))$ is finite, hence so is its dual $\mathrm{Hom}(B/(B \cap \mathrm{Br}(X)), \mathbf{Q}/\mathbf{Z})$. There thus exists an infinite subset of T_α such that the linear maps $B/(B \cap \mathrm{Br}(X)) \to \mathbf{Q}/\mathbf{Z}$ given by $\gamma \to \gamma(N_v)$ for v in that subset coïncide. Replacing T_α by this subset, we may thus assume that there exists a linear map $\varphi_\alpha : B/(B \cap \mathrm{Br}(X)) \to \mathbf{Q}/\mathbf{Z}$ with the property $\varphi_\alpha(\alpha) \neq 0$, such that for any $\beta \in B/(B \cap \mathrm{Br}(X))$, and any $v \in T_\alpha$, one has

$$\varphi_\alpha(\beta) = \beta(N_v) \in \mathbf{Q}/\mathbf{Z},$$

with $N_v \in U(k_v) \cap \mathcal{X}(O_v)$ as above.

In the dual of the finite group $B/(B \cap \mathrm{Br}(X))$, the sums of such maps φ_α as α varies in $B/(B \cap \mathrm{Br}(X))$ (repetitions allowed) build up a subgroup. Let us denote it C; this is a simple consequence of the fact that $B/(B \cap \mathrm{Br}(X))$, just like B, is a torsion group. Let us consider the natural bilinear pairing

$$B/(B \cap \mathrm{Br}(X)) \times C \to \mathbf{Q}/\mathbf{Z}.$$

The above discussion shows that this pairing of finite abelian groups is nondegenerate on the LHS. Thus $B/(B \cap \mathrm{Br}(X))$ injects into the dual of C. Counting then shows that this possibly only if C coincides with the dual of $B/(B \cap \mathrm{Br}(X))$.

The assumption in the theorem ensures that the family $\{P_v\}_{v \in S}$ defines a linear map $B/(B \cap \mathrm{Br}(X)) \to \mathbf{Q}/\mathbf{Z}$, which is given by

$$\beta \mapsto -\sum_{v \in S} \beta(P_v).$$

We have just seen that this map may be written as a sum of maps φ_α (possibly with repetitions). Each of the terms in this last sum may be written as $\beta \mapsto \beta(N_v)$, this time without repetition on the places v – since each time we have an infinite set of places v at our disposal. We have thus found a finite set T of places $v \notin S$ and points $N_v \in U(k_v) \cap \mathcal{X}(O_v)$ for $v \in T$ such that

$$\sum_{v \in S} \beta(P_v) + \sum_{v \in T} \beta(N_v) = 0$$

for each $\beta \in B/(B \cap \mathrm{Br}(X))$, that is

$$\sum_{v \in S} \beta(P_v) + \sum_{v \in T} \beta(N_v) = 0$$

for each $\beta \in B$. We then have

$$\sum_{v \in S} \beta(P_v) + \sum_{v \in T} \beta(N_v) + \sum_{v \notin S \cup T} \beta(P_v) = 0$$

for each $\beta \in B$. This completes the proof, upon setting $S_1 = S \cup T$ and choosing $M_v = N_v$ for $v \in T$ and $M_v = P_v$ for $v \notin S \cup T$. □

5.2. **The formal lemma for torsors under a torus.** The following statement and proof appear in [17, Prop. 3.1]

Theorem 5.3. *Let U be a smooth, geometrically integral variety over a number field k. Let T be a k-torus. Let $Y \to U$ be a torsor over U under T, and let $\theta \in H^1(U,T)$ be its cohomology class. Let $B \subset \mathrm{Br}(U)$ be the finite subgroup consisting of cup-products $\theta \cup \gamma$ for γ runnning through the finite group $H^1(k,\hat{T})$. Let $\{M_v\} \in U(A_k)$ be a point which is orthogonal to $B \cap \mathrm{Br}_{nr}(k(U))$. Let $S \subset \Omega$ be a finite set. Then there exists $\alpha \in H^1(k,T)$ such that the twisted torsor Y^α has points in all completions of k and such that for each $v \in S$, the point M_v lies in the image of $Y^\alpha(k_v) \to U(k_v)$.*

Proof. According to Theorem 5.2, there exists another adèle $\{P_v\} \in U(A_k)$ with $M_v = P_v$ for $v \in S$ such that

$$\forall \gamma \in H^1(k,\hat{T}), \ \sum_{v\in\Omega} [\theta(P_v) \cup \gamma] = 0 \in \mathbf{Q}/\mathbf{Z}.$$

Thus $\{\theta(P_v)\} \in \oplus_{v\in\Omega} H^1(k_v,T)$ is orthogonal to $H^1(k,\hat{T})$, hence by the Poitou-Tate exact sequence for tori is the image of an element $-\alpha \in H^1(k,T)$ under the diagonal map $H^1(k,T) \to \oplus_{v\in\Omega} H^1(k_v,T)$. Twisting Y by α yields a torsor over Y under T which over each point P_v posssesses a k_v-point. □

Remark 5.2. In [31], Proof of Thm. 3.1, there is a similar argument with a stronger hypothesis and a stronger conclusion. There we have the extra condition $\overline{k}^\times = \overline{k}[U]^\times$. Starting from an element in $X(A_k)^{\mathrm{Br}}$, the outcome is then more powerful, in the situation described there we manage to produce an adèle on a suitable Y^α with the added property that it is orthogonal to (a suitable subgroup) of the unramified Brauer group of Y^α.

6. The Brauer-Manin obstruction for rational points on rationally connected varieties

Given a smooth, projective, geometrically integral variety X over a number field k, one would like to know whether $X(k)$ is dense in $X(A_k)^{\mathrm{Br}}$ – we shall then simply write $X(k)^{cl} = X(A_k)^{\mathrm{Br}}$.

In loose words, one asks whether the Brauer-Manin obstruction is the only obstruction to weak approximation – and in particular to the Hasse principle. More precisely, one would like to produce geometric types of varieties for which this holds.

The property $X(A_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k)$ not empty is preserved under k-birational invariance of smooth projective variety.

If $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite, then the property $X(k)^{cl} = X(A_k)^{\mathrm{Br}}$ is preserved under k-birational invariance.

When $\mathrm{Br}(X)/\mathrm{Br}(k)$ is infinite, the situation for weak approximaiton is unclear. A variant of the statement, where the connected component is smashed down, is wrong ! See [35, Remark 6.2 (2)].

6.1. **Rationally connected varieties.** Let k denote a field of characteristic zero.

By definition, a "rationally connected variety over k" is a smooth, projective, geometrically connected variety over k with the following property :

Over any algebraically closed field Ω containing k, any two Ω-points are connected by a rational curve, i.e. lie in the image of a morphism $\mathbf{P}^1_\Omega \to X_\Omega$.

These varieties, studied by Kollár, Miyaoka and Mori, and by Campana, are actually characterized by many equivalent properties.

In particular, in the above definition, one may simply assume that any two points are connected by a chain of rational curves, or even that two "general" points are connected by such a chain.

A standard reference is Kollár's book on rational curves on higher dimensional varieties.

A rationally connected variety of dimension 1 is a smooth conic.

A rationally connected variety of dimension 2 is a geometrically rational surface.

Any geometrically unirational variety is a rationally connected variety (the converse is an open question).

By a theorem of Campana and Kollár, Miyaoka, Mori, any Fano variety (smooth projective variety with ample anticanonical bundle) is rationally connected. This is thus the case for smooth hypersurfaces of degree $d \leq n$ in projective space $\mathbf{P}^n$.

By a theorem of Enriques, Manin, Iskovskikh, Mori, any rational k-surface is k-birational to at least one of :

(i) Smooth del Pezzo surface of degree d, with $1 \leq d \leq 9$.

(ii) Conic bundle (with degeneracies) over a conic.

Del Pezzo surfaces of degree $d \geq 5$ are arithmetically simple. They satisfy the Hasse principle, and they are k-rational as soon as they have a k-point. The conjecture thus holds for them.

For a rationally connected variety X over an arbitrary field k, the quotient group $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.

For surfaces, the following conjecture was put forward as an open question by Sansuc and myself (1980). The general question was raised in lectures of mine in 1990.

Conjecture **Let X be a smooth, projective, geometrically integral variety over a number field k. If X is geometrically rationally connected, then the image of $X(k)$ is dense in $X(A_k)^{\mathrm{Br}}$.**

The conjecture is birational invariant.

For conic bundles over a conic, there is theoretical evidence for the conjecture. Schinzel's hypothesis implies the conjecture (see below).

For conic bundles over the projective line with $r \leq 5$ geometric degenerate fibres, the conjecture is known. The case $r \leq 3$ is trivial. For Châtelet surfaces, which satisfy $r = 4$, the conjecture was proven by CT-Sansuc-Swinnerton-Dyer. The general case with $r = 4$ is due to CT (and Salberger unpublished), the case $r = 5$ is due to Salberger and Skorobogatov. Swinnerton-Dyer also discusses this case, as well as some specific cases with $r = 6$.

For Del Pezzo surfaces of degree 4 with a k-point the conjecture is known (Salberger and Skorobogatov). This is one case where theorems about zero-cycles ultimately lead to results on rational points.

For the local-global problem for Del Pezzo surfaces of degree 4, Wittenberg's thesis, developing a method of Swinnerton-Dyer, produces strong evidence – depending on Schinzel's hypothesis and finiteness of Tate-Shafarevich groups of elliptic curves.

In higher dimension, the case of intersections of two quadrics has been much discussed (CT-Sansuc-Swinnerton-Dyer, Heath-Brown). When the number of variables is large with respect to the degree, the circle method applies. The circle method also gives good results in relatively low dimension for cubic hypersurfaces (Heath-Brown, Hooley). See also recent work of Tim Browning and his collaborators.

For X birational to a homogeneous space E under a connected linear algebraic group with connected geometric isotropy groups, it is a theorem of Borovoi, building upon earlier work of Sansuc, that $X(k)$ is dense in $X(A_k)^{\mathrm{Br}}$.

6.2. **What about other types of varieties ?** Let X be an arbitrary smooth, projective, geometrically integral variety. Let $X(A_k)_{\bullet}^{\mathrm{Br}}$ be defined by replacing each $X(k_v)$ for v archimedean by $\pi_0(X(k_v))$ (the set of connected components).

For X a curve, it is an open question whether the image of $X(k)$ is dense in $X(A_k)_{\bullet}^{\mathrm{Br}}$. For X a curve of genus one, this is the case if the Tate-Shafarevich group of the jacobian of X is finite. For X a curve of higher genus whose jacobian variety J satisfies $Sha(J)$ finite (expected) and also $J(k)$ finite, it is a theorem of Scharashkin and (independently) Skorobogatov that $X(k) = X(A_k)_{\bullet}^{\mathrm{Br}}$.

In higher dimension, there are by now many examples for which $X(k)$ is not dense in $X(A_k)_{\bullet}^{\mathrm{Br}}$, in particular examples where $X(A_k)^{\mathrm{Br}} \neq \emptyset$ but $X(k) = \emptyset$. The first examples (Skorobogatov, Harari) could be explained by the refined étale Brauer-Manin obstruction, which produces a closed subset

$$X(A_k)^{\mathrm{Br},et} \subset X(A_k)^{\mathrm{Br}}.$$

Further examples (Poonen, Harpaz-Skorobogatov, CT-Pál-Skorobogatov, Smeets) cannot even be explained by the étale Brauer-Manin obstruction.

For $K3$ surfaces, it is an open question whether $X(k)$ is dense in $X(A_k)^{\mathrm{Br}}$. There is work going on in this direction, particularly for surfaces which are geometrically Kummer.

For Enriques surfaces, for which examples with $X(A_k)^{\mathrm{Br},et} = \emptyset$, hence $X(k) = \emptyset$ and $X(A_k)^{\mathrm{Br}} \neq \emptyset$ have very recently been produced
(`http://arxiv.org/abs/1501.04974v1`)
one may ask whether $X(k)$ is dense in $X(A_k)^{\mathrm{Br},et}$.

If X is rationally connected, then the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite. The closed set $X(A_k)^{\mathrm{Br}} \subset X(A_k)$ is thus open. In particular, if the conjecture holds, then weak weak approximation holds for X.

If a smooth, projective, geometrically connected variety X over a number field satisfies weak weak approximation over any finite extension of k, then its geometric fundamental group is trivial (Harari). In particular $H^1(X, O_X) = 0$.

7. Rational points : applications of Schinzel's hypothesis, of a new hypothesis by Harpaz and Wittenberg, and of recent results in additive number theory

7.1. **Using Schinzel's hypothesis.** In a series of papers (CT-Sansuc [39], Serre (cf. [47]), Swinnerton-Dyer [136], CT-Swinnerton-Dyer [47], CT-Skorobogatov-Swinnerton-Dyer [45]), consequences of Schinzel's hypothesis were explored.

Let us recall the statement of Schinzel's hypothesis, which is an elaboration on conjectures of Bouniakowsky, Dickson, and also Hardy and Littlewood.

Hypothesis (H) Let $P_i(x) \in \mathbf{Z}[x]$, $i = 1, \ldots, n$, be irreducible polynomials with positive leading coefficients. Assume that no prime divides all $\prod_{i=1}^n P_i(m)$ for $m \in \mathbf{Z}$. Then there exists infinitely many $m \in \mathbb{N}$ such that each $P_i(m)$ is a prime number.

Here is a simple case, taken from [39].

Theorem 7.1. *(CT–Sansuc, 1979) Let $a \in \mathbf{Q}$, $a > 0$ and $P(x) \in \mathbf{Q}[x]$. Under Schinzel's hypothesis, if $P(x)$ is irreducible, then Hasse principle and weak approximation hold for the variety X defined by*

$$y^2 - az^2 = P(x) \neq 0$$

Proof. Let S be a the following finite set of finite primes : $p = 2$, primes p with $v_p(a) \neq 0$, primes p with $P(x) \notin \mathbf{Z}_p[x]$, other primes for which the reduction modulo p has smaller degree than the degree of $P(x)$ or is not separable, and finally primes $p \leq \deg(P)$. For each $p \in S$, one fixes $\lambda_p \in \mathbf{Q}_p$ such that $0 \neq P(\lambda_p)$ is represented by $y^2 - az^2$ over $\mathbf{Q}_p$.

Using the irreducibility of $P(x)$ and Schinzel's hypothesis, one finds $\lambda \in \mathbf{Q}$ very close to each λ_p for $p \in S$ and such that

$$P(\lambda) = \prod_{p \in S} p^{n_p}.q \in \mathbf{Q}$$

with $n_p \in \mathbf{Z}$ and q a prime not in S ("the Schinzel prime").

The rational number $0 \neq P(\lambda)$ is then represented by the quadratic form $y^2 - az^2$ over each completion of $\mathbf{Q}$ (including the reals, since we assumed $a > 0$), except possibly in $\mathbf{Q}_q$. The law of quadratic reciprocity then implies that $P(\lambda)$ is represented by this form over $\mathbf{Q}_q$ and over $\mathbf{Q}$. Using weak approximation on the affine conic $y^2 - az^2 = P(\lambda)$ and the implicit function theorem, one concludes that weak approximation holds for X. □

It is clear how this proof is a direct generalization of Hasse's proof of the local-global principle for zeros of quadratic forms in 4 variables, starting from the case of 3 variables.

A general result is the following ([45, Thm. 1.1]) :

Theorem 7.2. *Let k be a number field. Let $X \to \mathbf{P}^1_k$ be a dominant map of smooth, projective, geometrically integral varieties. Assume that the generic fibre is geometrically integral and that each special fibre $X_m/k(m)$ contains a component of multiplicity one Y such that the integral closure of $k(m)$ in the function field of Y is an abelian extension. Under Schinzel's hypothesis (H), if $X(A_k)^{\mathrm{Br}_{vert}} \neq \emptyset$, then there exists $c \in \mathbf{P}^1(k)$ such that X_c is smooth and has points in all completions of k.*

In the proof, the same reciprocity argument as in Hasse's proof is used : at some point, one produces a cyclic extension L/K of number fields and an element in $K^\times$ which is a norm locally at all places of K except possibly one, and then one concludes that the element is a global norm. The cyclic extensions in the argument come from the abelian extensions L/K mentioned in the statement of the theorem.

7.2. **Additive combinatorics come in.** An enormous breakthrough happened in 2010. Work of B.Green and T.Tao, followed by further work with T. Ziegler (2012), proves something which is essentially a two variable version of the Schinzel hypothesis, when restricted to a system of polynomials of total degree one over $\mathbf{Z}$.

The initial results of Green and Tao, together with further work by L. Matthiesen on additive combinatorics, first led to unconditional results in the spirit of "Schinzel implies Hasse". This is the work of Browning, Matthiesen and Skorobogatov [17] A typical result is the proof of (WABM) for conic bundles over $\mathbf{P}^1_{\mathbf{Q}}$ when all the singular fibres are above $\mathbf{Q}$-rational points of $\mathbf{P}^1_{\mathbf{Q}}$. They also prove a similar result for the total space of quadric bundles of relative dimension 2 over $\mathbf{P}^1_{\mathbf{Q}}$. These results were spectacular, indeed up till then, for most such $\mathbf{Q}$-varieties, we did not know that existence of one rational point implies that the rational points are Zariski dense – unless one was willing to accept Schinzel's hypothesis.

The work of Green, Tao and Ziegler, led to further progress. Here is the exact result used, which I reproduce from [79] :

Theorem 7.3. *(Green, Tao, Ziegler) Let $L_1(x,y),\dots,L_r(x,y) \in \mathbf{Z}[x,y]$ be pairwise nonproportional linear forms, and let $c_1,\dots,c_r \in \mathbf{Z}$. Assume that for each prime p, there exists $(m,n) \in \mathbf{Z}^2$ such that p does not divide $L_i(m,n)+c_i$ for any $i=1,\dots,r$. Let $K \subset \mathbb{R}^2$ be an open convex cone containing a point $(m,n) \in \mathbf{Z}^2$ such that $L_i(m,n) > 0$ for $i=1,\dots,r$. Then there exist infinitely many pairs $(m,n) \in K \cap \mathbf{Z}^2$ such that each $L_i(m,n)+c_i$ is a prime.*

Harpaz, Skorobogatov and Wittenberg [79] deduced a host of results on (WABM) from that theorem. Let me describe the argument in a simple case.

Theorem 7.4. *Let $k = \mathbf{Q}$. Let U be the surface*

$$y^2 - az^2 = b.\prod_{i=1}^{2n}(t-e_i) \neq 0,$$

where $a,b \in \mathbf{Q}^\times$ and the e_i are distinct elements in $\mathbf{Q}$. Let X be a smooth projective model, let $\{M_v\} \in X(A_k)^{Br}$, and let $W \subset X(A_k)$ be a neighbourhood of that *point. Then there exists a k-point of U in that neighbourhood. A consequence is that $\mathbf{Q}$-points are Zariski-dense on U, and that weak-weak approximation holds.*

Proof. The argument used to prove Theorem 7.2 reduces to Schinzel's hypothesis (H) for the system of polynomials $t - e_i$, which is not available.

We shall resort to a simple but slightly mysterious trick to get two variables.

Introduce two variables (u, v) and set $t = u/v$.

Consider the variety V given by

$$Y^2 - aZ^2 = b.\prod_i (u - e_i v) \neq 0$$

$$v \neq 0$$

The formulas $y = Y/v^n$, $z = Z/v^n$, $t = u/v$ produce an isomorphism between V and $U \times \mathbf{G}_m$, where the coordinate on the last factor is v.

For $V \subset V_c$ a smooth compactification, we get a k-birational isomorphism between $X \times \mathbf{P}^1_k$ and V_c. Brauer groups do not change under multiplication by $\mathbf{P}^1$.

For X as considered here, $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite. After moving $\{M_v\}$ in the given neighbourhood, we may assume that each M_v is in U and we may find find an adèle $\{N_v\} \in V_c(\mathbf{A}_k)^{Br}$ all components of which are in V above $\{M_v\}$.

One then uses Harari's formal lemma for the finite family of quaternion algebras $(a, u - e_i v)$ on V.

This produces elements $c_i \in k^\times$ and an adèle on the variety given by the system

$$Y^2 - aZ^2 = b.\prod_i (u - e_i v) \neq 0$$

$$y_i^2 - az_i^2 = c_i(u - e_i v) \neq 0, i = 1, \ldots, 2n$$

above the point $\{N_v\}$.

That system is isomorphic to the product of the conic $Y^2 - aZ^2 = b.\prod_i c_i$, which satsifies HP and WA, and the variety given by

$$y_i^2 - az_i^2 = c_i(u - e_i v) \neq 0, i = 1, \ldots, 2n.$$

Now we use Green-Tao-Ziegler for the family of linear forms $\{u - e_i v\}$ in place of the unproved Schinzel's hypothesis (H) for the family $\{t - e_i\}$ to prove – unconditionnally – Hasse principle and weak approximation for this last variety. □

Harpaz, Skorobogatov and Wittenberg [79] prove the following general result.

Theorem 7.5. *Let X be a smooth, proper, geometrically connected $\mathbf{Q}$-variety equipped with a dominant morphism $f : X \to \mathbf{P}^1_{\mathbf{Q}}$ such that*

(i) the generic fibre of f is geometrically integral;

(ii) the only nonsplit fibres X_m are above $\mathbf{Q}$-rational points m of $\mathbf{P}^1_{\mathbf{Q}}$ and each such fibre contains a component Y of multiplicity one such that the integral closure of $\mathbf{Q}$ in $\mathbf{Q}(Y)$ is an abelian extension of $\mathbf{Q}$;

(iii) the Hasse principle and weak approximation hold for the smooth fibres.

Then $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}}$.

Here are concrete examples :

Corollary 7.6. *Let $K_i/\mathbf{Q}$, $i = 1, \ldots, r$ be cyclic extensions. Let $P_i(t)$, $i = 1, \ldots, r$, be nonzero polynomials all roots of which are in $\mathbf{Q}$. Let $X/\mathbf{Q}$ be a smooth projective model of the $\mathbf{Q}$-variety given by the system of equations*

$$N_{K_i/\mathbf{Q}}(\Xi_i) = P_i(t) \neq 0, \ \ i = 1, \ldots, r.$$

Then X satisfies (WABM) : $X(\mathbf{Q})$ is dense in $X(A_{\mathbf{Q}})^{\mathrm{Br}}$.

Corollary 7.7. *Let $K_i/\mathbf{Q}$, $i = 1, \ldots, r$ be cyclic extensions. Let $b_i \in \mathbf{Q}^\times$ and $e_i \in \mathbf{Q}$, $i = 1, \ldots, r$. Then the $\mathbf{Q}$-variety given by the system of equations*

$$N_{K_i/\mathbf{Q}}(\Xi_i) = b_i(t - e_i) \neq 0, \ \ i = 1, \ldots, r$$

satisfies the Hasse principle and weak approximation.

To put this result in perspective, here is what was known before 2010. The case $r = 1$ is obvious. The case $r = 2$ and K_1 and K_2 both of degree 2 reduces to quadrics. An ancient result obtained by the circle method (Birch, Davenport, Lewis) gave this for $r = 2$ and $K_1 = K_2$ of arbitrary degree over $\mathbf{Q}$. The case $r = 3$ and $K_1 = K_2 = K_3$ of degree 2 over $\mathbf{Q}$ was covered by CT, Sansuc, Swinnerton-Dyer [41]. Much more was not known.

All the results above ultimately discussed the total space of a one-parameter family $X \to \mathbf{P}^1_k$ which satisfied :

(i) *The smooth fibres satisfy the Hasse principle and weak approximation*.

(ii) Each nonsplit fibre X_m contains a component of multiplicity one Y such that the algebraic closure of $k(m)$ in $k(Y)$ is *abelian*.

(iii) $k = \mathbf{Q}$ and the nonsplit fibres are over $\mathbf{Q}$-rational points. (this last result to be able to use Green, Tao, Ziegler).

[A fibre $X_m/k(m)$ over a closed point m with residue field $k(m)$ is called *split* if it contains a component of multiplicity one $Y/k(m)$ which is geometrically integral, in other words such that the field $k(m)$ is algebraically closed in the function field $k(Y)$.]s

7.3. Harpaz and Wittenberg's theorems on rational points.

Harpaz and Wittenberg [80] have produced a conjecture which leads to a result where conditions (i) and (ii) have disappeared, conjecture which work of Browning and Matthiesen has established in situations analogous to that of Green, Tao, Ziegler.

Conjecture H^{**} *Let k be a number field. Let $n \geq 1$ be an integer and $P_1, \ldots, P_n \in k[t]$ denote pairwise distinct irreducible monic polynomials. Let $k_i = k[t]/(P_i(t))$ and let $a_i \in k_i$ denote the class of t. For each $i \in \{1, \ldots, n\}$ suppose given a finite extension L_i/k_i and an element $b_i \in k_i^\times$. Let S be a finite set of places of k containing the real places of k and the finite places above which, for some i, either b_i is not a unit or L_i/k_i is ramified. Finally, for each $v \in S$, fix an element $t_v \in k_v$. Assume that for every $i \in \{1, \ldots, n\}$ and every $v \in S$, there exists $x_{i,v} \in (L_i \otimes_k k_v)^\times$ such that*

$$t_v - a_i = b_i.N_{L_i \otimes_k k_v/k_i \otimes_k k_v}(x_{i,v}) \in k_i \otimes_k k_v.$$

Then there exists $t_0 \in k$ satisfying the following conditions:

(1) t_0 is arbitrarily close to t_v for $v \in S$;

(2) for every $i \in \{1, ..., n\}$ and every finite place w of k_i with $w(t_0 - a_i) > 0$, either w lies above a place of S or the field L_i possesses a place of degree 1 over w.

This is conjecture 9.1 in [80]. Let $\varepsilon = \sum_i [k_i : k]$.

Over an arbitrary number field, the conjecture is known under any of the following hypotheses :

(i) $\varepsilon \leq 2$. The essential ingredient is strong approximation. In the case $k_1 = k_2 = k$, one may also give a proof using Dirichlet's theorem in a suitable field extension of the ground field. If one wants to control the situation at the real places, this method requires the use of a theorem of Waldschmidt.

(ii) $\varepsilon = 3$ and $[L_i : k_i] = 2$ for each i.

An important recently established case is :

(iii) $k = \mathbf{Q}$, all $k_i = \mathbf{Q}$, $\varepsilon = n$ arbitrary, each $L_i/k_i = \mathbf{Q}$ arbitrary.

This last case is a theorem of Lilian Matthiesen, in the wake of the results in additive combinatorics of Green, Tao, Ziegler and of her joint work with Tim Browning [16]. See [80, Theorem 9.14].

In [80, Prop. 9.9, Cor. 9.10], we find closely related conjectures which are possibly more appealing than conjecture H^{**}. The authors produce specific quasi-affine varieties W. If strong approximation off any finite place v_0 holds for these varieties, then conjecture H^{**} holds.

In the particular case where each $k_i = k$, W is an open set of a variety with equation

$$u - a_i v = b_i Norm_{L_i/k}(\Xi_i), i = 1 \ldots, r.$$

Here the a_i and b_i are in k, $a_i \neq a_j$ for $i \neq j$ and $b_i \neq 0$. The open set W is defined by deleting points with $(u, v) = (0, 0)$ and deleting points $(u, v, \Xi_1, \ldots, \Xi_{r)}$ for which, for some i, one has Ξ_i belongs to the singular locus of $R_{L_i/k}G_a \setminus R_{L_i/k}G_m$ (which implies $u - a_i v = 0$).

Harpaz and Wittenberg [80, Thm. 9.17] prove :

Theorem 7.8. *Let k be a number field. Let $X \to \mathbf{P}^1_k$ be a dominant map of smooth, projective, geometrically integral varieties. Assume that the generic fibre is geometrically integral and that the generic fibre is a rationally connected variety. In particular (Graber, Harris, Starr [60]) each special fibre contains a component of multiplicity one. Assume $X(A_k)^{\mathrm{Br}} \neq \emptyset$. Then, under conjecture H^{**}, there exists $t_0 \in \mathbf{P}^1(k)$ with smooth fibre X_{t_0} such that $X_{t_0}(A_k)^{\mathrm{Br}}$ is non-empty and satisfies an approximation condition at finitely many places v with respect to any given point in $X(A_k)^{\mathrm{Br}}$.*

Corollary 7.9. *Let k be a number field. Let X/k be a smooth, projective, geometrically connected variety equipped with a morphism $X \to \mathbf{P}^1_k$ such that the generic* *fibre is birational over $k(\mathbf{P}^1)$ to a homogeneous space of a connected linear $k(\mathbf{P}^1)$-algebraic group, with connected geometric stabilizers. Assume hypothesis H^{**}. Then $X(k)$ is dense in $X(A_k)^{\mathrm{Br}}$.*

That result applies in particular to smooth projective models X of varieties given by an arbitrary system of equations

$$Norm_{K_i/k}(\Xi_i) = P_i(t) \neq 0 \ \ i = 1, \ldots, n$$

which have been considered in many special situations.

Remark 7.1. One nontrivial algebraic problem is to decide when such a system satisfies $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$. If the $P_i(t)$ are all of degree 1 and no two of them are proportional, do we have $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$?

7.4. **Main steps of a proof of Theorem 7.8.** As mentioned by the authors in [80, Remark 9.18 (i)], if one is willing to take conjecture H^{**} for granted for an arbitrary set of polynomials $P_i(t)$, one may write down a lighter proof than the one they offer. This is what I shall do here, under some simplifying assumptions. The proof I give slightly differs from the proof in [80], it uses Severi-Brauer schemes.

Theorem 7.10. *Let k be a number field. Let $X \to \mathbf{P}^1_k$ be a dominant map of smooth, projective, geometrically integral varieties. Assume that the generic fibre is geometrically integral and that the fibration is geometrically split, I.e. each geometric special fibre contains a component of multiplicity one. Under conjecture H^{**}, if $X(\mathbf{A}_k)^{\mathrm{Br}_{vert}} \neq \emptyset$, then there exists $t_0 \in k = \mathbf{A}^1(k)$ such that X_{t_0} is smooth and has points in all completions of k. Moreover, given a finite set S of places of k, and a point $\{M_v\} \in X(\mathbf{A}_k)^{\mathrm{Br}_{vert}}$, one may find a t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Proof. For simplicity of notation, let us consider only the case where the fibre at infinity is smooth and the nonsplit fibres occur only above k-points of $\mathbf{A}^1_k = \mathrm{Spec} k[t]$, given by $t = a_i$, $i = 1 \ldots, n$. We also simply discuss the existence of a t_0 with $X_{t_0}(\mathbf{A}_k) \neq \emptyset$.

For each i, fix an irreducible component E_i of multiplicity one and let L_i be the integral closure of k in the field of functions of that component. Let $U \subset X$ be the complement of the fibre at infinity and the fibres over the points $t = a_i$. Let T be the k-torus $\prod_i R^1_{L_i/k}\mathbf{G}_m$. Consider the torsor over U under T given by the equations

$$0 \neq t - a_i = Norm_{L_i/k}(\Xi_i), i = 1, \ldots, n.$$

Applying the formal lemma for torsors (Theorem 5.3), and noticing that the given torsor over U comes from a torsor over an open set of $\mathbf{P}^1_k$, hence the group B mentioned in the proof of Theorem 5.3 consists in this particular case of elements coming from $\mathrm{Br}(k(\mathbf{P}^1))$, hence are vertical elements, we find elements $b_i \in k^\times$ and a family $\{M_v\} \in U(\mathbf{A}_k)$, with projections $t_v \in k_v, v \in \Omega$, such that for each v the system

$$0 \neq t_v - a_i = b_i.Norm_{L_i/k}(\Xi_i), i = 1, \ldots, n$$

has solutions over k_v.

Now we inject Conjecture H^{**}. We choose a *finite* set S of places of k, large enough for various purposes. We include all archimedean places and all finite places v with $v(a_i) < 0$ for some i. We demand that each L_i/k be unramified at $v \notin S$, that each b_i be a unit at $v \notin S$, that the fibre at infinity has good reduction at $v \notin S$ and has points in all k_v for $v \notin S$ (possible by Lang-Weil-Nisnevich, since that fibre is smooth and geometrically integral). We demand that $E_{i,smooth}$, which is geometrically integral over L_i, has points in all $L_{i,w}$ for w place of L_i not above a place of S.

Conjecture H^{**} then produces a $t_0 \in k$ very close to t_v for $v \in S$, and such that for any i and any $v \notin S$ either $v(t_0 - a_i) \leq 0$ or there exists a place of L_i of degree 1 over v.

Claim : the fibre X_{t_0} has points in all completions k_v of k.

For $v \in S$, this is a consequence of the implicit function theorem.

If v is not in S and $v(t_0 - a_i) < 0$ for some i, then $v(t_0) < 0$, and t_0 specializes at v to the same smooth $\kappa(v)$-variety as the fibre X_∞, hence has k_v-points.

If v is not in S and $v(t_0 - a_i) = 0$ for each i, then t_0 does not specialize to any of the specialisations of the points a_i and hence (provided S had been chosen reasonable at the beginning) specializes to a smooth, geometrically integral variety over the finite field $\kappa(v)$, of a given "type" (Hilbert polynomial), hence by Lang-Weil-Nisnevich, has a $\kappa(v)$-point, hence X_{t_0} has a k_v-point.

Finally, let us assume that for some $v \notin S$, we have $v(t_0 - a_i) > 0$ for some i. Then X_{t_0} specializes as X_{a_i} over the field $\kappa(v)$. However, by the conclusion of conjecture H^{**}, v has an extension w of degree 1 to the field L_i over which the component E_i is geometrically integral. Again provided that S was chosen big enough, that component admits a reduction over the field $\kappa(w) = \kappa(v)$ which is geometrically integral and hence (Lang-Weil-Nisnevich) possesses a smooth $\kappa(v)$-point. Using Hensel's lemma, we conclude that X_{t_0} contains a k_v-point. □

Assume that the smooth fibres satisfy the Hasse principle and weak approximation. Then, granting H^{**}, the above result immediately implies that $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}_{vert}}$, hence in $X(\mathbf{A}_k)^{\mathrm{Br}}$ – which then coincides with $X(\mathbf{A}_k)^{\mathrm{Br}_{vert}}$. Such a general result was out of reach of the theorems based on Schinzel's hypothesis.

The following corollary had already been obtained via a descent method [42, Theorem A]. In its turn, the latter result improved upon [45, §2.2].

Corollary 7.11. *Let k be a number field. Let $f : X \to \mathbf{P}^1_k$ be a dominant map of smooth, projective, geometrically integral varieties. Assume that the generic fibre is geometrically integral and that each special fibre contains a component of multiplicity one. Assume that $\delta(f) \leq 2$. If $X(\mathbf{A}_k)^{\mathrm{Br}_{vert}} \neq \emptyset$, then there exists $t_0 \in k = \mathbf{A}^1(k)$ such that X_{t_0} is smooth and has points in all completions of k. Moreover, given a finite set S of places of k, and a point $\{M_v\} \in X(\mathbf{A}_k)^{\mathrm{Br}_{vert}}$, one may find a t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Proof. In the case where the nonsplit fibres correspond to two k-rational points, in the proof of Theorem 7.10, this corresponds to the case $n = 2$, hence $\varepsilon = 2$, in which case H^{**} is known. □

In the sequel we make a free use of resolution of singularities in characteric zero.

Lemma 7.12. *Let X be a smooth, projective, geometrically connected variety. Let Y and Z be two smooth projective varieties, with dominant morphisms $Y \to X$ and $Z \to X$. Assume that the generic fibre of each of these morphisms is a Severi-Brauer variety, with associated class $\alpha_Y \in \mathrm{Br}(k(X))$ and $\alpha_Z \in \mathrm{Br}(k(X))$. Suppose $\alpha_Y - \alpha_Z \in \mathrm{Br}(k)$. For a class $\beta \in \mathrm{Br}(k(X))$, the following are equivalent :*
(i)The image of β in $\mathrm{Br}(k(Y))$ is unramified.
(ii) The image of β in $\mathrm{Br}(k(Z))$ is unramified.

Proof. The fibre product $Y \times_X Z$ is stably Z-birational to a constant Severi-Brauer scheme $Y' \to Z$ and stably Y-birational to a constant Severi-Brauer scheme $Z' \to Y$. The k-varieties Y' and Z' are stably k-birational to each other.

A class in $\mathrm{Br}(k(Y))$ is unramified over Y if and only if its image is unramified over Z'. Indeed for a constant Severi-Brauer scheme $Y' \to Z$, all fibres at codimension 1 points of Z are geometrically integral.

A class in $\mathrm{Br}(k(Z))$ is unramified over Z if and only if its image is unramified over Y'. Now Y' and Z' are stably k-birational to each other. Thus a class $\beta \in \mathrm{Br}(k(X))$ has unramified image in $\mathrm{Br}(k(Y))$ if and only if it has unramified image in $\mathrm{Br}(k(Z))$. □

There is a similar result with a finite product $\prod_X Y_i$ and a finite product $\prod_X Z_i$ of varieties Y_i, Z_i generically Severi-Brauer over X such that each class $\alpha_{Y_i} - \alpha_{Z_i} \in \mathrm{Br}(k(X))$ comes from $\mathrm{Br}(k)$.

Theorem 7.13. *Let k be a number field. Let $f : X \to \mathbf{P}^1_k$ be a dominant map of smooth, projective, geometrically integral varieties. Assume that the generic fibre is geometrically integral and that the fibration is geometrically split. Let $U \subset \mathbf{P}^1_k$ be a non-empty open set such that X_U/U is smooth, and let $B \subset \mathrm{Br}(X_U)$ be a finite subgroup. Let $\{M_v\} \in X(\mathbf{A}_k)$ be orthogonal to $[B + f^*(\mathrm{Br}(k(\mathbf{P}^1)))] \cap \mathrm{Br}(X)$. Let S be a finite set of places. Then, under conjecture H^{**}, there exists $t_0 \in U(k)$ such that $X_{t_0}(\mathbf{A}_k)^B$ is non-empty and contains a point $\{S_v\}$ with S_v close to M_v for each $v \in S$.*

Proof. Let $B \subset \mathrm{Br}(X_U)$ be spanned by the classes of finitely many Azumaya algebras A_i over X_U.

Let $Y_U \to X_U$ be the fibre product of the corresponding Severi-Brauer schemes. Upon resolution of singularities, this may be completed to a $g : Y \to X$, with Y/k smooth and projective. The composite fibration $h : Y \to X \to \mathbf{P}^1_k$ is geometrically split.

Indeed, this condition is equivalent with the property that h is locally split for the étale topology on $\mathbf{P}^1_k$. To check this, we may assume $k = \overline{k}$. Given a point $m \in \mathbf{P}^1$, there exist a connected étale neighbourhood $V \to \mathbf{P}^1$ whose image contains m and over which $X_V \to V$ admits a section $V \to X_V$, whose image is an integral curve $W \subset X_V$ whose image in V is dense. The map $Y \to X$ restricts to a map $Y_V \to X_V$. The restriction of this map to $W \subset X_V$, over the generic point of the curve W, is a product of Severi-Brauer varieties. Tsen's theorem then gives a rational section from $W \to Y_W$, which must be a morphism since W is a regular curve and $Y_W \to W$ is proper.

Let us go back to the case where k is a number field.

Since the composite fibration $h : Y \to X \to \mathbf{P}^1_k$ is geometrically split, the set of elements of $\mathrm{Br}(U)/\mathrm{Br}(k)$ which become unramified over Y is finite. Let $\gamma_j \in \mathrm{Br}(U)$ be a finite set of representatives.

Assume that we have an adelic point $\{M_v\} \in X(A_k)$ which is orthogonal to $B' = [B + f^*(\mathrm{Br}(k(\mathbf{P}^1)))] \cap \mathrm{Br}(X)$. Since $B'/\mathrm{Br}(k))$ is finite modulo $\mathrm{Br}(k)$, we may replace this adèle by an adèle $\{M_v\} \in X_U(A_k)$, still orthogonal to B', and with each new M_v close to the previous M_v.

By the formal lemma we may assume $\{M_v\} \in X_U(A_k)$ and

$$\sum_v A_i(M_v) = 0$$

and

$$\sum_v \gamma_j(M_v) = 0.$$

By class field theory, there exists $\rho_i \in \mathrm{Br}(k)$ with $\rho_{i,v} = A_i(M_v)$ for each place v.

Let $A'_i = A_i - \rho_i \in \mathrm{Br}(X_U)$ and choose Azumaya representatives over X_U. Consider the associated $Y'_U \to X_U$ and let $Y' \to X$ be some extension as above.

We now have points $\{N_v\} \in Y'_U(A_k)$ above the point $\{M_v\} \in X_U(A_k)$. Claim : such points are orthogonal to $\mathrm{Br}_{vert}(Y')$ (where "vert" here refers to the projection $Y' \to \mathbf{P}^1_k$). Indeed $\mathrm{Br}_{vert}(Y')$ is included in the image of the elements of $\mathrm{Br}(U)$ which become unramified on Y'. By Lemma 7.12, these elements of $\mathrm{Br}(U)$ are exactly those which become unramified on Y. Modulo $\mathrm{Br}(k)$, this group is spanned by the classes γ_j, and we had

$$\sum_v \gamma_j(M_v) = 0,$$

which over Y'_U gives

$$\sum_v \gamma_j(N_v) = 0,$$

If we now assume hypothesis H^{**} and apply[1] Theorem 7.10 to the fibration $Y' \to \mathbf{P}^1$, we find that there exists $t_0 \in U(k)$ such that the fibre Y'_{t_0} has an adelic point $\{R_v\}$, with R_v close to N_v for $v \in S$.

Let $S_v \in X_{t_0}(k_v)$ be the projection of R_v under $Y'_{t_0} \to X_{t_0}$.

For each i and each v we have $A'_i(S_v) = 0$, hence $A_i(S_v) = \rho_{i,v}$. Thus

$$\sum_v A_i(S_v) = 0,$$

with S_v close to M_v for $v \in S$. □

[1]Here we cannot restrict to the simplifying assumption made in the proof given above for that theorem.

To conclude, let us sketch the proof of the general theorem by Harpaz and Wittenberg, which we here repeat for the convenience of the reader.

Theorem 7.14. *(Harpaz and Wittenberg)* [80, Thm. 9.17] *Let k be a number field. Let $X \to \mathbf{P}^1_k$ be a dominant map of smooth, projective, geometrically integral varieties. Assume that the generic fibre is geometrically integral and that the generic fibre is a rationally connected variety, and that the fibration is smooth over the Zariski open set $U \subset P^1_k$. Assume $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. Then, under conjecture H^{**}, there exists $t_0 \in U(k)$ such that $X_{t_0}(\mathbf{A}_k)^{\mathrm{Br}}$ is non-empty and satisfies an approximation condition at finitely many places v with respect to any given point in $X(\mathbf{A}_k)^{\mathrm{Br}}$.*

Proof. (Sketch) By the Graber, Harris, Starr theorem [60]), each special fibre contains a component of multiplicity one. Also, the group $\mathrm{Br}(X_\eta)/\mathrm{Br}(k(t))$ is finite. One fixes an open set $U \subset \mathbf{P}^1_k$ such that X_U/U is smooth and there is a finite group $B \subset \mathrm{Br}(X_U)$ which spans $\mathrm{Br}(X_\eta)/\mathrm{Br}(k(t))$. Then one looks for a t_0 as in the previous theorem, with the extra condition that the image of B spans the finite group $\mathrm{Br}(X_{t_0})/\mathrm{Br}(k)$. By Harari's specialisation result ([62, §3] and [64, Thm. 2.3.1], see also [80, Prop. 4.1]), the set of k-points such that the last condition is fulfilled is a Hilbert set. The question is thus to show that in the previous theorem one may require t_0 to lie in a Hilbert set. We refer here to [80, Thm. 9.22], which uses [130, Prop. 6.1]. □

Building upon the results in additive combinatorics one then obtains the following *unconditional* result, first proven by Skorobogatov [125]. Skorobogatov's proof (of a slightly more general result) also uses the result of Browning and Matthiesen [16] on systems of equations

$$u - a_i v = b_i Norm_{L_i/k}(\Xi_i), i = 1 \ldots, r,$$

obtained using additive combinatorics, but his argument looks somewhat different. He uses descent and universal torsors ([40]). In the present argument, descent has been replaced by the use of the formal lemma for torsors.

Theorem 7.15. *(Skorobogatov) Let $X/\mathbf{Q}$, with a dominant morphism $X \to \mathbf{P}^1_{\mathbf{Q}}$ be a smooth projective model of a variety given by a system*

$$Norm_{K_i/\mathbf{Q}}(\Xi_i) = P_i(t) \neq 0 \;\; i = 1, \ldots, n,$$

where the $K_i/\mathbf{Q}$ are arbitrary field extensions, and each polynomial $P_i(t)$ has all its solutions in $\mathbf{Q}$, and $X \to \mathbf{P}^1_{\mathbf{Q}}$ extends projection to the t-coordinate. Then $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}}$. In particular, if $X(\mathbf{Q})$ is not empty, then it is Zariski dense in X, and weak weak approximation holds for X : there exists a finite set T of places such that for any finite set S of places disjoint from T, $X(\mathbf{Q})$ is dense in the product $\prod_{v \in S} X(\mathbf{Q}_v)$.

Proof. Let U be the complement of ∞ and the zeros of the polynomials $P_i(t)$. To prove the theorem, we may choose the model. Since tori admit smooth equivariant compactifications, we may produce a model $X \to \mathbf{P}^1_{\mathbf{Q}}$ such that the only singular fibres lie over the point at infinity and the $\mathbf{Q}$-points defined by the zeros of the polynomials $P_i(t)$ (see [31]).

Let $m \in U$ be a closed point, let $X_m/k(m)$ be the smooth, geometrically integral fibre at m.

Since X_m is a smooth, geometrically rational variety, one has

$$H^1(k(m), \mathbf{Q}/\mathbf{Z}) = H^1_{et}(X_m, \mathbf{Q}/\mathbf{Z}).$$

For $m \in U$ and any $\beta \in \mathrm{Br}(X_\eta)$, the residue of β at the generic point of X_m lies in

$$H^1_{et}(X_m, \mathbf{Q}/\mathbf{Z}) \subset H^1(k(m)(X_m), \mathbf{Q}/\mathbf{Z}).$$

Thus this residue comes from $H^1(k(m), \mathbf{Q}/\mathbf{Z})$.

Using the Faddeev exact sequence, one sees that any element of $\mathrm{Br}(X_\eta)$ is the sum of an element coming from $\mathrm{Br}(\mathbf{Q}(\mathbf{P}^1))$ and an element in $\mathrm{Br}(X_U)$.

From this one deduces that there exists a finite subgroup $B \subset \mathrm{Br}(X_U)$ which surjects onto $\mathrm{Br}(X_\eta)/\mathrm{Br}(k(\mathbf{P}^1))$. We may thus choose this U as the open set of $\mathbf{P}^1_{\mathbf{Q}}$ in the previous two theorems. The composite fibration $Y'_U \to X_U \to U$ is smooth, the complement of U consists of rational points. Since $k = \mathbf{Q}$, Matthiesen's theorem (see subsection 7.3) guarantees the validity of Hypothesis H^{**} in the present situation. The proof of Theorem 7.8 given in this subsection 7.4 thus specializes to an unconditional proof in the present case. □

Harpaz and Wittenberg [80], using more elaborate arguments, some of them coming from Harari's thesis [62], actually prove the following general, unconditional result.

Theorem 7.16. *Let $X/\mathbf{Q}$ be a smooth, connected, projective variety with a morphism $X \to \mathbf{P}^1_{\mathbf{Q}}$ with rationally connected generic fibre. Assume that the only non-split fibres are above $\mathbf{Q}$-rational points of $\mathbf{P}^1_{\mathbf{Q}}$. If $X_P(\mathbf{Q})$ is dense in $X_P(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}(X_P)}$ for smooth fibres X_P over rational points of $\mathbf{P}^1_{\mathbf{Q}}$, then $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}}$.*

For this, just like Harari in [62], they have to discuss what happens when representatives of $\mathrm{Br}(X_\eta)$ have nontrivial residues above split fibres.

Remark 7.17. Over $k = \mathbf{Q}$, Theorem 7.16 goes beyond most results which had been obtained by the descent method.

One exception is the work of Derenthal, Smeets, Wei [56] based on the sieve method result of Browning and Heath-Brown [15]. They get unconditional results for an equation

$$Norm_{K/\mathbf{Q}}(\Xi) = P(t)$$

with $P(t)$ irreducible of degree 2 and $K/\mathbf{Q}$ arbitrary. Here the sum of the degrees $\sum_{i=1}^n [k_i : k] = 3$.

Remark 7.18. A very special example for Corollary 7.11 is the Hasse principle for quadratic forms in 4 variables. One considers a system

$$0 \neq t = b_1(x_1^2 - a_1 y_1^2)$$
$$0 \neq t = b_2(x_2^2 - a_2 y_2^2).$$

Assume this has solutions everywhere locally. Hasse's method is to define an obvious set S of bad places, then to use Dirichlet's theorem to produce a value t_0 of t which is a unit away from S and one finite place v_0 (where t_0 has valuation one). Then each conic $t_0 = b_i(x_i^2 - a_1 y_i^2)$ has solutions in all completions except possibly at v_0. One uses reciprocity to conclude that it has a solution also in v_0 and then globally.

The present proof for this result is "different".

The argument based on Tate-Nakayama duality and the formal lemma for torsors directly produces a point t_0 such that each of the two equations $t_0 = b_i(x_i^2 - a_1 y_i^2)$ has solutions in *all* completions.

Since this is a situation where the fibre at infinity is not smooth, let us do the argument directly. One considers the 3-dimensional variety W given by

$$0 \neq t = b_1(x_1^2 - a_1 y_1^2)$$
$$0 \neq t = b_2(x_2^2 - a_2 y_2^2).$$

Assume this has solutions everywhere locally. Let $L = k(\sqrt{a_1}, \sqrt{b_1})$. Introduce the torsor given by

$$t = Norm_{L/k}(\Xi).$$

There is no vertical Brauer-Manin obstruction, since the given variety is k-birational to the product of $\mathbf{P}^1_k$ and a quadric. The formal lemma for torsors then produces an element $c \in k^\times$ such that the system

$$0 \neq t = b_1(x_1^2 - a_1 y_1^2)$$
$$0 \neq t = b_2(x_2^2 - a_2 y_2^2).$$
$$0 \neq t = c.Norm_{L/k}(\Xi)$$

has solutions in all completions of k. This now implies that the system

$$b_1(x_1^2 - a_1 y_1^2) = c = b_2(x_2^2 - a_2 y_2^2).$$

has solutions in all completions of k : there is a fibre of $W \to \mathbf{A}^1$ over a k-point which has points in all k_v. That fibre is the product of two conics, we can use the Hasse principle for each of them.

The same arguments would produce a suitable t_0 if one started from a system

$$0 \neq t = b_i Norm_{k_i/k}(\Xi_i)$$

with arbitrary field extensions k_i/k. But for arbitrary field extensions k_i/k one would not be able to conclude that the system has rational solutions. The question will be discussed further below : here one must take the whole Brauer group $\mathrm{Br}(X)$ into account. ■

Remark 7.19. In theorem 7.10, there is no geometric assumption on the generic fibre beyond the fact that it is geometrically integral.

Here is an interesting case, still with $\varepsilon = 2$, hence with H^{**} known.

Let $n > 1$ be any integer. Let $a, b, c, d \in k^\times$. If the projective surface $X \subset \mathbf{P}^3_k$ given by

$$ax^n + by^n = cz^n + dw^n$$

has solutions in all completions of k, and there is no Brauer-Manin obstruction with respect to the finite subgroup of $\mathrm{Br}(X)$ corresponding to the vertical part for (a projective model of) the fibration of $W \to \mathbf{A}^1_k$ defined by mapping

$$ax^n + by^n = t = cz^n + dw^n \neq 0$$

to t, then there exists $\rho \in k^\times$ such that the affine variety given by

$$ax^n + by^n = \rho = cz^n + dt^n$$

has solutions in all completions of k. For $n = p$ a prime, the relevant vertical Brauer group is reduced to $\mathrm{Br}(k)$.

For $n = 3$, this statement is a starting point in Swinnerton-Dyer's paper [140, Lemma 2, p. 901], see also a similar situation in [126]. Exercise : rephrase Swinnerton-Dyer's argument from the present point of view. The challenge here, assuming $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, is to produce a ρ such that for each of the two curves $ax^n + by^n = \rho$ and $cz^n + dt^n = \rho$, there is no Brauer-Manin obstruction to the existence of a rational point, or at least to the existence of a zero-cycle of degree 1. ■

8. Zero-cycles

8.1. **The conjectures.** Let X be a smooth, projective, geometrically integral variety over a number field k.

For any field F containing k, there is a natural bilinear pairing

$$CH_0(X_F) \times \mathrm{Br}(X) \to \mathrm{Br}(F)$$

between the Chow group of zero-cycles on X_F (group of zero-cycles modulo rational equivalence) and the Brauer group of X.

In particular, for each place v of k, there is a pairing

$$CH_0(X_{k_v}) \times \mathrm{Br}(X) \to \mathrm{Br}(k_v) \subset \mathbf{Q}/\mathbf{Z}.$$

For v archimedean, this pairing vanishes on $Norm_{k'_v/k_v} CH_0(X_{k'_v})$, where k'_v is an algebraic closure of k_v. Let then $CH'_0(X_{k_v})$ be the quotient $CH_0(X_{k_v})/Norm(CH_0(X_{k'_v}))$ for v archimedean, and $CH'_0(X_{k_v}) = CH_0(X_{k_v})$ for v finite.

Class field reciprocity gives rise to a complex

$$CH_0(X) \to \prod_v CH'_0(X_{k_v}) \to Hom(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z}).$$

For A an abelian group, let $\widehat{A}$ denote $\varprojlim A/n$. There is an induced complex

$$\widehat{CH_0}(X) \to \prod_v \widehat{CH}'_0(X_{k_v}) \to Hom(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z}).$$

Work of Sansuc and mine on rational surfaces [38] and of Kato and Saito on higher class field theory has led to the following general conjecture.

Conjecture (E) : **For any smooth, projective variety X over a number field k, the complex**

$$\widehat{CH}_0(X) \to \prod_v \widehat{CH}'_0(X_{k_v}) \to Hom(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z})$$

is exact.

This conjecture subsumes

Conjecture (E_1) : **For any smooth, projective variety X over a number field k, If there exists a family $\{z_v\}$ of local zero-cycles of degree 1 on X such that, for all $A \in \mathrm{Br}(X)$**

$$\sum_v A(z_v) = 0 \in \mathbf{Q}/\mathbf{Z},$$

then there exists a zero-cycle of degree one on X.
as well as a Conjecture (E_0), where $CH_0(X)$ is replaced by the subgroup $A_0(X)$ of zero-cycles classes of degree zero.

For more on this, and precise references, see [38], Kato–Saito, Saito, [20], [22], [141], and the introduction of Wittenberg's paper [152].

Note that this is a conjecture for *all* smooth, projective, geometrically connected varieties over a number field.

For curves, classical results imply the conjecture – modulo finiteness of Tate-Shafarevich groups.

For Châtelet surfaces, conjecture (E) was proved in [41].

Then Salberger [108] by a very ingenious method proved the conjecture for arbitrary conic bundles over $\mathbf{P}^1$.

Further progress was achieved in papers by myself, Swinnerton-Dyer, Skorobogatov, Salberger [110], Frossard, van Hamel [141], and more recently Wittenberg [152] and Yongqi Liang (see the references).

The following simple case gives a good idea of Salberger's method.

Theorem 8.1. *Assume that the equation*

$$y^2 - az^2 = P(t) \neq 0$$

with $P(t) \in k[t]$ irreducible of degree d has solutions in all completions k_v. Then for any integer $N > d$ there exists a solution in an extension of degree N of k. Taking N and $N+1$, we find that there exists a zero-cycle of degree 1 on X.

Proof. [Here I am cheating on the real places. There are various methods to fix the problem.]

Let U be the k-variety defined by the equation.

One fixes an obvious set S of bad places of k for the given equation.

For each $v \in S$, one fixes a polynomial $G_v(t) \in k_v[t]$, monic, separable, of degree N with all roots in k_v corresponding to projections of points of $U(k_v)$, in particular $G_v(t)$ is prime to $P(t)$.

One also fixes a place v_0 outside of S, such that a is a square in k_{v_0}, and a monic irreducible polynomial of degree N over k_{v_0}.

One then applies euclidean division :

$$G_v(t) = P(t)Q_v(t) + R_v(t)$$

with degree of R_v smaller than N.

Let K be the field $k[t]/P(t)$. Let $\xi_v \in K \otimes_k k_v$ be the image of $R_v(t)$.

Using Dirichlet's theorem in the field K, one produces an element $\xi \in K^\times$ which is very close to each ξ_v for $v \in S \cup \{v_0\}$ and such that its prime decomposition in K involves only places above $S \cup \{v_0\}$ and one place w of K such that $w(\xi) = 1$, this place being of degree 1 over k. The element $\xi \in k[t]/P(t)$ lifts to a unique element $R(t)$ of $k[t]$ of degree strictly smaller than the degree of $P(t)$.

Fix a place v_1 outside the previous set of places, such that a is a square at v_1. Using strong approximation in k away from v_1 one produces a monic polynomial $Q(t) \in k[t]$ with integral coefficients aways from v_1 and very close coefficientwise to each $Q_v(t)$ for $v \in S \cup \{v_0\}$.

One then defines

$$G(t) := P(t)Q(t) + R(t).$$

This polynomial is irreducible (because close to G_{v_0}). It is monic and has integral coefficients away from $S \cup v_0 \cup v_1$.

Let $L = k[t]/G(t)$. This is a field extension of degree N of k. Let $\theta \in L$ be the class of t. The element θ is integral over places not in $S \cup v_1 \cup v_0$. Let $\rho = P(\theta)$ denote the class of $P(t)$ in L.

Claim : The L-conic given by $y^2 - az^2 = \rho$ has an L-point.

At places w of L above S, the conic has an L_w-point because $G(t)$ is very close to $G_v(t)$.

The formula for the resultant of two polynomials shows that the product of the conjugates of $P(\theta)$ is $\pm$ the product of the conjugates of $G(\alpha)$, for α the class of t in $k[t]/P(t)$. Now

$$G(\alpha) := P(\alpha)Q(\alpha) + R(\alpha) = R(\alpha) = \xi.$$

And the degree 1 condition on the Dirichlet prime implies that the norm of ξ, away from $S \cup \{v_0\}$ has in its factorisation only one prime, and of valuation 1. Since $P(\theta)$ is integral away from $S \cup v_0 \cup v_1$, this implies that the prime decomposition of $P(\theta)$ away from $S \cup v_0 \cup v_1$ involves only one prime w' of L. Thus the L-conic has points in all completions of L except possibly at the prime w', and we conclude that it has an L-point by the reciprocity argument. □

In the above proof, we did not assume that we had started from an adelic point orthogonal to the unramified Brauer group. But this is automatic, since $P(t)$ irreducible implies that the Brauer group of a smooth projective model of $y^2 - az^2 = P(t)$ is reduced to $Br(k)$.

8.2. **From results on rational points to results on zero-cycles : work of Yongqi Liang.** The following proposition is a baby case (Liang, [93, Prop. 3.2.3]).

Proposition 8.2. *Let k be a number field and X/k be a smooth, proper, geometrically integral variety. Assume that for any field extension K/k, the Hasse principle holds for X_K. Then the Hasse principle holds for zero-cycles of degree one on X.*

Proof. By the Lang-Weil-Ninevich estimates, there exists a finite set S of places of k such that for any place $v \notin S$, one has $X(k_v) \neq \emptyset$. Fix a closed point m of some degree N over X. For each $v \in S$, let $z_v = z_v^+ - z_v^-$ be a local zero-cycle of degree one, where z_v^+ and z_v^- are effective zero-cycles. Let $z_v^1 = z_v^+ + (N-1)z_v^-$. This is an effective zero-cycle of degree congruent to 1 modulo N. Since there are finitely many v's, we can add to each z_v^1 a suitable positive multiple $n_v m$ of the closed point m and ensure that all the effective cycles $z_v^2 = z_v^1 + n_v m$, $v \in S$, have the same common degree d congruent to 1 modulo N.

Here comes the basic trick. Let $Y = X \times \mathbf{P}^1$ and let $f : Y \to \mathbf{P}^1_k$ be the natural projection. Fix a rational point $q \in \mathbf{P}^1(k)$. On Y we have the effective zero-cycles $z_v^2 \times q$, of degree d.

A moving lemma which builds upon the implicit function theorem ensures that each $z_v^2 \times q$ is rationally equivalent on Y_{k_v} to an effective cycle z_v^3 without multiplicty and such that the projected $f_*(z_v^3)$ is also without multiplicity. This amounts to saying that $z_v^3 = \sum_j R_j$ with all R_j distinct and with $k(f(R_j)) = k(R_j)$ for each j. We may assume that all $f_*(z_v^3)$ lie in $\mathrm{Spec} k[t] = \mathbf{A}^1_k \subset \mathbf{P}^1_k$. Each $f_*(z_v)$ is defined by a separable monic polynomial $P_v(t)$. We pick a place v_0 outside S and an arbitrary monic irreducible polynomial $P_{v_0}(t) \in k_{v_0}[t]$. By weak approximation on the coefficients, we then approximate the $P_v(t)$, $v \in S \cup v_0$, by a monic polynomial $P(t) \in k[t]$. This defines a closed point $M \in \mathbf{A}^1_k$ of degree d.

If the approximation is close enough, Krasner's lemma and the implicit function theorem imply that the fibre $X \times_k k(M)$ has points in all completions of $k(M)$ at the places above $v \in S$. By the definition of S, $X \times_k k(M)$ has points in all the other completions. By assumption, $X \times_k k(M)$ satisfies the Hasse principle over $k(M)$, hence it has a $k(M)$-point. Thus X has a point in an extension of degree d. As d is congruent to 1 mod N, we conclude that the k-variety X possesses a zero-cycle of degree 1. ∎

Theorem 8.3. *(Y. Liang* [93]*) Let k be a number field and X a smooth, projective, geometrically connected variety over k. Assume that $H^i(X, O_X) = 0$ for $i = 1, 2$ and that the geometric Picard group $\mathrm{Pic}\,\overline{X}$ is torsion free. For any finite field extension K of k, assume that the Brauer-Manin obstruction to the Hasse principle for X_K is the only obstruction. Then the Brauer-Manin obstruction to the existence of a zero-cycle of degree one on X is the only obstruction : conjecture (E_1) holds.*

Proof. Over any field k of char. zero, the assumptions on the geometry of the variety X imply that the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite. Let $A_1, \ldots, A_n \in \mathrm{Br}(X)$ span $\mathrm{Br}(X)/\mathrm{Br}(k)$.

Let S be a finite set of places such that away from S, X and the A_i have good reduction, and $X(k_v) \neq \emptyset$ vanishes when evaluated on any zero-cycle of X_{k_v} when v is not in S.

One starts with a family z_v of zero-cycles of degree one on X which is orthogonal to $\mathrm{Br}(X)$ for the Brauer-Manin pairing.

This boils down to

$$\forall i \quad \sum_v A_i(z_v) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

Let N be an integer which is a multiple of the degree of a closed point of X and also annihilates each $A_i \in \mathrm{Br}(X)$.

Let $Y = X \times \mathbf{P}^1_k$ and $f : Y \to \mathbf{P}^1_k$ be the projection.

Proceeding as in the previous proof, one produces reduced effective zero-cycles z_v on Y of the same degree d congruent to 1 modulo N, with the property that $f_*(z_v)$ is reduced. We may choose coordinates so that the support of these zero-cycles lies in $\mathrm{Spec} k[t] = \mathbf{A}^1_k \subset \mathbf{P}^1_k$. They are then defined by the vanishing of separable, monic polynomials $P_v(t)$ of degree d.

One then approximates the $P_v(t)$ for $v \in S$ and at another place v_0 by a suitable monic irreducible polynomial $P(t) \in k[t]$. Just as before, this defines a closed point $M \in \mathbf{P}^1_k$.

For each place $v \in S$, there exists an effective zero-cycle z'_v close to z_v on $X_M \otimes_k k_v$, this corresponds to $k(M)_w$-rational points R_w of the $k(M)$-variety $X_{k(M)}$ over the various completions w of $k(M)$ above the places in S.

At each place $w \notin S_{k(M)}$ above a place $v \notin S$, we take an arbitrary $k(M)_w$-rational point, for instance one coming from a k_v-point on X.

We then have

$$\forall i \sum_{w \in \Omega_{k(M)}} A_i(R_w) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

Now this is not enough to ensure that the adèle $\{R_w\} \in X_{k(M}(A_{k(M)})$ is orthogonal to $\mathrm{Br}(X_{k(M)})$. This is enough if we can choose the point M, i.e. the polynomial $P(t)$, in such a way that the map

$$\mathrm{Br}(X)/\mathrm{Br}(k) \to \mathrm{Br}(X_{k(M)})/\mathrm{Br}(k(M))$$

is onto.

The geometric hypotheses made on X imply, by [93, Prop. 3.1.1] (a special and easier case of a more general theorem of Harari [64, Thm. 2.3.1] that there exists a fixed finite Galois extension L/k such that the above surjectivity holds for any closed point M as soon as the tensor product $L \otimes_k k(M)$ is a field. But this last condition is easy to ensure. just impose on N from the very beginning that it is also a multiple of $[L : k]$. Then $d = [k(M) : k]$, congruent to 1 modulo N, is prime to $[L : k]$. □

The paper by Yongqi Liang proves more :

Theorem 8.4. *(Y. Liang) Let k be a number field. Let X/k be a smooth, projective, geometrically connected, geometrically rationally connected variety. Assume that for any finite field extension K/k, $X(K)$ is dense in $X(A_K)^{\mathrm{Br}}$. Then conjecture*

(E) holds, i.e. the sequence

$$\widehat{CH}_0(X) \to \prod_v \widehat{CH}'_0(X_{k_v}) \to Hom(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z})$$

is exact. That is, conjecture (E) holds.

To prove this, Yongqi Liang first proves a version of the previous theorem for zero-cycles of degree one, keeping track of "approximation" modulo a positive integer. Here z is said to be close to z_v modulo n if they have the same image in $CH_0(X_{k_v})/n$. Here work of Wittenberg [152] is used.

Then one goes from this statement to the long exact sequence above for rationally connected varieties. This more stringent geometric condition is imposed, indeed at some point one uses results of Kollár and Szabó on Chow groups of zero-cycles of such varieties over local fields – in the good reduction case.

The results of Yongqi Liang thus establish conjecture (E), (E_0) and (E_1) for smooth projective varieties which are birational to a homogeneous space of a connected linear algebraic group with connected homogeneous stabilizers, since the standard conjecture for rational points has been proved for such varieties (Sansuc [111] when the stabilizers are trivial, Borovoi [4] in general).

8.3. **Harpaz and Wittenberg's general theorem on zero-cycles : statement and proof of a very special case.** In the various papers quoted in subsection 8.1, inspiration came from Salberger's paper [108], which was later seen [45] to build upon a zero-cycle, unconditional version of Schinzel's hypothesis (Salberger's trick).

Just as in the case of rational points, one got results for fibrations $X \to \mathbf{P}^1_k$ if one assumed :

– Over any closed point m, the fibre contains a component Y of multiplicity one Y such that the integral closure of $k(m)$ in $k(Y)$ is *abelian*.

– The Hasse principle and weak approximation hold for the smooth closed fibres.

These restrictions on the algebra and arithmetic of fibres have now been lifted, and we have the unconditional result :

Theorem 8.5. *(Harpaz et Wittenberg)*[80] *Let X be a smooth, projective, geometrically connected variety over a number field, equipped with a dominant morphism $f : X \to \mathbf{P}^1_k$. Assume the geometric generic fibre is a rationally connected variety. If the smooth fibres satisfy conjecture (E), then X satisfies conjecture (E).*

Corollary 8.6. *Let X be a smooth, projective, geometrically connected variety over a number field, equipped with a dominant morphism $f : X \to \mathbf{P}^1_k$. Assume that the generic fibre is birational to a homogeneous space of a connected , linear $k(\mathbf{P}^1)$-algebraic group G with connected geometric stabilizers. Then conjecture (E) holds for X.*

Harpaz and Wittenberg actually prove their result for varieties fibred over a curve of arbitrary genus, under the assumption that conjecture (E) holds for the curve, for instance when the Tate-Shafarevich group of the jacobian of the curve is finite. For the time being I shall not attempt to present the proof of their result, not even in the case where the bottom curve is $\mathbf{P}^1_k$.

I shall describe one idea in the proof of Theorem 8.5. This is a zero-cycle analogue of Theorem 7.10.

Theorem 8.7. *Let $f : X \to \mathbf{P}^1_k$ be a fibration. Assume that the nonsplit fibres all lie over k-points of $\mathbf{A}^1_k = Speck[t]$, that they are given by $t = e_i, i = 1, \dots, n$, and that for each i there is a component E_i of multiplicity one in the fibre at e_i. Assume that there exists an adèle $\{P_v\} \in X(A_k)$ which is orthogonal to the finite group $\mathrm{Br}_{vert}(X)/\mathrm{Br}(k)$. Then for any integer $N > n$ there exists a closed point m of $\mathbf{P}^1_k$ of degree N such that the fibre $X_m/k(m)$ has points in all the completions of $k(m)$.*

Proof. Let U be the complement of the points $t = e_i$ in $\mathbf{A}^1_k$. Let $V = f^{-1}(U)$. Let k_i be the integral closure of k in $k(E_i)$.

We assume that there exists an adèle $P_v \in X(A_k)$ which is orthogonal to the finite group $\mathrm{Br}_{vert}(X)/\mathrm{Br}(k)$. One may assume that the point actually lies in $V(A_k)$. Let $t_v \in k_v$ denote the projection of P_v. Let $f_i(t) := t - e_i \in k[t]$.

Over any field K containing k, there is an obvious homomorphism of abelian groups

$$f_i : Z_0(U_K) \to K^\times$$

defined by f_i using norms.

Let $N > n$ be an integer. By using the implicit function theorem, out of the P_v's one constructs effective 0-cycles z_v of degree N such that $f_*(z_v)$ is reduced, sum of N points of $U(k_v)$.

Applying the formal lemma for torsors and for rational points, using the fact that there is no vertical Brauer-Manin obstruction, one produces other z_v's (identical to the original ones at a preassigned finite set of places) and global elements $b_i \in k^\times$ such that for each v,

$$f_i(z_v) \in b_i.Norm_{k_i/k}(k_{iv}^\times).$$

We now use Proposition 2.13 (Lemma 5.2 of [80]).

Let S denote the obvious finite set of bad places, in particular all places where b_i is not a unit, all infinite places, all places where one k_i/k is ramified.

That proposition produces elements $c_i \in k^\times$ such that c_i is close to $b_i^{-1}f_i(z_v) \in k_v^\times$ for $v \in S$, and such that at any place $v \notin S$ either c_i is a unit or k_i/k possesses a place of degree 1 over v. Moreover if we fix a place $v_0 \notin S$ we may take c_i integer away from $S \cup v_0$. We fix such a place v_0 which is split in all extensions k_i/k (possible by Tchebotarev).

For $v \in S$, let $G_v(t) \in k_v[t]$ be the monic polynomial vanishing on z_v.

By interpolation, as in Salberger's trick, one builds a monic irreducible polynomial $G(t) \in k[t]$ of degree N, integral away from $S \cup v_0$, such that $G(t)$ is close to $G_v(t)$ for $v \in S$ and such that $G(e_i) = b_i c_i \in k^\times$.

Let m be the closed point defined by $G(t) = 0$. We claim that X_m has points in all completions F_w of $F := k(m)$.

For w a place which lies over S, this is a consequence of the implicit function theorem.

For the other places, one must only discuss the finitely many closed points of $\mathbf{P}^1_{O_S}$ where the closure of m in $\mathbf{P}^1_{O_S}$ meets the closure of one of the e_i's. Indeed, provided S was chosen big enough at the outset, at any other closed point the fibre of a model over $\mathbf{P}^1_{O_S}$ is split, and provided again S was chosen big enough, has a smooth rational point over its residue field.

Let us consider one of the above closed points, of degree one above a place $v \notin S \cup \{v_0\}$. One has

$$O_S[t]/(t-e_i, G(t)) = O_S/G(e_i) = O_S/(b_ic_i).$$

This is nonzero only if $v(b_ic_i) = v(c_i) > 0$. In that case, v admits an extension w of degree one in the extension k_i/k. The smooth locus of the component $E_i \subset X_m$ possesses a a smooth point in its reduction modulo w. This produces a smooth point in the reduction of X_m at the place of $k(m)$ intersection of the closure of m and the closure of e_i. the fibre E_i/k_i possesses a smooth point over the reduction of k_i at w, hence also does the reduction of $X_m/k(m)$ at the closed point under consideration, which defines a place of degree one w' on $F = k(m)$ hence X_m has a point in the completion $F_{w'}$.

As for the place v_0, the fibration $X \to \mathbf{P}^1_k$ has all its fibres split over k_{v_0}, hence there is no difficulty : provided v_0 was chosen big enough, over any finite field extension F of k_{v_0}, the map $X(F) \to \mathbf{P}^1(F)$ is onto. [If the generic fibre is a rationally connected variety, the fibration $X \to \mathbf{P}^1_k$ admits a section over a finite field extension of k (Graber-Harris-Starr [60]). It is then enough to take a v_0 which splits in this extension, and the existence of such a v_0 is guaranteed by Tchebotarev's theorem.] □

References

[1] A. Beauville, On the Brauer group of Enriques surfaces. Math. Res. Lett. 16 (2009), no. 6, 927–934.

[2] F. A. Bogomolov, Le groupe de Brauer des espaces quotients de représentations linéaires (en russe), The Brauer group of quotient spaces of linear representations, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987) 485–516, 688.

[3] F. A. Bogomolov, Brauer groups of the fields of invariants of algebraic groups, Mat. Sb. **180** (1989), 279–293; English transl.: Math. USSR Sb. 66 (1990), 285–299.

[4] M. Borovoi, The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer. J. reine angew. Math. (Crelle) **473** (1996), 181–194. 47

[5] M. Borovoi, A note on the unramified Brauer group of a homogeneous space, http://arxiv.org/abs/1206.1023v1. 23

[6] M. Borovoi, J.-L. Colliot-Thélène, A. Skorobogatov, The elementary obstruction and homogeneous spaces, Duke math. J. **141** no. 2 (2008) 321–364. 16

[7] M. Borovoi, C. Demarche et D. Harari, Complexes de groupes de type multiplicatif et groupe de Brauer non ramifié des espaces homogènes, http://arxiv.org/abs/1203.5964v1, à paraître dans Ann. Sc. Éc. Norm. Sup. 23

[8] M. Borovoi, B. Kunyavskiĭ, Formulas for the unramified Brauer group of a principal homogeneous space of a linear algebraic group. J. Algebra **225** (2000), no. 2, 804821. 23

[9] M. Borovoi et B. Kunyavskiĭ, with an appendix by P. Gille, Arithmetical birational invariants of linear algebraic groups over two-dimensional geometric fields. J. of Algebra **276** (2004) 292-339. 23

[10] M. Bright, Brauer groups of diagonal quartic surfaces, J. Symb. Computat. 41(5) (2006), 544–558.
[11] M. Bright, Efficient evaluation of the Brauer–Manin obstruction, Math. Proc. Cambridge Philos. Soc. **142** (2007) 13–23.
[12] M. Bright, Evaluating Azumaya algebras on cubic surfaces, Manuscripta math. **134** (2011) 405–421.
[13] M. Bright, The Brauer-Manin obstruction on a general diagonal quartic surface, Acta Arith. 147 (2011), no. 3, 291–302.
[14] M. Bright et P. Swinnerton-Dyer, Computing the Brauer-Manin obstructions. Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 1, 1–16.
[15] T. D. Browning et D. R. Heath-Brown, Quadratic polynomials represented by norm forms, `http://arxiv.org/abs/1109.0232v1` 40
[16] T. D. Browning et L. Matthiesen, Norm forms for arbitrary number fields as products of linear polynomials, arXiv:1307.7641, 2013. 34, 39
[17] T. D. Browning, L. Matthiesen et A. N. Skorobogatov, Rational points on pencils of conics and quadrics with many degenerate fibres, Ann. of Math. 180 (2014) 381-402. 27, 31
[18] J.-L. Colliot-Thélène, L'arithmétique des variétés rationnelles, Annales de la Faculté des Sciences de Toulouse (6) **I** (1992) 295-336.
[19] J.-L. Colliot-Thélène, Birational invariants, purity and the Gersten conjecture, K- theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Amer. Math. Soc., Providence, RI, 1995, pp. 1–64.
[20] J.-L. Colliot-Thélène, L'arithmétique du groupe de Chow des zro-cycles (exposé aux Journées arithmétiques de Bordeaux, Septembre 93), Journal de théorie des nombres de Bordeaux 7 (1995) 51–73. 43
[21] J.-L. Colliot-Thélène, The Hasse principle in a pencil of algebraic varieties, in *Number theory (Tiruchirapalli, 1996)*, K. Murty and M. Waldschmidt ed, Contemp. Math. vol. **210**, Amer. Math. Soc. (Providence, RI, 1998), pp. 19–39.
[22] J.-L. Colliot-Thélène, Conjectures de type local-global sur l'image de l'application cycle en cohomologie étale, in Algebraic K-Theory (1997), W. Raskind and C. Weibel ed., Proceedings of Symposia in Pure Mathematics 67, Amer. Math.Soc. (1999) 1–12. 43
[23] J.-L. Colliot-Thélène, Principe local-global pour les zéro-cycles sur les surfaces réglées, (avec un appendice par E. Frossard et V. Suresh). J. Amer. Math. Soc. 13 (1) (2000), p. 101–124.
[24] J.-L. Colliot-Thélène, Points rationnels sur les fibrations (notes d'un cours donné à Budapest en septembre 2001), in *Higher Dimensional Varieties and Rational Points*, Bolyai Society Mathematical Series, vol 12, Springer-Verlag, 2003, edited by K. J. Böröczky, J. Kollár and T. Szamuely, 171–221. 23
[25] J.-L. Colliot-Thélène, Zéro-cycles de degré 1 sur les solides de Poonen, Bull. Soc. math. France. 138 (2) (2010) 249–257.
[26] J.-L. Colliot-Thélène, Groupe de Brauer non ramifié des hypersurfaces cubiques singulières (d'après P. Salberger) Appendice un article de Tim Browning, Compositio mathematica 146 (2010) 882–883. 23
[27] J.-L. Colliot-Thélène, Variétés presque rationnelles, leurs points rationnels et leurs dégénérescences (cours au CIME, Cetraro, septembre 2007), in Arithmetic Geometry (CIME 2007), Springer LNM 2009 (2011), p. 1–44.
[28] J.-L. Colliot-Thélène, Groupe de Brauer non ramifié de quotients par un groupe fini, prépublication janvier 2012 (sur arXiv), à paraître, Proc. Amer. Math. Soc. 23
[29] J.-L. Colliot-Thélène, Groupe de Brauer non ramifié d'espaces homogènes de tores, prépublication, 2012, sur arXiv. 17, 23
[30] J.-L. Colliot-Thélène, D. Coray et J.-J. Sansuc, Descente et principe de Hasse pour certaines variétés rationnelles. J. für die reine und ang. Math. (Crelle) **320** (1980) 150–191. 13, 17

[31] J.-L. Colliot-Thélène, D. Harari et A. N. Skorobogatov, Valeurs d'un polynôme à une variable représentées par une norme, in Number theory and algebraic geometry (M. Reid, A. Skorobogatov editors), London Math. Society Lecture Notes Series 303, Cambridge University Press (2003). 69-89. 23, 27, 40

[32] J.-L. Colliot-Thélène, D. Kanevsky et J.-J. Sansuc, Arithmétique des surfaces cubiques diagonales, in *Diophantine approximation and transcendence theory*, Bonn, 1985 (ed. G. Wüstholz), Lecture Notes in Mathematics **1290** (Springer, Berlin, 1987), pp. 1–108. 12, 19

[33] J.-L. Colliot-Thélène et B. È. Kunyavskiĭ, Groupe de Brauer non ramifié des espaces principaux homogènes de groupes linéaires, J. Ramanujan Math. Soc. 13 (1998), 37-49. 23

[34] J.-L. Colliot-Thélène et B. Kunyavskiĭ, Groupe de Picard et groupe de Brauer des compactifications lisses d'espaces homogènes, J. Algebraic Geom. 15 (2006), 733–752. 23

[35] J.-L. Colliot-Thélène, A. Pál. et A. N. Skorobogatov, Pathologies of the Brauer-Manin obstruction. Math. Zeitschrift, to appear. arXiv:1310.5055 18, 27

[36] J.-L. Colliot-Thélène et P. Salberger, Arithmetic on some singular cubic hypersurfaces, Proceedings of the London Mathematical Society (3) **58** (1989) 519-549.

[37] J.-L. Colliot-Thélène et J.-J. Sansuc, La R-équivalence sur les tores. Ann. Sc. Éc. Norm. Sup. **10** (1977) 175–229.

[38] J.-L. Colliot-Thélène et J.-J. Sansuc, On the Chow groups of certain rational surfaces : a sequel to a paper of S.Bloch. Duke Math. J. 48 (1981) 421–447. 42, 43

[39] J.-L. Colliot-Thélène et J.-J. Sansuc, Sur le principe de Hasse et l'approximation faible, et sur une hypothèse de Schinzel, Acta Arithmetica **41** (1982) 33-53. 30

[40] J.-L. Colliot-Thélène et J.-J. Sansuc, La descente sur les variétés rationnelles, II, Duke Math. J. **54** (1987) 375-492. 39

[41] J.-L. Colliot-Thélène, J.-J. Sansuc and Sir Peter Swinnerton-Dyer, Intersections of two quadrics and Châtelet surfaces, I, J. für die reine und angew. Math. (Crelle) **373** (1987) 37-107; II, ibid. **374** (1987) 72-168. 4, 33, 43

[42] J-L. Colliot-Thélène et A.N. Skorobogatov, Descent on fibrations over $\mathbf{P}^1_k$ revisited, Math. Proc. Camb. Phil. Soc. **128** (3) (2000) 383–393. 25, 36

[43] J-L. Colliot-Thélène et A.N. Skorobogatov, Good reduction of the Brauer–Manin obstruction, Trans. Amer. Math. Soc., à paraître. `http://arxiv.org/abs/1009.0247`.

[44] J-L. Colliot-Thélène et A.N. Skorobogatov, Descente galoisienne sur le groupe de Brauer, J. für die reine und ang. Math. (Crelle), à paraître. 19

[45] J.-L. Colliot-Thélène, A. N. Skorobogatov and Sir Peter Swinnerton-Dyer, Rational points and zero-cycles on fibred varieties : Schinzel's hypothesis and Salberger's device, Journal für die reine und angewandte Mathematik **495** (1998) 1–28. 30, 36, 47

[46] J.-L. Colliot-Thélène, A. N. Skorobogatov and Sir Peter Swinnerton-Dyer, Hasse principle for pencils of curves of genus one whose Jacobians have rational 2-division points. Inventiones math. **134** (1998) 579–650.

[47] J.-L. Colliot-Thélène and Sir Peter Swinnerton-Dyer, Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties, J. für die reine und angew. Math. (Crelle) **453** (1994) 49-112. 30

[48] J.-L. Colliot-Thélène et O. Wittenberg, Groupe de Brauer et points entiers de deux familles de surfaces cubiques affines (23 pages), à paraître dans Amer. J. Math. 23

[49] D. F. Coray, M. A. Tsfasman, Arithmetic on singular Del Pezzo surfaces. Proc. London Math. Soc. (3) 57 (1988), no. 1, 25–87. 13

[50] P. Corn, The Brauer-Manin obstruction on del Pezzo surfaces of degree 2. (Proc. Lond. Math. Soc. (3) 95 (2007), no. 3, 735–777.

[51] A. J. de Jong et J. Starr, Every rationally connected variety over the function field of a curve has a rational point, Amer. J. Math. **125** (2003) 567-580.

[52] C. Demarche, Obstruction de descente et obstruction de Brauer-Manin étale, Algebra and Number Theory 3 (2009), no. 2, 237–254. 18

[53] C. Demarche, Groupe de Brauer non ramifié d'espaces homogènes à stabilisateurs finis, Math. Annalen **346** (2010) 949–968. 23

[54] C. Demarche, Une formule pour le groupe de Brauer algébrique d'un torseur, http://arxiv.org/abs/1011.5100v1. 23

[55] U. Derenthal et D. Wei, Strong approximation and descent, http://arxiv.org/abs/1311.3914v2

[56] U. Derenthal, A. Smeets et D. Wei, Universal torsors and values of quadratic polynomials represented by norms, arXiv:1202.3567v2 [math.NT]. 40

[57] T. Ekedahl, An effective version of Hilbert's irreducibility theorem. Séminaire de Théorie des Nombres, Paris 1988–1989, 241–249, Progr. Math. **91**, Birkhäuser Boston, Boston, MA, 1990. 8

[58] E. Frossard, Obstruction de Brauer-Manin pour les zéro-cycles sur des fibrations en variétés de Severi–Brauer. J. reine angew. Math. 557 (2003), 81–101.

[59] P. Gille, T. Szamuely, Central Simple Algebras and Galois Cohomology, Cambridge Studies in Advanced Mathematics 101, Cambridge University Press, 2006.

[60] T. Graber, J. Harris, et J. Starr, Families of rationally connected varieties, J. Amer. Math. Soc. **16** (2003) 57-67. 5, 34, 39, 49

[61] A. Grothendieck, Le groupe de Brauer, I, II, III, in *Dix exposés sur la cohomologie des schémas*, J. Giraud et al., Adv. Stud. Pure Math., **3**, Masson & North-Holland, Amsterdam, 1968, pp. 46–188.

[62] D. Harari, Méthode des fibrations et obstruction de Manin, Duke Math. J. **75** (1994) 221–260. 9, 23, 24, 25, 39, 40

[63] D. Harari, Obstructions de Manin transcendantes. Number theory (Paris, 19931994), 75–87, London Math. Soc. Lecture Note Ser., 235, Cambridge Univ. Press, Cambridge, 1996.

[64] D. Harari, Flèches de spécialisation en cohomologie étale et applications arithmétiques, Bull. Soc. Math. France 125 (1997), no. 2, 143–166. 39, 46

[65] D. Harari, Weak approximation and non-abelian fundamental groups., Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 4, 467–484. 18

[66] D. Harari, Groupes algébriques et points rationnels. Math. Ann. 322 (2002), no. 4, 811–826.

[67] D. Harari, The Manin obstruction for torsors under connected algebraic groups. Int. Math. Res. Not. 2006, Art. ID 68632, 1–13.

[68] D. Harari, Quelques propriétés d'approximation reliées à la cohomologie galoisienne d'un groupe algébrique fini. Bull. Soc. Math. France 135 (2007), no. 4.

[69] D. Harari, Spécialisation des conditions de Manin pour les variétés fibrées au-dessus de l'espace projectif, Compositio Math. 143, no 3, 603-617 (2007).

[70] D. Harari, Le défaut d'approximation forte pour les groupes algébriques commutatifs, Algebra & Number Theory 2 (2008), no. 5, 595–611.

[71] D. Harari et A. N. Skorobogatov, Non-abelian cohomology and rational points, Compositio Math. 130 (2002), no. 3, 241–273. 18

[72] D. Harari et A. N. Skorobogatov, The Brauer group of torsors and its arithmetic applications, Ann. Inst. Fourier (Grenoble) 53 (2003), no. 7, 1987–2019.

[73] D. Harari et A. N. Skorobogatov, Non-abelian descent and the arithmetic of Enriques surfaces, Int. Math. Res. Not. 52 (2005), 3203–3228. 18

[74] D. Harari et A. N. Skorobogatov, Descent theory for open varieties, in Torsors, étale homotopy and applications to rational points (Edinburgh, 2011).

[75] D. Harari et J. Stix, Descent obstruction and fundamental exact sequence, dans The Arithmetic of Fundamental Groups – PIA 2010, Contributions in Mathematical and Computational Sciences, Vol. 2, Springer-Verlag Berlin Heidelberg, 2012.

[76] D. Harari et T. Szamuely, Arithmetic duality theorems for 1-motives, J. reine angew. Math. **578**, 93–128 (2005). Voir les errata sur la page des auteurs.

[77] D. Harari et T. Szamuely, Local-global principles for 1-motives, Duke Math. J. **143**, no 3, 531–557 (2008). 16

[78] Y. Harpaz et A. N. Skorobogatov, Singular curves and the étale Brauer-Manin obstruction for surfaces. Ann. Sci. E.N.S. 47 (2014) 765–778. 18
[79] Yonatan Harpaz, Alexei N. Skorobogatov, Olivier Wittenberg, The Hardy–Littlewood conjecture and rational points, The Hardy-Littlewood conjecture and rational points. Compositio Math. 150 (2014) 2095-2111. 31, 32
[80] Yonatan Harpaz et Olivier Wittenberg, On the fibration method for zero-cycles and rational points, preprint, 2014. `http://www.math.ens.fr/ wittenberg/zcfib.pdf` 9, 10, 33, 34, 35, 39, 40, 47, 48
[81] B. Hasssett et Yu. Tschinkel, Effective computation of Picard groups and Brauer–Manin obstructions of degree two $K3$ surfaces over number fields, arXiv 1203.2214v1 [math.AG] 10 Mar 2012.
[82] B. Hassett, A. Várilly-Alvarado et P. Varilly, Transcendental obstructions to weak approximation on general K3 surfaces. Adv. Math. 228 (2011), no. 3, 13771404. 19
[83] B. Hassett et A. Várilly-Alvarado, Failure of the Hasse principle on general K3 surfaces, arXiv:1110.1738v1 [math.NT].
[84] R. Heath-Brown et A. N. Skorobogatov, Rational solutions of certain equations involving norms, Acta Math. **189** (2002), no. 2, 161–177.
[85] E. Ieronymou, Diagonal quartic surfaces and transcendental elements of the Brauer groups. J. Inst. Math. Jussieu 9 (2010), no. 4, 769–798. 19
[86] E. Ieronymou, A.N. Skorobogatov et Yu.G. Zarhin, On the Brauer group of diagonal quartic surfaces. J. London Math. Soc. 83 (2011) 659-672. 19
[87] J. Jahnel et D. Loughran, The Hasse principle for lines on Del Pezzo surfaces, arXiv:1410.5671 [math.NT] 11
[88] A. Kresch et Yu. Tschinkel, On the arithmetic of del Pezzo surfaces of degree 2, Proc. London Math. Soc. (3) 89 (2004), no. 3, 545–569. 19
[89] A. Kresch et Yu. Tschinkel, Effectivity of Brauer-Manin obstructions on surfaces. Adv. Math. 226 (2011), no. 5, 4131–4144.
[90] S. Lang and A. Weil, Number of points of varieties over finite fields, Amer. J. Math. **76** (1954) 819–827.
[91] Yongqi Liang, Principe local-global pour les zéro-cycles sur certaines fibrations au-dessus d'une courbe : I, à paraître dans Mathematische Annalen.
[92] Yongqi Liang, Astuce de Salberger et zéro-cycles sur certaines fibrations, à paraître dans International Mathematics Research Notices.
[93] Yongqi Liang, Arithmetic of 0-cycles on varieties defined over number fields, Annales Scientifiques de l'École Normale Supérieure. 44, 45, 46
[94] Yongqi Liang, Principe local-global pour les zéro-cycles sur certaines fibrations au-dessus de l'espace projectif, Bull. Soc. Math. France, 2012.
[95] Yongqi Liang, The local-global exact sequence for Chow groups of zero-cycles, preprint 2012 http://arxiv.org/abs/1206.1486
[96] Qing Liu et Fei Xu, Very strong approximation for certain algebraic varietes, Math. Ann., à paraître, 11
[97] Manin, Ju. I. Rational surfaces over perfect fields (en russe). Inst. Hautes tudes Sci. Publ. Math. No. 30 (1966) 55–113.
[98] Yu. I. Manin, Le groupe de Brauer-Grothendieck en géométrie diophantienne, in: Actes du Congrès International des Mathématiciens, Nice, 1970, Gauthier-Villars, Paris, 1971, pp. 401–411.
[99] Yu. I. Manin, Cubic Forms: Algebra, Geometry, Arithmetic (North-Holland Publishing Co., Amsterdam, 1974); Translated from Russian by M. Hazewinkel.
[100] A. S. Merkurjev, Zero-cycles on algebraic tori. The geometry of algebraic cycles, 119–122, Clay Math. Proc., 9, Amer. Math. Soc., Providence, RI, 2010.
[101] J.S. Milne. *Étale cohomology*. Princeton University Press, 1980.

[102] J.S. Milne. *Arithmetic Duality Theorems*. 2nd edition. Kea Books, BookSurge, 2004.

[103] V. P. Platonov et A. S. Rapinchuk, Groupes algébriques et théorie des nombres (en russe), Nauka, Moscou 1991. Trad. ang. Algebraic groups and Number Theory, Academic Press, 1994.

[104] B. Poonen, Heuristics for the Brauer-Manin obstruction for curves. Experiment. Math. 15 (2006), no. 4, 415–420.

[105] B. Poonen, The set of nonsquares in a number field is Diophantine. Math. Res. Lett. 16 (2009), no. 1, 165–170.

[106] B. Poonen, Existence of rational points on smooth projective varieties. J. Eur. Math. Soc. (JEMS) 11 (2009), no. 3, 529–543.

[107] B. Poonen, Insufficiency of the Brauer-Manin obstruction applied to étale covers. Ann. of Math. (2) 171 (2010), no. 3, 2157–2169. 18

[108] P. Salberger, Zero-cycles on rational surfaces over number fields, Invent. math. **91** (1988) 505–524. 43, 47

[109] P. Salberger et A. N. Skorobogatov, Weak approximation for surfaces defined by two quadratic forms. Duke Math. J. 63 (1991), no. 2, 517–536.

[110] P. Salberger, On obstructions to the Hasse principle. Number theory and algebraic geometry, 251277, London Math. Soc. Lecture Note Ser., 303, Cambridge Univ. Press, Cambridge, 2003. 43

[111] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires, J. für die reine und angew. Math. (Crelle) 327 (1981) 12–80. 16, 22, 47

[112] J.-J. Sansuc, Descente et principe de Hasse pour certaines variétés rationnelles, Séminaire de théorie des nombres (Delange–Pisot–Poitou) Paris 1980-1981, Progr. Math. Birkhäuser **22** (1982) 253-272. 8

[113] J.-J. Sansuc, À propos d'une conjecture arithmétique sur le groupe de Chow d'une surface rationnelle, Séminaire de théorie des nombres de Bordeaux, 1981/1982, Exp. No. **33**, 38 pp., Univ. Bordeaux I, Talence, 1982. 23

[114] D. Schindler et A. N. Skorobogatov, Norms as products of linear polynomials. J. London Math. Soc. 89 (2014) 559–580.

[115] A. Schinzel et W. Sierpiński, Sur certaines hypothèses concernant les nombres premiers, Acta Arith. **4** (1958) 185-208.

[116] J-P. Serre. *Corps locaux*. Hermann, 1968.

[117] J.-P. Serre, *Cohomologie galoisienne*, 5ème édition, révisée et complétée, Lecture Notes in Mathematics **5**, Springer-Verlag, Berlin Heidelberg New York, 1994. 12

[118] J-P. Serre, *Lie Algebras and Lie groups*, LNM **1500**, Springer-Verlag.

[119] A. N. Skorobogatov, On the fibration method for proving the Hasse principle and weak approximation, in *Séminaire de théorie des nombres de Paris 1988-1989*, éd. C. Goldstein, Progress in Math. **91** (1990) 205-219.

[120] A. N. Skorobogatov, Arithmetic on certain quadric bundles of relative dimension 2. I, J. für die reine und angew. Math. **407** (1990) 57-74 .

[121] A. N. Skorobogatov, Descent on fibrations over the projective line, Amer. J. Math. **118** (1996) 905-923.

[122] A. N. Skorobogatov, Beyond the Manin obstruction, Invent. math. 135 (1999), 399424. 18

[123] A. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics **144**, Cambridge University Press, Cambridge, 2001.

[124] A. N. Skorobogatov, Descent obstruction is equivalent to étale Brauer-Manin obstruction, Math. Ann. 344 (2009), 501–510. 18

[125] A. N. Skorobogatov, Descent on toric fibrations, To appear in "Arithmetic Geometry" (Proceedings of the Hausdorff trimester, 2013). Dieulefait, L.; Faltings, G.; Heath-Brown, R.; Manin, Y.; Moroz, B.; Wintenberger, J.-P. (eds.), C.U.P. http://arxiv.org/abs/1312.7539 39

[126] A. N. Skorobogatov and P. Swinnerton-Dyer, 2-descent on elliptic curves and rational points on certain Kummer surfaces, Adv. Math. 198(2) (2005), 448–483. 42

[127] A.N. Skorobogatov and Yu.G. Zarhin, A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces, *J. Alg. Geom.* **17** (2008) 481–502. 19

[128] A.N. Skorobogatov and Yu.G. Zarhin, The Brauer group of Kummer surfaces and torsion of elliptic curves, J. reine angew. Math., 666 (2012). 19

[129] A.N. Skorobogatov and Yu.G. Zarhin, The Brauer group and the Brauer–Manin set of products of varieties, arXiv:1112.3089 J. Eur. Math. Soc., to appear. 19

[130] A. Smeets, Principes locaux-globaux pour certaines fibrations en torseurs sous un tore. Math. Proc. Cambridge Philos. Soc. 158 (2015), no. 1, 131–145. 8, 39

[131] A. Smeets, Insufficiency of the étale Brauer–Manin obstruction : towards a simply connected example, http://arxiv.org/abs/1409.6706 18

[132] M. Stoll, Finite descent obstructions and rational points on curves. Algebra Number Theory 1 (2007), no. 4, 349–391. 11

[133] M. Swarbrick Jones, Weak approximation for cubic hypersurfaces of large dimension http://arxiv.org/abs/1111.4082v1.

[134] M. Swarbrick Jones, A note on a theorem of Heath-Brown and Skorobogatov, http://arxiv.org/abs/1111.4089v1

[135] P. Swinnerton-Dyer, The Brauer group of cubic surfaces, Math. Proc. Cambridge Philos. Soc. 113(3) (1993) 449460.

[136] Sir Peter Swinnerton-Dyer, Rational points on pencils of conics and on pencils of quadrics, J. London Math. Soc. (2) **50** (1994) 231–242. 30

[137] Sir Peter Swinnerton-Dyer, Rational points on certain intersections of two quadrics, in *Abelian Varieties*, Proceedings of a conference held at Egloffstein, Barth, Hulek, Lange ed., Walter de Gruyter, Berlin – New York 1995, p. 273–292.

[138] P. Swinnerton-Dyer, Brauer-Manin obstructions on some Del Pezzo surfaces. Math. Proc. Cambridge Philos. Soc. 125 (1999), no. 2, 193–198.

[139] P. Swinnerton-Dyer, Arithmetic of diagonal quartic surfaces, II, Proc. Lond. Math. Soc. 80(3) (2000), 513–544. Corrigenda: Proc. London Math. Soc. (3) **85** (2002), no. 3, 564.

[140] Sir Peter Swinnerton-Dyer, The solubility of diagonal cubic surfaces. Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 6, 891–912. 42

[141] J. van Hamel, The Brauer-Manin obstruction for zero-cycles on Severi-Brauer fibrations over curves, J. London Math. Soc. (2) **68** (2003) no. 2, 317–337. 43

[142] A. Várilly-Alvarado, B. Viray, Failure of the Hasse principle for Enriques surfaces. Adv. Math. 226 (2011), no. 6, 4884–4901. 18

[143] A. Várilly-Alvarado and B. Viray, Higher-dimensional analogs of Châtelet surfaces. Bull. Lond. Math. Soc. 44 (2012), no. 1, 125–135.

[144] V. E. Voskresenskiĭi, Maximal tori without affect in semisimple algebraic groups, Mat. Zametki 44 (1988), no. 3, 309–318, 410.

[145] V. E. Voskresenskiĭ, *Algebraic groups and their birational invariants*, Amer. Math. Soc., Translations of mathematical monographs 179 (1998).

[146] D. Wei, On the equation $N_{K/k}(\Xi) = P(t)$, prépublication, 2011. arXiv:1202.4115, to appear in Proc. London Math. Soc.

[147] D. Wei, The unramified Brauer group of norm one tori, arXiv. 1202.4714v1 [math.NT], Advances in Mathematics. 23

[148] O. Wittenberg, Transcendental Brauer-Manin obstruction on a pencil of elliptic curves. Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 259–267, Progr. Math., 226, Birkhäuser Boston, Boston, MA, 2004. 23

[149] O. Wittenberg, Intersections de deux quadriques et pinceaux de courbes de genre 1/Intersections of two quadrics and pencils of curves of genus 1. Lecture Notes in Mathematics **1901**. Springer, Berlin, 2007. viii+218 pp.

[150] O. Wittenberg, On Albanese torsors and the elementary obstruction. Math. Ann. **340** (2008), no. 4, 805–838. 16

[151] O. Wittenberg, La connexité rationnelle en arithmétique, in Variétés rationnellement connexes : aspects géométriques et arithmétiques, p. 61114, Panoramas & Synthèses **31**, Société Mathématique de France, 2010.

[152] O. Wittenberg, Zéro-cycles sur les fibrations au-dessus d'une courbe de genre quelconque, Duke Math. J. **161** (2012) no. 11, 2113–2166. 43, 47

[153] O. Wittenberg, Une remarque sur les courbes de Reichardt–Lind et de Schinzel, in The arithmetic of fundamental groups, PIA 2010 (ed. J. Stix), 329–337, Contributions in Mathematical and Computational Sciences 2, Springer-Verlag, Heidelberg, 2012.

[154] *Théorie des topos et cohomologie étale des schémas* (SGA 4). Séminaire de géométrie algébrique du Bois-Marie 1963-1964. Dirigé par M. Artin, A. Grothendieck et J.-L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Math. **269**, **270**, **305**. Springer-Verlag, 1972, 1973.

[155] *Cohomologie étale* (SGA $4\frac{1}{2}$). Séminaire de géométrie algébrique du Bois-Marie. Avec la collaboration de J.F. Boutot, A. Grothendieck, L. Illusie et J.-L. Verdier. Lecture Notes in Math. **569**. Springer-Verlag, 1977.

CNRS & UNIVERSITÉ PARIS SUD, MATHÉMATIQUES, BÂTIMENT 425, F-91405 ORSAY CEDEX, FRANCE

E-mail address: jlct@math.u-psud.fr