General quartic threefolds are not stably rational

(Joint work with Alena Pirutka)

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§1. Rationality, stable rationality, unirationality

X a reduced irreducible variety over $\mathbf{C}$, $d = \dim(X)$.

rational : X is birational to $\mathbf{P}^d$
$\Longrightarrow$ **stably rational** : There exists $n \geq 0$ with $X \times \mathbf{P}^n$ rational
$\Longrightarrow$ **unirational** : There exists a dominant rational map from $\mathbf{P}^n$ to X – one can here assume $n = d$.
$\Longrightarrow$ **rationally connected** : Through any two points of X there is a curve of genus zero. Example : Fano varieties. (Campana; Kollár, Miyaoka, Mori).

For $d = 1$ (Lüroth) and $d = 2$ (Castelnuovo) these are all equivalent properties.

The situation changes as soon as $d \geq 3$.

$X/\mathbf{C}$ smooth and projective, unirational and not rational

1972 Clemens–Griffiths
X smooth cubic hypersurface in $\mathbf{P}^4$.
[X is birational to a conic bundle over $\mathbf{P}^2$]
Any such X is unirational
Method : Intermediate jacobian, Prym varieties (Mumford)
Shows X is not rational. Leaves open stable rationality.

1972 Iskovskikh–Manin
X quartic hypersurface in $\mathbf{P}^4$
Some of them are unirational (all of them ?)
Method: Rigidity : $BirAut(X) = Aut(X)$, finite, hence X not rational.
Shows X is not rational. Leaves open stable rationality.

1972 Artin–Mumford
X smooth projective over $\mathbf{C}$, birational to $z^2 = f_4(u, v, w)$, a double covering of $\mathbf{P}^3$ ramified along a certain (singular) quartic surface
X can also be viewed as a conic bundle over $\mathbf{P}^2$
X is unirational
Invariant detecting nonrationality : $H^3(X, \mathbf{Z})_{tors} \neq 0$, in other words $Br(X) \neq 0$.
This implies that X is not stably rational.

Rational $\neq$ stably rational (Beauville, CT, Sansuc, Swinnerton-Dyer 1985)
Method for nonrationality : Intermediate jacobian, Prym varieties (Clemens-Griffiths 1972, Mumford, Beauville 1977)

stably rational $\neq$ unirational : Brauer group, Artin–Mumford

unirational $\neq$ rationally connected ? Unknown.

$X \subset \mathbf{P}^n$ smooth cubic hypersurface, $n \geq 4$.
All unirational. Artin-Mumford Invariant $Br(X) = 0$.
$n = 4$. Never rational (Clemens–Griffiths). Are some, are all stably rational ? Open problem.
$n = 5$: some are rational (classical; Hassett). Is this an exception ?
n arbitrary. Are all stably rational ? Open problem.

$X \subset \mathbf{P}^4$ smooth quartic hypersurface.
Iskovskikh–Manin : X is never rational.
Artin-Mumford Invariant $Br(X) = 0$.

Is X stably rational ?

A basic stable birational invariant : the Chow group of zero-cycles over any field

k a field, $Char.(k) = 0$.

X/k, smooth, projective, irreducible, stably rational

$\Longrightarrow$ For any field extension F/k, $Grad_F : CH_0(X_F) \to \mathbf{Z}$ is an isomorphism. We then say that X is universally CH_0-trivial.

Let $k = \mathbf{C}$.
Bloch-Srinivas (1983) and others have studied the consequences of $CH_0(X_\Omega) = \mathbf{Z}$, where Ω is an arbitrary algebraically closed field containing $\mathbf{C}$.
Let $\Delta \subset X \times X$ be the diagonal.
The above hypothesis is equivalent to : There exists an integer $N > 0$ and a point $x \in X$, such that
$N\Delta = Z_1 + Z_2 \in CH^d(X \times X)$, with the support of Z_1 in $x \times X$ the support of Z_2 in $X \times Y$, $Y \subset X$, $Y \neq X$ closed.
Under this hypothesis $H^i(X, O_X) = 0$ for all $i \geq 1$.

$X/\mathbf{C}$ is universally CH_0-trivial if and only if there exists such a decomposition of the diagonal with $N = 1$.

Warning

There exist surfaces $X/\mathbf{C}$ which are universally CH_0-trivial while being of general type, hence in particular not stably rational.
For these surfaces $H^0(X, K) = 0$, but some have $H^0(X, 2K) \neq 0$.

$\S$2. Trying to prove non stable rationality using invariants beyond the Brauer group

Unramified cohomology

X/k, smooth, projective, irreducible, function field $k(X)$
$H^i_{nr}(k(X)/k, \mu_n^{\otimes j}) :=$
$\cap_{x \in X^{(1)}} Ker[\partial_x : H^i(k(X), \mu_n^{\otimes j}) \to H^{i-1}(\kappa(x), \mu_n^{\otimes j-1})]$

$=$ (Bloch-Ogus)
$\cap_v Ker[\partial_v : H^i(k(X), \mu_n^{\otimes j}) \to H^{i-1}(\kappa_v, \mu_n^{\otimes j-1})]$
where v runs through all discrete rank one valuations of $(k(X)$ trivial on k.

These are k-birational invariants. If X/k is stably rational, then $H^i(k, \bullet) = H^i_{nr}(k(X)/k, \bullet)$ (CT-Ojanguren 1989)

For algebraically closed fields $k \subset K$, rigidity :
$H^i_{nr}(k(X)/k, \bullet) = H^i_{nr}(K(X)/K, \bullet)$ (CT, Jannsen, method due to Suslin)

$X/\mathbf{C}$ smooth, projective

X stably rational
$\Longrightarrow$ X universally CH_0-trivial
$\Longrightarrow$ For any overfield F/k, any $i, n \in \mathbf{N}_{>0}$, any $j \in \mathbf{Z}$,

$$H^i(F, \mu_n^{\otimes j}) \to H^i_{nr}(F(X)/F, \mu_n^{\otimes j})$$

is an isomorphism

$$H^1_{nr}(k(X)/k, \mathbf{Q}/\mathbf{Z}) = H^1_{\acute{e}t}(X, \mathbf{Q}/\mathbf{Z}) = Hom(\pi_1(X), \mathbf{Q}/\mathbf{Z}).$$

For $k = \mathbf{C}$, this group is an extension of the finite group $NS(X)_{tors} = H^2_B(X, \mathbf{Z})_{tors}$ by $(\mathbf{Q}/\mathbf{Z})^{b_1}$.
If X is rationally connected, then $\pi_1(X) = 0$ (Kollár).

$$H^2_{nr}(k(X)/k, \mathbf{Q}/\mathbf{Z}(1)) = Br(X), \text{ Brauer group of } X.$$

For $k = \mathbf{C}$, this group is an extension of the finite group $H^3_B(X, \mathbf{Z})_{tors}$ by the group $(\mathbf{Q}/\mathbf{Z})^{b_2-\rho}$.
If $X$is rationally connected, then $b_2 - \rho = 0$.

$H^3_{nr}(k(X)/k, \mathbf{Q}/\mathbf{Z}(2))$
$k = \mathbf{C}$
Let $Z^4(X) := Hdg^4(X, \mathbf{Z})/Im(CH^2(X))$ (conjecturally finite).

Theorem (CT-Voisin 2012)
For any smooth projective variety X over $k = \mathbf{C}$, the group $H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))$ is an extension of the finite group $Z_4(X)_{tors}$ by a divisible group.
If $CH_0(X) = \mathbf{Z}$ (e.g. if X is rationally connected), then $Z_4(X)$ is finite and

$$H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = Z^4(X).$$

Theorem (Voisin 2006). *For X rationally connected of dimension 3, $Z^4(X) = 0$ and $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$.*

There exist examples of smooth, projective, unirational varieties X of dimension ≥ 6 with $Br(X) = 0$, $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) \neq 0$, hence also $Z^4(X) \neq 0$, CT-Ojanguren 1989, via Arason 1975 (pre-Merkurjev-Suslin 1983).

X rationally connected, $dim(X) = 4, 5$. Is $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) \neq 0$ possible? Open question.

A “new” stable birational invariant

$X/\mathbf{C}$ smooth and projective, $F/\mathbf{C}$ an overfield, $\overline{F}$ algebraic closure of F, $G = \mathrm{Gal}(\overline{F}/F)$.

Theorem : Coker $[CH^2(X_F) \to CH^2(X_{\overline{F}})^G]$ *is a birational invariant of* $X/\mathbf{C}$*, trivial on stably rational varieties.*

Proof : behaviour of Chow groups under blow up of smooth $\mathbf{C}$-subvariety.

Proposition

Assume $CH_0(X) = \mathbf{Z}$, *and* $H^i(X, \mathbf{Z})_{tors} = 0$ *for* $i = 2, 3$, *and* $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$. *Then for any field* $F/\mathbf{C}$ *there is an exact sequence*

$$0 \to H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to$$
$$\to Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^G]$$
$$\to H^2(G, \mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^{\times}).$$

(Bloch, CT-Raskind 1985, Kahn 1996, CT 2013)

Theorem. *Let $X \subset \mathbf{P}^n_{\mathbf{C}}$, $n \geq 4$, be a smooth cubic hypersurface.*
(i) $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$.
(ii) For $n \geq 5$, for any field $F/\mathbf{C}$, the map
$H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))$ *is an isomorphism.*
(iii) For $n \geq 5$, for any field $F/\mathbf{C}$, the map $CH^2(X_F) \to CH^2(X_{\overline{F}})^G$ ist onto.

Proofs (2013) : Auel, CT, Parimala (for $n = 5$, when X contains a plane); Voisin ($n = 5$, general case).
Many cases may be handled by K-Theoretical methods (Use of Merkurjev-Suslin, results of Kahn, Rost, Sujatha).
For the time being, the case $n = 5$ requires Hodge-theoretic arguments of C. Voisin.

The classical case of cubic hypersurfaces $X \subset \mathbf{P}^4_{\mathbf{C}}$ remains open.
For any such X and any overfield $F/\mathbf{C}$, one may prove

$$H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \simeq Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^G]$$

Can this this group be non-zero (hence X not stably rational) ?
The key case is when F is the function field of the intermediate jacobian $J^3(X)$. This has been very recently investigated by C. Voisin. She has found families of cubic threefolds for which the above group ... vanishes.
In fact she has found families of such threefolds with $CH_0(X_F) = \mathbf{Z}$ for any $F/\mathbf{C}$.

§3. Proving non-rationality by specialisation arguments

In unequal characteristic, a specialisation method to prove non-ruledness (based on a specialisation result of Matsusaka) was used by J. Kollár (1995).

In equal characteristic zero, a different method was used by C. Voisin (Dec. 2013) : use of correspondances on Chow groups and on Betti cohomology.

A variant was applied by CT-Pirutka (Feb. 2014) : use of the specialisation map (Fulton) on the Chow group of zero-cycles

Claire Voisin' s specialisation argument (Dec. 2013)
$B/\mathbf{C}$ smooth curve; $X/\mathbf{C}$ smooth irreducible; $X \to B$ flat, projective, fibres of dimension d; general fibre smooth, special fibre X_0 with at most isolated ordinary singularities; desingularisation $\tilde{X}_0 \to X_0$.
(a) If there is a Chow decomposition of the diagonal for the general X_t, then this also holds for $\tilde{X}_0$.
(b) Assume that for the general X_t there is a cohomological decomposition of the image of the diagonal in $H^{2d}_{Betti}(X_t \times X_t, \mathbf{Z})$ and the even integral Betti cohomology of $\tilde{X}_0$ is entirely algebraic ($=$ image of algebraic cycles). Then there is a cohomological decomposition of the image of the diagonal in $H^{2d}_{Betti}(\tilde{X}_0 \times \tilde{X}_0, \mathbf{Z})$.

$X \subset \mathbf{P}(2,1,1,1,1)$ double cover of $\mathbf{P}^3$ ramified along a quartic surface $S \subset \mathbf{P}^3$.

Theorem (Voisin) *If S has $n \leq 7$ ordinary singularities, and is otherwise very general, then X is not rational, but it satisfies $Br(X) = 0$ and $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$.*

Method : Specialisation to an Artin-Mumford threefold Y. This is a double cover of $\mathbf{P}^3$ ramified along a suitable quartic surface $S \subset \mathbf{P}^3$ with 10 ordinary singularities.

Artin and Mumford (1972) produced such a Y with a desingularisation $Z \to Y$ satisfying $Br(Z) \neq 0$.

$X \subset \mathbf{P}(2,1,1,1,1)$ double cover of $\mathbf{P}^3$ ramified along a quartic surface $S \subset \mathbf{P}^3$.

Theorem (Voisin) *If S is very general and has 7 ordinary singularities, then there exists a field $F/\mathbf{C}$ with $H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \neq 0$.*

This second theorem comes together with :

There exists a 3-dimensional rationally connected variety whose intermediate jacobian $J^3(X)$ has no universal codimension 2 cycle. That is, there is no codimension 2 cycle on $J^3(X) \times X$ which parametrizes all cycles in $CH^2_{hom}(X)$.

Specialisation theorem (CT-Pirutka 2014)

A = discrete valuation ring, K = field of fractions, k = residue class field, algebraically closed, $Char(k) = 0$. Let $\mathcal{X}/A$ be flat and projective over A. Assume :
1) The generic geometric fibre $\mathcal{X} \times_A \overline{K}$ is smooth, integral and stably rational.
2) The special fibre $Y := \mathcal{X} \times_A k$ is integral, and there exists a desingularisation morphism $p : Z \to Y$ which is universally CH_0-trivial, that is, for any overfield F/k the projection map $p_{F,*} : CH_0(Z_F) \to CH_0(Y_F)$ is an isomorphism.

Then Z/k is universally CH_0-trivial, that is for any overfield F/k, deg : $CH_0(Z_F) \to \mathbf{Z}$ is an isomorphism, which implies $Br(Z) = 0$.

Proof. One reduces to the case $k = F$ (not necessarily algebraically closed), $A = F[[t]]$, $K = F((t))$ and $X := \mathcal{X} \times_A K$ is stably rational over K. Then $deg_K : CH_0(X) \to \mathbf{Z}$ is an isomorphism.
Let $U \subset Y$ be a nonempty Zariski open set such that $p : p^{-1}(U) \to U$ is an isomorphism. Let $V = p^{-1}(U)$. Let z be a zero-cycle of degree 0 on Z. Since Z/F is smooth, z is rationally equivalent on Z to a degree 0 zero-cycle z_1 with support in V. The degree 0 zero-cycle $p_*(z_1)$ then has its support in U. Since U is smooth and $A = F[[t]]$ is complete, one may lift the degree 0 zero-cycle $p_*(z_1)$ to a 1-cycle on $\mathcal{X}$ of relative degree 0 over A.
Fact (Fulton) : *The homomorphism $CH_1(\mathcal{X}) \to CH_0(Y)$ induces a homomorphism $CH_0(X) \to CH_0(Y)$.*
From $deg_K : CH_0(X) \simeq \mathbf{Z}$ we then deduce $p_*(z) = p_*(z_1) = 0 \in CH_0(Y)$. Since $p_* : CH_0(Z) \to CH_0(Y)$ is an isomorphism, we conclude $z = 0 \in CH_0(Z)$. QED

Note :

- We do not impose any regularity condition on $\mathcal{X}$.
- One need not even assume that $\mathcal{X} \times_A \overline{K}$ is smooth.

Hypotheses on the special fibre

• The requirement *the morphism* $p : Z \to Y$ *universally* CH_0*-trivial* depends only on Y.

• If all fibres $Z_M/\kappa(M)$, $M \in Y$, residue field $\kappa(M)$, are universally CH_0-trivial, then the morphism $p : Z \to Y$ is universally CH_0-trivial.

• Simple example for a universally CH_0-trivial morphism $p : Z \to Y$:
k algebraically closed, $dim(Y) \geq 2$, and all singularities of Y are ordinary quadratic singularities.

Two examples of quartic hypersurfaces $Y \subset \mathbf{P}^4_{\mathbf{C}}$ whose singular locus is of dimension 1 and whose desingularisations $p : Z \to Y$ have the two properties :
(i) the morphism $p : Z \to Y$ is universally CH_0-trivial :
(ii) $Br(Z) \neq 0$, i.e. $H^3(Z, \mathbf{Z})_{tors} \neq 0$, i.e. $H^4(X, \mathbf{Z})_{tors} \neq 0$.

One starts with an Artin-Mumford surface in $\mathbf{P}^3_{\mathbf{C}}$:

$$\alpha_2(z_0, z_1, z_2)z_3^2 + \beta_3(z_0, z_1, z_2)z_3 + \gamma_4(z_0, z_1, z_2) = 0$$

$\beta^2 - \alpha\gamma = \varepsilon_1.\varepsilon_2$, with each $\varepsilon_i = 0$ an elliptic curve in $\mathbf{P}^2_{\mathbf{C}}$, and with the smooth conic $\alpha = 0$ tangent to each of them.

• June Huh (A counterexample to the geometric Chevalley-Lang conjecture, 2013)
$\alpha_2(z_0,z_1,z_2)z_3^2+\beta_3(z_0,z_1,z_2)z_3+\gamma_4(z_0,z_1,z_2)+\delta_2(z_0,z_1,z_2)z_4^2=0$
Here $\delta_2(z_0,z_1,z_2)=0$ is smooth and general enough.
J. Huh constructs a desingularisation $p: Z \to Y$, shows : $\mathbb{L}$-desingularisation (akin to CH_0-trivial desingularisaton).
Computes $H^4(Z,\mathbf{Z})_{tors}$, finds it is not zero (delicate computation, as in Artin-Mumford).

• CT-Pirutka (2014)
$\alpha_2(z_0,z_1,z_2)z_3^2+\beta_3(z_0,z_1,z_2)z_3+\gamma_4(z_0,z_1,z_2)+z_0^2z_4^2=0$
We compute a desingularisation $p: Z \to Y$ and show p is universally CH_0-trivial. We need not compute $H^4(Z,\mathbf{Z})_{tors}$, because our variety is birational to the 3-dimensional Artin-Mumford variety, hence one already knows $Br(Z)\neq 0$.

One may take these singular quartics with coefficients in $\overline{\mathbf{Q}}$.
In the projective space P parametrizing quartics in $\mathbf{P}^4$, take a line L defined over $\overline{\mathbf{Q}}$ and a point R in $L(\mathbf{C})$ which does not belong to $L(\overline{\mathbf{Q}})$.
By application of the specialisation theorem we get the main theorem of the talk (CT et Pirutka, Februar 2014) :

The quartic threefold with parameter R is is not stably rational.

The method produces smooth quartic hypersurfaces defined on an algebraic closure of $\mathbf{Q}(t)$ which are not stably rational.

To be compared with Iskovkikh-Manin (1972) : no smooth quartic hypersurface in $\mathbf{P}^4_{\mathbf{C}}$ is rational.

Appendix : Smooth cubic hypersurfaces $X \subset \mathbf{P}^n_k$, $n \geq 3$, over a non-algebraically closed field k

There exist cubic hypersurfaces

$X \subset \mathbf{P}^3_k$ with $Br(X) \neq Br(k)$ hence X not retract rational over $k = \mathbf{C}((t))$ and over $k = \mathbf{F}$ a finite field (Shafarevich, Manin).

$X \subset \mathbf{P}^3_{\mathbf{Q}}$ which are stably rational over $\mathbf{Q}$, but not rational over $\mathbf{Q}$. Also over $\mathbf{Q}_p$ (Beauville, CT, Sansuc, Swinnerton-Dyer)

$X \subset \mathbf{P}^n_{\mathbf{R}}$ with $X(\mathbf{R})$ not connected, hence X not stably rational, and $A_0(X) = \mathbf{Z}/2$, any $n \geq 3$. Deformation of

$$x(x-z)(x+z) + (\sum_{i=1}^{i=m} y_i^2)z = 0.$$

$X \subset \mathbf{P}^4_k$ with $k = \mathbf{C}((x))((y))$ and $A_0(X) \neq 0$, hence X not stably rational

$$T_0^3 + T_1^3 + xT_2^3 + yT_3^3 + xyXT_4^3 = 0.$$

(Specialisation on Chow groups, D. Madore 2008).

$X \subset \mathbf{P}^4_k$ not stably rational, with $k = \mathbf{Q}_p$, $K = k(\omega)$ cubic, unramified.

$$Norm_{K/k}(u + \omega.y + \omega^2.z) + x.y.(x + y) + p.\Phi(u, v, w, x, y) = 0$$

with suitable $\mathbf{Z}_p$-smooth $\Phi \in \mathbf{Z}_p[u, v, w, x, y] = 0$.
(Same technique as in the present talk)

Open problems

In any of the following cases does there exist a smooth cubic hypersurface $X \subset \mathbf{P}^n_k$ which is not stably rational ?

Over $\mathbf{C}$ (any $n \geq 4$)

Over $\mathbf{C}((t))$ or $\mathbf{C}(t)$ (any $n \geq 4$)

Over a finite field (any $n \geq 4$)

Over $\mathbf{C}((x))((y))$ (any $n \geq 5$)

Over $\mathbf{Q}_p$ (any $n \geq 5$)