Low degree unramified cohomology of generic diagonal hypersurfaces

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Abstract

We prove that the i-th unramified cohomology group of the generic diagonal hypersurface in the projective space of dimension $n \geq i+1$ is trivial for $i \leq 3$.

1 Introduction

Let k be a field with separable closure k_{s} and absolute Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let μ be a finite étale commutative group k-scheme of order not divisible by $\mathrm{char}(k)$. The datum of such a group k-scheme μ is equivalent to the datum of the finite Γ-module $\mu(k_{\mathrm{s}})$. For an integer $m \geq 2$ we denote by μ_m the group k-scheme of m-th roots of unity. If N is a positive integer not divisible by $\mathrm{char}(k)$ such that $N\mu = 0$, then $\mu(-1)$ denotes the commutative group k-scheme $\mathbf{Hom}_{k-\mathrm{gps}}(\mu_N, \mu)$. In terms of Galois modules, $\mu(-1)$ is $\mathrm{Hom}_{\mathbb{Z}}(\mu_N(k_{\mathrm{s}}), \mu(k_{\mathrm{s}}))$ with the natural Galois action.

Let X be a smooth integral variety over k. We denote by $X^{(n)}$ the set of points of X of codimension n. In this paper, the unramified cohomology group $\mathrm{H}^i_{\mathrm{nr}}(X, \mu)$, where i is a positive integer, is defined as the intersection of kernels of the residue maps

$$\partial_x \colon \mathrm{H}^i(k(X), \mu) \to \mathrm{H}^{i-1}(k(x), \mu(-1)),$$

for all $x \in X^{(1)}$. For equivalent definitions, see [CT95, Thm. 4.1.1]. Restriction to the generic point of X gives rise to a natural map

$$\mathrm{H}^i_{\text{ét}}(X, \mu) \to \mathrm{H}^i_{\mathrm{nr}}(X, \mu).$$

Purity for étale cohomology implies that it is an isomorphism for $i = 1$ and surjective for $i = 2$, see [CT95, §3.4]. In the case $i = 2$ with $\mu = \mu_m$, where m is not divisible by $\mathrm{char}(k)$, this gives a canonical isomorphism

$$\mathrm{Br}(X)[m] \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{nr}}(X, \mu_m),$$

see [CT95, Prop. 4.2.1 (a), Prop. 4.2.3 (a)]. If X/k is smooth, proper, and integral, then $\mathrm{H}^i_{\mathrm{nr}}(X, \mu)$ does not depend on the choice of X in its birational equivalence class, see [CT95, Prop. 4.1.5] and [R96, Remark (5.2), Cor. (12.10)].

Let $n \geq 2$ and let $K = k(a_1, \ldots, a_n)$ be the field of rational functions in the variables $a_1, \ldots, a_n$. Let $X_K \subset \mathbb{P}^n_K$ be the hypersurface with equation

$$x_0^d + a_1 x_1^d + \ldots + a_n x_n^d = 0,$$

where d is not divisible by $\mathrm{char}(k)$. In this paper, for $i = 1, 2, 3$ and $n \geq i + 1$, we prove that the natural map

$$\mathrm{H}^i(K, \mu) \to \mathrm{H}^i_{\mathrm{nr}}(X_K, \mu)$$

is an isomorphism, see Theorem 4.8. In the case when $i = 2$ and $\mu = \mu_m$ with $m \geq 2$, this gives that the natural map of Brauer groups $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ is an isomorphism of subgroups of elements of order not divisible by $\mathrm{char}(k)$, see Corollary 4.9. In the case when k has characteristic zero, this result was obtained as [GS, Thm. 1.5] by a completely different method, using results on the topology of the Fermat surface as a complex manifold.

In this paper we use the formalism proposed by M. Rost in [R96] which applies *inter alia* to Galois cohomology [R96, Remarks (1.11), (2.5)]. We do not use the Gersten conjecture for étale cohomology [BO74].

Let us describe the structure of this note. In Section 2 we recall some basic facts about unramified cohomology including a functoriality property of the Bloch–Ogus complex with respect to faithfully flat morphisms with integral fibres. In Section 3 we show that for smooth complete intersections $X \subset \mathbb{P}^n_k$ there are canonical isomorphisms $\mathrm{H}^i(k, \mu) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{nr}}(X, \mu)$ for $i = 1, 2$ when $\dim(X) \geq i + 1$. Generic diagonal hypersurfaces are studied in Section 4. The easy proof of the main theorem in the case $i = 1$ is given in Section 4.1. This is used in the proof in the case $i = 2$ in Section 4.3, after some preparations in Section 4.2. Finally, in Section 5 we use a similar idea to give a short proof of the triviality of the Brauer group of certain surfaces in $\mathbb{P}^3_{k(t)}$ defined by a pair of polynomials with coefficients in k, see Theorem 5.1, which was proved in [GS] when $\mathrm{char}(k) = 0$.

Our proof in this note develops a geometric idea suggested by Mathieu Florence during the second author's talk at the seminar "Variétés rationnelles" in November 2022. The authors are very grateful to Mathieu Florence for his suggestion.

2 Functoriality of the Bloch–Ogus complex

For any smooth integral variety X over k and any $i \geq 2$ there is a complex

$$0 \longrightarrow \mathrm{H}^i(k(X), \mu) \xrightarrow{(\partial_x)} \bigoplus_{x \in X^{(1)}} \mathrm{H}^{i-1}(k(x), \mu(-1)) \xrightarrow{(\partial_y)} \bigoplus_{y \in X^{(2)}} \mathrm{H}^{i-2}(k(y), \mu(-2)),$$

which we call the *Bloch–Ogus complex*. The maps in this complex are defined in [R96, (2.1.0)]. (The map ∂_x is the residue defined for discrete valuation rings by Serre [S03], see also [CTS21, Def. 1.4.3].) The proof that the resulting sequence is a complex is given in [R96, Section 2]. If $y \in X^{(2)}$ is a regular point of the closure of $x \in X^{(1)}$, then the map $\partial_y \colon \mathrm{H}^{i-1}(k(x), \mu(-1)) \to \mathrm{H}^{i-2}(k(y), \mu(-2))$ is the residue map for the local ring of y in the closure of x, which is a discrete valuation ring.

The unramified cohomology group $\mathrm{H}^i_{\mathrm{nr}}(X, \mu)$ is the homology group of this complex at the term $\mathrm{H}^i(k(X), \mu)$, i.e., the intersection of $\mathrm{Ker}(\partial_x)$ for all $x \in X^{(1)}$.

Let $p \colon X \to Y$ be a faithfully flat morphism of smooth integral k-varieties with integral fibres. By [R96, Section (3.5); Prop. (4.6)(2)], there is a chain map of complexes

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^i(k(X), \mu) & \longrightarrow & \bigoplus_{v \in X^{(1)}} \mathrm{H}^{i-1}(k(x), \mu(-1)) & \longrightarrow & \bigoplus_{x \in X^{(2)}} \mathrm{H}^{i-2}(k(x), \mu(-2)) \\ & \uparrow & & \uparrow & & \uparrow \\ 0 \longrightarrow & \mathrm{H}^i(k(Y), \mu) & \longrightarrow & \bigoplus_{y \in Y^{(1)}} \mathrm{H}^{i-1}(k(y), \mu(-1)) & \longrightarrow & \bigoplus_{y \in Y^{(2)}} \mathrm{H}^{i-2}(k(y), \mu(-2)) \end{array}$$

The middle vertical map is the natural one if $p(x) = y$, otherwise it is zero, and similarly for the right-hand vertical map.

The morphism $X \to Y$ is called an *affine bundle* if Zariski locally on Y, it is isomorphic to $Y \times_k \mathbb{A}^n \to Y$ with affine transition morphisms. In this case the vertical maps in the above diagram induce isomorphisms on the left-hand and middle homology groups, see [R96, Prop. (8.6)]. In particular, we have an isomorphism

$$\mathrm{H}^i_{\mathrm{nr}}(X, \mu) \cong \mathrm{H}^i_{\mathrm{nr}}(Y, \mu). \tag{1}$$

Combined with [R96, Cor. (12.10)], this implies that $\mathrm{H}^i_{\mathrm{nr}}(X, \mu)$ is a stable birational invariant of smooth and proper integral k-varieties.

3 Low degree unramified cohomology of complete intersections

For a variety X over a field k we write $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$.

Proposition 3.1 *Let X be a smooth, projective, geometrically integral variety over a field k such that the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism of finitely generated free abelian groups. Then for any k-group of multiplicative type M the natural map*

$$\mathrm{H}^2(k, M) \to \mathrm{H}^2(k(X), M)$$

is injective.

Proof. We have a commutative diagram with exact rows and natural vertical maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k_{\mathrm{s}}^{\times} & \longrightarrow & k_{\mathrm{s}}(X)^{\times} & \longrightarrow & \mathrm{Div}(X^{\mathrm{s}}) & \longrightarrow & \mathrm{Pic}(X^{\mathrm{s}}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \cong\uparrow & & \\ 0 & \longrightarrow & k^{\times} & \longrightarrow & k(X)^{\times} & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \end{array} \qquad (2)$$

The abelian group $\mathrm{Pic}(X)$ is free, so the homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ has a section. Then our assumption implies that the map of Γ-modules $\mathrm{Div}(X^{\mathrm{s}}) \to \mathrm{Pic}(X^{\mathrm{s}})$ has a section. By definition, the elementary obstruction $e(X) \in \mathrm{Ext}^2_k(\mathrm{Pic}(X_{\bar{k}}), \bar{k}^{\times})$ is the class of the 2-extension of Γ-modules given by the upper row of (2). Thus we have $e(X) = 0$. The result now follows from [CTS87, Prop. 2.2.5]. ☐

Lemma 3.2 *Let $X \subset \mathbb{P}^n_k$ be a complete intersection.*

(a) *If* $\dim(X) \geq 2$*, then the natural map* $\mathrm{H}^1(k, \mu) \to \mathrm{H}^1_{\text{ét}}(X, \mu)$ *is an isomorphism.*

(b) *If* $\dim(X) \geq 3$*, then the natural map* $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k, \mu) \to \mathrm{H}^2_{\text{ét}}(X, \mu)$ *is an isomorphism.*

Proof. A combination of the weak Lefschetz theorem with Poincaré duality gives that the map $\mathrm{H}^i_{\text{ét}}(\mathbb{P}^n_{k_{\mathrm{s}}}, \mu) \to \mathrm{H}^i_{\text{ét}}(X^{\mathrm{s}}, \mu)$ is an isomorphism for $i < \dim(X)$, see [K04, Cor. B.6]. In particular, if $\dim(X) \geq 2$, then $\mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}}, \mu) = 0$. Then the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, \mu)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mu)$$

implies the first claim.

If $\dim(X) \geq 3$, then $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_{k_{\mathrm{s}}}, \mu) \to \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu)$ is an isomorphism of Γ-modules. The above spectral sequence gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^2(k, \mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X, \mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu)^{\Gamma} & \longrightarrow & \mathrm{H}^3(k, \mu) \\ & & \mathrm{id}\uparrow & & \uparrow & & \cong\uparrow & & \mathrm{id}\uparrow \\ 0 & \longrightarrow & \mathrm{H}^2(k, \mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k, \mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_{k_{\mathrm{s}}}, \mu)^{\Gamma} & \longrightarrow & \mathrm{H}^3(k, \mu) \end{array}$$

By the 5-lemma we deduce that $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k, \mu) \to \mathrm{H}^2_{\text{ét}}(X, \mu)$ is an isomorphism. ☐

Proposition 3.3 *Let $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of dimension* $\dim(X) \geq 3$*. Then the natural map*

$$\mathrm{H}^2(k, \mu) \to \mathrm{H}^2_{\mathrm{nr}}(X, \mu)$$

is an isomorphism.

Proof. The map $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n_{k_s}) \to \mathrm{Pic}(X^s)$ is an isomorphism by [H70, Ch. IV, Cor. 3.2], hence $\mathrm{Pic}(X) \to \mathrm{Pic}(X^s)$ is an isomorphism. By Proposition 3.1 it is thus enough to prove that the map $\mathrm{H}^2(k,\mu) \to \mathrm{H}^2_{\mathrm{nr}}(X,\mu)$ is surjective.

Choose an affine subspace $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ such that $X \cap \mathbb{A}^n_k \neq \emptyset$. Our map is the composition of maps in the top row of the following natural commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^2(k,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k,\mu) & \xrightarrow{\cong} & \mathrm{H}^2_{\text{ét}}(X,\mu) & \longrightarrow & \mathrm{H}^2_{\mathrm{nr}}(X,\mu) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \hookdownarrow \\
\mathrm{H}^2(k,\mu) & \xrightarrow{\cong} & \mathrm{H}^2_{\text{ét}}(\mathbb{A}^n_k,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X\cap\mathbb{A}^n_k,\mu) & \longrightarrow & \mathrm{H}^2(k(X),\mu)
\end{array}$$

In the top row, the middle map is an isomorphism by Lemma 3.2 (b), and the right-hand map is surjective, as was recalled in the introduction. Thus any $a \in \mathrm{H}^2_{\mathrm{nr}}(X,\mu)$ can be lifted to an element $b \in \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k,\mu)$. The image of b in $\mathrm{H}^2_{\text{ét}}(\mathbb{A}^n_k,\mu)$ comes from a unique element $c \in \mathrm{H}^2(k,\mu)$. The commutativity of the diagram gives that the image of c in $\mathrm{H}^2(k(X),\mu)$ is equal to the image of a. But the right-hand vertical map is injective, hence c is a desired lifting of a to $\mathrm{H}^2(k,\mu)$. □

4 Generic diagonal hypersurfaces

Let Π_1 (respectively, Π_2) be the projective space with homogeneous coordinates $x_0,\ldots,x_n$ (respectively, $t_0,\ldots,t_n$). Write $K = k(\Pi_2)$. Let $X \subset \Pi_1 \times \Pi_2$ be the hypersurface

$$t_0x_0^d + \ldots + t_nx_n^d = 0. \tag{3}$$

Let p be the projection $X \to \Pi_1$, and let f be the projection $X \to \Pi_2$. The generic fibre X_K of f is a smooth diagonal hypersurface of degree d in the projective space $(\Pi_1)_K \cong \mathbb{P}^n_K$.

Lemma 4.1 *With notation as above, the following statements hold.*

(i) *The fibres of f at codimension 1 points of Π_2 are integral if $n \geq 2$ and geometrically integral if $n \geq 3$.*

(ii) *The fibres of f at codimension 2 points of Π_2 are integral if $n \geq 3$ and geometrically integral if $n \geq 4$.*

Proof. One only needs to check this for the singular fibres, which are the fibres above the generic points of the projective subspaces given by $t_i = 0$ or by $t_i = t_j = 0$. □

4.1 Unramified cohomology in degree 1

Lemma 4.2 *Let $f\colon X \to Y$ be a proper, dominant morphism of smooth and geometrically integral varieties over a field k. Write $K = k(Y)$ and let X_K be the generic* *fibre of f. Assume that the fibres of f over the points of Y of codimension 1 are integral and X_K is geometrically integral. Let $m \geq 2$ be an integer. Then the map $f^*\colon \mathrm{Pic}(Y)/m \to \mathrm{Pic}(X)/m$ is injective if and only if $\mathrm{Pic}(X)[m] \to \mathrm{Pic}(X_K)[m]$ is surjective.*

Proof. In our situation we have an exact sequence

$$0 \to \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \to \mathrm{Pic}(X_K) \to 0. \tag{4}$$

Exactness at $\mathrm{Pic}(X_K)$: since X is smooth, the Zariski closure in X of a Cartier divisor in X_K is a Cartier divisor in X. Exactness at $\mathrm{Pic}(X)$: if $D \in \mathrm{Div}(X)$ restricts to a principal divisor in X_K, then D is the sum of a principal divisor in X and a divisor contained in the fibres of f, which by our assumption is contained in $f^*\mathrm{Div}(Y)$. Exactness at $\mathrm{Pic}(Y)$: if $D \in \mathrm{Div}(Y)$ is such that $f^*D = \mathrm{div}_X(\phi)$, where $\phi \in k(X)^\times$, then the restriction of ϕ to X_K is a regular function. Since X_K proper and geometrically integral, we must have $\phi \in K^\times$. Then $D - \mathrm{div}_Y(\phi) \in \mathrm{Div}(Y)$ goes to zero in $\mathrm{Div}(X)$, so $D = \mathrm{div}_Y(\phi)$ is a principal divisor in Y.

Applying the snake lemma to the commutative diagram obtained from (4) and multiplication by m, proves the lemma. □

Proposition 4.3 *Let $m \geq 2$ be an integer. Let k be a field of characteristic exponent coprime to m. Let $f\colon X \to Y$ be a proper, dominant morphism of smooth and geometrically integral varieties over k such that*

(i) *the fibres of f over the points of Y of codimension 1 are integral and the generic fibre X_K is geometrically integral (where $K = k(Y)$);*

(ii) *$\mathrm{Pic}(X)$ is torsion-free;*

(iii) *$f^*\colon \mathrm{Pic}(Y)/m \to \mathrm{Pic}(X)/m$ is injective.*

Then $\mathrm{H}^1(K, \mu_m) \to \mathrm{H}^1_{\text{ét}}(X_K, \mu_m)$ is an isomorphism.

Proof. The Kummer sequence gives rise to an exact sequence

$$0 \to K^\times/K^{\times m} \to \mathrm{H}^1_{\text{ét}}(X_K, \mu_m) \to \mathrm{Pic}(X_K)[m] \to 0.$$

By Lemma 4.2 we have $\mathrm{Pic}(X_K)[m] = 0$. □

Theorem 4.4 *Let $n \geq 2$. Let Π_1, Π_2, X, $K = k(\Pi_2)$ be as above. Then the map $\mathrm{H}^1(K, \mu) \to \mathrm{H}^1_{\text{ét}}(X_K, \mu)$ is an isomorphism.*

Proof. Let us first prove the statement for $\mu = \mu_m$ with m not divisible by $\mathrm{char}(k)$. Let us check the assumptions of Proposition 4.3 for $f\colon X \to \Pi_2$. By Lemma 4.1, assumption (i) is satisfied. The projection $p\colon X \to \Pi_1$ is a projective bundle over Π_1. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(\Pi_1) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(\mathbb{P}^{n-1}_{k(\Pi_1)}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \mathrm{Pic}(\Pi_1) & \longrightarrow & \mathrm{Pic}(\Pi_1 \times \Pi_2) & \longrightarrow & \mathrm{Pic}((\Pi_2)_{k(\Pi_1)}) & \longrightarrow & 0
\end{array}$$

The right-hand vertical map is induced by the inclusion of a projective hyperplane in a projective space, so it is an isomorphism. Hence (ii) holds and the restriction map $\mathrm{Pic}(\Pi_1 \times \Pi_2) \to \mathrm{Pic}(X)$ is an isomorphism. It follows that $\mathrm{Pic}(\Pi_2) \to \mathrm{Pic}(X)$ is split injective, hence (iii) holds.

For an arbitrary group μ, let E/k be a finite Galois extension, with Galois group G, such that $\mu_E = \mu \times_k E$ is isomorphic to a finite product of groups $\mu_{m,E}$ where m is coprime to $\mathrm{char}(k)$. Let L be the compositum of the linearly disjoint field extensions K/k and E/k. We have $\mu(E) = \mu(L) = \mathrm{H}^0_{\text{ét}}(X_L, \mu)$. The Hochschild–Serre spectral sequence gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^1(G, \mu(L)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(X_K, \mu) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(X_L, \mu)^G & \longrightarrow & \mathrm{H}^2(G, \mu(L)) \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & \mathrm{H}^1(G, \mu(L)) & \longrightarrow & \mathrm{H}^1(K, \mu) & \longrightarrow & \mathrm{H}^1(L, \mu)^G & \longrightarrow & \mathrm{H}^2(G, \mu(L))
\end{array}$$

Since the result is already proved for μ_m, all vertical maps, except possibly the map $\mathrm{H}^1(K, \mu) \to \mathrm{H}^1_{\text{ét}}(X_K, \mu)$, are isomorphisms. Hence so is this map. □

Remark 4.5 The geometric argument based on the projective bundle structure of $X \subset \Pi_1 \times \Pi_2$ over Π_1 in the proof of Theorem 4.4 is needed only in the case $n = 2$, that is, when the hypersurface $X_K \subset \mathbb{P}^2_K$ is a smooth curve of degree d. When $n \geq 3$ and $X \subset \mathbb{P}^n_K$ is an *arbitrary* smooth hypersurface, we have $\mathrm{H}^1(K, \mu) \cong \mathrm{H}^1(X_K, \mu)$ by Lemma 3.2 (a).

4.2 Basic diagram

We now assume $n \geq 3$ and $i \geq 2$. Recall the Bloch–Ogus complex from Section 2:

$$\mathrm{H}^i(k(X), \mu) \xrightarrow{(\partial_x)} \bigoplus_{x \in X^{(1)}} \mathrm{H}^{i-1}(k(x), \mu(-1)) \to \bigoplus_{x \in X^{(2)}} \mathrm{H}^{i-2}(k(x), \mu(-2)).$$

Since the fibres $X_y = f^{-1}(y)$ over $y \in \Pi_2^{(1)}$ are integral (which holds for $n \geq 2$, see Lemma 4.1) we obtain a complex

$$\mathrm{H}^i_{\mathrm{nr}}(X_K, \mu) \xrightarrow{(\partial_y)} \bigoplus_{y \in \Pi_2^{(1)}} \mathrm{H}^{i-1}(k(X_y), \mu(-1)) \to \bigoplus_{x \in X^{(2)}} \mathrm{H}^{i-2}(k(x), \mu(-2)).$$

To simplify notation, in what follows we do not write the coefficients of cohomology groups. One should bear in mind that there is a change of twist when the codimension of points increases.

Since this is a complex, the image of ∂_y is unramified over the smooth locus of X_y. If X_y is smooth we write $X'_y = X_y$. In the opposite case, X_y is the projective cone over the hyperplane section of X given by some $t_i = 0$, and then we denote by

X'_y this hyperplane section, which is smooth since $n \geq 3$. In this case, the smooth locus $X_{y,\mathrm{sm}} \subset X_y$ is an affine bundle over X'_y, so we have $\mathrm{H}^{i-1}_{\mathrm{nr}}(X_{y,\mathrm{sm}}) \cong \mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y)$ by (1). Thus $\mathrm{Im}(\partial_y)$ is contained in $\mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y)$. Since the fibres X_y over $y \in \Pi_2^{(2)}$ are integral (note that they need not be geometrically integral if $n = 3$), from the diagram in Section 2 we obtain a commutative diagram of complexes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^i_{\mathrm{nr}}(X_K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(1)}} \mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(2)}} \mathrm{H}^{i-2}(k(X_y)) \\
& \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & \mathrm{H}^i(K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(1)}} \mathrm{H}^{i-1}(k(y)) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(2)}} \mathrm{H}^{i-2}(k(y))
\end{array}$$

where the vertical maps are induced by f. Note that since X is a projective bundle over the projective space Π_1, the maps $\mathrm{H}^i(k) \to \mathrm{H}^i(k(X))$ is injective. So is the map $\mathrm{H}^i(k) \to \mathrm{H}^i(K) = \mathrm{H}^i(k(\Pi_2))$.

Let $Y = \mathbb{A}^n_k \subset \Pi_2$ be the affine space given by $t_0 \neq 0$. From the previous diagram we then get a commutative diagram of complexes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^i_{\mathrm{nr}}(X_K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in Y^{(1)}} \mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y) & \longrightarrow & \bigoplus_{y\in Y^{(2)}} \mathrm{H}^{i-2}(k(X_y)) \\
& \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & \mathrm{H}^i(K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in Y^{(1)}} \mathrm{H}^{i-1}(k(y)) & \longrightarrow & \bigoplus_{y\in Y^{(2)}} \mathrm{H}^{i-2}(k(y))
\end{array} \tag{5}$$

Since $Y \cong \mathbb{A}^n_k$, the bottom complex is exact by [R96, Prop. 8.6].

The homology group of the top complex at the first term is $\mathrm{H}^i_{\mathrm{nr}}(X_Y)/\mathrm{H}^i(k)$, where $X_Y = f^{-1}(Y) \subset X$. Let us show that this group is zero. The fibres of $p\colon X \to \Pi_1$ are hyperplanes in Π_2. The map $p\colon X_Y \to U$ is an affine bundle, and $p(X_Y) = U$, where $U = \mathbb{P}^n_k \setminus \{(1:0:\ldots:0)\}$. By (1) the map $p^*\colon \mathrm{H}^i_{\mathrm{nr}}(U) \to \mathrm{H}^i_{\mathrm{nr}}(X_Y)$ is an isomorphism. Since U is the complement to a k-point in $\Pi_1 \cong \mathbb{P}^n_k$, and $n \geq 2$, we have

$$\mathrm{H}^i(k,\mu) \cong \mathrm{H}^i_{\mathrm{nr}}(\Pi_1,\mu) \cong \mathrm{H}^i_{\mathrm{nr}}(U,\mu).$$

The following lemma is proved by a straightforward diagram chase.

Lemma 4.6 *Suppose that we have a commutative diagram of abelian groups*

$$\begin{array}{ccccccc}
 & A & \overset{i}{\hookrightarrow} & B & \overset{j}{\longrightarrow} & C \\
 & a\uparrow & & b\uparrow\cong & & c\uparrow \\
0 \longrightarrow & D & \longrightarrow & E & \longrightarrow & F
\end{array}$$

where i is injective, b is an isomorphism, c is injective, the top row is a complex, and the bottom row is exact. Then a is an isomorphism.

From Lemma 4.6 we conclude:

Proposition 4.7 *With notation as above, if the middle vertical map in diagram (5) is an isomorphism and the right-hand vertical map is injective, then $f^*\colon \mathrm{H}^i(K,\mu) \to \mathrm{H}^i_{\mathrm{nr}}(X_K,\mu)$ is an isomorphism.*

4.3 Unramified cohomology in degrees 2 and 3

The main result of this paper is the following

Theorem 4.8 *Let Π_1 (respectively, Π_2) be the projective space with homogeneous coordinates $x_0, \ldots, x_n$ (respectively, $t_0, \ldots, t_n$). Write $K = k(\Pi_2)$. Let $X \subset \Pi_1 \times \Pi_2$ be the hypersurface*

$$t_0 x_0^d + \ldots + t_n x_n^d = 0. \tag{6}$$

Let $f\colon X \to \Pi_2$ be the natural projection, and let X_K be the generic fibre of f. Let μ be a finite étale commutative group k-scheme of order not divisible by $\mathrm{char}(k)$*.*

(i) *If $n \geq 3$, then $f^*\colon \mathrm{H}^2(K, \mu) \to \mathrm{H}^2_{\mathrm{nr}}(X_K, \mu)$ is an isomorphism.*

(ii) *If $n \geq 4$, then $f^*\colon \mathrm{H}^3(K, \mu) \to \mathrm{H}^3_{\mathrm{nr}}(X_K, \mu)$ is an isomorphism.*

Proof. (i) Consider diagram (5) for $i = 2$. Then the middle vertical map of the diagram is an isomorphism. This follows from Theorem 4.4 when X_y is singular, which happens exactly when the codimension 1 point y is given by $t_i = 0$ for some $i = 1, \ldots, n$. (Note that if $n = 3$ we then need Theorem 4.4 in the case $n = 2$.) If X_y is smooth, the isomorphism follows from Lemma 3.2 (a). By Lemma 4.1, each fibre X_y at a codimension 2 point y is integral, hence the right hand vertical map is injective. By Proposition 4.7, this proves (i).

(ii) Consider diagram (5) for $i = 3$. For $y \in Y^{(1)}$ such that X_y is singular, the vertical map $\mathrm{H}^2(k(y)), \mu(-1)) \to \mathrm{H}^2_{\mathrm{nr}}(X'_y, \mu(-1))$ is an isomorphism by (i). For $y \in Y^{(1)}$ such that X_y is smooth, the map $\mathrm{H}^2(k(y), \mu(-1)) \to \mathrm{H}^2_{\mathrm{nr}}(X_y, \mu(-1))$ is an isomorphism by Proposition 3.3. For $y \in \Pi_2^{(2)}$ the fibre X_y is geometrically integral over $k(y)$ by Lemma 4.1, hence $k(y)$ is separably closed in $k(X_y)$. Thus the restriction map $\mathrm{H}^1(k(y), \mu(-2)) \to \mathrm{H}^1(k(X_y), \mu(-2))$ is injective, so the right-hand vertical map in the diagram is injective. By Proposition 4.7, this proves (ii). □

Corollary 4.9 *For $n \geq 3$, the map* $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ *induces an isomorphism of subgroups of elements of order not divisible by* $\mathrm{char}(k)$*.*

Proof. This follows from Theorem 4.8 (i) by taking $\mu = \mu_m$ for each integer m not divisible by $\mathrm{char}(k)$. □

Remark 4.10 Only the case $n = 3$ of this corollary requires the above proof. For $n \geq 4$ and any smooth hypersurface in $\mathbb{P}^n$, we have the general Proposition 3.3.

5 Pairs of polynomials

In this section we give a short elementary proof that the Brauer group of the surface given by the equation (7) below over the field of rational functions $K = k(t)$, with $t = \lambda/\mu$, is naturally isomorphic to $\mathrm{Br}(K)$ (away from p-primary torsion if $\mathrm{char}(k) = p$). In the case when k has characteristic zero, this follows from more general results of [GS], namely, the combination of [GS, Thm. 1.1 (i)] and [GS, Thm. 1.4].

Theorem 5.1 *Let k be a field. Let d be a positive integer. Let $f(x,y)$ and $g(z,t)$ be products of d pairwise non-proportional linear forms. Let $X \subset \mathbb{P}^1_k \times_k \mathbb{P}^3_k$ be the hypersurface given by*

$$\lambda f(x,y) = \mu g(z,t), \tag{7}$$

where $(\lambda : \mu)$ are homogeneous coordinates in $\mathbb{P}^1_k$ and $(x : y : z : t)$ are homogeneous coordinates in $\mathbb{P}^3_k$. Let $K = k(\mathbb{P}^1_k)$ and let X_K be the generic fibre of the projection $f\colon X \to \mathbb{P}^1_k$. Then the natural map $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ induces an isomorphism of subgroups of elements of order not divisible by $\mathrm{char}(k)$.

Proof. The singular locus X_{sing} is contained in the union of fibres of f above $\lambda = 0$ and $\mu = 0$. The fibre above $\mu = 0$ is given by $f(x,y) = 0$. It is a union of d planes in $\mathbb{P}^3_k$ through the line $x = y = 0$. The intersection of X_{sing} with the fibre above $\mu = 0$ is the zero-dimensional scheme given by $x = y = g(z,t) = 0$. The situation above $\lambda = 0$ is entirely similar. Let $Y = X \setminus X_{\mathrm{sing}}$ be the smooth locus of X/k. The projection $p\colon X \to \mathbb{P}^3_k$ is a birational morphism which restricts to an isomorphism $Y_V \xrightarrow{\sim} V$ on the complement V to the curve in $\mathbb{P}^3_k$ given by $f(x,y) = g(z,t) = 0$. We have

$$\mathrm{Br}(k) \cong \mathrm{Br}(\mathbb{P}^3_k) \cong \mathrm{Br}(V) \cong \mathrm{Br}(Y_V),$$

where the first isomorphism is by [CTS21, Thm. 6.1.3] and the second one is by purity for the Brauer group [CTS21, Thm. 3.7.6]. Since $Y(k) \neq \emptyset$, we have $\mathrm{Br}(k) \subset \mathrm{Br}(Y) \subset \mathrm{Br}(Y_V)$ where the second inclusion is by [CTS21, Thm. 3.5.5]. We conclude that $\mathrm{Br}(Y) \cong \mathrm{Br}(k)$.

Let $m \geq 2$ be an integer not divisible by $\mathrm{char}(k)$. If a closed fibre $X_M = f^{-1}(M)$ is smooth, then X_M is a smooth surface in $\mathbb{P}^3_{k(M)}$, thus we have

$$\mathrm{H}^1_{\text{ét}}(X_M, \mathbb{Z}/m) \cong \mathrm{H}^1(k(M), \mathbb{Z}/m) \tag{8}$$

by Lemma 3.2 (a). The smooth locus of the fibre of f above $\mu = 0$ is a disjoint union of d affine planes $\mathbb{A}^2_k$. We have

$$\mathrm{H}^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{Z}/m) \cong \mathrm{H}^1(k, \mathbb{Z}/m) \tag{9}$$

since $\mathrm{char}(k)$ does not divide m.

Without loss of generality we can write

$$f(x,y) = c\prod_{i=1}^{d}(x - \xi_i y), \qquad g(z,t) = c'\prod_{j=1}^{d}(z - \rho_j t),$$

where $c, c' \in k^\times$ and $\xi_i, \rho_j \in k$ for $i, j = 1, \ldots, d$. We note that for each pair (i,j) the map $s_{ij}\colon (\lambda : \mu) \to \big((\lambda : \mu), (\xi_i : 1 : \rho_j : 1)\big)$ is a section of the morphism $f\colon X \to \mathbb{P}^1_k$.

Each section s_{ij} gives a K-point of X_K. Thus the natural map $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ is injective.

Let $\alpha \in \mathrm{Br}(X_K)[m]$. Evaluating α at the K-point of X_K given by $s_{1,1}$ gives an element $\beta \in \mathrm{Br}(K)[m]$. We replace α by $\alpha - \beta$.

Note that each section $s_{ij}(\mathbb{P}^1_k)$ meets every closed fibre of f at a smooth point. The new element $\alpha \in \mathrm{Br}(X_K)[m]$ has trivial residue on the irreducible component of the smooth locus of every fibre of f that $s_{1,1}(\mathbb{P}^1_k)$ intersects. Indeed, by (8) and (9) this residue is constant, but specialises to zero at the intersection point with $s_{1,1}(\mathbb{P}^1_k)$. In particular, α has trivial residues at the smooth fibres of f, as well as at the affine plane given by $x - \xi_1 y = 0$ in the fibre $\mu = 0$ and the affine plane given by $z - \rho_1 t = 0$ in the fibre $\lambda = 0$.

We now evaluate α at the K-point of X_K given by $s_{1,j}$, where $j = 2, \ldots, d$. The result is an element of $\mathrm{Br}(K)$ which is unramified everywhere except possibly at the k-point of $\mathbb{P}^1_k$ given by $\lambda = 0$. By Faddeev reciprocity, the residue at that point must be zero, too. This implies that α is unramified at the smooth locus of the fibre at $\lambda = 0$. A similar argument using sections $s_{i,1}$ for $i = 2, \ldots, d$ shows that α is unramified at the smooth locus of the fibre at $\mu = 0$.

We see that the residue of α at every point of codimension 1 of Y is zero. Thus α belongs to $\mathrm{Br}(Y)$, hence to $\mathrm{Br}(k)$. We conclude that $\mathrm{Br}(K)[m] \xrightarrow{\sim} \mathrm{Br}(X_K)[m]$. □

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