Contemporary Mathematics
Volume, 199?

The Hasse principle in a pencil of algebraic varieties

J.-L. Colliot-Thélène

ABSTRACT. Let k be a number field, X/k a smooth, projective, geometrically connected variety and $p : X \to \mathbf{P}^1_k$ a flat morphism from X to the projective line, with (smooth) geometrically integral generic fibre. Assume that X has points in all completions of k. Does there exist a k-rational point m of $\mathbf{P}^1_k$ with smooth fibre $X_m = p^{-1}(m)$ having points in all completions of k ? This may fail, but known counterexamples can be interpreted by means of the subgroup of the Brauer group of X whose restriction to the generic fibre of p comes from the Brauer group of the function field of $\mathbf{P}^1_k$. This fibred version of the Brauer–Manin obstruction has been at the heart of recent investigations on the Hasse principle and weak approximation. Work in this area is surveyed in the present text, which develops the talk I gave at Tiruchirapalli.

1. The Hasse principle, weak approximation and the Brauer–Manin obstruction

Let k be a number field, let Ω be the set of its places and Ω_∞ the set of archimedean places. Let $\overline{k}$ denote an algebraic closure of k, and let k_v the completion of k at the place v. Let X/k be an algebraic variety, i.e. a separated k-scheme of finite type. Given an arbitrary field extension K/k, one lets $X(K) = \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathrm{Spec}(K), X)$ be the set of K-rational points of the k-variety X, and one writes $X_K = X \times_k K$ and $\overline{X} = X \times_k \overline{k}$. We have obvious inclusions

$$X(k) \hookrightarrow X(\mathbb{A}_k) \subset \prod_{v \in \Omega} X(k_v),$$

the first one being the diagonal embedding into the set $X(\mathbb{A}_k)$ of adèles of X. The set $X(\mathbb{A}_k)$ is empty if and only if the product $\prod_{v\in\Omega} X(k_v)$ is empty. When X/k is proper, e.g. projective, we have $X(\mathbb{A}_k) = \prod_{v\in\Omega} X(k_v)$. If X is smooth over k and irreducible, and if $U \subset X$ is a non-empty Zariski open set of X, the conditions $X(\mathbb{A}_k) \neq \emptyset$ and $U(\mathbb{A}_k) \neq \emptyset$ are equivalent. Deciding whether the set $X(\mathbb{A}_k)$ is empty or not is a finite task. This prompts:

1991 *Mathematics Subject Classification.* Primary 14G05 Secondary 11G35, 14G25, 14D10, 14C25.

This text has benefited from numerous discussions with Alexei Skorobogatov and Peter Swinnerton-Dyer over a number of years.

Definition 1. *The condition $X(\mathbb{A}_k) = \emptyset$ is the local obstruction to the existence of a rational point on X. A class $\mathcal{C}$ of algebraic varieties defined over k is said to satisfy the Hasse principle (local-to-global principle) if the local obstruction to the existence of a rational point on a variety of $\mathcal{C}$ is the only obstruction, i.e. if for X in $\mathcal{C}$ the (necessary) condition $X(\mathbb{A}_k) \neq \emptyset$ implies $X(k) \neq \emptyset$.*

A 'counterexample to the Hasse principle' is a variety X/k such that $X(\mathbb{A}_k) \neq \emptyset$ but $X(k) = \emptyset$.

Classes of varieties known to satisfy the Hasse principle include: quadrics (Legendre, Minkowski, Hasse), principal homogeneous spaces of semisimple, simply connected (linear) algebraic groups (Kneser, Harder, Tchernousov), projective varieties which are homogenous spaces under connected linear algebraic groups (a corollary of the previous case, as shown by Harder), varieties defined by a norm equation $N_{K/k}(\Xi) = c$ for $c \in k^*$ and K/k a finite, cyclic extension (Hasse). The arithmetic part of the proof of these results is encapsulated in the injection $\mathrm{Br}(k) \hookrightarrow \oplus_{v\in\Omega} \mathrm{Br}(k_v)$, which is part of the basic reciprocity sequence from class field theory:

$$0 \to \mathrm{Br}(k) \longrightarrow \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \xrightarrow{\sum_{v\in\Omega} \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0 \tag{1}$$

Here $\mathrm{Br}(F)$ denotes the Brauer group of a field F, i.e. the second (continuous) cohomology group of the absolute Galois group $\mathrm{Gal}(F_s/F)$ acting on the multiplicative group F_s^* of a separable closure F_s of F, viewed as a discrete module. For each place v, the map $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the embedding provided by local class field theory (an isomorphism with $\mathbb{Q}/\mathbb{Z}$ if v is a finite place, an isomorphism with $\mathbb{Z}/2$ if v is a real place, zero if v is complex).

Given a variety X over the number field k, for each place v, the set $X(k_v)$ is naturally equipped with the topology coming from the topology on k_v. Given any set $\Omega_1 \subset \Omega$ of places of k, we then have a topology on the product $\prod_{v\in\Omega_1} X(k_v)$, a basis of which is given by open sets of the shape $\prod_{v\in S} U_v \times \prod_{v\in\Omega_1\setminus S} X(k_v)$, where S is a finite subset of Ω_1 and $U_v \subset X(k_v)$, $v \in S$ is open. If X/k is proper, hence $X(\mathbb{A}_k) = \prod_{v\in\Omega} X(k_v)$, this gives the usual topology on the set of adèles $X(\mathbb{A}_k)$. For any finite set $S \subset \Omega$, the projection map $\prod_{v\in\Omega} X(k_v) \to \prod_{v\in S} X(k_v)$ is open.

Definition 2. *Let X/k be a variety over the number field k. One says that weak approximation holds for X if the diagonal map $X(k) \to \prod_{v\in\Omega} X(k_v)$ has dense image, which is the same as requiring: for any finite set S of places of k, the diagonal map $X(k) \to \prod_{v\in S} X(k_v)$ has dense image.*

If X/k is proper, this condition is also equivalent to the density of $X(k)$ in $X(\mathbb{A}_k)$ for the adèle topology (but if X/k is not proper, these conditions are far from being equivalent – think of the case where X is the additive group $\mathbf{G}_a$ or the multiplicative group $\mathbf{G}_m$). If X is smooth over k and irreducible, and if $U \subset X$ is a non-empty Zariski open set of X, then X satisfies weak approximation if and only if U does.

According to this definition, if X satisfies weak approximation, then it satisfies the Hasse principle. The reader should be aware that in many earlier papers, weak approximation is defined only under the additional assumption $X(k) \neq \emptyset$.

In parallel with the Hasse principle, one studies weak approximation, for a number of reasons:

(a) the techniques are similar ;

(b) proofs of the Hasse principle often yield a proof of weak approximation at the same stroke (taking $S = \Omega_\infty$, for specific classes of varieties, one thus gets positive answers to questions raised by Mazur [Maz2], even in cases where the group of birational automorphisms does not act transitively on rational points);

(c) weak approximation on a smooth irreducible variety implies Zariski density of rational points, as soon as there is at least one such point;

(d) proofs of the Hasse principle often rely on the weak approximation property for some auxiliary variety.

Weak approximation holds for affine space $\mathbf{A}^n_k$ and projective space $\mathbf{P}^n_k$. More generally, it holds for any smooth irreducible k-variety which is k-birational to affine space, for instance for varieties defined by a norm equation $N_{K/k}(\Xi) = 1$ when K/k is a finite, cyclic extension of k (this is a consequence of Hilbert's theorem 90). It also holds for semisimple, simply connected (linear) algebraic groups ([Pl/Ra] VII.3, Prop. 9).

Once the Hasse principle has been defined, one cannot but admit: it very rarely holds ! Indeed, (subtle) counterexamples to the Hasse principle for smooth, irreducible varieties have been exhibited among: varieties defined by a norm equation $N_{K/k}(\Xi) = c$ when K/k is not cyclic, e.g. when K/k is Galois with group $(\mathbb{Z}/2)^2$ (Hasse and Witt in the 30's), curves of genus one (Reichardt and Lind in the 40's), in particular the curve defined by the famous diagonal equation $3x^3 + 4y^3 + 5z^3 = 0$ (Selmer in the 50's), principal homogeneous spaces under some semisimple algebraic groups (Serre in the 60's, [Se1], III.4.7), smooth cubic surfaces (Swinnerton-Dyer, 1962), then the diagonal cubic surface $5x^3 + 9y^3 + 10z^3 + 12t^3 = 0$ (Cassels and Guy, 1966), conic bundles over the projective line, such as the surface given by $y^2 + z^2 = (3 - x^2)(x^2 - 2)$ (Iskovskikh, 1970), smooth intersections of two quadrics in $\mathbf{P}^4_k$ (Birch and Swinnerton-Dyer, 1975), singular intersections of two quadrics in $\mathbf{P}^5_k$.

Similarly, even under the additional assumption $X(k) \neq \emptyset$ and X smooth and irreducible, weak approximation quite often fails. Counterexamples have been found among varieties defined by a norm equation $N_{K/k}(\Xi) = 1$ when K/k is not cyclic, e.g. when K/k is Galois with group $(\mathbb{Z}/2)^2$, curves of genus one (even when rational points are Zariski dense on them), principal homogeneous spaces under some semisimple algebraic groups (Serre), smooth cubic surfaces (Swinnerton-Dyer). After Faltings' theorem (Mordell's conjecture), weak approximation clearly fails for any curve of genus at least 2 possessing at least one rational point.

As it turns out, the arguments underlying the counterexamples just listed can all be cast in a common mould, namely the Brauer–Manin obstruction to the Hasse principle, described by Manin [Ma1] in his talk at the ICM in 1970. A similar obstruction (the same indeed) accounts for the quoted counterexamples to weak approximation ([CT/San3]).

Let $\mathrm{Br}(X)$ denote the (Grothendieck) Brauer group of a scheme X, namely $H^2_{\text{ét}}(X, \mathbf{G}_m)$. If $X = \mathrm{Spec}(F)$ is the spectrum of a field F, then $\mathrm{Br}(X) = \mathrm{Br}(F)$. If X is a k-variety, F/k a field extension, and $A \in \mathrm{Br}(X)$ an element of the Brauer group, functoriality yields an evaluation map $\mathrm{ev}_A : X(F) \to \mathrm{Br}(F)$, sending the point $P \in X(F)$ to the fibre $A(P)$ of A at P.

Lemma 1. *Let X/k be a smooth, proper, irreducible variety over the number field k, and let $A \in \mathrm{Br}(X)$.*

(i) For each place $v \in \Omega$, the evaluation map $\mathrm{ev}_A : X(k_v) \to \mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}$ *is continuous and has finite image.*

(ii) There exists a finite set $S_A \subset \Omega$ of places of k such that for $v \notin S_A$, the evaluation map $\mathrm{ev}_A : X(k_v) \to \mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}$ *is zero.*

Let X/k be a variety over k, and let $A \in \mathrm{Br}(X)$. We have the basic commutative diagramme:

$$\begin{tikzcd}
X(k) \arrow[r, hook] \arrow[d, "\mathrm{ev}_A"] & X(\mathbb{A}_k) \arrow[d, "\mathrm{ev}_A"] \arrow[dr, "\theta_A"] & \\
\mathrm{Br}(k) \arrow[r] & \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

where the vertical map lands in the direct sum by the lemma. That sequence (1) is exact says in particular that the bottom composite map is zero (this is a generalization of the classical quadratic reciprocity law). As indicated in the diagramme, we denote by θ_A the composite map

$$\theta_A : X(\mathbb{A}_k) \to \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}.$$

We let $\mathrm{Ker}(\theta_A) \subset X(\mathbb{A}_k)$ denote the inverse image of $0 \in \mathbb{Q}/\mathbb{Z}$. By lemma 1, this is a closed and open set of $X(\mathbb{A}_k)$.

Let $B \subset \mathrm{Br}(X)$ be a subgroup of the Brauer group. Let us define

$$X(\mathbb{A}_k)^B = \bigcap_{A\in B} \mathrm{Ker}(\theta_A) \subset X(\mathbb{A}_k)$$

and $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}(X)}$. Because of the continuity statement in Lemma 1, the closure $X(k)^{cl}$ of $X(k)$ in $X(\mathbb{A}_k)$ is contained in $X(\mathbb{A}_k)^{\mathrm{Br}}$. Let us state this as:

Proposition 2. *Let X/k be a smooth, proper, irreducible variety over a number field k, and let $B \subset \mathrm{Br}(X)$ be a subgroup of the Brauer group of X. We have the natural inclusions*

$$X(k)^{cl} \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)^B \subset X(\mathbb{A}_k)$$

of closed subsets in $X(\mathbb{A}_k) = \prod_{v\in\Omega} X(k_v)$.

Clearly, if $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A}_k)^B = \emptyset$ for some B, then we have a counterexample to the Hasse principle. What Manin [Ma1] noticed in 1970 was that this simple proposition accounts for most counterexamples to the Hasse principle hitherto known – the Cassels and Guy example awaited 1985 (work of Kanevsky, Sansuc and the author) to be fit into the Procustean bed without damage. In these counterexamples, the rôle of sequence (1) is played by some explicit form of the reciprocity law. Clearly again, if X/k is proper and the inclusion $X(\mathbb{A}_k)^B \subset X(\mathbb{A}_k)$ is strict, i.e. if $X(\mathbb{A}_k)^B \neq X(\mathbb{A}_k)$, then weak approximation fails for X : that most known counterexamples to weak approximation can be explained in this fashion was pointed out in 1977 (see [CT/San3]).

Several remarks are in order.

(i) The set $X(\mathbb{A}_k)^B$ only depends on the image of B under the projection map $\mathrm{Br}(X) \to \mathrm{Br}(X)/\mathrm{Br}(k)$. More precisely, if $A_i \in B$, $i \in I$, generate the image of B in $\mathrm{Br}(X)/\mathrm{Br}(k)$, then $X(\mathbb{A}_k)^B = \cap_{i\in I}\mathrm{Ker}(\theta_{A_i})$. If the set I is finite, then the closed subset $X(\mathbb{A}_k)^B \subset X(\mathbb{A}_k)$ is also open. If the geometric Picard group $\mathrm{Pic}(\overline{X})$ is torsionfree, which implies that the coherent cohomology group $H^1(X, O_X)$ vanishes, and if the coherent cohomology group $H^2(X, O_X)$ also vanishes, then the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite, hence $X(\mathbb{A}_k)^{\mathrm{Br}}$ is open in $X(\mathbb{A}_k)$.

(ii) From a theoretical point of view, the best choice for $B \subset \mathrm{Br}(X)$ is $\mathrm{Br}(X)$ itself. However, for concrete computations, one prefers to use only finitely many elements in $\mathrm{Br}(X)$.

(iii) Suppose $X(\mathbb{A}_k)^B \neq X(\mathbb{A}_k)$. Then we may be more precise about the set of places where weak approximation fails. Indeed, there then exists a family $\{M_v\}_{v\in\Omega} \in X(\mathbb{A}_k)$, and an $A \in B \subset \mathrm{Br}(X)$ such that $\sum_{v\in\Omega}\mathrm{inv}_v(A(M_v)) \neq 0$. If we let S_A be as in lemma 1 (ii), then the diagonal map $X(k) \to \prod_{v\in S_A} X(k_v)$ does not have dense image.

(iv) One may formulate a version of Lemma 1 and Proposition 2 for the set of adèles $X(\mathbb{A}_k)$ of a not necessarily proper variety X/k, but until now this has been of little use. Even when one studies homogeneous spaces of a linear algebraic group, as Sansuc [San2] and Borovoi [Bo] do, the elements of the Brauer group of such a space E which turn out to play the main rôle may be shown to lie in the image of the Brauer group of smooth compactifications of E.

From now on we shall assume X/k smooth, proper, and geometrically irreducible. Let us recall some terminology. The condition $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ is the Brauer–Manin obstruction to the existence of a rational point on X. The condition $X(\mathbb{A}_k)^B = \emptyset$ is the Brauer–Manin obstruction to the existence of a rational point on X attached to $B \subset \mathrm{Br}(X)$. We shall sometimes refer to it as the B-obstruction to the existence of a rational point on X. When we already know $X(\mathbb{A}_k) \neq \emptyset$, the condition $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ is also referred to as the Brauer–Manin obstruction to the Hasse principle. For a class $\mathcal{C}$ of algebraic varieties over k, suppose that we have a standard way of defining a subgroup $B(X) \subset \mathrm{Br}(X)$ for any $X \in \mathcal{C}$ (e.g. $B(X) = \mathrm{Br}(X)$). We say that the Brauer–Manin obstruction to the existence of a rational point attached to B is the only obstruction for $\mathcal{C}$ if, for $X \in \mathcal{C}$, the conditions $X(k) \neq \emptyset$ and $X(\mathbb{A}_k)^{B(X)} \neq \emptyset$ are equivalent.

Similarly, the condition $X(\mathbb{A}_k)^{\mathrm{Br}} \neq X(\mathbb{A}_k)$ is the Brauer–Manin obstruction to weak approximation on X. The condition $X(\mathbb{A}_k)^B \neq X(\mathbb{A}_k)$ is the Brauer–Manin obstruction to weak approximation on X attached to B. We shall sometimes refer to it as the B-obstruction to weak approximation on X. We say that the Brauer–Manin obstruction attached to B is the only obstruction to weak approximation for X if the inclusion $X(k)^{cl} \subset X(\mathbb{A}_k)^B$ is an equality. This then implies $X(k)^{cl} = X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^B$. It also implies that the B-obstruction to the existence of a rational point on X is the only obstruction.

Suppose that the image of B in $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite and $X(k)^{cl} = X(\mathbb{A}_k)^B$. Assume $X(k) \neq \emptyset$. Then:

(i) For any finite set S of places of k, the closure of the image of the diagonal map $X(k) \to \prod_{v\in S} X(k_v)$ is open. In particular $X(k)$ is Zariski-dense in X.

(ii) There exists a finite set of places S_0 of k such that for any finite set of places S of k with $S \cap S_0 = \emptyset$, the image of the diagonal map $X(k) \to \prod_{v \in S} X(k_v)$ is dense, i.e. weak weak approximation holds (cf. [Se2]).

Two theorems are particularly noteworthy. The first one is Manin's reinterpretation ([Ma2], VI.41.24, p. 228) of results of Cassels and Tate.

Theorem 3 (Manin). *Let X/k be a curve of genus one. Assume that the Tate–Shafarevich group of the Jacobian of X is a finite group. Then the Brauer–Manin obstruction to the existence of a rational point on X is the only obstruction: if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, then $X(k) \neq \emptyset$. More precisely, let $\mathrm{Br}_0(X) \subset \mathrm{Br}(X)$ be the kernel of the map $\mathrm{Br}(X) \to \prod_{v \in \Omega} \mathrm{Br}(X_{k_v})/\mathrm{Br}(k_v)$. The $\mathrm{Br}_0(X)$-obstruction is the only obstruction: if $X(\mathbb{A}_k)^{\mathrm{Br}_0(X)} \neq \emptyset$, then $X(k) \neq \emptyset$.*

Conjecturally, the Tate–Shafarevich group of any abelian variety is a finite group (remarkable results in this direction are due to Rubin and to Kolyvagin, see [Maz1].) The theorem can be extended to cover principal homogeneous spaces of abelian varieties. Lan Wang [W] has established an analogous result for weak approximation on an abelian variety A, namely that the closure of $A(k)$ in $A(\mathbb{A}_k)$ coincides with $A(\mathbb{A}_k)^{\mathrm{Br}}$, under some additional condition. She still assumes the finiteness of the Tate–Shafarevich group of A, but there is a well-known additional difficulty here. Over the real or the complex field k_v, the Brauer group of a variety X over k_v cannot make any difference between two points in the same connected component of $X(k_v)$. Thus for an abelian variety A over a number field k, a statement like Theorem 4 below can hold only if one makes the additional assumption that the closure of the image of the diagonal map $A(k) \to \prod_{v \in \Omega_\infty} A(k_v)$ contains the connected component of identity (if A were a connected linear algebraic group, this would be automatic). This is precisely the additional assumption which L. Wang makes. Waldschmidt [Wald] has given sufficient conditions for this to hold. This problem is related to questions raised by Mazur [Maz2] [Maz3].

The second theorem builds upon work of Kneser, Harder, and Tchernousov for principal homogeneous spaces under semisimple, simply connected groups, and of Sansuc [San2], who handled the case of principal homogeneous spaces under arbitrary connected linear groups.

Theorem 4 (Borovoi) [Bo]. *Let G be a connected linear algebraic group over k, let Y be a homogeneous space under G, and let X be a smooth, projective compactification of Y (i.e. Y is a dense open set in the smooth, projective variety X). Assume that the geometric stabilizer (isotropy group of an arbitrary $\overline{k}$-point of Y) is a connected group. Then the Brauer–Manin obstruction to weak approximation, and in particular to the existence of a rational point on X, is the only obstruction for X. More precisely, let $\mathrm{Br}_0(X) \subset \mathrm{Br}(X)$ be the kernel of the map*

$$\mathrm{Br}(X) \to \prod_{v \in \Omega} \mathrm{Br}(X_{k_v})/\mathrm{Br}(k_v).$$

The $\mathrm{Br}_0(X)$-obstruction to weak approximation, and in particular to the existence of a rational point on X, is the only obstruction.

These two theorems should not lead one to hasty generalizations: many counterexamples, such as Swinnerton-Dyer's cubic surface, or Iskovskih's conic bundle, depend on bigger subgroups of the Brauer group. This is clearly also the case for

Harari's 'transcendental example' [Ha3]: here the obstruction involves an element of $\mathrm{Br}(X)$ which does not vanish in $\mathrm{Br}(\overline{X})$.

These two theorems could nevertheless induce us into asking the question:

Is the Brauer–Manin obstruction the only obstruction to the existence of a rational point (resp. to weak approximation) for arbitrary smooth projective varieties ?

For arbitrary varieties, the answer to that question is presumably NO. Indeed, a positive answer would imply that the Hasse principle and weak approximation hold for smooth complete intersections of dimension at least 3 in projective space, e.g. non-singular hypersurfaces $f(x_0, x_1, x_2, x_3, x_4) = 0$ in projective space $\mathbf{P}^4$, of arbitrary degree d. The weak approximation statement would contradict the generalization of Mordell's conjecture to higher dimension, which predicts in particular that rational points on such hypersurfaces are not Zariski-dense as soon as $d \geq 6$. As for the Hasse principle, Sarnak and Wang [S/W], using the (elementary) fibration method (Theorem 6 below), have shown that this would contradict Lang's conjecture (a variation on the Mordell conjecture theme) that a smooth, projective variety $X/\mathbb{Q}$ such that $X(\mathbb{C})$ is hyperbolic has only finitely many $\mathbb{Q}$-rational points.

One should also bear in mind some examples of curves of genus at least two given by Coray and Manoil [Co/Ma]. The principle of their counterexamples to the Hasse principle is simple: they produce curves X/k equipped with a dominant k-morphism $f : X \to Y$ to a curve of genus one such that $Y(k)$ is non-empty, finite and explicitly known, $f^{-1}(Y(k)) = \emptyset$ by inspection, and $X(\mathbb{A}_k) \neq \emptyset$ (easy to check). The problem here is that the group $\mathrm{Br}(X)/\mathrm{Br}(k)$ is huge, and we do not have a finite algorithm for computing the Brauer–Manin obstruction.

On the basis of Theorems 3 and 4, and of the various results to be discussed later, it nevertheless makes sense to put forward:

CONJECTURE 1. *Let X/k be a smooth, projective, geometrically irreducible variety, and let $p : X \to \mathbf{P}^1_k$ be a dominant (flat) morphism. Assume:*

(α) the generic fibre of p is birational to a homogeneous space Y of a connected algebraic group G over $k(\mathbf{P}^1)$, and the geometric stabilizer of the action of G on Y is connected.

(β) For any closed point $M \in \mathbf{P}^1_k$, the fibre $X_M = p^{-1}(M)$ contains a component of multiplicity one.

Then the Brauer–Manin obstruction to the existence of a rational point on X is the only obstruction. If G is a linear group, the Brauer–Manin obstruction to weak approximation for X is the only obstruction.

The geometric stabilizer is the isotropy group of an arbitrary geometric point of the homogeneous space. The connectedness assumption for this stabilizer may be necessary. Indeed, Borovoi and Kunyavskiĭ [Bo/Ku] have recently produced a homogeneous space of a connected linear algebraic group with (non-commutative) finite geometric stabilizer, which is a counterexample to the Hasse principle, and for which it is unclear whether the Brauer–Manin obstruction holds.

Condition (β) is equivalent to condition

(β') *For any point $m \in \mathbf{P}^1(\overline{k})$, the fibre X_m contains a component of multiplicity one.*

This condition in turn is implied by condition

(γ) *The map p has a section over $\overline{k}$.*

This last condition is automatically satisfied if G in (α) is a connected linear algebraic group (indeed, in that case, the fibration p admits a section over $\overline{k}$, by a theorem of Serre). On the other hand, if one drops condition (β), and G is an elliptic curve, then it may happen that $X(k)$ is not Zariski-dense in X ([CT/Sk/SD2]).

2. Fibrations

We shall explore ways of proving the Hasse principle (and weak approximation) by fibring a variety into hopefully simpler smaller dimensional varieties. Let us first fix standard assumptions. Let X/k be a smooth, projective, geometrically irreducible variety over the field k. In this survey, we shall say that $p : X \to \mathbf{P}^1_k$ is a *fibration* if the map p is dominant (hence flat) and the generic fibre X_η over the field $k(\mathbf{P}^1)$ is smooth (automatic if $\mathrm{char}(k)= 0$) and geometrically irreducible.

Let k be a number field and $p : X \to \mathbf{P}^1_k$ be a fibration. A naïve question would be: Assume that the Hasse principle (resp. weak approximation) holds for (smooth) fibres $X_m = p^{-1}(m)$ for $m \in \mathbf{P}^1(k)$; does it follow that the Hasse principle (resp. weak approximation) holds for X ?

Iskovskih's example, a one parameter family of conics, shows that the answer in general is in the negative: in this case, X has points in all completions of k, but for each fibre X_m over a k-point $m \in \mathbf{P}^1(k)$, there is at least one completion k_v (depending on m) such that $X_m(k_v) = \emptyset$ (otherwise, from the Hasse principle for conics, we would conclude $X(k) \neq \emptyset$). There are similar examples which give a negative answer to the question on weak approximation even when $X(k) \neq \emptyset$.

Before we raise what we feel are the relevant questions, we need a definition (see [Sk2]). Let $p : X \to \mathbf{P}^1_k$ a fibration. Let $\mathrm{Br}_{\mathrm{vert}}(X) \subset \mathrm{Br}(X)$ be the subgroup consisting of elements $A \in \mathrm{Br}(X)$ whose restriction to the generic fibre X_η lies in the image of the map $\mathrm{Br}(k(\mathbf{P}^1_k)) \to \mathrm{Br}(X_\eta)$. If the map p does not have a section (over k), this group may be bigger than the image of $\mathrm{Br}(k) = \mathrm{Br}(\mathbf{P}^1_k)$ under p^*. By 'vertical' Brauer–Manin obstruction to the existence of a rational point (resp. weak approximation), we shall mean the obstruction attached to the subgroup $\mathrm{Br}_{\mathrm{vert}}(X) \subset \mathrm{Br}(X)$. We have a fibred version of Proposition 2 :

PROPOSITION 5. *Let $p : X \to \mathbf{P}^1_k$ be a fibration. Let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k) \neq \emptyset$. Let $\mathcal{R}_1 \subset \mathcal{R}$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k)^{\mathrm{Br}(X_m)} \neq \emptyset$.*

Let $\mathcal{R}^{cl}$, resp. $\mathcal{R}_1^{cl}$, be the closure of $\mathcal{R}$, resp. $\mathcal{R}_1$, under the diagonal embedding $\mathbf{P}^1(k) \hookrightarrow \mathbf{P}^1(\mathbb{A}_k)$. We then have inclusions of closed subsets:

$$\mathcal{R}^{cl} \subset p(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}) \subset p(X(\mathbb{A}_k)) \subset \mathbf{P}^1(\mathbb{A}_k)$$

and

$$\mathcal{R}_1^{cl} \subset p(X(\mathbb{A}_k)^{\mathrm{Br}}) \subset p(X(\mathbb{A}_k)) \subset \mathbf{P}^1(\mathbb{A}_k).$$

In particular, if there exists a point $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k) \neq \emptyset$, then $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$: there is no vertical Brauer–Manin obstruction to the existence of a rational point on X.

In the light of the work to be discussed in the next subsections, and of Proposition 5, it seems natural to raise the following general questions (for pencils of Severi-Brauer varieties, see also [Se1, p. 125]). We shall restrict attention to fibrations satisfying condition

(β) *For any closed point $M \in \mathbf{P}^1_k$, the fibre X_M contains a component of multiplicity one.*

Using the Faddeev exact sequence for the Brauer group of the function field of $k(\mathbf{P}^1_k)$, we see that, under (β), the quotient $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k)$ is finite and essentially computable (cf. [Sk2]). Finiteness of that quotient then implies that the (closed) set $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$ is open in $X(\mathbb{A}_k)$.

Question 1. *Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration satisfying (β). Let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k) \neq \emptyset$. Do we have $\mathcal{R}^{cl} = p(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$?*

In other words, let $\{M_v\} \in X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, and let $m_v = p(M_v)$. Is the family $\{m_v\}$ in the closure of the set T of $m \in \mathbf{P}^1(k)$ such that X_m is smooth and has points in all completions ? In particular, if $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, i.e. if there is no vertical Brauer–Manin obstruction to the existence of a rational point on X, does there exist a rational point $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and has points in all completions of k ?

Question 2. *Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration satisfying (β). Let $\mathcal{R}_1 \subset \mathbf{P}^1(k)$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k)^{\mathrm{Br}(X_m)} \neq \emptyset$. Do we have $\mathcal{R}_1^{cl} = p(X(\mathbb{A}_k)^{\mathrm{Br}})$?*

In particular, if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$, i.e. if there is no Brauer–Manin obstruction to the existence of a rational point on X, does there then exist an $m \in \mathbf{P}^1(k)$ with smooth fibre X_m, such that $X_m(\mathbb{A}_k)^{\mathrm{Br}(X_m)} \neq \emptyset$, i.e. such that there is no Brauer–Manin obstruction to the existence of a rational point on X_m?

Giving even partial answers to these questions becomes harder and harder as the number of reducible geometric fibres of p grows. Let us be more precise. Following Skorobogatov [Sk2], given an arbitrary closed point $M \in \mathbf{P}^1_k$, let us say that the fibre X_M over the field k_M (residue field of $\mathbf{P}^1_k$ at M) is *split* if there exists at least one component Y of X_M in the divisor $X_M = p^{-1}(M) \subset X$, satisfying the two conditions:

(i) its multiplicity in X_M is one ;

(ii) the k_M-variety Y/k_M is geometrically irreducible, i.e. k_M is algebraically closed in the function field $k_M(Y)$ of Y.

As we shall now see, the difficulty to answer the above questions grows with the integer

$$\delta = \delta(p) = \sum_{M \in \mathbf{P}^1_k, X_M \text{non-split}} [k_M : k].$$

Here M runs through the set of closed points of $\mathbf{P}^1_k$, and $[k_M : k]$ is the degree of the finite extension k_M/k. If we let $\delta_1(p)$ be the number of geometric fibres which are reducible, then we have $\delta \leq \delta_1$. In several papers, the invariant δ_1 was used. However, as pointed out by Skorobogatov, it is really δ which measures the arithmetic difficulty.

2.1 The case $\delta \leq 1$.

Let $p : X \to \mathbf{P}^1_k$ be a fibration. If $\delta(p) = 0$, all fibres are split. If $\delta(p) = 1$, there is one non-split fibre, and it lies over a k-rational point m_0 (usually taken as the point at infinity).

The following theorem admits several variants ([CT/San/SD1] p. 43; Skorobogatov [Sk1]; [CT2]). There are useful extensions of the result over $\mathbf{P}^n_k$ rather than $\mathbf{P}^1_k$, see [Sk1].

Theorem 6. *Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration. Assume:*

(i) $\delta \leq 1$;

(ii) the map p has a section over $\overline{k}$.

Then Question 1 has a positive answer. More precisely, let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k) \neq \emptyset$. Let $\mathcal{H} \subset \mathbf{P}^1(k)$ be a Hilbert subset of $\mathbf{P}^1(k)$. Then:

(a) The closure of $\mathcal{R} \cap \mathcal{H}$ in $\mathbf{P}^1(\mathbb{A}_k)$ coincides with $p(X(\mathbb{A}_k))$.

(b) If smooth fibres of p over $\mathcal{H} \subset \mathbf{P}^1(k)$ satisfy the Hasse principle, then X satisfies the Hasse principle: it has a k-point provided $X(\mathbb{A}_k) \neq \emptyset$; more precisely, $p(X(k))$ is then dense in $p(X(\mathbb{A}_k)) \subset \mathbf{P}^1(\mathbb{A}_k)$.

(c) If smooth fibres of p over $\mathcal{H} \subset \mathbf{P}^1(k)$ satisfy weak approximation, then X satisfies weak approximation: $X(k)$ is dense in $X(\mathbb{A}_k)$.

As a matter of fact, it was recently realized that assumption (ii) can be replaced by the weaker assumption (β) (see section 2).The same remark most certainly also holds for a number of results down below, but the (easy) details have not yet been written down. As already mentioned, some such condition is necessary (see [CT/Sk/SD2]).

In the case under consideration here, assumption (β) holds automatically for $M \neq m_0$. We could therefore replace (ii) by the simple assumption: there exists a component $Y_0 \subset X_{m_0}$ of multiplicity one (we do not require that k be algebraically closed in the function field $k(Y_0)$).

This 'fibration technique' was first used in [CT/San/SD1,2] to prove the Hasse principle and weak approximation for certain intersections of two quadrics. It was then used by Salberger and the author to prove the Hasse principle and weak approximation for certain cubic hypersurfaces, and formalized by Skorobogatov [Sk1], who studied weak approximation on certain intersections of three quadrics. The problem in these various papers is to produce fibrations such that the fibres, which are the basic building blocks, satisfy the Hasse principle (and possibly weak approximation). In [CT/San/SD1,2], the main building blocks are non-conical, integral, complete intersections of two quadrics in $\mathbf{P}^4_k$, containing a set of two skew conjugate lines. For such surfaces, the Hasse principle and weak approximation hold ([Ma2], IV.30.3.1). In my joint paper with Salberger, the basic building blocks are cubic surfaces with a set of three conjugate singular points – here the Hasse principle is a result of Skolem (1955). In [Sk1], the building blocks are the intersections of two quadrics for which the Hasse principle and weak approximation had been proved in [CT/San/SD1,2]. As is clear on these examples, when discussing concrete cases, one is soon led to consider 'fibrations' whose generic fibre need not be smooth. This can be obviated in a number of ways, for which we refer the reader to the original papers.

In the theorem above, no mention is made of the Brauer group. As a matter of fact, the assumption $\delta \leq 1$ implies $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k) = 0$, and the conclusion of the theorem is concerned with the existence of fibres X_m with points everywhere locally (compare Question 1).

In the next theorem, we still have $\delta \leq 1$, hence $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k) = 0$, but this time the whole Brauer group $\mathrm{Br}(X)$ is taken into account, and the theorem provides a positive answer to Question 2 for some fibrations. The following statement is a slight reformulation of Harari's result.

Theorem 7 (Harari) [Ha2], [Ha4] . *Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration. Let X_{η_s} denote the geometric generic fibre of p over a separable closure of $k(\mathbf{P}^1)$. Assume:*

(i) $\delta \leq 1$;

(ii) p admits a section over $\overline{k}$;

(iii) $\mathrm{Pic}(X_{\eta_s})$ *is torsionfree ;*

(iv) the Brauer group of X_{η_s} is finite ;

Let $\mathcal{R}_1 \subset \mathbf{P}^1(k)$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. Let $\mathcal{H} \subset \mathbf{P}^1(k)$ be a Hilbert subset. Then:

(a) $(\mathcal{R}_1 \cap \mathcal{H})^{cl} = p(X(\mathbb{A}_k)^{\mathrm{Br}})$.

(b) If the Brauer–Manin obstruction to the existence of a rational point is the only obstruction for smooth fibres over $\mathcal{H}$, then $p(X(k))$ is dense in $p(X(\mathbb{A}_k)^{\mathrm{Br}})$. In particular the Brauer–Manin obstruction to the existence of a rational point on X is the only obstruction: X has a k-point provided $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

(c) If the Brauer–Manin obstruction to weak approximation on X_m for $m \in \mathcal{H}$ is the only obstruction, then the Brauer–Manin obstruction to weak approximation on X is the only obstruction: $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$.

It is worth noticing that hypotheses (ii), (iii) and (iv) together imply that the group $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.

Hypotheses (iii) and (iv) in Theorem 7 hold for instance if the generic fibre is geometrically unirational, or if it is a smooth complete intersection of dimension at least three in projective space.

In specific cases, for purely algebraic reasons, the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ vanishes (whereas it need not vanish for the fibres X_m for general $m \in \mathbf{P}^1(k)$). In this case, to assume that there is no Brauer–Manin obstruction to the existence of a rational point on X is simply to assume that there is no local obstruction: $X(\mathbb{A}_k) \neq \emptyset$. One thus gets the corollary (a special case of which had been obtained by D. Kanevsky in 1985):

Corollary 8 [Ha2]. *If the Brauer–Manin obstruction to the existence of a rational point (resp. to weak approximation) is the only obstruction for smooth cubic surfaces in $\mathbf{P}^3_k$, then the Hasse principle (resp. weak approximation) holds for smooth cubic hypersurfaces in $\mathbf{P}^n_k$ for $n \geq 4$.*

The assumption made in this corollary is an open question, and the conclusion a well-known conjecture for $n \leq 7$ (using the circle method, Heath-Brown and Hooley have proved such results for $n \geq 8$).

Similarly, one obtains the Corollary (Sansuc and the author, 1986):

Corollary 9 (see [Ha2]). *If the Brauer–Manin obstruction to the existence of a rational point is the only obstruction for smooth complete intersections of two quadrics in* $\mathbf{P}^4_k$*, then the Hasse principle holds for smooth complete intersections of two quadrics in* $\mathbf{P}^n_k$ *for* $n \geq 5$.

The assumption made in this corollary is an open question (but see section 2.4 below). The conclusion is known to hold for $n \geq 8$ ([CT/San/SD2]). Assume $X(k) \neq \emptyset$. Then weak approximation for a smooth complete intersection X in $\mathbf{P}^n_k$ is completely under control: for $n \geq 5$, weak approximation holds by an easy application of Theorem 6 ([CT/San/SD1]), and for $n = 4$, the Brauer–Manin obstruction to weak approximation is the only obstruction. This last statement is a delicate theorem of Salberger and Skorobogatov [Sal/Sk], which builds upon [Sal1] and [CT/San3].

The proof of Theorem 7 involves a number of novel ideas. One is a systematic use of Hilbert's irreducibility theorem: one looks for points $m \in \mathbf{P}^1(k)$ such that the action of $\mathrm{Gal}(\overline{k}/k)$ on the geometric Picard group of the fibre X_m is controlled by the situation over the generic fibre. Another one is a very useful 'formal lemma' formulated by Harari ([Ha2], 2.6.1), which is a variant, for arbitrary (ramified) classes of the Brauer group of the function field $k(X)$ of X, of the Brauer–Manin condition $X(\mathbb{A}_k) \subset \mathrm{Ker}(\theta_A)$ for unramified classes $A \in \mathrm{Br}(X)$, and which elaborates on the following theorem, a kind of converse to Lemma 1 above:

Theorem 10 [Ha2]. *Let* X/k *be a smooth, geometrically irreducible variety over a number field* k*. Let* $U \subset X$ *be a non-empty open set of* X*. If* $A \in \mathrm{Br}(U)$ *is not the restriction of an element of* $\mathrm{Br}(X)$*, then there exists infinitely many places* $v \in \Omega$ *such that the evaluation map* $\mathrm{ev}_A : U(k_v) \to \mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}$ *is not constant.*

2.2 The case where δ is small.

When the number of 'degenerate' fibres is small, a combination of the fibration method just described and of the descent method developed by Sansuc and the author ([CT/San2], [CT/San3]) has led to some general results.

Theorem 11 [CT4] [Sal2] . *Let* k *be a number field, and let* $p : X \to \mathbf{P}^1_k$ *be a fibration. Assume*

(i) The generic fibre X_η *is a smooth conic ;*

(ii) $\delta \leq 4$.

Then the Brauer–Manin obstruction to weak approximation (hence to the existence of a rational point) on X *is the only obstruction:* $X(k)$ *is dense in* $X(\mathbb{A}_k)^{\mathrm{Br}}$.

For Châtelet surfaces, given by an affine equation $y^2 - az^2 = P(x)$, with $a \in k^*$ and $P(x) \in k[x]$ a separable polynomial of degree 4, this is a result of [CT/San/SD2]. For the general case, the proof in [CT4] is also based on descent. Salberger's results in [Sal1] (independent of the descent method) led him to an earlier proof (see [Sal2]) of the result on the existence of a rational point. As for the weak approximation statement when existence of a k-point is known, see also [Sal/Sk].

Skorobogatov used the descent technique to study pencils of 2-dimensional quadrics when the number of reducible fibres is small (general results for $\delta \leq 2$, special results for $\delta = 3$). He more recently proved a general result for $\delta = 2$, which may be reformulated as a positive answer to Question 1 in the case under study:

THEOREM 12 (Skorobogatov) [Sk2]. *Let k be a number field and $p : X \to \mathbf{P}^1_k$ a fibration. Assume:*

(i) p admits a section over $\overline{k}$;

(ii) $\delta = 2$;

(iii) if a fibre X_M/k_M over a closed point M is not split, then all its components have multiplicity one.

Let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k) \neq \emptyset$. Let $\mathcal{H} \subset \mathbf{P}^1(k)$ be a Hilbert set. Then:

(a) The inclusion $(\mathcal{R} \cap \mathcal{H})^{cl} \subset p(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$ is an equality.

(b) If smooth fibres of p over $\mathcal{H}$ satisfy the Hasse principle, then $p(X(k))$ is dense in $p(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$. In particular, the vertical Brauer–Manin obstruction to the existence of rational points on X is the only obstruction: if $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then $X(k) \neq \emptyset$.

(c) If smooth fibres of p over $\mathcal{H}$ also satisfy weak approximation, then $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, which then coincides with $X(\mathbb{A}_k)^{\mathrm{Br}}$: the vertical Brauer–Manin obstruction to weak approximation on X is the only obstruction.

When the set of closed points with non-split fibre consists of two rational points of $\mathbf{P}^1(k)$, most of the above results can also be obtained by an (unconditional) application of the technique of the next section (see [CT/Sk/SD1]) – except that assumption (iii) above has to be replaced by assumption (i) in Theorem 13. The application is unconditional, because the special case of Schinzel's hypothesis (H) it uses is Dirichlet's theorem on primes in an arithmetic progression.

It would be nice to find a common generalization of Theorem 7 and Theorem 12.

2.3 δ arbitrary (conditional results).

To prove that four-dimensional quadratic forms over a number field k have a non-trivial zero over k as soon as they have one over each k_v for $v \in \Omega$, Hasse used the result for three-dimensional quadratic forms (a consequence of exact sequence (1)) combined with the generalization for the number field k of Dirichlet's theorem on primes in an arithmetic progression. In 1979, Sansuc and the author [CT/San1] noticed that if one is willing to use a bold generalization of Dirichlet's theorem, and of the conjecture on twin primes, namely the conjecture known as Schinzel's hypothesis (H), then Hasse's technique could be pushed further. For instance, this would show that equations of the shape $y^2 - az^2 = P(x)$ with $a \in \mathbb{Q}^*$ and $P(x)$ an arbitrary *irreducible* polynomial in $\mathbb{Q}[x]$ satisfy the Hasse principle. Further developments are due to Swinnerton-Dyer [SD2], Serre (a lecture at Collège de France, see [Se1], p. 125), and Swinnerton-Dyer and the author [CT/SD].

Schinzel's hypothesis (H) claims the following. Let $P_i(t)$, $i = 1, \dots, n$ be irreducible polynomials in $\mathbb{Z}[t]$, with positive leading coefficients. Assume that the g.c.d. of the $\prod_{i=1}^n P_i(m)$ for $m \in \mathbb{Z}$ is equal to one. Then there exist infinitely many integers $m \in \mathbb{N}$ such that each $P_i(m)$ is a prime number. In his talk, Serre formulated a convenient analogue of Schinzel's hypothesis (H) over an arbitrary number field, and showed that the hypothesis over $\mathbb{Q}$ implies this generalization over any number field (see [CT/SD]).

The most general result is:

Theorem 13 [CT/Sk/SD1]. *Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration. Assume:*

(i) for each closed point $M \in \mathbf{P}^1_k$, there exists a multiplicity one component $Y_M \subset X_M$ such that the algebraic closure of k_M in the function field $k(Y_M)$ is abelian over k_M;

(ii) Schinzel's hypothesis (H) holds over $\mathbb{Q}$.

Let $\mathcal{R} \subset \mathbf{P}^1(k)$ be the set of points $m \in \mathbf{P}^1(k)$ whose fibre X_m is smooth and satisfies $X_m(\mathbb{A}_k) \neq \emptyset$. Then:

(a) The inclusion $\mathcal{R}^{cl} \subset p(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$ is an equality.

(b) If the Hasse principle holds for smooth fibres of p, then $p(X(k))$ is dense in $p(X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}})$. In particular, the vertical Brauer–Manin obstruction to the existence of rational points on X is the only obstruction: if $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then $X(k) \neq \emptyset$.

(c) If weak approximation holds for smooth fibres of p, then $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, which then coincides with $X(\mathbb{A}_k)^{\mathrm{Br}}$: the vertical Brauer–Manin obstruction to weak approximation on X is the only obstruction.

A Severi–Brauer variety over a field F is an F-variety which becomes isomorphic to a projective space after a finite separable extension of the ground field. These varieties were studied by Severi and by F. Châtelet. One-dimensional Severi–Brauer varieties are just smooth projective conics in $\mathbf{P}^2_F$. Severi and Châtelet proved that a Severi–Brauer variety Y over F is isomorphic to projective space over F as soon as it has an F-point. Châtelet proved that Severi–Brauer varieties over a number field satisfy the Hasse principle. Weak approximation then follows. Degenerate fibres of standard models of pencils of Severi–Brauer variety have multiplicity one, and they split over a cyclic extension of the ground field. We thus have the following corollary (see [CT/SD]), which extends the original result of [CT/San1] and [SD2]:

Corollary 14 (Serre). *Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration. Assume that the generic fibre X_η is a Severi–Brauer variety. Under Schinzel's hypothesis (H), the vertical Brauer–Manin obstruction to weak approximation on X is the only obstruction. In particular the vertical Brauer–Manin obstruction to the existence of a rational point is the only obstruction to the existence of a rational point.*

For Y/F a Severi–Brauer variety, the map $\mathrm{Br}(F) \to \mathrm{Br}(Y)$ is surjective. Thus in the case discussed in the Corollary, the inclusion $\mathrm{Br}_{\mathrm{vert}}(X) \subset \mathrm{Br}(X)$ is an equality.

We do not see how to dispense with the abelianness requirement in assumption (i) of Theorem 13, and this is quite a nuisance. Indeed, this prevents us from producing a (conditional) extension of Theorem 7 (Harari's result, which takes into account the whole Brauer group of X) to the case where δ is arbitrary. The simplest case which we cannot handle, even under (H), is the following. Let K/k be a biquadratic extension of number fields (i.e. K/k is Galois, and $\mathrm{Gal}(K/k) = (\mathbb{Z}/2)^2$). Let $P(t) \in k[t]$ be a polynomial. Is the Brauer–Manin obstruction to the existence of a rational point the only obstruction for a smooth projective model of the variety given by the equation $N_{K/k}(\Xi) = P(t)$? The abelianness condition also prevents us from stating the analogue of Corollary 14 for fibrations whose generic fibre is an arbitrary projective homogeneous space of a connected linear algebraic group.

Before closing this subsection, let us recall how far we are from unconditional proofs. If we grant Schinzel's hypothesis, then for X as in Corollary 14, the assumption $X(k) \neq \emptyset$ implies that $p(X(k))$ is Zariski-dense in $\mathbf{P}^1$. But already for pencils of conics, there is an unconditional proof of this result only under the assumption $\delta(p) \leq 5$; in special cases, this may be extended to $\delta(p) \leq 8$, as shown by Mestre [Mes]. Upper bounds for the number of points of $p(X(k))$ of a given height have been given by Serre (see [Se1] p. 126).

2.4 Pencils of curves of genus one (Swinnerton-Dyer's recent programme).

Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration. Assume $X(\mathbb{A}_k) \neq \emptyset$. Suppose the generic fibre of p is a curve of genus one, and Hypothesis (i) of Theorem 13 is fulfilled. If all the geometric fibres are irreducible, or if one is willing to take Schinzel's hypothesis (H) for granted as in the previous section, and one assumes that there is no Brauer–Manin obstruction to the existence of a rational point on X, the previous techniques lead us to the existence of k-points $m \in \mathbf{P}^1(k)$ such that the fibre X_m/k is smooth, hence is a curve of genus one, and has points everywhere locally. Since the Hasse principle need not hold for curves of genus one (the Tate–Shafarevich group of the Jacobian of X_m is in the way), we cannot conclude that X_m has a k-point. One may however observe that the previous techniques only take into account the 'vertical' part of the Brauer group. One exception is Harari's technique, but until now that technique only applies to pencils of varieties such that the generic fibre $Y = X_\eta$ satisfies $H^1(Y, O_Y) = 0$ (among other conditions), and the dimension of the vector space $H^1(Y, O_Y)$ is one for a curve of genus one.

In the course of the study of a special case, taking some difficult conjectures for granted, Swinnerton-Dyer [SD3] managed to overcome the difficulty. His method has very recently been expanded in the joint work [CT/Sk/SD3], and many of the mysterious aspects of [SD3] have now been put in a more general context. In the present report, I will satisfy myself with an informal description of the results in [SD3], and with a very short indication of the progress accomplished in [CT/Sk/SD3].

In [SD3], the ground field k is the rational field $\mathbb{Q}$. Let a_i, b_i, $i = 0, \dots, 4$ be elements of $\mathbb{Q}^*$ such that $a_i b_j - a_j b_i \neq 0$ for $i \neq j$. Let $S \subset \mathbf{P}^4_{\mathbb{Q}}$ be the smooth surface defined by the system of homogeneous equations

$$\sum_{i=0}^{4} a_i x_i^2 = 0, \quad \sum_{i=0}^{4} b_i x_i^2 = 0.$$

Setting $x_4 = t x_0$ defines a pencil of curves of genus one on S. A suitable blowing-up transforms the surface S into a surface X equipped with a fibration $p : X \to \mathbf{P}^1_{\mathbb{Q}}$, the general fibre of which is precisely the hyperplane section of Y given by $x_4 = t x_0$ (for $t \in \mathbf{A}^1(\mathbb{Q})$).

The general fibre of this fibration is thus a curve of genus one. Associated to such a fibration we have a jacobian fibration $q : \mathcal{E} \to \mathbf{P}^1_{\mathbb{Q}}$. For a closed point $M \in \mathbf{P}^1_{\mathbb{Q}}$ with smooth fibre $X_M = p^{-1}(M)$, the fibre $\mathcal{E}_M/k_M$ of the jacobian fibration is an elliptic curve, which is the jacobian of X_M. The special shape of the intersection of two quadrics (simultaneously diagonal equations) ensures that the generic fibre (hence the general fibre) of q has its 2-torsion points rational. Thus the 2-torsion subgroup of the generic fibre $\mathcal{E}_{k(\mathbf{P}^1)}$ is $(\mathbb{Z}/2)^2$, with trivial Galois action. For $m \in \mathbf{P}^1(\mathbb{Q})$ with smooth fibre X_m, there is a natural map $X_m \to \mathcal{E}_m$, which makes X_m into a 2-covering of $\mathcal{E}_m$; here we are using the classical language of descent on elliptic curves, as in Cassels' survey [Cass].

Swinnerton-Dyer throws in two hypotheses which we have already encountered:

(i) The hypothesis that the Tate–Shafarevich group of elliptic curves over $k = \mathbb{Q}$ is finite;

(ii) Schinzel's hypothesis (H).

He then makes a further algebraic assumption, referred to as (D), on the $a_i \in \mathbb{Q}^*$. Roughly speaking, these elements are supposed to be in general position (some field extensions of $\mathbb{Q}$ which depend on them are linearly independent). This condition implies the vanishing of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$, but it is not equivalent to the vanishing of this quotient.

Under these three assumptions, assuming that S has points in all completions of $\mathbb{Q}$, he proves that there exists at least one $m \in \mathbf{P}^1(\mathbb{Q})$ such that the fibre X_m is smooth and has points in all completions of $\mathbb{Q}$ (this is shown by the general technique described in section 2.3), and such that moreover the 2-Selmer group $\mathrm{Sel}(2, \mathcal{E}_m)$ of $\mathcal{E}_m$ has order at most 8 (the argument here is extremely elaborate). By the general theory of elliptic curves, the 2-Selmer group fits into an exact sequence

$$0 \to \mathcal{E}_m(\mathbb{Q})/2 \to \mathrm{Sel}(2, \mathcal{E}_m) \to {}_2\text{Ш}(\mathcal{E}_m) \to 0$$

where ${}_2\text{Ш}(\mathcal{E}_m)$ denotes the 2-torsion subgroup of the Tate–Shafarevich group of the elliptic curve $\mathcal{E}_m$. Now the order of $\mathcal{E}_m(\mathbb{Q})/2$ is at least 4, since all 2-torsion points are rational. Thus the order of ${}_2\text{Ш}(\mathcal{E}_m)$ is at most 2. If we assume that the Tate–Shafarevich group $\text{Ш}(\mathcal{E}_m)$ is finite, then, by a result of Cassels, this finite abelian group is equipped with a non-degenerate alternate bilinear form. In particular the order of its 2-torsion subgroup must be a square. Since this order here is at most 2, it must be 1, and ${}_2\text{Ш}(\mathcal{E}_m) = 0$. But the 2-covering X_m has points in all completions of $\mathbb{Q}$ and defines an element of ${}_2\text{Ш}(\mathcal{E}_m)$. This element must therefore be trivial: X_m has a $\mathbb{Q}$-rational point. The argument actually yields infinitely many such m, and it also shows that for such m the rank of $\mathcal{E}_m(\mathbb{Q})$ is one, hence $X_m(\mathbb{Q})$ is infinite.Thus $\mathbb{Q}$-rational points are Zariski-dense on X (in the case under study in [SD3], this follows from general properties of intersections of two quadrics as soon as $X(\mathbb{Q}) \neq \emptyset$, but the argument is general.)

In [CT/Sk/SD3], with input from class field theory and the duality theory of elliptic curves over local fields, we unravel some of the ad hoc computations of [SD3]. This enables us to extend the argument to arbitrary number fields, and to obtain similar conditional results (under Schinzel's hypothesis (H) and the assumption that Tate–Shafarevich groups of elliptic curves over number fields are finite) for many pencils of curves of genus one such that the (generic) jacobian has all its 2-torsion points rational. Among the surfaces controlled by these methods there are in particular some $K3$-surfaces. For such surfaces virtually nothing in the direction of the Hasse principle was known or conjectured until now.

We have a generalized condition (D). The theory of Néron minimal models and computations of Grothendieck enable us to relate condition (D) to the vanishing of the 2-torsion subgroup of $\mathrm{Br}(X)/\mathrm{Br}_{\mathrm{vert}}(X)$; however the two conditions are not exactly equivalent.

3. Zero-cycles of degree one

Let k be a field and X a k-variety. A zero-cycle on X is a finite linear combination with integral coefficients of closed points of X. It is thus an element $\sum n_M M \in \oplus_{M \in X_0} \mathbf{Z}$, where X_0 denotes the set of closed points of X. The residue field k_M of a closed point M is a finite field extension of k. One defines the degree (over k) of a zero-cycle $\sum_{M \in X_0} n_M M$ by the formula

$$\deg(\sum_{M \in X_0} n_M M) = \sum_{M \in X_0} n_M [k_M : k] \in \mathbb{Z}.$$

The following conditions are (trivially) equivalent:

(i) There exists a zero-cycle of degree one on X.

(ii) The greatest common divisor of the degrees of finite field extensions K/k such that $X(K) \neq \emptyset$ is equal to one.

Thus the condition that there exists a zero-cycle of degree one is a weakening of the condition that X possesses a k-rational point. For certain varieties (Severi–Brauer varieties, quadrics, curves of genus one and more generally principal homogeneous spaces of commutative algebraic groups, pencils of conics over $\mathbf{P}^1$ with at most 5 reducible geometric fibres), these conditions are equivalent, but in general they are not.

Let X/k be a smooth, projective, irreducible variety over a number field k. Using the corestriction (norm map) on Brauer groups, one may easily define a Brauer–Manin obstruction to the existence of a zero-cycle of degree one on X (see [Sai], [CT/SD]). As explained in section 1, we do not expect the Brauer–Manin obstruction to the existence of a rational point to be the only obstruction for arbitrary (smooth, projective) varieties. However, for zero-cycles of degree one, the following general conjecture still looks reasonable:

CONJECTURE 2. *Let X/k be an arbitrary smooth, projective, irreducible variety over the number field k. If there is no Brauer–Manin obstruction to the existence of a zero-cycle of degree one on X, then there exists a zero-cycle of degree one on X.*

For curves of genus one, this conjecture amounts to the conjecture on rational points, as discussed by Manin (Theorem 3 above). For rational surfaces, the special case where $\operatorname{Br}(X)/\operatorname{Br}(k) = 0$ was conjectured by Sansuc and the author in 1981. For conic bundles over $\mathbf{P}^1_k$ with at most 4 degenerate geometric fibres, it follows from the result on rational points (Theorem 11 above). In 1986, Kato and Saito put forward a very general conjecture (see [Sai]). The statement above itself is raised as a question by Saito ([Sai], § 8), and as a conjecture in the survey [CT3], to which I refer for zero-cycles analogues of weak approximation.

Here is some evidence for the conjecture. The first theorem specializes to Theorem 3 when X is a curve of genus one.

THEOREM 15 (S. Saito) [Sai]. *Let X/k be a smooth, projective, geometrically irreducible curve over the number field k. Assume that the Tate–Shafarevich group of the jacobian of X is a finite group. Then the Brauer–Manin obstruction to the existence of a zero-cycle of degree one on X is the only obstruction.*

Theorem 16 [CT/Sk/SD1]. *Let k be a number field, and let $p : X \to \mathbf{P}^1_k$ be a fibration. Assume that X has a zero-cycle of degree one over each completion k_v. Assume:*

(i) for each closed point $M \in \mathbf{P}^1_k$, there exists a multiplicity one component $Y_M \subset X_M$ such that the algebraic closure of k_M in the function field $k(Y_M)$ is abelian over k_M;

(ii) the Hasse principle for zero-cycles of degree one holds for smooth fibres X_M/k_M (where M is a closed point of $\mathbf{P}^1_k$);

(iii) there is no vertical Brauer–Manin obstruction to the existence of a zero-cycle of degree one on X.

Then there exists a zero-cycle of degree one on X.

The main technique used in the proof of Theorem 16 is due to Salberger [Sal1], who proved the theorem for conic bundles (in [Sal1], the theorem is proved explicitly under the additional assumption $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$, in which case it reduces to a Hasse principle for zero-cycles of degree one). The theorem was then generalized to pencils of Severi–Brauer varieties and 'similar' varieties (including quadrics) in [CT/SD] and independently in [Sal2]. The above statement encompasses these previous results.

The reader will notice the striking analogy between Theorem 13, which is a statement for rational points but is conditional on Schinzel's hypothesis, and Theorem 16, which is a statement on zero-cycles of degree one, and is unconditional. As a matter of fact, the proofs of these two theorems run parallel. With hindsight, it appears that the key arithmetic idea of Salberger in [Sal1] is a substitute for Schinzel's hypothesis. Salberger's trick is explained in detail in [CT/Sk/SD1], to which I refer for a general statement. Let me here explain this simple but powerful idea on a special case, the case of twin primes.

Take $k = \mathbb{Q}$. The conjecture on twin primes predicts that there are infinitely many integers n such that n and $n+2$ are both prime numbers.

Proposition 17. *For any integer $N \geq 2$, there exist infinitely many field extensions $K/\mathbb{Q}$ of degree N for which there exists an integer $\theta \in K$, prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ in the ring of integers O_K of K, and prime ideals $\mathfrak{p}_2$ and $\mathfrak{q}_2$ of O_K above the prime 2, such that we have the prime decompositions*

$$(\theta) = \mathfrak{p}\mathfrak{p}_2,\ (\theta+2) = \mathfrak{q}\mathfrak{q}_2$$

in O_K.

Proof. Choose arbitrary prime numbers p and q. Let $R(t) \in \mathbb{Z}[t]$ be a monic polynomial of degree $N-2$. Let $P(t) = R(t)t(t+2) + qt + p(t+2)$. For R, p and q general enough, this is an irreducible polynomial. Let $K = \mathbb{Q}[t]/P(t)$, and let $\theta \in K$ be the integer which is the class of t. Let $N_{K/\mathbb{Q}}$ denote the norm map from K to $\mathbb{Q}$. We clearly have $N_{K/\mathbb{Q}}(\theta) = \pm 2p$ and $N_{K/\mathbb{Q}}(\theta+2) = \pm 2q$. □

In other words, in the field K, each of θ and $\theta+2$ is a prime up to multiplication by primes in a fixed bad set. The bad factors $\mathfrak{p}_2$ and $\mathfrak{q}_2$ above 2 are not a serious problem, but for the actual twin prime conjecture, we would want to take $N = 1$!

In the proof, the 2 coming in the inequality $N \geq 2$ is the sum of the degrees of the polynomials t and $t+2$. As for the second 2, the one coming in the result (the bad set), it appears because $t(t+2)$ is not separable modulo 2, and it also appears as a 'small' prime (smaller than the sum of the degrees of t and $t+2$).

In many cases, Salberger's trick replaces Schinzel's hypothesis, if one is satisfied with finding zero-cycles of degree one rather than rational points. Our next hope is that we shall be able to proceed along this way with the results of [CT/Sk/SD3]. In the original case considered by Swinnerton-Dyer [SD3], namely smooth complete intersections of two simultaneously diagonal quadrics in $\mathbf{P}^4_k$, with some algebraic conditions on the coefficients, this would ultimately lead to a proof of the existence of rational points which would 'only' depend on the finiteness assumption of Tate–Shafarevich groups.

Acknowledgements

Along with the organisers of the Tirichurapalli meeting, I also thank the following institutions, which invited me to lecture on various aspects of the material presented here: Université de Lausanne-Dorigny; University of Southern California, Los Angeles; Université de Caen; Panjab University at Chandigarh; Tata Institute of Fundamental Research, Bombay; Matscience, Madras; Independent University, Moscow; Ohio State University; C.I.R.M., Luminy; Universités Paris 6, Paris 7 et Paris-Sud; C.U.N.Y., New York; Technion Institute, Haïfa.

References

Books and surveys

[Cass] J. W. S. Cassels, *Diophantine equations with special reference to elliptic curves*, J. London Math. Soc. **41** (1966), 193-291.

[CT1] J.-L. Colliot-Thélène, *Arithmétique des variétés rationnelles et problèmes birationnels*, Proc. ICM 1986, vol. **I**, Amer. Math. Soc., Berkeley, California, 1987, pp. 641-653.

[CT2] ———, *L'arithmétique des variétés rationnelles*, Ann. Fac. Sci. Toulouse (6) **I** (1992), 295-336.

[CT3] ———, *L'arithmétique du groupe de Chow des zéro-cycles*, Journal de théorie des nombres de Bordeaux **7** (1995), 51-73.

[Kol] J. Kollár, *Low degree polynomial equations : arithmetic, geometry and topology*, 2nd European Mathematical Congress, Budapest 1996, to appear.

[Ma1] Yu. I. Manin, *Le groupe de Brauer-Grothendieck en géométrie diophantienne*, Actes Congrès intern. math. Nice, 1970, tome **I**, Gauthier-Villars, Paris, 1971, pp. 401-411.

[Ma2] ———, *Cubic Forms : algebra, geometry, arithmetic*, 2nd engl. edition, North Holland Mathematical Library, vol. **4**, Amsterdam, 1986.

[Ma/Ts] Yu. I . Manin and M. A. Tsfasman, *Rational varieties : algebra, geometry, arithmetic*, Russian math. surveys (=Uspekhi Mat. Nauk) **41** (1986), 51-116.

[Maz1] B. Mazur, *On the passage from local to global in number theory*, Bull. Amer. Math. Soc. **29** (1993), 14-50.

[Pl/R] V. P. Platonov and A. S. Rapinchuk, *Algebraic groups and number theory*, Pure and applied mathematics **139**, Academic Press, London and New York, 1994.

[San1] J.-J. Sansuc, *Principe de Hasse, surfaces cubiques et intersections de deux quadriques*, Journées arithmétiques de Besançon 1985, Astérisque **147-148**, Société mathématique de France, 1987, pp. 183-207.

[SD1] Sir Peter Swinnerton-Dyer, *Diophantine Equations : the geometric approach*, Jber. d. Dt. Math.-Verein. **98** (1996), 146-164.

[V] V. E. Voskresenskiĭ, *Algebraicheskie tory*, Nauka, Moskwa, 1977.

I refer to these books and surveys for literature prior to 1992, except for the following papers which are quoted in the text:

[CT4] J.-L. Colliot-Thélène, *Surfaces rationnelles fibrées en coniques de degré 4*, Séminaire de théorie des nombres de Paris 1988-1989, éd. C. Goldstein, Progress in Math. **91**, Birkhäuser, Boston, Basel, Berlin, 1990, pp. 43-55.

[CT/San1] J.-L. Colliot-Thélène et J.-J. Sansuc, *Sur le principe de Hasse et l'approximation faible, et sur une hypothèse de Schinzel*, Acta Arithmetica **41** (1982), 33-53.

[CT/San2] ———, *La descente sur les variétés rationnelles, I*, Journées de géométrie algébrique d'Angers, 1979, éd. A. Beauville, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980, pp. 223-237.

[CT/San3] ———, *La descente sur les variétés rationnelles, II*, Duke Math. J. **54** (1987), 375-492.

[CT/San/SD1] J.-L. Colliot-Thélène, J.-J. Sansuc and Sir Peter Swinnerton-Dyer, *Intersections of two quadrics and Châtelet surfaces, I*, J. für die reine und ang. Math. (Crelle) **373** (1987), 37-107.

[CT/San/SD2] ———, *Intersections of two quadrics and Châtelet surfaces, II*, J. für die reine und ang. Math. (Crelle) **374** (1987), 72-168.

[Sai] S. Saito, *Some observations on motivic cohomology of arithmetic schemes*, Invent. math. **98** (1989), 371-414.

[Sal1] P. Salberger, *Zero-cycles on rational surfaces over number fields*, Invent. math. **91** (1988), 505-524.

[Sal/Sk] P. Salberger and A. N. Skorobogatov, *Weak approximation for surfaces defined by two quadratic forms*, Duke Math. J. **63** (1991), 517-536.

[San2] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. für die reine und ang. Math. (Crelle) **327** (1981), 12-80.

[Sk1] A. N. Skorobogatov, *On the fibration method for proving the Hasse principle and weak approximation*, Séminaire de théorie des nombres de Paris 1988-1989, éd. C. Goldstein, Progress in Math. **91**, Birkhäuser, Boston, Basel, Berlin, 1990, pp. 205-219.

Recent papers and books

[Bo] M. Borovoi, *The Brauer–Manin obstructions for homogeneous spaces with connected or abelian stabilizer*, J. für die reine und angew. Math. **473** (1996), 181-194.

[Bo/Ku] M. Borovoi and B. Kunyavskiĭ, *On the Hasse principle for homogeneous spaces with finite stabilizers*, Ann. Fac. Sci. Toulouse, à paraître.

[CT/SD] J.-L. Colliot-Thélène and Sir Peter Swinnerton-Dyer, *Hasse principle and weak approximation for pencils of Severi–Brauer and similar varieties*, J. für die reine und ang. Math. (Crelle) **453** (1994), 49-112.

[CT/Sk/SD1] J.-L. Colliot-Thélène, A. N. Skorobogatov and Sir Peter Swinnerton-Dyer, *Rational points and zero-cycles on fibred varieties: Schinzel's hypothesis and Salberger's device*, in preparation.

[CT/Sk/SD2] ———, *Double fibres and double covers : paucity of rational points*, Prépublication **96-44**, Université de Paris-Sud, to appear in Acta Arithmetica.

[CT/Sk/SD3] ———, *Hasse principle for pencils of curves of genus one whose Jacobians have rational 2-division points*, in preparation.

[Co/Ma] D. Coray and C. Manoil, *On large Picard groups and the Hasse principle for curves and K3 surfaces*, Acta Arithmetica **LXXVI.2** (1996), 165-189.

[Ha1] D. Harari, *Principe de Hasse et approximation faible sur certaines hypersurfaces*, Ann. Fac. Sci. Toulouse (6) **IV** (1995), 731-762.

[Ha2] ———, *Méthode des fibrations et obstruction de Manin*, Duke Math. J. **75** (1994), 221-260.

[Ha3] ———, *Obstructions de Manin "transcendantes"*, Number Theory 1993-1994, ed. S. David, Cambridge University Press, Cambridge, 1996.

[Ha4] ———, *Flèches de spécialisation en cohomologie étale et applications arithmétiques*, prépublication.

[Maz2] B. Mazur, *The Topology of Rational Points*, J. Experimental Math. **1** (1992), 35-45.

[Maz3] ———, *Speculations about the topology of rational points: an up-date*, Columbia University Number Theory Seminar 1992, Astérisque **228**, Société mathématique de France, 1995, pp. 165-181.

[Mes] J.-F. Mestre, A series of notes, C. R. Acad. Sc. Paris : **319** (1994), 529-532; **319** (1994), 1147-1149; **322** (1996), 503-505.

[Sal2] P. Salberger, unpublished manuscript (1993).

[S/W] P. Sarnak and L. Wang, *Some hypersurfaces in* $\mathbf{P}^4$ *and the Hasse-principle*, C. R. Acad. Sci. Paris **321** (1995), 319-322.

[Se1] J.-P. Serre, *Cohomologie galoisienne*, 5ème édition, Lecture Notes in Mathematics **5**, Springer-Verlag, Berlin Heidelberg New York, 1994.

[Se2] ———, *Topics in Galois Theory*, Notes written by H. Darmon, Jones and Bartlett Publishers, Boston, 1992.

[Sk2] A. N. Skorobogatov, *Descent on fibrations over the projective line*, Amer. J. Math. **118** (1996), 905-923.

[SD2] Sir Peter Swinnerton-Dyer, *Rational points on pencils of conics and on pencils of quadrics*, J. London Math. Soc. (2) **50** (1994), 231-242.

[SD3] ———, *Rational points on certain intersections of two quadrics*, Abelian Varieties, Proceedings of a conference held at Egloffstein, ed. Barth, Hulek and Lange, de Gruyter, Berlin New York, 1995, pp. 273-292.

[Wald] M. Waldschmidt, *Densité des points rationnels sur un groupe algébrique*, J. Experimental Math. **3** (1994), 329-352.

[W] L. Wang, *Brauer–Manin obstruction to weak approximation on abelian varieties*, Israel Journal of Mathematics **94** (1996), 189-200.

J.-L. Colliot-Thélène

C.N.R.S., U.R.A. D0752,
Mathématiques,
Bâtiment 425,
Université de Paris-Sud,
F-91405 Orsay
France

e–mail : colliot@math.u-psud.fr