

# Fields of cohomological dimension 1 versus $C_1$ -fields

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ABSTRACT. Ax gave examples of fields of cohomological dimension 1 which are not  $C_1$ -fields. Kato and Kuzumaki asked whether a weak form of the  $C_1$ -property holds for all fields of cohomological dimension 1 (existence of solutions in extensions of coprime degree rather than existence of a solution in the ground field). Using work of Merkur'ev and Suslin, and of Rost, D. Madore and I recently produced examples which show that the answer is in the negative. In the present note, I produce examples which require less work than the original ones. In the original paper, some of the examples were given by forms of degree 3 in 4 variables. Here, for an arbitrary prime  $p \geq 5$ , I use forms of degree  $p$  in  $p + 1$  variables.

## 1. Introduction

Let  $X$  be an algebraic variety over a field  $k$ . The index  $I(X)$  of  $X/k$  is the greatest common divisor of the degrees over  $k$  of the residue fields at closed points of  $X$ :

$$I(X) = \text{g.c.d.}_{x \in X_0} [k(x) : k]$$

This is also the greatest common divisor of the degrees of finite field extensions  $K/k$  such that the set  $X(K)$  of  $K$ -rational points of  $X$  is not empty. If  $X$  has a  $k$ -rational point, then  $I(X) = 1$ , but the converse does not generally hold.

A field  $k$  is said to be  $C_1$  if any homogeneous polynomial

$$F(x_0, x_1, \dots, x_n) \in k[x_0, x_1, \dots, x_n]$$

of degree  $d$  in  $n+1 > d$  variables has a nontrivial zero, that is, there exist  $\alpha_0, \alpha_1, \dots, \alpha_n$  in  $k$  not all of them zero, such that  $F(\alpha_0, \alpha_1, \dots, \alpha_n) = 0$ .

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The three basic examples of such fields are:

- Finite fields (Chevalley, Warning, 1935)
- Function fields in one variable over an algebraically closed field (Tsen, 1933)
- Formal power series field in one variable over an algebraically closed field (Lang, 1952)

Let  $k$  be a perfect field,  $\bar{k}$  an algebraic closure,  $\mathfrak{g}_k = \text{Gal}(\bar{k}/k)$ . A perfect field  $k$  is of (Galois) cohomological dimension  $\leq 1$  if any of the following equivalent properties hold (see [5]):

- (i) For any (continuous) finite  $\mathfrak{g}_k$ -module  $M$  and any integer  $i \geq 2$ ,  $H^i(\mathfrak{g}_k, M) = 0$ .
- (ii) For any finite field extension  $K \subset \bar{k}$ , the Brauer group  $\text{Br}(K) = H^2(\mathfrak{g}_K, \bar{k}^*) = 0$ .
- (iii) Any homogeneous space  $X/k$  under a connected linear algebraic group has a  $k$ -rational point.

If the group  $\mathfrak{g}_k$  is a pro- $p$ -group, and  $p \neq \text{char.}(k)$ , then these conditions are equivalent to the simple condition:

- (iv) The  $p$ -torsion subgroup  ${}_p\text{Br}(k)$  of  $\text{Br}(k)$  is trivial.

If a (perfect) field  $k$  is  $C_1$ , then it is of cohomological dimension at most 1. In 1965, Ax [1] showed that the converse does not hold. He produced an example of a field  $k$  which is of cohomological dimension 1, and a form  $F$  of degree 5 in 10 variables over that field with no nontrivial zero. However, the very construction of that form shows that it possesses a zero in field extensions of degrees 2, 3 and 5. If we let  $X$  be the hypersurface in 9-dimensional projective space defined by this form, we have  $I(X) = 1$ . Ax gave other examples, but they all have the property that the index of the associated hypersurface is 1.

In 1986, Kato and Kuzumaki [3] asked: If a field  $k$  is of cohomological dimension at most 1, and  $X \subset \mathbf{P}_k^n$  is a hypersurface defined by a form of degree at most  $n$ , does it follow that  $I(X) = 1$ ? In other words, is the field  $k$   $C_1$  as far as zero-cycles of degree 1 are concerned (as opposed to rational points)?

In a recent article [2], David Madore and I show that the answer to this question is in the negative. We produce a field  $k$  of cohomological dimension 1 and a cubic surface  $X \subset \mathbf{P}_k^3$  such that  $I(X) = 3$ . The geometric Picard group of a smooth cubic surface is of rank 7. This creates some technical difficulties.

In the present note, where I review the method of [2], I give new examples which are easier to discuss. The result is the following:

**THEOREM 1.1.** *For each prime  $p \geq 5$ , there exists a field  $F$  of characteristic zero, of cohomological dimension 1 and a smooth hypersurface  $X \subset \mathbf{P}_F^p$  of degree  $p$  such that  $I(X) = p$ .*

**2. How to produce fields of cohomological dimension  $\leq 1$**

Let  $k$  be a field,  $n \geq 2$  an integer. A Severi-Brauer variety of index  $n$  over  $k$  is a twisted form of projective space  $\mathbf{P}^{n-1}$ , that is, it is a  $k$ -variety which after a suitable extension  $K/k$  becomes isomorphic to  $\mathbf{P}_K^{n-1}$ .

There is a bijection, due to F. Châtelet (1944), between the set of isomorphism classes of such  $k$ -varieties and the set of isomorphism classes of central simple  $k$ -algebras of index  $n$ . In this bijection, projective space corresponds to matrix algebra; this is referred to as the trivial class. For  $n = 2$ , this bijection is induced by the well known correspondence between conics and quaternion algebras.

**THEOREM 2.1.** *Let  $k$  be a field. Let  $p$  be a prime which does not divide the characteristic of  $k$ . If one starts from  $k$  and one iterates the following two operations*

- 1) *go from a field  $K$  to the fixed field of a pro- $p$ -Sylow subgroup of the absolute Galois group of  $K$ ,*
  - 2) *go from a field  $K$  to the function field of a nontrivial Severi-Brauer variety over  $K$  of index  $p$ ,*
- then one ultimately<sup>1</sup> obtains a field  $F$  containing  $k$  whose cohomological dimension is at most 1.*

*Sketch of proof* For the field  $F$  one obtains in the limit, the operations in 1) ensure that the absolute Galois group of  $F$  is a pro- $p$ -group. To show that  $F$  is of cohomological dimension 1, it thus suffices to show that the  $p$ -torsion of the Brauer group of  $F$  is trivial. The Merkur'ev-Suslin theorem (1983) implies that for any field  $K$  of characteristic different from  $p$  containing the  $p$ -th roots of 1, the  $p$ -torsion of the Brauer group is generated by the classes of central simple algebras of index  $p$ . But the operations in 2) ensure that over the field  $F$  there is no such nontrivial central simple algebra.

**3. Galois action and the Picard group**

**PROPOSITION 3.1.** *Let  $k$  be a field,  $\bar{k}$  a separable closure of  $k$  and  $\mathfrak{g} = \text{Gal}(\bar{k}/k)$ . Let  $X/k$  be a smooth, projective, geometrically integral*

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<sup>1</sup>(Added October 1st, 2014) As Serre points out, “ultimately” does not make sense. I refer to section 2 of my paper [2] with D. Madore for a rigorous statement and proof.

variety. Write  $\bar{X} = X \times_k \bar{k}$ . Let  $k(X)$  be the function field of  $X$ . One then has a natural exact sequence

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(\bar{X})^{\mathfrak{g}} \rightarrow \text{Br}(k) \rightarrow \text{Br}(k(X)).$$

The sequence is functorial contravariant with respect to dominant  $k$ -morphisms.

Here  $\text{Pic}(X)$  is the Picard group of  $X$ . The second and fourth maps are the obvious restriction maps.

When  $X$  is a Severi-Brauer variety corresponding to a central simple algebra  $A$  of index  $n$  then  $\text{Pic}(\bar{X}) = \text{Pic}(\mathbf{P}^{n-1}) = \mathbf{Z}\mathcal{O}_{\mathbf{P}^{n-1}}(1)$  and the image of  $\mathcal{O}_{\mathbf{P}^{n-1}}(1)$  is the class of  $A$  in the Brauer group of  $k$ . The kernel of the restriction map  $\text{Br}(k) \rightarrow \text{Br}(k(X))$  is the finite cyclic group spanned by the class of  $A$ .

*Exercise* Use the above sequence and its functoriality to establish the following frequently rediscovered result:

**PROPOSITION 3.2.** *Let  $f : C \rightarrow D$  be a  $k$ -morphism of smooth, projective, geometrically integral  $k$ -curves. If  $C/k$  is a smooth conic and if the degree of  $f$  is even, then  $D(k) \neq 0$ .*

(If  $f$  is constant, the result is clear. If  $f$  is not constant, then by Lüroth's theorem,  $D$  is of genus zero, hence is a smooth conic.)

**PROPOSITION 3.3.** *Let  $X \subset \mathbf{P}_k^n$  be a smooth hypersurface. If  $n \geq 4$ , then the restriction map  $\text{Br}(k) \rightarrow \text{Br}(k(X))$  is one-to-one.*

*Proof* Indeed, it is a theorem of Max Noether that under the above assumptions the group  $\text{Pic}(\bar{X})$  is free of rank one, spanned by the class of a hyperplane section. Since such hyperplanes are defined over  $k$ , the result follows from Proposition 3.1.

*Remark* For surfaces in  $\mathbf{P}^3$ , the situation is more complicated. This accounts for the more elaborate arguments used in [2] to produce examples with cubic surfaces.

#### 4. Rost's degree formula

To any prime  $p$  and any projective irreducible variety  $X$  over a field  $k$ , Rost associates a class  $\eta_p(X) \in \mathbf{Z}/I(X)$ . This class is killed by  $p$ . If  $X$  is a nontrivial Severi-Brauer variety of dimension  $p-1$ , then  $I(X) = p$  and  $\eta_p(X) = 1 \in \mathbf{Z}/p$ .

The construction of this invariant belongs to the world of coherent modules and intersection theory. There is no Galois cohomology here.

**THEOREM 4.1.** (Rost, cf. [4]). *Let  $f : Z \rightarrow X$  be a  $k$ -morphism of proper integral  $k$ -varieties of the same dimension. Then  $I(X)$  divides  $I(Z)$  and*

$$\eta_p(Z) = \deg(f)\eta_p(X) \in \mathbf{Z}/I(X).$$

*When  $Z$  is moreover smooth, the same result holds under the mere assumption that  $f$  is a rational map from  $Z$  to  $X$ .*

This formula implies in particular that  $\eta_p(X) = 0$  for any variety which can be written as a product  $X = Y \times_k Z$  where  $Z$  is a  $k$ -variety of dimension at least one and  $I(Z) = 1$ .

*Exercise* Use Rost's degree formula to give an alternate proof of Proposition 3.2.

The formula actually yields the following more general result:

**PROPOSITION 4.2.** *If  $p$  is a prime and  $f : Z \rightarrow X$  is a dominant rational map from a Severi-Brauer variety  $Z/k$  of dimension  $p - 1$  to a projective integral  $k$ -variety  $X$ , and  $p$  divides the degree of  $f$ , then  $I(X) = 1$ . This is in particular so if the dimension of  $X$  is less than that of  $Z$ .*

## 5. The example

We can now prove Theorem 1.1. More precisely, we shall establish the following result:

**THEOREM 5.1.** *Let  $p \geq 5$  and  $l$  be distinct primes such that  $p$  divides  $l - 1$ , that is  $\mathbf{F}_l^*/\mathbf{F}_l^{*p} \neq 1$ . Let  $\alpha \in \mathbf{Z}$  be such that the class of  $\alpha$  in  $\mathbf{F}_l$  is not a  $p$ -th power. Let  $X \subset \mathbf{P}_{\mathbf{Q}}^p$  be the smooth hypersurface over  $\mathbf{Q}$  defined by the equation*

$$x_1^p + lx_2^p + \dots + l^{p-1}x_p^p - \alpha x_0^p = 0.$$

*Then there exists a field  $F$  of characteristic zero, of cohomological dimension 1, such that the index  $I(X_F)$  of the  $F$ -variety  $X_F = X \times_{\mathbf{Q}} F$  is equal to  $p$ .*

*Proof* It is clear that  $I(X) = I(X/\mathbf{Q})$  divides  $p$ . Let  $K/\mathbf{Q}_l$  be an extension of the  $l$ -adic field  $\mathbf{Q}_l$  of degree prime to  $p$ . One easily checks that  $X(K) = \emptyset$ . Thus  $p = I(X_{\mathbf{Q}_l})$ , hence also  $p = I(X)$ .

Starting from  $k = \mathbf{Q}$  (or if one wishes  $k = \mathbf{Q}_l$ ), one then applies the process described in Theorem 2.1. To achieve the announced result, one must check that under each of the changes of fields described in that theorem, the condition  $p = I(X)$  is preserved. This is obvious for the change 1), which consists in going over to the fixed field of a pro- $p$ -Sylow subgroup.

Let  $p = I(X/K)$ , and let  $Y/K$  be a nontrivial Severi-Brauer variety of dimension  $p - 1$  with associated class  $A_Y \in {}_p\text{Br}(K)$ . Assume  $I(X_{K(Y)}) = 1$ . Then there exist a projective integral  $K$ -variety  $Z$  of dimension  $p - 1$ , a dominant  $K$ -morphism  $f : Z \rightarrow Y$  of degree prime to  $p$  and a morphism  $h : Z \rightarrow X$ . Then  $I(X)$  divides  $I(Z)$  and  $p = I(Y)$  divides  $I(Z)$ . By Rost's degree formula (Theorem 4.1) we have

$$\eta_p(Z) = \deg(f)\eta_p(Y) \in \mathbf{Z}/I(Y) = \mathbf{Z}/p,$$

and

$$\eta_p(Z) = \deg(h)\eta_p(X) \in \mathbf{Z}/I(X) = \mathbf{Z}/p.$$

Since  $\eta_p(Y) = 1 \in \mathbf{Z}/p$  and the degree of  $f$  is prime to  $p$ , the first equality implies

$$\eta_p(Z) \neq 0 \in \mathbf{Z}/p.$$

The second equality then implies that  $p$  does not divide the degree of  $h$ . The restriction map of  $p$ -torsion groups  ${}_p\text{Br}(K) \rightarrow {}_p\text{Br}(K(Z))$  factorizes as  ${}_p\text{Br}(K) \rightarrow {}_p\text{Br}(K(X)) \rightarrow {}_p\text{Br}(K(Z))$ . The first map is injective by Proposition 3.3. Since the degree of  $h$  is prime to  $p$ , a corestriction-restriction argument shows that the second map is also an injection. On the other hand the factorization  ${}_p\text{Br}(K) \rightarrow {}_p\text{Br}(K(Y)) \rightarrow {}_p\text{Br}(K(Z))$  shows that the nonzero class  $A_Y \in \text{Br}(K)$  vanishes in  $\text{Br}(K(Z))$ , since it vanishes in  $\text{Br}(K(Y))$ . This contradiction shows that  $I(X_{K(Y)}) = p$ , and this completes the proof that for the field  $F$  of cohomological dimension at most 1 which one obtains in the limit, one has  $p = I(X_F)$ .

I refer to [2] for more detailed literature references and for comments on the general context in which such problems arise.

## References

- [1] J. Ax, *A field of cohomological dimension 1 which is not  $C_1$* , Bulletin Amer. Math. Soc. **71** (1965), 7–17.
- [2] J.-L. Colliot-Thélène et D. Madore, *Surfaces de Pezzo sans point rationnel sur un corps de dimension cohomologique 1*, Journal de l'Institut Mathématique de Jussieu **3** (2004), 1–16.
- [3] K. Kato and T. Kuzumaki, *The dimension of fields and algebraic K-theory*, J. Number Theory **24** (1986), 229–244.
- [4] A. S. Merkur'ev, *Rost's degree formula*, <http://www.math.ucla.edu/~merkurev/>
- [5] J.-P. Serre, *Cohomologie galoisienne*, cinquième édition, révisée et complétée, Springer Lecture Notes in Mathematics **5** (1994).

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