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1 Introduction

It had long been suspected that varieties X over a number field k without k-points but with a non-empty Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ are fairly common. The first examples were found in [27] and then in [2]. An earlier, conditional example is given in [21]. One should also expect that there are many varieties X without k-points for which the étale Brauer–Manin set $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathrm{Br}}$ is non-empty (we refer to [19] or [29] for the definition of these subsets of the space $X(\mathbf{A}_k)$ of adèles of X). Different methods to construct such varieties have been found recently. In [19] Poonen constructs a threefold X with a surjective morphism to a curve C that has exactly one k-point P and the fibre X_P has points everywhere locally but not globally. In Poonen's example X_P is a smooth Châtelet surface. The trick with a curve with just one rational point was also used in [11] where the fibres of $X \to C$ are curves of high genus and X_P is a singular curve which geometrically is a union of projective lines. In retrospect one could note that the examples in [27] and [2] are families of genus 1 curves parameterised by elliptic curves of Mordell–Weil rank 0.

In this paper we propose more flexible methods to construct such examples. We show that the varieties X such that $X(k) = \emptyset$ and $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$ include the following:

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a conic bundle surface $X \to E$ over a real quadratic field k, where E is an elliptic curve such that $E(k) = \{0\}$, see Sect. 5.2;
a threefold over an arbitrary real number field $k \subset \mathbb{R}$, which is a family $X \to C$ of 2-dimensional quadrics parameterised by a curve C with exactly one k-point (one can choose C to be an elliptic curve), see Sect. 3.1;
a threefold over an arbitrary number field k, which is a family $X \to C$ of geometrically rational surfaces parameterised by a curve C with exactly one k-point, the fibre above which is singular, see Sect. 3.2.

In the first and second examples, in contrast to those previously known, the smooth fibres satisfy the Hasse principle and weak approximation. To put this into a historical perspective let us note that soon after Manin [17] introduced the obstruction now bearing his name, Iskovskikh [13] constructed a counterexample to the Hasse principle on a conic bundle over the projective line over $\mathbb{Q}$. His intention was, as he pointed out to one of us, to give a counterexample to the Hasse principle that could not be explained by the Brauer–Manin obstruction. It is well known nowadays that Iskovskikh's counterexample can be explained by the Brauer–Manin obstruction, and conjecturally the same should be true for all counterexamples to the Hasse principle on geometrically rational surfaces, see [4,5].

The examples we construct in this paper show that this is no longer the case for conic or quadric bundles over curves of genus at least 1.

In a nutshell, the idea is this. Let k be a number field. Following Poonen we use a base variety B such that $B(k) = \{P\}$. By a continuous deformation of the adèle attached to P at an archimedean component we see that $B(k)$ is not dense in $B(\mathbf{A}_k)^{\mathrm{Br}}$. Density may also fail due to places of k that need not be archimedean. Suppose B contains an irreducible singular conic S so that $P = S_{\mathrm{sing}}$. If a place v of k splits in the quadratic extension given by the discriminant of the binary quadratic form that defines S, then $B \times_k k_v$ contains two copies of $\mathbb{P}^1_{k_v}$ meeting at P. Since $\mathrm{Br}(\mathbb{P}^1_{k_v}) = \mathrm{Br}(k_v)$, we can modify the adèle of P at v while staying inside $B(\mathbf{A}_k)^{\mathrm{Br}}$. However, the k-point P cannot be moved in B, so $B(k)$ is not dense in $B(\mathbf{A}_k)^{\mathrm{Br}}$.

Next, one constructs a surjective morphism $X \to B$ for which the fibre X_P has local points in all but one or two completions of k, and ensures that X has k_v-points for missing places v such the resulting adelic point of X projects to $B(\mathbf{A}_k)^{\mathrm{Br}}$. Now, if the natural map $\mathrm{Br}(B) \to \mathrm{Br}(X)$ is surjective we have found an adelic point in $X(\mathbf{A}_k)^{\mathrm{Br}}$. But since $X(k) \subset X_P$ we have $X(k) = \emptyset$. With more work one can find examples such that $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$ is non-empty, too.

In this paper we have nothing to say about the important open question whether the implication

$$X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset \quad \Rightarrow \quad X(k) \neq \emptyset$$

holds if X is a surface with finite geometric fundamental group, e.g. a K3 surface or an Enriques surface.

The paper is organised as follows. After some preparations in Sect. 2 we realise the aforementioned programme for threefolds in Sect. 3. Making it work for surfaces requires rather more effort. For this purpose in Sect. 4 we establish some Bashmakov-style properties of elliptic curves with a large Galois image on torsion points. These properties are used in the proof of our main result in the case of surfaces in Sect. 5. Some general observations on the Brauer–Manin set are collected in Sect. 6.

The authors are grateful to Nathan Jones and Chris Wuthrich for their helpful advice on Serre curves. We used `sage` [30] in our calculations with elliptic curves. This work started

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2 Brauer groups and torsors on quadric bundles

For the convenience of the reader we recall the following well known lemma.

Lemma 2.1 *Let k be a field of characteristic zero. Let X be a smooth projective quadric over k of dimension at least 1. Then the natural map* $\mathrm{Br}(k) \to \mathrm{Br}(X)$ *is surjective. If* $\dim(X) \geq 3$, *then this map is an isomorphism.*

Proof Let $\bar{k}$ be an algebraic closure of k, and let $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$. For any smooth, projective and geometrically integral variety X over k there is a well known exact sequence

$$0 \to \mathrm{Pic}\,(X) \to \mathrm{Pic}\,\left(\overline{X}\right)^{\Gamma_k} \to \mathrm{Br}\,(k) \to \mathrm{Ker}\left[\mathrm{Br}\,(X) \to \mathrm{Br}\,\left(\overline{X}\right)\right] \to H^1\left(k, \mathrm{Pic}\,\left(\overline{X}\right)\right),$$

where $\overline{X} = X \times_k \bar{k}$. If X is a quadric of dimension at least 1, then $\mathrm{Pic}(\overline{X})$ is a permutation Γ_k-module and $\mathrm{Br}(\overline{X}) = 0$ (since $\overline{X}$ is rational and the Brauer group is a birational invariant of smooth projective varieties). By Shapiro’s lemma we have $H^1(k, \mathrm{Pic}(\overline{X})) = 0$, so the exact sequence implies the surjectivity of the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$. When $\dim(X) \geq 3$, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})$ is an isomorphism, because both groups are generated by the hyperplane section class, so in this case $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is an isomorphism. □

In this paper a *quadric bundle* is a surjective flat morphism $f : X \to B$ of smooth, projective, geometrically integral varieties over a field k, the generic fibre of which is a smooth quadric of dimension at least 1, and all geometric fibres are reduced.

We denote by $k(B)$ the function field of B, and by $X_{k(B)}$ the generic fibre of $f : X \to B$. If $\dim(X_{k(B)}) = 1$, then $f : X \to B$ is called a *conic bundle*.

The following proposition is essentially well known, at least when $B = \mathbb{P}^1_k$, see [26, Cor. 3.2], [6, Thm. 2.2.1, Thm. 2.3.1], [3, Prop. 2.1].

Proposition 2.2 *Let $f : X \to B$ be a quadric bundle over a field k of characteristic zero. In each of the following cases the map* $f^* : \mathrm{Br}(B) \to \mathrm{Br}(X)$ *is surjective.*

(i) $\dim(X_{k(B)}) = 1$ *and there is a point $P \in B$ of codimension 1 such that for each point $Q \neq P$ of codimension 1 in B the fibre X_Q contains a geometrically integral component of multiplicity 1;*
(ii) $\dim(X_{k(B)}) = 2$ *and for each point $Q \in B$ of codimension 1 the fibre X_Q contains a geometrically integral component of multiplicity 1;*
(iii) $\dim(X_{k(B)}) \geq 3$.

Proof (i) Let $\gamma \in \mathrm{Br}(k(B))$ be the class of the conic $X_{k(B)}$. Since γ is in the kernel of the natural map $f^* : \mathrm{Br}(k(B)) \to \mathrm{Br}(X_{k(B)})$, the assumption of (i) implies that the residue $\mathrm{res}_Q(\gamma) \in H^1(k(Q), \mathbb{Q}/\mathbb{Z})$ is zero if $Q \neq P$. Take any $\alpha \in \mathrm{Br}(X)$. By Lemma 2.1 the map $f^* : \mathrm{Br}(k(B)) \to \mathrm{Br}(X_{k(B)})$ is surjective, so the image of α in $\mathrm{Br}(X_{k(B)})$ comes from some $\beta \in \mathrm{Br}(k(B))$. Again, by the assumption of (i) we have $\mathrm{res}_Q(\beta) = 0$ if $Q \neq P$. Moreover, $\mathrm{res}_P(\beta) = 0$ or $\mathrm{res}_P(\beta) = \mathrm{res}_P(\gamma)$. By the purity theorem for the Brauer group [10, III, Thm. 6.1, p. 134] we conclude that $\beta \in \mathrm{Br}(B)$ or $\beta - \gamma \in \mathrm{Br}(B)$. Since $f^*(\beta) = f^*(\beta - \gamma) = \alpha$ in $\mathrm{Br}(X_{k(B)})$, and the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{k(B)})$ is injective, we have proved (i).

The proof of (ii) uses Lemma 2.1 and the arguments from the proof of (i).

In case (iii) it is well known that each fibre of f at a point of B of codimension 1 contains a geometrically integral component of multiplicity 1. Then (iii) follows from the last statement of Lemma 2.1. □

Proposition 2.3 *Let $f : X \to B$ be a quadric bundle over a field k of characteristic zero. Then any torsor $X' \to X$ of a finite k-group scheme G is the inverse image under f of a torsor $B' \to B$ of G.*

Proof By our definition of quadric bundles, the morphism f is flat and all its geometric fibres are connected and reduced. The generic geometric fibre of f is simply connected. By [25, X, Cor. 2.4] this implies that each geometric fibre of such a fibration is simply connected. The result then follows from [25, IX, Cor. 6.8]. □

3 Threefolds

3.1 Example based on real deformation

Let k be a number field with a real place. We fix a real place v, so we can think of k as a subfield of $k_v = \mathbb{R}$.

Let C be a smooth, projective, geometrically integral curve over k such that $C(k)$ consists of just one point, $C(k) = \{P\}$. By [20] such a curve exists for any number field k, and by [18, Thm. 1.1] we can take C to be an elliptic curve over k. Let $\Pi \subset C(\mathbb{R})$ be an open interval containing P. Let $f : C \to \mathbb{P}^1_k$ be a surjective morphism that is unramified at P. Choose a coordinate function t on $\mathbb{A}^1_k = \mathbb{P}^1_k \backslash f(P)$ such that f is unramified above $t = 0$. We have $f(P) = \infty$. Take any $a > 0$ in k such that a is an interior point of the interval $f(\Pi)$ and f is unramified above $t = a$.

Let w be a finite place of k. There exists a quadratic form $Q(x_0, x_1, x_2)$ of rank 3 that represents zero in all completions of k other than k_v and k_w, but not in k_v or k_w. We can assume that Q is positive definite over $k_v = \mathbb{R}$. Choose $n \in k$ with $n > 0$ in k_v and $-nQ(1, 0, 0) \in k_w^{*2}$. Let $Y_1 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(x_0, x_1, x_2) + nt(t - a)x_3^2 = 0$, and let $Y_2 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(X_0, X_1, X_2) + n(1 - aT)X_3^2 = 0$. We glue Y_1 and Y_2 by identifying $T = t^{-1}$, $X_3 = tx_3$, and $X_i = x_i$ for $i = 0, 1, 2$. This produces a quadric bundle $Y \to \mathbb{P}^1_k$ with exactly two degenerate fibres (over $t = a$ and $t = 0$), each given by the quadratic form $Q(x_0, x_1, x_2)$ of rank 3. Define $X = Y \times_{\mathbb{P}^1_k} C$. This is a quadric bundle $X \to C$ with geometrically integral fibres.

For example, if $k = \mathbb{Q}$, we can take $k_w = \mathbb{Q}_2$ and consider Y defined by

$$x_0^2 + x_1^2 + x_2^2 + 7t(t - a)x_3^2 = 0.$$

Proposition 3.1 *In the above notation we have $X(\mathbf{A}_k)^{\text{ét,Br}} \neq \emptyset$ and $X(k) = \emptyset$.*

Proof Since $C(k) = \{P\}$ we have $X(k) \subset X_P$. The fibre X_P is the smooth quadric $Q(x_0, x_1, x_2) + nx_3^2 = 0$. This quadratic form is positive definite thus X_P has no points in $k_v = \mathbb{R}$ and so $X(k) = \emptyset$. By assumption X_P has local points in all completions of k other than k_v and k_w. The condition $-nQ(1, 0, 0) \in k_w^{*2}$ implies that X_P contains k_w-points, so X_P has local points in all completions of k but one. Choose $N_u \in X_P(k_u)$ for each place $u \neq v$. Consider a small real $\varepsilon > 0$ such that $a - \varepsilon \in f(\Pi)$ and $\varepsilon < a$. Let $M \in \Pi$ be such that $f(M) = a - \varepsilon$. Then the smooth real fibre X_M is given by an indefinite quadratic form and so $X_M(k_v) \neq \emptyset$. Choose any $N_v \in X_M(k_v)$. We now have an adelic point (N_u), where we allow $u = v$.

We claim that $(N_u) \in X(\mathbf{A}_k)^{\text{ét,Br}}$.

Let G be a finite k-group scheme. Proposition 2.3 implies that any torsor X'/X of G comes from a torsor C'/C of G, in the sense that $X \times_C C' \to X$ and $X' \to X$ are isomorphic as X-torsors with the structure group G. Let $\sigma \in Z^1(k, G)$ be a 1-cocycle defining the k-torsor which is the fibre of $C' \to C$ at P. Twisting X'/X and C'/C by σ and replacing the group G by the twisted group G^σ and changing notation, we can assume that C' contains a k-point P' that maps to P in C. The irreducible component C'' of C' that contains P' is a geometrically integral curve over k. Let $X'' \subset X'$ denote the inverse image of C'' in X'. The fibres of the morphism $X \to C$ are geometrically integral, hence such are also the fibres of $X' \to C'$ and $X'' \to C''$. Thus X'' is a geometrically integral variety over k.

There are natural isomorphisms $X''_{P'} \cong X'_{P'} \cong X_P$, so we can define $N'_u \in X''(k_u)$ as the point that maps to $N_u \in X(k_u)$ for each $u \neq v$. The map $C'' \to C$ is finite and étale. The image of $C''(\mathbb{R})$ in $C(\mathbb{R})$ is thus closed and open. The image of the connected component of $P' \in C''(\mathbb{R})$ is the whole connected component of $P \in C(\mathbb{R})$, hence contains Π. The inverse image of the interval Π in $C''(\mathbb{R})$ is a disjoint union of intervals, one of which contains P' and maps bijectively onto Π. Let us call this interval Π'. Let M' be the unique point of Π' over M. Let $N'_v \in X''_{M'}(\mathbb{R})$ be the point that maps to $N_v \in X_M(\mathbb{R})$. Thus the adelic point $(N'_u) \in X''(\mathbf{A}_k) \subset X'(\mathbf{A}_k)$ projects to the adelic point $(N_u) \in X(\mathbf{A}_k)$.

By the definition of the étale Brauer–Manin obstruction, to prove that (N_u) is contained in $X(\mathbf{A}_k)^{\text{ét,Br}}$ it suffices to show that (N'_u) is orthogonal to $\mathrm{Br}(X')$. For this it is enough to show that (N'_u) is orthogonal to $\mathrm{Br}(X'')$. By Proposition 2.2 (ii) applied to $X'' \to C''$ we know that the natural map $\mathrm{Br}(C'') \to \mathrm{Br}(X'')$ is surjective. Thus it is enough to show that the adèle on C'' such that its u-adic component is P' when $u \neq v$ and and its v-component is M', is orthogonal to $\mathrm{Br}(C'')$. The real point M' is path-connected to P', so this adèle is in the connected component of the diagonal image of the k-point P' in $C''(\mathbf{A}_k)$. But the latter adèle is certainly in $C''(\mathbf{A}_k)^{\mathrm{Br}}$, and the proposition follows. □

Remark 3.2 (1) Our method gives simple examples of threefolds with points everywhere locally but not globally and no Brauer–Manin obstruction. An even simpler proof is available in the case of fibrations into quadrics of dimension at least 3 over a curve.

(2) By a theorem of Wittenberg [32, Thm. 1.3] the variety X has a 0-cycle of degree 1 over k, that is, there exist field extensions $k_1, \ldots, k_r$ of k whose degrees have no common factor such that $X(k_i) \neq \emptyset$ for $i = 1, \ldots, r$. Although [32, Thm. 1.3] requires the finiteness of the Shafarevich–Tate group of the Jacobian of C, Wittenberg pointed out that in the proof of his theorem this assumption is only used to ensure the existence of a suitable 0-cycle of degree 1 on C. In our case such a 0-cycle is directly provided by the k-point P, so the assumption on the Shafarevich–Tate group is not needed. For more details see Remark 5.7 below.

3.2 Examples based on deformation along a rational curve defined over a completion of *k*

Lemma 3.3 *Let k be a number field. There exists a smooth, projective, geometrically integral surface B over k with the following properties:*

B contains a curve S isomorphic to an irreducible singular projective conic;
the singular point of S is the unique k-point of B;
there is a surjective morphism $\pi : B \to \mathbb{P}^1_k$ with smooth and geometrically integral generic fibre such that $\pi(S) = \mathbb{P}^1_k$.

Proof According to [20] there is a smooth, projective and geometrically integral curve C over k with exactly one k-point, $C(k) = \{O\}$. Moreover, by [18, Thm 1.1] there is an elliptic curve over k with this property. Let $f : Z \to C$ be any conic bundle such that the fibre Z_O over $O \in C(k)$ is an irreducible singular conic. The singular point of $S = Z_O$ is then the unique k-point of Z.

There is a closed embedding $Z \subset \mathbb{P}^n_k$ for some $n \geq 1$. By the Bertini theorem [12, II.8.18, III.7.9], there exists a hyperplane $H_1 \subset \mathbb{P}^n_k$ such that $Z \cap H_1$ is a smooth and geometrically integral curve. This implies that S is not a subset of H_1.

Let d be the degree of Z in $\mathbb{P}^n_k$. We can find a hyperplane $H_2 \subset \mathbb{P}^n_k$ such that $Z \cap H_1 \cap H_2$ is a set of d distinct $\bar{k}$-points not in S. It follows that no geometric irreducible component of S is contained in a hyperplane passing through $H_1 \cap H_2$.

Let $\tilde{\mathbb{P}}^n_k$ be the blowing-up of $\mathbb{P}^n_k$ at $H_1 \cap H_2 \simeq \mathbb{P}^{n-2}_k$. The projection from $H_1 \cap H_2$ defines a morphism $\tilde{\mathbb{P}}^n_k \to \mathbb{P}^1_k$. Let B be the Zariski closure of $Z \backslash (Z \cap H_1 \cap H_2)$ in $\tilde{\mathbb{P}}^n_k$, so that B is the blowing-up of Z in d distinct points. Thus B is a smooth, projective, geometrically integral surface with a unique k-point, equipped with a surjective morphism $\pi : B \to \mathbb{P}^1_k$ with smooth and geometrically integral generic fibre. Moreover, S is contained in B, and $\pi(S) = \mathbb{P}^1_k$. □

Let P be the unique k-point of B, and let $Q = \pi(P) \in \mathbb{P}^1_k(k)$. Let K be the quadratic extension of k over which the components of S are defined. If w is a place of k that splits in K, then the k_w-variety $S \times_k k_w$ is the union of two projective lines meeting at P. Let $L_w \subset B \times_k k_w$ be one of these rational curves. Since $\pi(L_w) = \mathbb{P}^1_{k_w}$, there is a point $N_w \in L_w(k_w)$ such that $\pi(N_w) \neq Q$.

Proposition 3.4 *Let w_1 and w_2 be places of k that split in K, $w_1 \neq w_2$. Let $Y \to \mathbb{P}^1_k$ be a conic bundle satisfying the following conditions:*

there exists a closed point $R \in \mathbb{P}^1_k$, $R \neq Q$, such that the restriction $Y \backslash Y_R \to \mathbb{P}^1_k \backslash R$ is a smooth morphism, and the fibre of $\pi : B \to \mathbb{P}^1_k$ at R is smooth;
the fibre Y_Q is a smooth conic that has k_v-points for all completions of k except w_1 and w_2, in particular $Y_Q(k) = \emptyset$;

$$Y_{\pi(N_w)}(k_w) \neq \emptyset \quad \textit{for } w = w_1, w_2.$$

Then for the smooth threefold $X = Y \times_{\mathbb{P}^1_k} B$ we have $X(\mathbf{A}_k)^{\text{ét,Br}} \neq \emptyset$ and $X(k) = \emptyset$.

Proof Let $p : X \to B$ be the natural projection. Since $B(k) = \{P\}$, $\pi(P) = Q$ and $Y_Q(k) = \emptyset$, we see that $X(k) = \emptyset$.

The fibre X_P is naturally isomorphic to Y_Q. For a place v such that $v \neq w_1$, $v \neq w_2$ choose $M_v \in X_P(k_v)$. For $w = w_1, w_2$, choose a k_w-point M_w in the k_w-fibre X_{N_w} (which is isomorphic to $Y_{\pi(N_w)}$). We claim that $(M_v) \in X(\mathbf{A}_k)^{\text{ét,Br}}$.

One easily checks that the projection map $X \to B$ is a quadric bundle as defined in this paper: both X and B are smooth and projective over k, the generic fibre is a geometrically integral conic, the morphism $X \to B$ is flat and all its geometric fibres are reduced. Let G be a finite k-group scheme. By Proposition 2.3 every torsor $X' \to X$ of G is the pullback of a torsor $B' \to B$ of G. After a twist by a k-torsor of G, as detailed in the proof of Proposition 3.1, we may assume that B' has a k-point P' over P. Let B'' be the irreducible component of B' that contains P'. Then B'' is geometrically integral. Let X'' be the inverse image of B'' under the map $X' \to B'$. The k-variety X'' is geometrically integral, and $X'' \to B''$ is a conic bundle.

For all $v \neq w_1, w_2$ the fibre $X''_{P'}$ contains a k_v-point M'_v that maps to M_v in X_P. Since $B'' \to B$ is finite and étale, for $w = w_1, w_2$ the inverse image of L_w in B'' contains a rational k_w-curve L'_w through P' that maps isomorphically to L_w. Let N'_w be the k_w-point of L'_w that maps to N_w. Then there is a k_w-point M'_w in $X''_{N'_w}$ that maps to $M_w \in X_{N_w}(k_w)$. To prove our claim it is enough to show that the adelic point (M'_v) in X'' is orthogonal to $\mathrm{Br}(X'')$.

Let $\bar{x} \in R$ be a $\bar{k}$-point. Since the singular loci of the morphisms $\overline{Y} \to \mathbb{P}^1_{\bar{k}}$ and $\pi : \overline{B} \to \mathbb{P}^1_{\bar{k}}$ are disjoint, the fibre $\overline{B}_{\bar{x}} = \pi^{-1}(\bar{x})$ is a smooth integral curve over $\bar{k}$. The fundamental group of $\mathbb{P}^1_{\bar{k}}$ is trivial. Therefore, by [25, Chap. X, Cor. 1.4], the map $\pi_1(\overline{B}_{\bar{x}}) \to \pi_1(\overline{B})$ is surjective. Since $\overline{B}''$ is integral, this implies that $B'' \times_B B_{\bar{x}}$ is also integral. As a consequence, the inverse image of R in B'' is a geometrically integral $k(R)$-curve D. Hence D is integral.

Since the restriction of the conic bundle $X'' \to B''$ to the complement $B'' \setminus D$ of the integral curve D is a smooth morphism, this conic bundle satisfies the condition of Proposition 2.2 (i). It follows that the induced map $\mathrm{Br}(B'') \to \mathrm{Br}(X'')$ is surjective. Thus it remains to prove that the image of (M'_v) in B'' is orthogonal to $\mathrm{Br}(B'')$.

This image is the adelic point such that for all $v \neq w_1, w_2$ the k_v-component is P'. For $w = w_1, w_2$ the k_w-component is N'_w. But L'_w is a projective line over k_w containing both P' and N'_w. The natural map $\mathrm{Br}(k_w) \to \mathrm{Br}(L'_w)$ is an isomorphism. Thus when pairing the image of (M'_v) in B'' with $\mathrm{Br}(B'')$ we may replace (M'_v) by the diagonal adèle (P'), which by the reciprocity law is orthogonal to $\mathrm{Br}(B'')$. □

Remark 3.5 Consider a surface B with a morphism $\pi : B \to \mathbb{P}^1_k$ as in Lemma 3.3. Let $Y \to \mathbb{P}^1_k$ be a quadric bundle all fibres of which are of dimension $d \geq 2$ and contain a geometrically integral component of multiplicity one (which is automatic if $d \geq 3$). Assume that the singular loci of $B \to \mathbb{P}^1_k$ and $Y \to \mathbb{P}^1_k$ do not intersect. Let $X = Y \times_{\mathbb{P}^1_k} B$. Suppose that the fibre Y_Q, where $Q = \pi(P) \in \mathbb{P}^1(k)$, is a smooth quadric such that $Y_Q(k_v) \neq \emptyset$ for all $v \neq w$ but $Y_Q(k_w) = \emptyset$ (such quadrics exist in dimension 2 and higher, but not in dimension 1). Assume that L_w has a non-empty intersection with the image of $X(k_w) \to B(k_w)$. In view of Proposition 2.2 (ii), (iii), an argument similar but shorter than the one above shows that $X(k) = \emptyset$ and $X(\mathbf{A}_k)^{\text{ét,Br}} \neq \emptyset$. In this example $\dim(X) = 2 + d \geq 4$.

Example 3.6 One can construct a threefold over $k = \mathbb{Q}$ as in Proposition 3.4 as follows. Let E be the elliptic curve $y^2 = x^3 - 5$. Then $E(\mathbb{Q}) = \{O\}$, where O is the point at infinity. Let $(r : s : u : v : t)$ be homogeneous coordinates in $\mathbb{P}^4_{\mathbb{Q}}$. The first of the equations

$$xyt^2 = u^2 + v^2, \qquad r^2 - 5s^2 - 17t^2 - u^2 = 0 \tag{1}$$

defines a closed subset of $(E \setminus \{O\}) \times \mathbb{P}^2_{\mathbb{Q}}$ which extends to a conic bundle surface $B \to E$. The fibre B_O over O is the singular conic S with equation $u^2 + v^2 = 0$. The unique $\mathbb{Q}$-point P of B is the singular point of S. The morphism $\pi : B \to \mathbb{P}^1_{\mathbb{Q}}$ given by the projection to $(u : t)$ satisfies the conclusions of Lemma 3.3. The second equation of (1) defines a smooth quadric $Q \subset \mathbb{P}^3_{\mathbb{Q}}$. Let $Y \to Q$ be the blowing-up of the closed point of Q given by $u = t = 0$. The projection via the coordinates $(u : t)$ is a morphism $Y \to \mathbb{P}^1_{\mathbb{Q}}$ which makes Y a conic bundle as in Proposition 3.4. Let $X = Y \times_{\mathbb{P}^1_k} B$. The fibre X_P over P is the conic $r^2 - 5s^2 - 17t^2 = 0$ over $\mathbb{Q}$, so for the places w_1 and w_2 one takes the primes 5 and 17. For $p = 5, 17$ choose a suitable point $N_p \in S(\mathbb{Q}_p)$ with $v = 1$ and $u = \alpha_p \in \mathbb{Q}_p$ such that $\alpha_p^2 = -1$.

One can give a different proof of the non-emptyness of the set $X(\mathbf{A}_{\mathbb{Q}})^{\text{ét,Br}}$ using the method of [11]. (By Proposition 6.1 (ii) below it does not matter which birational model is used for this). Let $K = \mathbb{Q}(\sqrt{-1})$. Consider the fibre X_O of the composed morphism $X \to B \to E$

over $O \in E(\mathbb{Q})$. The singular surface X_O is fibred into conics over the singular conic S; the inverse image of the singular point $P \in S$ is a smooth conic $X_P \subset X_O$. Thus $X_O \times_{\mathbb{Q}} K$ is the union of two geometrically irreducible components permuted by $\mathrm{Gal}(K/\mathbb{Q})$ that intersect transversally in X_P, each of them isomorphic to $Y_K = Y \times_{\mathbb{Q}} K$. Since 5 and 17 split in K and the components of $X_O \times_{\mathbb{Q}} K$ have K-points, we see that $X_O(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$. We claim that

$$X_O(\mathbf{A}_{\mathbb{Q}}) \subset X(\mathbf{A}_{\mathbb{Q}})^{\text{ét,Br}}.$$

Indeed, let G be a finite k-group scheme. The generic fibre of $X \to E$ is a geometrically integral, smooth and geometrically rational surface, so it is geometrically simply connected. One checks that the morphism $X \to E$ is flat and all its geometric fibres are connected and reduced. Now [25, X, Cor. 2.4] implies that each geometric fibre of $X \to E$ is simply connected. As in the proof of Proposition 2.3 we see from [25, IX, Cor. 6.8] that any torsor X'/X of G is the pullback of a torsor E'/E of G. As in the proofs of Propositions 3.1 and 3.4 it is enough to assume that E' has a $\mathbb{Q}$-point O' over $O \in E(\mathbb{Q})$. Thus a natural isomorphism $X'_{O'} \cong X_O$ gives an identification $X'_{O'}(\mathbf{A}_{\mathbb{Q}}) \cong X_O(\mathbf{A}_{\mathbb{Q}})$, so to prove our claim it is enough to show that the natural map $\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(X_O)$ is an isomorphism.

Let $i : X_P \to X_O$ be the closed embedding. Let $\nu : Y_K \to X_O$ be the normalisation morphism and let $C = \nu^{-1}(X_P)$. Then $C = X_P \times_{\mathbb{Q}} K$ is the intersection of the quadric $Q_K = Q \times_{\mathbb{Q}} K$ given by the second equation in (1) with the plane $u = 0$. The morphism $\nu : C \to X_P$ is the natural projection $X_P \times_{\mathbb{Q}} K \to X_P$.

The exact sequence of étale sheaves on X_O

$$0 \to \mathbb{G}_{m,X_O} \to \nu_*\mathbb{G}_{m,Y_K} \oplus i_*\mathbb{G}_{m,X_P} \to i_*\nu_*\mathbb{G}_{m,C} \to 0$$

is similar to the exact sequence (2) in [11]. The normalisation morphism ν and the closed embedding i are finite morphisms, so ν_* and i_* are exact functors, hence on taking cohomology we obtain an exact sequence

$$\mathrm{Pic}(Y_K) \oplus \mathrm{Pic}(X_P) \to \mathrm{Pic}(C) \to \mathrm{Br}(X_O) \to \mathrm{Br}(Y_K) \oplus \mathrm{Br}(X_P) \to \mathrm{Br}(C).$$

The discriminant of the quadratic form defining Q is not a square in K, hence $\mathrm{Pic}(Q_K)$ is generated by the class of the hyperplane section, and the natural map $\mathrm{Br}(K) \to \mathrm{Br}(Q_K)$ is an isomorphism. By the birational invariance of the Brauer group we obtain that the natural map $\mathrm{Br}(K) \to \mathrm{Br}(Y_K)$ is also an isomorphism. It is well known that $\mathrm{Br}(X_P)$ is the quotient of $\mathrm{Br}(\mathbb{Q})$ by the subgroup generated by the class of the conic X_P, which is given by the symbol $(5, 17)$. This symbol remains non-zero in $\mathrm{Br}(K)$, hence $C(K) = \emptyset$ and so $\mathrm{Pic}(C)$ is also generated by the class of the hyperplane section. Since the composition of the embedding $C \to Y_K$ with the birational morphism $Y_K \to Q_K$ is the natural embedding of C as a plane section of Q_K, we see that the restriction map $\mathrm{Pic}(Y_K) \to \mathrm{Pic}(C)$ is surjective. Now using the fact that $\mathrm{Br}(C)$ is the quotient of $\mathrm{Br}(K)$ by the subgroup generated by the symbol $(5, 17)$ we easily deduce that $\mathrm{Br}(X_O) = \mathrm{Br}(\mathbb{Q})$.

To conclude, our example resembles that of Poonen [19] in that the fibres of $X \to E$ are birationally equivalent to intersections of two quadrics in $\mathbb{P}^4$. However, in our case the fibre above the unique $\mathbb{Q}$-point is geometrically simply connected and satisfies $\mathrm{Br}(X_O) = \mathrm{Br}(\mathbb{Q})$ and $X_O(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$, so we see that the étale Brauer–Manin set of X is non-empty without studying X any further. Note that it is essential that the fibres of $X \to E$ have dimension at least 2. Indeed, each everywhere locally solvable geometrically connected and simply connected *curve* over a number field k has a k-point [11, Remark 2.2].

4 Elliptic curves with a large Galois image

Let k be a field of characteristic zero, with an algebraic closure $\bar{k}$. For a field K such that $k \subset K \subset \bar{k}$ we denote by Γ_K the Galois group $\mathrm{Gal}(\bar{k}/K)$. Let $k^{\mathrm{cyc}} \subset \bar{k}$ be the cyclotomic extension of k, i.e. the abelian extension of k obtained by adjoining all roots of unity.

Let E be an elliptic curve over k, and let $\rho : \Gamma_k \to \mathrm{GL}_2(\hat{\mathbb{Z}})$ be the Galois representation in the torsion subgroup of E. The group $\mathrm{SL}_2(\mathbb{Z}/2)$ is isomorphic to the symmetric group S_3. Let us define $\mathrm{SL}_2^+(\hat{\mathbb{Z}})$ as the kernel of the composition of the reduction modulo 2 map $\mathrm{SL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/2)$ with the unique non-trivial homomorphism $\varepsilon : \mathrm{GL}_2(\mathbb{Z}/2) \to \{\pm 1\}$. (The finite group $\mathrm{GL}_2(\mathbb{Z}/2)$ can be identified with the symmetric group S_3, and then ε is the signature character.)

In this section we prove the following theorem using methods that go back to Bashmakov [1, Ch. 5].

Theorem 4.1 *Let E be an elliptic curve over a field k of characteristic zero such that $\mathrm{SL}_2^+(\hat{\mathbb{Z}})$ is a subgroup of $\rho(\Gamma_k) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$. Let K be a field such that $k \subset K \subset k^{\mathrm{cyc}}$, and let $\varphi : E' \to E \times_k K$ be an isogeny of elliptic curves over K. Then for any point $P \in E(K)$ that cannot be written as $P = mQ$ with $m > 1$ and $Q \in E(K)$, the scheme $\varphi^{-1}(P)$ is integral.*

In the assumption of the theorem, Lemma 4.7 (b) below shows that the isogeny $\varphi : E' \to E$ can be identified with the multiplication by n map $E \to E$ for some integer n. The theorem then follows from Proposition 4.8.

Serre [24, Prop. 22] proved that for any elliptic curve E over $\mathbb{Q}$ the image $\rho(\Gamma_{\mathbb{Q}})$ is contained in a certain subgroup H_Δ which only depends on the discriminant Δ of E. The group H_Δ has index 2 in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ and contains $\mathrm{SL}_2^+(\hat{\mathbb{Z}})$. N. Jones [14] showed that almost all elliptic curves over $\mathbb{Q}$ are *Serre curves* which means that $\rho(\Gamma_{\mathbb{Q}}) = H_\Delta$. Theorem 4.1 thus applies to almost all elliptic curves E over $\mathbb{Q}$.

4.1 Group cohomology

For an integer $n > 1$ we define $\mathrm{SL}_2^+(\mathbb{Z}/n)$ as $\mathrm{SL}_2(\mathbb{Z}/n)$ if n is odd, and as the kernel of the following composite map if n is even:

$$\mathrm{SL}_2(\mathbb{Z}/n) \to \mathrm{SL}_2(\mathbb{Z}/2) \to \mathbb{Z}/2,$$

where the first arrow is reduction modulo 2, and the second arrow is the signature $\mathrm{SL}_2(\mathbb{Z}/2) \simeq S_3 \to \{\pm 1\}$.

Proposition 4.2 *Let n be a positive integer, and let $G \subset \mathrm{GL}_2(\mathbb{Z}/n)$ be a subgroup containing $\mathrm{SL}_2^+(\mathbb{Z}/n)$. If n is odd, then $H^1(G, (\mathbb{Z}/n)^2) = 0$. For $n = 2^r m$, where m is odd and $r \geq 1$, the abelian group $H^1(G, (\mathbb{Z}/n)^2)$ is annihilated by 2^{r-1}.*

The proof of Proposition 4.2 is based on a few lemmas.

Lemma 4.3 *For any integer $n > 1$ we have $H^0(\mathrm{SL}_2^+(\mathbb{Z}/n), (\mathbb{Z}/n)^2) = 0$.*

Proof The group $\mathrm{SL}_2^+(\mathbb{Z}/n)$ contains the transformation $(x, y) \mapsto (x + y, -x)$. □

Lemma 4.4 *Let G_1 and G_2 be finite groups, let M_1 be a G_1-module and let M_2 be a G_2-module such that $(M_1)^{G_1} = (M_2)^{G_2} = 0$. The following natural map is injective:*

$$H^1(G_1 \times G_2, M_1 \oplus M_2) \longrightarrow H^1(G_1, M_1) \oplus H^1(G_2, M_2).$$

Proof It is enough to prove that $H^1(G_1 \times G_2, M_i)$ injects into $H^1(G_i, M_i)$ for $i = 1, 2$. To fix ideas, assume $i = 1$. We have an inflation-restriction exact sequence

$$0 \to H^1\left(G_2, (M_1)^{G_1}\right) \to H^1\left(G_1 \times G_2, M_1\right) \to H^1\left(G_1, M_1\right),$$

which implies the lemma. □

Lemma 4.5 *For an odd prime p we have $H^1(\mathrm{SL}_2(\mathbb{Z}/p^r), (\mathbb{Z}/p^r)^2) = 0$ for any positive integer r.*

Proof Let $C = \{\pm \mathrm{Id}\} \subset \mathrm{GL}_2(\mathbb{Z}/p^r)$. This is a central subgroup, so we have an inflation-restriction exact sequence

$$0 \to H^1\left(\mathrm{SL}_2\left(\mathbb{Z}/p^r\right)/C, \left(\left(\mathbb{Z}/p^r\right)^2\right)^C\right) \to H^1\left(\mathrm{SL}_2\left(\mathbb{Z}/p^r\right), \left(\mathbb{Z}/p^r\right)^2\right)$$
$$\to H^1\left(C, \left(\mathbb{Z}/p^r\right)^2\right).$$

The order of C is 2 but p is odd, so we have $H^1(C, (\mathbb{Z}/p^r)^2) = 0$. We also have $((\mathbb{Z}/p^r)^2)^C = 0$, and the lemma follows. □

Lemma 4.6 *For a positive integer r the group $H^1(\mathrm{SL}_2^+(\mathbb{Z}/2^r), (\mathbb{Z}/2^r)^2)$ is annihilated by 2^{r-1}.*

Proof When $r = 1$ the group $\mathrm{SL}_2^+(\mathbb{Z}/2^r)$ has order 3, so the claim is obvious.

Now suppose $r \geq 2$. We denote the tautological $\mathrm{SL}_2^+(\mathbb{Z}/2^r)$-module $(\mathbb{Z}/2^r)^2$ by M. Let σ be the scalar 2×2-matrix $(1 + 2^{r-1})\mathrm{Id}$, and let $H = \{\mathrm{Id}, \sigma\}$. It is clear that $H \simeq \mathbb{Z}/2$ is a central subgroup of $\mathrm{SL}_2^+(\mathbb{Z}/2^r)$ and $M^H = 2M$. Let $G = \mathrm{SL}_2^+(\mathbb{Z}/2^r)/H$. There is an inflation-restriction exact sequence

$$0 \to H^1(G, 2M) \to H^1\left(\mathrm{SL}_2^+\left(\mathbb{Z}/2^r\right), M\right) \to H^1(H, M)^G.$$

Since $2^{r-1}(2M) = 0$ the group $H^1(G, 2M)$ is annihilated by 2^{r-1}. We have

$$H^1(H, M) = \mathrm{Ker}\left[(1 + \sigma) : M \to M\right]/(1 - \sigma)M.$$

For $r \geq 3$ we have $1 + 2^{r-2} \in (\mathbb{Z}/2^r)^*$, hence the kernel of $1 + \sigma = 2(1 + 2^{r-2})\mathrm{Id}$ is $2^{r-1}M = (1 - \sigma)M$. Thus $H^1(H, M) = 0$ for $r \geq 3$. For $r = 2$ the map $(1 + \sigma) : M \to M$ is the multiplication by 4 on $M = (\mathbb{Z}/4)^2$, hence in this case $H^1(H, M) = M/2M$. Since H is central in $\mathrm{SL}_2^+(\mathbb{Z}/2^r)$, the action of G on H is trivial, hence G acts on $M/2M$ through its quotient $\mathrm{SL}_2^+(\mathbb{Z}/2)$. The only invariant element under this action is zero. Thus $H^1(H, M)^G = 0$ in all cases, so the lemma is proved. □

Proof of Proposition 4.2 If R_1 and R_2 are commutative rings with 1, then we have $\mathrm{SL}_2(R_1 \times R_2) \cong \mathrm{SL}_2(R_1) \times \mathrm{SL}_2(R_2)$. By Lemma 4.3 we have $H^0(\mathrm{SL}_2^+(\mathbb{Z}/m), (\mathbb{Z}/m)^2) = 0$ for every positive integer m. Writing n as a product of prime powers, and applying Lemmas 4.4, 4.5 and 4.6 we prove Proposition 4.2 in the case $G = \mathrm{SL}_2^+(\mathbb{Z}/n)$. In the general case we note that $\mathrm{SL}_2^+(\mathbb{Z}/n)$ is normal in G, and the only $\mathrm{SL}_2^+(\mathbb{Z}/n)$-invariant element in $(\mathbb{Z}/n)^2$ is zero. Hence the restriction from G to $\mathrm{SL}_2^+(\mathbb{Z}/n)$ gives an injective map

$$H^1\left(G, (\mathbb{Z}/n)^2\right) \hookrightarrow H^1\left(\mathrm{SL}_2^+\left(\mathbb{Z}/n\right), (\mathbb{Z}/n)^2\right),$$

and the proposition follows. □

4.2 Isogenies of elliptic curves

The multiplication by n map on an elliptic curve E is denoted by $[n] : E \to E$. Let $\rho_n : \Gamma_k \to \mathrm{GL}_2(\mathbb{Z}/n)$ be the Galois representation in the n-torsion subgroup $E[n]$.

Lemma 4.7 *Let E be an elliptic curve over a field k of characteristic zero such that $\mathrm{SL}_2^+(\mathbb{Z}/\ell) \subset \rho_\ell(\Gamma_k)$ for every prime ℓ. Then we have the following statements.*

(i) Let M be a finite Γ_k-submodule of $E(\bar{k})$. Then $M = E[m]$ for an integer $m \neq 0$. In particular $E(k)$ is torsion-free.
(ii) Let $\varphi : C \to E$ be an isogeny of elliptic curves. Then there is an integer $n > 0$ and an isomorphism of elliptic curves $\psi : E \xrightarrow{\sim} C$ such that $\varphi \circ \psi = [n]$.

Proof (i) Let n be the smallest positive integer such that $[n]M = 0$. We claim that $M = E[n]$. The Γ_k-module M is the direct sum of its ℓ-primary components $M\{\ell\}$. If ℓ^r is the highest power of the prime number ℓ that divides n, then ℓ^r is the smallest positive integer that annihilates $M\{\ell\}$. The tautological $\mathrm{SL}_2^+(\mathbb{Z}/\ell)$-module $(\mathbb{Z}/\ell)^2$ is simple, hence the Γ_k-module $E[\ell]$ is simple by assumption. But $\ell^{r-1}M\{\ell\} \neq 0$, so we have $\ell^{r-1}M\{\ell\} = E[\ell]$. The abelian group $E[\ell^r]$ is isomorphic to $(\mathbb{Z}/\ell^r\mathbb{Z})^2$, so any subgroup that is mapped by multiplication by ℓ^{r-1} onto $(\ell^{r-1}\mathbb{Z}/\ell^r\mathbb{Z})^2$ is equal to the whole group. Thus $M\{\ell\} = E[\ell^r]$, and hence $M = E[n]$.
(ii) Passing to the dual isogeny of $\varphi : C \to E$ we see that it is enough to prove that every isogeny $\alpha : E \to E'$ is $[m] : E \to E$ for some integer m. This follows from (i) by taking $M = \mathrm{Ker}(\alpha)$. □

Proposition 4.8 *Let E be an elliptic curve over a field k of characteristic zero such that $\rho(\Gamma_k) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ contains $\mathrm{SL}_2^+(\hat{\mathbb{Z}})$. For any field K such that $k \subset K \subset k^{\mathrm{cyc}}$ we have the following statements.*

(i) $\mathrm{SL}_2^+(\hat{\mathbb{Z}}) \subset \rho(\Gamma_K)$, hence $\mathrm{SL}_2^+(\mathbb{Z}/n) \subset \rho_n(\Gamma_K)$ for any positive integer n.
(ii) For any point $P \in E(K)$ that cannot be written as $P = mQ$ for $m > 1$ with $Q \in E(K)$, and for any integer $n > 0$, the scheme $[n]^{-1}(P)$ is integral.

Proof (i) The composition $\det \rho : \Gamma_k \to \mathrm{GL}_2(\hat{\mathbb{Z}}) \to \hat{\mathbb{Z}}^*$ is the cyclotomic character, so $\Gamma_{k^{\mathrm{cyc}}}$ is the subgroup of Γ_k given by the condition $\det \rho(x) = 1$. In view of the natural surjections $\mathrm{SL}_2(\hat{\mathbb{Z}}) \to \mathrm{SL}_2(\mathbb{Z}/n)$, this proves (i).
(ii) Let $K_n = K(E[n])$. In (i) we proved that $G = \Gamma_K/\Gamma_{K_n}$ is a subgroup of $\mathrm{GL}_2(\mathbb{Z}/n)$ containing $\mathrm{SL}_2^+(\mathbb{Z}/n)$. The scheme $[n]^{-1}(P)$ is a K-torsor of $E[n]$ of the class $\kappa(P) \in H^1(K, E[n])$, where $\kappa : E(K)/n \hookrightarrow H^1(K, E[n])$ is the Kummer map. There is an inflation-restriction exact sequence

$$0 \to H^1(G, E[n]) \to H^1(K, E[n]) \to H^1(K_n, E[n])^G. \tag{2}$$

The restriction of $\kappa(P)$ to $H^1(K_n, E[n])$ is a Γ_K-equivariant homomorphism $\varphi : \Gamma_{K_n} \to E[n]$ (where Γ_K acts on Γ_{K_n} by conjugations, and on $E[n]$ in the usual way). Its image is thus a Γ_K-submodule of $E[n]$. By (i) and by Lemma 4.7 (i), we have $\varphi(\Gamma_{K_n}) = E[m]$ for some $m|n$. The set of $\overline{K}$-points of $[n]^{-1}(P) \times_K K_n$ with a natural action of Γ_{K_n} can be identified, by a choice of the base point, with the set $E[n]$ on which $g \in \Gamma_{K_n}$ acts by translation by $\varphi(g)$.

Write $n = 2^r s$, where $r \geq 0$ and s is odd. We first deal with the case $r = 0$, and then proceed by induction in r. Since P is not divisible in $E(K)$ by assumption, and $E(K)$ is torsion-free, the order of $\kappa(P) \in H^1(K, E[n])$ is n.

If n is odd, by Proposition 4.2 the exact sequence (2) gives rise to an embedding of $H^1(K, E[n])$ into $\mathrm{Hom}(\Gamma_{K_n}, E[n])$, so that φ has order n. It follows that $\varphi(\Gamma_{K_n}) = E[n]$ in this case. This implies that $[n]^{-1}(P) \times_K K_n$ is irreducible, hence $[n]^{-1}(P)$ is irreducible.

Now suppose that $n = 2n'$ and the scheme $[n']^{-1}(P)$ is irreducible. The multiplication by 2 map defines a surjective morphism $[n]^{-1}(P) \to [n']^{-1}(P)$ which is a torsor of $E[2]$. We know that Γ_K acts transitively on $[n']^{-1}(P)(\overline{K})$ and we want to show that Γ_K acts transitively on $[n]^{-1}(P)(\overline{K})$. For this we must show that each $\overline{K}$-fibre of $[n]^{-1}(P) \to [n']^{-1}(P)$ is contained in one Γ_K-orbit. Recall that the Γ_{K_n}-set $[n]^{-1}(P)(\overline{K})$ is identified with $E[n]$ so that $g \in \Gamma_{K_n}$ acts as the translation by $\varphi(g)$. The $\overline{K}$-fibres of $[n]^{-1}(P) \to [n']^{-1}(P)$ are the $E[2]$-orbits in $E[n]$. Therefore, it is enough to show that $\varphi(\Gamma_{K_n})$ contains $E[2]$. As the order of $\kappa(P)$ is n, by Proposition 4.2 the exact sequence (2) shows that the order of φ is divisible by $2s$. Thus $\varphi(\Gamma_{K_n})$ contains $E[2s]$ and hence contains $E[2]$. This finishes the proof. □

5 Surfaces

5.1 An elliptic curve

For an elliptic curve E over a field k of characteristic zero we denote by E^c the quadratic twist of E by $c \in k^*$.

Lemma 5.1 *Let E be an elliptic curve over a field k of characteristic zero. For a quadratic extension $K = k(\sqrt{d})$ we have an exact sequence*

$$0 \to E^d(k) \to E(K) \to E(k).$$

Proof Let σ be the non-zero element of $\mathrm{Gal}(K/k)$. We have $E(k) \simeq E(K)^\sigma$. The choice of a square root of d in K defines an isomorphism $E^d \times_k K \simeq E \times_k K$. This gives an identification $E^d(k) \simeq \{x \in E(K) | \sigma(x) = -x\}$. Sending $x \in E(K)$ to $x + \sigma(x)$ defines a homomorphism $E(K) \to E(k)$ with kernel $E^d(k)$. □

Proposition 5.2 *There exist the following data:*

a real quadratic field $k = \mathbb{Q}(\sqrt{c})$, where c is a square-free positive integer not congruent to 1 modulo 8;
a totally real biquadratic field $K = \mathbb{Q}(\sqrt{c}, \sqrt{d})$, where d is a square-free positive integer;
an elliptic curve E over $\mathbb{Q}$ of discriminant $\Delta < 0$, such that $\mathrm{SL}_2^+(\hat{\mathbb{Z}}) \subset \rho(\Gamma_k)$, $E(k) = \{0\}$, and $E(K)$ is torsion-free of positive rank.

Proof Let E be the curve $y^2 + y = x^3 + x^2 - 12x - 21$ of conductor 67 and discriminant $\Delta = -67$, and take $c = 10$, $d = 2$. Using `sage` we check that $E(\mathbb{Q}) = E^{10}(\mathbb{Q}) = 0$. By Lemma 5.1 we have $E(k) = 0$. Using `sage` we check that $E^2(\mathbb{Q}) \simeq E^5(\mathbb{Q}) \simeq \mathbb{Z}$, so by Lemma 5.1 we conclude that $E(K)$ is torsion-free of positive rank. We claim that E is a Serre curve, which means that $\rho(\Gamma_\mathbb{Q}) = H_\Delta$, where H_Δ is a subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of index 2 containing $\mathrm{SL}_2^+(\hat{\mathbb{Z}})$. By a result of N. Jones [14, Lemma 5], for this it is enough to show that E satisfies the following conditions:

$\rho_\ell(\Gamma_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Z}/\ell)$ for all primes ℓ (this is checked using `sage`),
$\rho_8(\Gamma_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Z}/8)$ (this is checked using [7]), and
$\rho_9(\Gamma_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Z}/9)$ (this is checked using [8]).

By Proposition 4.8 (i), $\mathrm{SL}_2^+(\hat{\mathbb{Z}}) \subset \rho(\Gamma_\mathbb{Q})$ implies $\mathrm{SL}_2^+(\hat{\mathbb{Z}}) \subset \rho(\Gamma_k)$. This finishes the proof. □

Remark 5.3 For c and $k = \mathbb{Q}(\sqrt{c})$ as above, the conic $x^2 + y^2 + z^2 = 0$ has points in all non-Archimedean completions of k, but no points in the two real completions of k. Indeed, this conic has $\mathbb{Q}_p$-points for all odd primes p. Since 2 is ramified or inert in k, this conic also has points in the completion of k at the unique prime over 2.

5.2 A conic bundle over the elliptic curve

Let k, K and E be as in Proposition 5.2. The elliptic curve E can be given by its short Weierstraß equation

$$y^2 = r(x),$$

where $r(x) = x^3 + px + q$, for $p, q \in \mathbb{Q}$. Since $\Delta = -4p^3 - 27q^2 < 0$ the topological space $E(\mathbb{R})$ is connected. The neutral element of $E(k)$ is the point at infinity. We denote by $\pi : E \to \mathbb{P}^1_k$ the projection sending (x, y) to x.

Let $P \in E(K)$ be a point not divisible in $E(K)$. Let $\sigma \in \mathrm{Gal}(K/k)$ be the generator. Since $P + \sigma(P) \in E(k) = 0$ we obtain that $\pi(P)$ is a point of $\mathbb{A}^1_k(k) = k$, say $\pi(P) = a \in k$. The K-point P gives rise to a solution of $y^2 = r(a)$ in K, hence $r(a) \in k$ is totally positive. We have $r(a) \neq 0$ since $E(K)$ is torsion-free.

Let $b \in k$, $b \neq a$. We define a central simple algebra A over $k(\mathbb{P}^1_k) = k(x)$ as a tensor product of quaternion algebras

$$A = ((x - a)/(x - b), r(b)) \otimes (-1, -1). \tag{3}$$

The algebra A is unramified outside of the points $x = a$ and $x = b$. At each of these points the residue of A is given by the class of $r(b)$ in k^*/k^{*2}.

Proposition 5.4 (Albert) *Let F be a field, $char(F) \neq 2$, and let $\alpha, \beta, \gamma, \delta \in F^*$. The tensor product of quaternion algebras $(\alpha, \beta) \otimes (\gamma, \delta)$ is a division algebra if and only if the diagonal quadratic form $\langle\alpha, \beta, -\alpha\beta, -\gamma, -\delta, \gamma\delta\rangle$ is anisotropic. If it is isotropic, then $(\alpha, \beta) \otimes (\gamma, \delta)$ is similar to a quaternion algebra over F.*

For the proof see [15, § 16.A]. The quadratic form $\langle\alpha, \beta, -\alpha\beta, -\gamma, -\delta, \gamma\delta\rangle$ is called an *Albert form* associated to $(\alpha, \beta) \otimes (\gamma, \delta)$.

Lemma 5.5 *Let k, K, a, $r(t)$ be as above. If $b \in k$ is such that $r(b)$ is totally negative, then the algebra A over the field $k(\mathbb{P}^1_k) = k(x)$ is similar to a quaternion algebra.*

Proof The associated Albert form contains the subform $\Phi = \langle r(b), 1, 1, 1\rangle$. By Remark 5.3 the form $\langle 1, 1, 1\rangle$ is isotropic over all finite completions of k. Since $r(b)$ is totally negative, Φ is isotropic over both real completions of k. By the Hasse–Minkowski theorem the quadratic form Φ is isotropic over k. An application of Proposition 5.4 concludes the proof. □

We are now ready to state one of the main results of this paper. (For an explanation why we do not consider the case $k = \mathbb{Q}$ see Proposition 6.5 below.)

Theorem 5.6 *There exist a real quadratic field k, an elliptic curve E and a smooth, projective and geometrically integral surface X over k with a surjective morphism $f : X \to E$ satisfying the following properties:*

(i) the fibres of $f : X \to E$ are conics;
(ii) there exists a closed point $P \in E$ such that the field $k(P)$ is a totally real biquadratic extension of $\mathbb{Q}$ and the restriction $X \setminus f^{-1}(P) \to E \setminus P$ is a smooth morphism;
(iii) $X(\mathbf{A}_k)^{\text{ét,Br}} \neq \emptyset$ and $X(k) = \emptyset$.

Proof We keep the above notation: $k = \mathbb{Q}(\sqrt{10})$, E is the curve of Proposition 5.2 given by its short Weierstraß equation $y^2 = r(x)$, viewed as a curve over k, and $K = k(P) = \mathbb{Q}(\sqrt{10}, \sqrt{2})$. We fix $b \in k$ such that $r(b)$ is totally negative, and define a central simple algebra A by (3). By Lemma 5.5, A is similar to a quaternion algebra B over $k(\mathbb{P}^1_k)$. Let $f : X \to E$ be a relatively minimal conic bundle such that the generic fibre $X_{k(E)}$ is the conic over $k(E)$ defined by the quaternion algebra $B \otimes_{k(\mathbb{P}^1)} k(E)$. (We call a conic bundle $X \to E$ relatively minimal if for every conic bundle $Y \to E$ any birational morphism $X \to Y$ compatible with the projections to E, is an isomorphism. See [16, Thm. 1.6] for the well known description of the fibres of $X \to E$.) The closed point $P = \pi^{-1}(a) \simeq \operatorname{Spec} K$ is a solution to $y^2 = r(a)$. The residue of $B \otimes_{k(\mathbb{P}^1)} k(E)$ at P is the class of $r(b)$ in $K^\times/K^{\times 2}$, which is non-trivial since $r(b)$ is totally negative. We have $\pi^{-1}(b) = \operatorname{Spec} k(\sqrt{r(b)})$, and the residue at this closed point is the class of $r(b)$ in $k(\sqrt{r(b)})^*/k(\sqrt{r(b)})^{*2}$ which is trivial. Thus $f : X \to E$ is smooth away from P. This gives (i) and (ii).

If we go over to one of the two real completions k_v of k, the point P breaks up into two real points P_1 and P_2. The residue at each of these points is an element of $\mathbb{R}^*/\mathbb{R}^{*2}$ given by the image of $r(b)$, and that element is totally negative. Thus the algebra A is ramified at both P_1 and P_2. The fibre of $X \to E$ above each point $P_i \in E(k_v)$ is thus a singular conic, hence has a k_v-point. In particular, $X(k_v) \neq \emptyset$ for each of the real completions of k. The condition $\Delta < 0$ implies that $E(k_v)$ is connected, thus any point in the image of $X(k_v) \to E(k_v)$ is path connected to the point 0, which is the unique point of $E(k)$. (In fact, the image of $X(\mathbb{R})$ in $E(\mathbb{R})$ is the closed interval between P_1 and P_2 which does not contain 0.) Let M_v be any point of $X(k_v)$, and let $I_v \subset E(k_v) \simeq S^1$ be a real interval linking $f(M_v)$ and 0.

The value of A at $\infty \in \mathbb{P}^1_k$, hence also at $0 \in E(k)$, is $(-1, -1)$, hence the fibre $X_0 = f^{-1}(0)$ is the conic $x^2 + y^2 + z^2 = 0$. By Remark 5.3, X_0 has points in all completions of k except the two real completions, hence $X_0(k) = \emptyset$. Since $E(k) = \{0\}$ it follows that $X(\mathbf{A}_k) \neq \emptyset$, but $X(k) = \emptyset$. For each finite place v of k choose $M_v \in X_0(k_v)$.

We now prove that $(M_v) \in X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$. By Proposition 2.3 every torsor $X' \to X$ of a finite k-group scheme G is the pullback $X' = X \times_E E \to X$ of a torsor $E' \to E$ of G. By twisting E' and X' with a k-torsor of G (and replacing G with the corresponding inner form) we can assume that E' has a k-point $0'$ over $0 \in E(k)$. The connected component C of E' containing this k-point is a smooth, projective, geometrically integral curve. The k-morphism $\varphi : C \to E$ is finite and étale, hence C has genus 1. Choosing $0'$ for the origin of the group law on C we make $\varphi : C \to E$ into an isogeny of elliptic curves and write $0 = 0'$.

Let $Y = X \times_E C$. Then Y is a smooth, projective, geometrically integral surface over k which is an irreducible component of X'. The morphism $g : Y \to C$ is a conic bundle. By Proposition 5.2 (3) we have $\mathrm{SL}_2^+(\hat{\mathbb{Z}}) \subset \rho(\Gamma_k)$. Since the point $P \in E(K)$ is not divisible, we can apply Theorem 4.1 to the elliptic curve E and the isogeny $\varphi : C \to E$. It follows that the inverse image $Q = \varphi^{-1}(P)$ of the closed point P of the k-curve E is a closed point of the k-curve C. We see that $g : Y \to C$ is a conic bundle such that $Y \backslash Y_Q \to C \backslash Q$ is a smooth morphism. By Proposition 2.2 (i), the induced map $g^* : \mathrm{Br}(C) \to \mathrm{Br}(Y)$ is surjective.

For each finite place v of k let M'_v be the k_v-point in the fibre Y_0 over $0 \in C(k)$ that projects to $M_v \in X_0(k_v)$. Now let v be a real place of k. By Lemma 4.7 (ii) the isogeny $\varphi : C \to E$ is identified with $[n] : E \to E$ for some n. Since $\Delta < 0$, the induced map $C(k_v) \simeq E(k_v) \to E(k_v)$ is a surjective étale map $S^1 \to S^1$. Since I_v is contractible, we see that $\varphi^{-1}(I_v)$ is a disjoint union of copies of I_v exactly one of which contains 0. Let us call this interval I'_v. One of its ends is 0 and the other end is a point $R \in C(k_v)$ such that $\varphi(R) = f(M_v)$. Hence the real fibre Y_R is naturally isomorphic to the fibre of $f : X \to E$ that contains M_v. Let M'_v be the point in $Y_R(\mathbb{R})$ that projects to M_v. Since $\mathrm{Br}(Y) = g^*(\mathrm{Br}(C))$, we see that $(M'_v) \in Y(\mathbf{A}_k)^{\mathrm{Br}}$. But Y is an irreducible component of X' hence $(M'_v) \in X'(\mathbf{A}_k)^{\mathrm{Br}}$. This is a lifting of the adelic point (M_v), and therefore $(M_v) \in X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$. □

Remark 5.7 In the proof above the adèle $(f(M_v)) \in E(\mathbf{A}_k)$ and the k-point $0 \in E(k)$ have the following property: for each place v the class of the 0-cycle $f(M_v) - 0$ is infinitely divisible in $\mathrm{Pic}(E \times_k k_v)$, in fact it is even zero if v is non-archimedean. Methods initiated by one of the authors and developed by Frossard [9, Thm. 0.3], by van Hamel, and by Wittenberg [32, Thm. 1.3], then show that there exists a 0-cycle of degree 1 on X. In these various papers, finiteness of the Shafarevich–Tate group of the Jacobian of the base curve is assumed in order to appeal to the Cassels–Tate dual exact sequence, which guarantees the existence of a 0-cycle of degree 1 on the curve satisfying a divisibility property analogous to the one above. In our case we can take this 0-cycle of degree 1 to be the k-point 0, so there is no need to assume the finiteness of the Shafarevich–Tate group of E.

6 Remarks on the Brauer–Manin set

6.1 Birational invariance

Recall that $\mathrm{Br}_0(X)$ denotes the image of the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$.

Proposition 6.1 *Let k be a number field, and let X and Y be smooth, projective, geometrically integral varieties over k that are birationally equivalent.*

(i) If $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then $Y(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$.
(ii) If $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$, then $Y(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$.
(iii) Assume, in addition, that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Then the density of $X(k)$ in $X(\mathbf{A}_k)^{\mathrm{Br}}$ implies the density of $Y(k)$ in $Y(\mathbf{A}_k)^{\mathrm{Br}}$.

Proof (i) By Hironaka's theorem, there exist a smooth, projective variety Z over k and birational morphisms $f : Z \to X$ and $g : Z \to Y$. Let v be a place of k. Since Z is proper, $f(Z(k_v))$ is closed in $X(k_v)$. Let us show that $X(k_v) = f(Z(k_v))$, for which it is enough to show that $f(Z(k_v))$ is dense in $X(k_v)$. There exists a non-empty Zariski open set $U \subset X$ such that f induces an isomorphism $f^{-1}(U) \xrightarrow{\sim} U$. Then $U(k_v) \subset f(Z(k_v))$, but $U(k_v)$ is dense in $X(k_v)$ by the implicit function theorem.

Thus for any $(M_v) \in X(\mathbf{A}_k)$ there exists $(N_v) \in Z(\mathbf{A}_k)$ such that $(f(M_v)) = (N_v)$. The birational morphism f induces an isomorphism $f^* : \mathrm{Br}(X) \xrightarrow{\sim} \mathrm{Br}(Z)$. From the projection formula we conclude that if $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$, then $(N_v) \in Z(\mathbf{A}_k)^{\mathrm{Br}}$. By the covariant functoriality of the Brauer–Manin set we have $g\left(Z(\mathbf{A}_k)^{\mathrm{Br}}\right) \subset Y(\mathbf{A}_k)^{\mathrm{Br}}$.

(ii) Let G be a finite k-group scheme. By the birational equivalence of the fundamental group [25, X, Cor. 3.4] there is a natural bijection between X-torsors and Y-torsors of G in which a torsor X'/X corresponds to Y'/Y if $X' \times_X Z = Y' \times_Y Z$. (This bijection respects the twisting by a k-torsor of G.) Let us denote this Z-torsor of G by Z'. The natural morphism $Z' \to X'$ is a componentwise birational morphism of smooth and projective varieties, so it induces an isomorphism $\mathrm{Br}(X') \xrightarrow{\sim} \mathrm{Br}(Z')$.

Consider any $(M_v) \in X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$. Let $(N_v) \in Z(\mathbf{A}_k)$ be a lifting of (M_v) as in part (i). By the definition of $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$, for any torsor X'/X the adèle (M_v) is the image of some $(M'_v) \in X'(\mathbf{A}_k)^{\mathrm{Br}}$ (after twisting X'/X by a k-torsor of G and replacing G by the corresponding inner form). If $N'_v \in Z'(k_v)$ is the point $M'_v \times_{M_v} N_v$ for each place v, then $(N'_v) \in Z'(\mathbf{A}_k)^{\mathrm{Br}}$ by the projection formula. This implies that

$(N_v) \in Z(\mathbf{A}_k)^{\text{ét,Br}}$. By the covariant functoriality of the étale Brauer–Manin set we have

$$g(Z(\mathbf{A}_k)^{\text{ét,Br}}) \subset Y(\mathbf{A}_k)^{\text{ét,Br}}.$$

(iii) Let $U \subset X$ and $V \subset Y$ be non-empty Zariski open sets such that there is an isomorphism $h : V \xrightarrow{\sim} U$. Under the assumption of (ii), the group $\mathrm{Br}(Y)/\mathrm{Br}_0(Y) \simeq \mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Let $\beta_1, \ldots, \beta_n \in \mathrm{Br}(Y)$ be coset representatives. There exists a finite set Σ of places of k such that each β_i vanishes on $Y(k_v)$ for each $v \notin \Sigma$. Thus every adèle $(M_v) \in Y(\mathbf{A}_k)^{\mathrm{Br}}$ can be approximated by an adèle $(N_v) \in Y(\mathbf{A}_k)^{\mathrm{Br}}$ such that $N_v \in V(k_v)$ for all completions k_v of k. Let $M_v = h(N_v)$. Since h induces an isomorphism $\mathrm{Br}(Y) \simeq \mathrm{Br}(X)$, the adèle (M_v) is in $X(\mathbf{A}_k)^{\mathrm{Br}}$. If $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}}$, we can approximate (M_v) by a k-point M in U. Then $N = h^{-1}(M) \in V(k)$ is close to $(N_v) \in Y(\mathbf{A}_k)^{\mathrm{Br}}$. □

Remark 6.2 (1) We do not know if the analogue of Proposition 6.1 (iii) holds for the étale Brauer–Manin set.

(2) We do not know if Proposition 6.1 (iii) still holds when $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is infinite, but we can make the following observation. Recall that $X(\mathbf{A}_k)_\bullet$ denotes the quotient of $X(\mathbf{A}_k)$ by the relation which identifies two points in the same connected component. *For smooth, projective, geometrically integral varieties X and Y that are birationally equivalent, if one does not assume the finiteness of* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$, *then $Y(k)$ can be dense in $Y(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$ without $X(k)$ being dense in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$*. Indeed, let $Y = A$ be an abelian surface over $\mathbb{Q}$ such that $A(\mathbb{Q}) = \{0\}$ and the Shafarevich–Tate group of A is finite. Let Z be the blowing-up of A at the $\mathbb{Q}$-point 0, and let X be the blowing-up of Z at some $\mathbb{Q}$-point (all $\mathbb{Q}$-points of Z are contained in the exceptional divisor). The surface X contains two copies of $\mathbb{P}^1_{\mathbb{Q}}$ meeting at a $\mathbb{Q}$-point P; let us call them E and F. The finiteness of the Shafarevich–Tate group of A implies [28, Prop. 6.2.4] that $A(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$ is the connected component of 0 in $A(\mathbf{A}_{\mathbb{Q}})$, which is isomorphic to the real connected component of 0 in $A(\mathbb{R})$. Choose a $\mathbb{Q}$-point $M \neq P$ in E, and a $\mathbb{Q}$-point $N \neq P$ in F. Let q be a prime. Consider the adelic point (M_p) of X, where $M_p = M$ for all $p \neq q$ (including $p = \infty$) and $M_q = N$. Since the morphism $f : X \to A$ induces an isomorphism $f^* : \mathrm{Br}(A) \xrightarrow{\sim} \mathrm{Br}(X)$ and $f(M_p) = 0$ for all p, we see that $(M_p) \in X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$. However, the connected component of (M_p) in $X(\mathbf{A}_{\mathbb{Q}})$ contains no $\mathbb{Q}$-points. Indeed, a $\mathbb{Q}$-point of X is either in E or in F. In the first case it cannot approximate M_q in the q-adic topology, and in the second case it cannot approximate M_p in the p-adic topology where p is any prime different from q. It is easy to see that $(M_p) \in X(\mathbf{A}_{\mathbb{Q}})^{\text{ét,Br}}$, so the previous discussion applies to the set of connected components of the étale Brauer–Manin set as well.

6.2 Cases where the Brauer–Manin obstruction suffices

The following result shows that for the conclusion of Theorem 5.6 to hold, the conic bundle $f : X \to E$ must contain degenerate fibres.

Proposition 6.3 *Let E be an elliptic curve over a number field k with a finite Shafarevich–Tate group. Let $f : X \to E$ be a Severi–Brauer scheme over E. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$. Moreover, $X(k)$ is dense in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$.*

Proof Since $f : X \to E$ is a projective morphism with smooth geometrically integral fibres, there exists a finite set of places Σ such that $E(k_v) = f(X(k_v))$ for $v \notin \Sigma$. We may assume that Σ contains the archimedean places of k. At an arbitrary place v the set

$f(X(k_v))$ is open and closed in $E(k_v)$. Let $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. By functoriality we then have $(f(M_v)) \in E(\mathbf{A}_k)^{\mathrm{Br}}$. The finiteness of the Shafarevich–Tate group of E implies [28, Prop. 6.2.4] the exactness of the Cassels–Tate dual sequence

$$0 \to E(k) \otimes \hat{\mathbb{Z}} \to \prod E(k_v)_\bullet \to \mathrm{Hom}(\mathrm{Br}(E), \mathbb{Q}/\mathbb{Z}), \tag{4}$$

where $E(k_v)_\bullet = E(k_v)$ if v is a finite place of k, and $E(k_v)_\bullet = \pi_0(E(k_v))$ if v is an archimedean place. By a theorem of Serre, the image of $E(k)\otimes\hat{\mathbb{Z}}$ in that product coincides with the topological closure of $E(k)$, see [22,23,31]. Approximating at the places of Σ, we find a k-point $M \in E(k)$ such that the fibre $X_M = f^{-1}(M)$ is a Severi–Brauer variety with points in all k_v for $v \in \Sigma$, hence also for all places v. Since X_M is a Severi–Brauer variety over k, it contains a k-point, hence $X(k) \neq \emptyset$. For the last statement of the theorem we include into Σ the places where we want to approximate. If $k_v \simeq \mathbb{R}$, each connected component of $X(k_v)$ maps surjectively onto a connected component of $E(k_v)$. The Severi–Brauer varieties satisfy the Hasse principle and weak approximation, so an application of the implicit function theorem finishes the proof. □

Remark 6.4 The same argument works more generally for a projective morphism $f : X \to E$ such that each fibre contains a geometrically integral component of multiplicity 1, provided that the smooth k-fibres satisfy the Hasse principle. For the last statement to hold, the smooth k-fibres also need to satisfy weak approximation.

The following proposition is a complement to Theorem 5.6 which explains why a similar counterexample cannot be constructed over $\mathbb{Q}$.

Proposition 6.5 *Let E be an elliptic curve over a number field k such that both $E(k)$ and the Shafarevich–Tate group of E are finite. Let $f : X \to E$ be a conic bundle. Suppose that there exists a real place v_0 of k such that for each real place $v \neq v_0$ no singular fibre of $f : X \to E$ is over a k_v-point of E. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$.*

Proof If a k-fibre of f is not smooth, then this fibre contains a k-point. We may thus assume that the fibres above $E(k)$ are smooth. Let $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. Then $(f(M_v)) \in E(\mathbf{A}_k)^{\mathrm{Br}}$. Set $N_v = f(M_v)$ for each place v. The finiteness of the Shafarevich–Tate group of E implies the exactness of the Cassels–Tate dual sequence (4). Hence there exists $N \in E(k)$ such that $N = N_v$ for each finite place v and such that N lies in the same connected component as N_v for v archimedean. The fibre X_N is a smooth conic with points in all finite completions of k. For an archimedean place $v \neq v_0$, the map $X(k_v) \to E(k_v)$ sends each connected component of $X(k_v)$ onto a connected component of $E(k_v)$. Since N and N_v are in the same connected component of $E(k_v)$, this implies $X_N(k_v) \neq \emptyset$. Thus the conic X_N has points in all completions of k except possibly k_{v_0}. By the reciprocity law it has points in all completions of k and hence in k. □

Remark 6.6 If $E(k)$ is finite we cannot in general expect $X(k)$ to be dense in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$ or even in $X(\mathbf{A}_k)_\bullet^{\text{ét},\mathrm{Br}}$. Indeed, if the fibre X_N over some k-point N of E is an *irreducible singular* conic, then the singular point P of X_N is the unique k-point of X_N. Let (M_v) be any adèle in $X_N(\mathbf{A}_k)$. Note that if v splits in the quadratic extension of k over which the components of X_N are defined, then $X_N \times_k k_v$ is a union of two projective lines meeting at P. Using the fact that $\mathrm{Br}(\mathbb{P}^1_{k_v}) = \mathrm{Br}(k_v)$ we see that $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. Furthermore, we have $(M_v) \in X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$, cf. [11, Remark 2.4]. On the other hand, (M_v) is not in the closure of $X(k)$ in $X(\mathbf{A}_k)_\bullet$ provided $M_v \neq P$ for at least one finite place v of k.

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