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R-EQUIVALENCE ON CONIC BUNDLES OF DEGREE 4

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To Yu. I. Manin on his fiftieth birthday

0. Introduction. Let k be a field of characteristic zero, $\bar{k}$ an algebraic closure of k, and $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$. Let X/k be a smooth projective surface which admits a dominant k-morphism p: $X \to \mathbb{P}_k^1$ with smooth generic fibre of genus zero and with 4 geometric degenerate fibres, and assume that p is relatively minimal. Then any geometric degenerate fibre is the transverse intersection of two exceptional curves of the first kind ([10], 1.6) and the degree $(\omega_X \cdot \omega_X)$ of X is 4. According to Iskovskikh [7], [8] (see also [9], §3), two cases may occur: either the anticanonical bundle ω_X^{-1} is not ample, in which case X is a generalized Châtelet surface as soon as $X(k) \neq \varnothing$ (see [9]), or ω_X^{-1} is ample, in which case X is a k-minimal Del Pezzo surface of degree 4 with Pic X free of rank 2 (see [8] and [9]).

These last surfaces, as soon as the set $X(k)$ of their rational points is not empty, may also be characterized [9] as those smooth complete intersections of two quadrics $X \subset \mathbb{P}_k^4$ which are given in homogeneous coordinates by a system of two quadratic forms with coefficients in k of the following type:

$$\begin{cases} f(x_0, \dots, x_4) = 0, \\ x_1 x_2 + x_3 x_4 = 0. \end{cases}$$

In Manin's description of the possible actions of $\mathfrak{g}$ on the 16 lines of a Del Pezzo surface of degree 4 ([10]; [12], 2nd edition, p. 178), they correspond to those actions such that no orbit crosses the middle vertical line in Table 2 (see [8] and [9]).

In descent theory over a given smooth proper rational k-surface X [3], [4], which was developed after work of F. Châtelet [1] and Yu. I. Manin ([12], chap. VI), J.-J. Sansuc and the first named author have raised two basic questions, an affirmative answer to which would solve most natural arithmetico-geometric problems concerning X:

(Q1) *If $\mathcal{T}$ is a universal torsor over X with $\mathcal{T}(k) \neq \varnothing$, is $\mathcal{T}$ a k-rational variety?*

(Q2) *If k is a number field and $\mathcal{T}$ a universal torsor over X, does $\mathcal{T}$ satisfy the Hasse principle?*

In [6], an affirmative answer was given to both questions when X is (a suitable model of) a generalized Châtelet surface, i.e. when X belongs to the first

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nontrivial class of rational surfaces from the point of view of the Enriques-Manin-Iskovskih classification ([10], [11], [8]). It is the purpose of this note to show that for the next class of nontrivial rational surfaces, question **(Q1)** still has a positive answer:

THEOREM 1. *Any universal torsor $\mathcal{T}$ with $\mathcal{T}(k) \neq \varnothing$ over a relatively minimal conic bundle $X/\mathbb{P}^1_k$ of degree 4 is a k-rational variety.*

For simplicity, we give the proof when $\mathrm{char}(k) = 0$, but the proof given works for any perfect field k with $\mathrm{char}(k) \neq 2$ and which is not too small. We refer the reader to [3], §5, and [4], §2.8 and §3.8, for the list of consequences which such a theorem has—and which can be neatly formulated when $\mathrm{char}(k) = 0$. We shall only repeat two of these consequences.

THEOREM 2. *For X over k as above with a k-rational point O, the natural map $X(k)/R \to A_0(X)$ from the set of R-equivalence classes to the Chow group of 0-cycles of degree 0 on X modulo rational equivalence, induced by $P \to (P - O)$, is a bijection.*

That the map is surjective was proved in [2]: it depends on the fact that p has less than 5 geometric degenerate fibres. The injectivity is a general consequence of Theorem 1.

THEOREM 3. *For X over k as above, each R-equivalence class is parametrized by the k-points of one smooth k-rational variety, and if k is finitely generated (as a field) over* $\mathbf{Q}$, *there are only finitely many R-equivalence classes on $X(k)$.*

When k is a number field, the finiteness of $X(k)/R$ in the last theorem was already obtained in [5], using some results of [6].

Let us point out that the paper of Salberger [14], which builds upon [6], may be interpreted as an affirmative answer to question **(Q2)** for all conic bundles X over $\mathbb{P}^1_k$ of degree 4 which are Del Pezzo surfaces of degree 4 and are not of type (XV) in Manin's classification ([12], p. 178).

The reader is referred to [3] and [4] for the basic concepts and results of descent theory on rational varieties which we shall freely use, in particular for the definition of universal torsors and the "local description" of these torsors. He is advised to consult the survey [13] for a comprehensive overview.

1. Complement to a previous paper. Let $p\colon X \to \mathbb{P}^1_k$ be a relatively minimal conic bundle. In [5] and [4] §2.6, down-to-earth equations for the universal torsors over X were given. In this section, we recall a few of the notations and results of [5], and we complement them in a few places, which are essential for the argument given in §2.

We shall assume that the fibre at infinity of $X/\mathbb{P}^1_k$ is smooth, and that it contains a k-point (hence is k-isomorphic to $\mathbb{P}^1_k$). The degeneracy locus of the fibration can be described as $Y = \operatorname{spec} A \subset \mathbb{A}^1_k = \operatorname{Spec} k[x]$ for A a finite étale k-algebra $A = k[x]/P(x) = k[\theta]$ of rank r. There is a finite free A-algebra A'

of rank 2, which may be described by $A' = A[t]/(t^2 - a)$ for a suitable unit $a(\theta) \in A^*$, which describes the local quadratic extensions which prevent the fibration $X/\mathbf{P}_k^1$ from being smooth at the points of Y. Let U_0 be the complement of Y in $\mathbf{A}^1_k$ and let $U = p^{-1}(U_0)$. Let $\hat{S}$ be the kernel of the restriction map $\hat{S}_0 = \operatorname{Pic} \overline{X} \to \operatorname{Pic} \overline{U} = \mathbf{Z}$, and let λ denote the inclusion $\hat{S} \subset \hat{S}_0$.

Let $W \subset \mathbf{A}_k^2 \times_k R_{A/k}\mathbf{A}_k^2$ ($\simeq \mathbf{A}_k^{2r+2}$) be the k-variety defined by the equation

$$u - \theta v = \left(u_0 + u_1\theta + \cdots + u_{r-1}\theta^{r-1}\right)^2 - a(\theta)\left(v_0 + v_1\theta + \cdots + v_{r-1}\theta^{r-1}\right)^2. \tag{1}$$

Note that this variety is a geometrically integral complete intersection of $(r-2)$ quadratic cones in affine space $\mathbf{A}_k^{2r}$ (with coordinates u_i, v_i), which is the affine cone over a geometrically integral nonconical complete intersection of $(r-2)$ quadrics $V \subset \mathbf{P}_k^{2r-1}$.

PROPOSITION 1. *The universal torsor over X whose fibre at the k-points at infinity is trivial is k-birational to $\mathbf{A}_k^2 \times_k W$, hence to $\mathbf{A}_k^3 \times_k V$.*

Proof. Let $\mathcal{T}$ be the universal torsor on X with trivial fibre at any of the k-points of X at infinity. Applying the change of structural group $S_0 \to S$ defined by λ we get a torsor $\mathcal{T}_1$ over X under S. This torsor is of type λ, has a trivial fibre at the k-points of X at infinity and $\mathcal{T}$ is a torsor over $\mathcal{T}_1$ under $\mathbf{G}_m$ hence $\mathcal{T}$ is k-birational to $\mathcal{T}_1 \times_k \mathbf{G}_m$. Given $\alpha(\theta) \in A^*$, let W_α be the k-variety defined in $\mathbf{A}_k^2 \times_k R_{A/k}\mathbf{A}_k^2$ ($\simeq \mathbf{A}_k^{2r+2}$) by the equation

$$\alpha(\theta)(u - \theta v) = \left(u_0 + u_1\theta + \cdots + u_{r-1}\theta^{r-1}\right)^2 - a(\theta)\left(v_0 + v_1\theta + \cdots + v_{r-1}\theta^{r-1}\right)^2. \tag{2}$$

As explained in [5] and [4], just as for any torsor of type $\lambda\colon S \to S_0$, the restriction of $\mathcal{T}_1$ to the open set U of X is given by a diagram of the following type:

$$\begin{tikzcd} \mathcal{T}_{1,U} \arrow[r] \arrow[d] & W_0 \arrow[r] \arrow[d] & M \arrow[d] \\ U \arrow[r, "p"] & U_0 \arrow[r] & R, \end{tikzcd}$$

where the two squares are fibre products, where $M \to R$ is the map

$$\mathbf{G}_{m,k} \times_k R_{A'/k}\mathbf{G}_m \to R_{A/k}\mathbf{G}_m$$

$$\left(t, X + \sqrt{a(\theta)}\,Y\right) \to t^{-1} \cdot \left(X^2 - a(\theta)Y^2\right),$$

(X and Y being "variables" in A), whose kernel is the k-torus S, and where the map $U_0 \to R$ is given by the function $\alpha(\theta) \cdot (x - \theta)$, for some $\alpha(\theta) \in A^*$ whose class in $A^*/k^* \cdot N_{A'/A}A'^*$ only depends on the choice of $\mathscr{T}_1$ among torsors of type λ.

Now W_0 is simply a nonempty open set of the variety W_α. In order to prove the proposition, it is enough to show that the hypothesis that $\mathscr{T}_1$ has trivial fibre at infinity implies the two facts:

(a) The conic fibration $\mathscr{T}_{1,U} \to W_0$ admits a section over an open set of W_0.

(b) In the above description, the class of $\alpha \in A^*/k^* \cdot N_{A'/A}A'^*$ is 1.

Let $U_1 \subset \mathbb{P}^1_k$ be the open set $\mathbb{P}^1_k - \{Y \cup 0\}$, let $y = 1/x$ be the usual parameter for $\mathbb{A}^1_k = \mathbb{P}^1_k - \{0\}$, and consider the torsor $\mathscr{T}_2$ under S over $V = p^{-1}(U_1)$ given by the diagram of fibre products:

$$\begin{tikzcd} \mathscr{T}_2 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & M \arrow[d] \\ V \arrow[r, "p"] & U_1 \arrow[r] & R, \end{tikzcd}$$

where the map $U_1 \to R$ is given by the function $\alpha(\theta) \cdot (1 - \theta y)$. An obvious change of variables reveals that the restrictions of $\mathscr{T}_1$ and $\mathscr{T}_2$ to $U \cap V$ are isomorphic torsors under S. But two torsors under a k-torus S over a smooth k-variety Z which are isomorphic when restricted to some open set of Z have isomorphic restriction to any local ring of this variety hence have equal fibres in $H^1(k, S)$ over any k-point of Z (see [4], 2.7.5). By definition, the fibre of $\mathscr{T}_1$ over a k-point of X at infinity is equal to 1. But the fibre of $\mathscr{T}_2$ over any k-point of V with $y = 0$ is clearly equal to the class of $\alpha(\theta)$ in $A^*/k^* \cdot N_{A'/A}A'^* = H^1(k, S)$. Thus $\alpha(\theta)$ in (2) may be taken equal to 1, which proves (b).

The same computation also gives (a). Indeed, [5] and [4] (proof of Theorem 2.6.4 and Remark 2.6.8) show that the class say ξ of the generic fibre of the conic fibration $\mathscr{T}_{1,U} \to W_0$ in the Brauer group $\operatorname{Br} k(W_0)$ of the function field $k(W_0)$ of W_0 comes from the Brauer group of k. Hence so does the generic fibre of the smooth conic fibration $\mathscr{T}_2 \to W_1$. But we have just seen that the fibres of this fibration over the k-points of W_1 which project down to the point at infinity of $\mathbb{P}^1_k$ are trivial. A specialization argument then shows $\xi = 0 \in \operatorname{Br} k(W_0)$, which implies that the conic fibration admits a section over an open set.

Remark. When $X/\mathbb{P}^1_k$ is a generalized Châtelet surface, as described in §7 of [6], a universal torsor $\mathscr{T}$ over X with $\mathscr{T}(k) \neq \varnothing$ is a k-rational variety (loc. cit., theorem 8.1). A slightly simpler proof has since been shown to us by P. Salberger. With the same notations as above, we here have $r = 4$ and $a(\theta) = a \in k^*$. The main theorem of [5] accounts for the change of variables in [6] which shows that $\mathscr{T}$ is k-birational to $\mathbb{P}^3_k \times_k V$, where V is a geometrically integral nonconical complete intersection of two quadrics in $\mathbb{P}^7_k$ which contains two skew linear spaces of dimension 3, Π_1 and Π_2, each defined over $k(\sqrt{a})$ and conjugate (Π_1 is defined by the equations $u_i = \sqrt{a}\, v_i$, $i = 0, \ldots, 3$, and Π_2 by $u_i = -\sqrt{a}\, v_i$, $i = 0, \ldots, 3$). Also, $\overline{V} = V \times_k \overline{k}$ contains exactly 8 singular points. For all this, see [6], Theorem 7.1. The hypothesis $\mathscr{T}(k) \neq \varnothing$ and a simple lemma imply $V(k) \neq \varnothing$. Let M be in $V(k)$.

Salberger's nice remark is that the line L which is the intersection of the 4-dimensional linear spans $\{M, \Pi_1\}$ and $\{M, \Pi_2\}$, which is clearly defined over k, lies on V. It is enough to show that L lies on any quadric Q which contains V, and this follows from the fact that Q contains at least three points of L: the point $\{M, \Pi_1\} \cap \Pi_2$, the point $\{M, \Pi_2\} \cap \Pi_1$ and the point M itself. Now V contains the k-rational line L, which does not belong entirely to the singular locus, hence V is a k-rational variety (cf. [6], Prop. 2.2).

2. The proof of Theorem 1. Let $p\colon X \to \mathbb{P}^1_k$ be a relatively minimal conic bundle of degree 4 with $X(k) \neq \varnothing$. In order to prove Theorem 1, we shall only consider the case where X is also a Del Pezzo surface of degree 4, since the case of Châtelet surfaces is handled in [6] and in the above remark. According to Iskovskih [8], X may be given a second conic fibration structure $q\colon X \to \mathbb{P}^1_k$, which is also k-minimal (actually, X itself is a k-minimal surface) and if F_1, resp. F_2 are geometric fibres of p, resp. q, their intersection number $(F_1 \cdot F_2)$ equals 2. Also, the 16 lines on the Del Pezzo surface of degree 4 $\overline{X} = X \times_k \overline{k}$ are none but the components of the 4 degenerate geometric fibres of each of the two fibrations.

Let now $\mathscr{T}$ be a universal torsor over X with $\mathscr{T}(k) \neq \varnothing$, and let M be a k-point in $X(k)$ which belongs to the projection of $\mathscr{T}(k)$ under the structural map.

The fibres $p^{-1}(p(M))$ and $q^{-1}(q(M))$ cannot both be degenerate: since both fibrations p and q are k-minimal, M would lie simultaneously on 4 lines of $\overline{X}$, and this is impossible for a Del Pezzo surface of degree 4. Changing p and q if need be, we may assume that the fibre $F_1 = p^{-1}(p(M))$ is a smooth curve, hence isomorphic to $\mathbb{P}^1_k$. Because k is infinite (actually, only small finite fields would be a problem), we may replace M by another k-point N on F_1 so that the fibre $F_2 = q^{-1}(q(N))$ is smooth, does not pass through the 4 geometric singular points of the degenerate fibres of the fibration p, and intersects F_1 in N and in a different k-point.

Performing a suitable k-automorphism of the basis of the first fibration, we find that there lies on X the smooth proper curve $C = F_2$ k-isomorphic to $\mathbb{P}^1_k$, which the map p makes into a double cover of $\mathbb{P}^1_k$, and which does not pass through the singular points of the degenerate fibres of p, and such that the (smooth) fibre of p at infinity intersects C in two distinct k-rational points. Note that the universal torsor $\mathscr{T}$ on X, which has trivial fibre at M, also has trivial fibre at N, since these two k-points are R-equivalent ([4], 2.7.2).

After another suitable change of variables, this implies that the restriction of X over $\mathbf{A}^1_k = \operatorname{Spec} k[x]$ splits when going over to the cover

$$\operatorname{Spec} k[x][y]/(y^2 - x^2 - ax - b)$$

for suitable $a, b \in k$. That the coefficient of x^2 may be chosen to be 1 follows from the fact that $C \cap p^{-1}(\infty)$ consists of two k-rational points.

The above choices now imply that the element $a(\theta)$ which defines the extension A'/A may be taken equal to $\theta^2 + a\theta + b$. Thus the equation (1) of the k-variety $W \subset \mathbf{A}^{10}_k$ here reads:

$$(3) \qquad u - \theta v = \left(u_0 + u_1\theta + u_2\theta^2 + u_3\theta^3\right)^2 - (\theta^2 + a\theta + b)\left(v_0 + v_1\theta + v_2\theta^2 + v_3\theta^3\right)^2.$$

This k-variety contains the k-variety defined by:

$$(4) \qquad u_2 = u_3 = v_1 = v_2 = v_3 = 0, \qquad u_1 = v_0,$$
$$u = u_0^2 - bv_0^2, \qquad v = av_0^2 - 2u_0u_1.$$

Now the projection of $\mathbf{A}^{10}_k$ onto the affine space $\mathbf{A}^8_k$ which forgets the coordinates u and v induces a k-isomorphism of W with an intersection of two quadrics in $\mathbf{A}^8_k$, which is itself a cone over a complete intersection Z of two quadrics in $\mathbf{P}^7_k$, whose equations are immediately deduced from (3). It now follows from (4) that Z contains the k-line given by

$$(5) \qquad u_2 = u_3 = v_1 = v_2 = v_3 = 0, \qquad u_1 = v_0,$$

and one checks that this line is not contained in the singular locus of Z. It now follows from [6], Prop. 2.2. that Z is a k-rational variety, hence also W, hence finally the universal torsor $\mathcal{T}$ over X with trivial fibre at the k-points of X at infinity.

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