ARIZONA PROJECTS 2015

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Here are the three projects which have been actively discussed during the problem sessions, and which led to presentations on the 18th of March.

There were further projects, but they were not pursued.

1. Computing the Brauer group of a generic affine quadric

Let b, c, d be independent variables over $\mathbb{C}$. Let $K = \mathbb{C}(b, c, d)$. Let X/K be the affine quadric in $\mathbf{A}^3_K$ given by the equation

$$x^2 + by^2 + cz^2 + d = 0.$$

In `arXiv:1412.1240`, T. Uematsu proves $\mathrm{Br}\ (X)/\mathrm{Br}\ (K) = 0$. He uses explicit cocycle computations. Can one show $\mathrm{Br}\ (Y)/\mathrm{Br}\ (K) = 0$ without cocycle computations ?

In a previous paper, `arxiv:1412.0087`, T. Uematsu shows $\mathrm{Br}\ (X)/\mathrm{Br}\ (K) = 0$ for the generic diagonal cubic surface X in $\mathbf{P}^3_K$, given by the homogeneous equation

$$x^3 + by^3 + cz^3 + dt^3 = 0.$$

It would be nice to find a cocycle-free proof, but this seems harder.

The point is that in both cases, we have exact sequences

$$\mathrm{Br}\ (K) \to \mathrm{Br}\ (X) \to H^1(K, \mathrm{Pic}(\overline{X})) \to H^3(K, \mathbf{G}_m)$$

with $H^1(K, \mathrm{Pic}(\overline{X})) \neq 0$. Computing the arrow $H^1(K, \mathrm{Pic}(\overline{X})) \to H^3(K, \mathbf{G}_m)$ is difficult.

2. Brauer-Manin obstruction for rational points

Let k be a number field.

Let X be a smooth projective model of 9-nodal cubic threefold $Y \subset \mathbf{P}^4_k$. In the paper `arXiv:1210.5178`, Shepherd-Barron shows that $X(A_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$.

1) Read that paper.

Warning : Y. Harpaz points out that the Brauer class proposed in Proposition 4.2 is trivial. One needs to consider another class.

2) Produce an explicit, numerical, example of a 9-nodal cubic threefold Y over a number field k with a smooth projective model X satisfying $X(A_k) \neq \emptyset$ but $X(A_k)^{\mathrm{Br}} = \emptyset$. This implies producing explicit nontrivial elements in $\mathrm{Br}\ (X)/\mathrm{Br}\ (k)$.

3) Is $X(k)$ dense in $X(A_k)^{\mathrm{Br}}$?

Date: March 22nd, 2015.

3. Integral points on generalized affine Châtelet surfaces

The purpose of this project is to analyse and extend the results in the paper :

F. Gundlach, Integral Brauer-Manin obstructions for sums of two squares and a power. J. Lond. Math. Soc. (2) 88 (2013), no. 2, 599–618.

In that paper, the author studies integral solutions, in x, y, z, of the affine equation

$$y^2 + z^2 = m - x^k$$

with fixed $m \in \mathbf{Z}$ and $k \in \mathbf{Z}, k > 0$.

More generally, one could try to study the integral solutions of an eqution

$$y^2 - az^2 = P(x),$$

with $a \in \mathbf{Z}$ and $P(x) \in \mathbf{Z}[x]$.

3.1. **Geometry.** Prove :

Proposition 3.1. *Let k be a field. Let $Q(x) \in k[x]$ be a separable polynomial with $Q(0) \neq 0$. Let X be the affine k-surface defined in $\mathbf{A}^3_k$ by the equation*

$$yz = xQ(x).$$

Let $F \subset X$ be the closed subset defined by the equations $y = 0 = Q(x)$. Let$V \subset X$ be the complementary open set of F. Then

(i) The k-variety V is k-isomorphic to affine plane $\mathbf{A}^2_k$.

(ii) $k[X]^\times = k[V]^\times = k^\times$.

(iii) $Pic(V) = 0$ et $Pic(X)$ is a finitely generated torsion-free group.

(iii) Assume $\mathrm{char}(k) = 0$. *Then* $\mathrm{Br}\ (k) \overset{\simeq}{\to} \mathrm{Br}\ (V)$ *and* $\mathrm{Br}\ (k) \overset{\simeq}{\to} \mathrm{Br}\ (X)$.

Remarque 3.1. (Only for the very algebraically inclined readers) The morphism $V \to \mathbf{A}^1 k$ given by the y coordinate has all its fibres isomorphic to $\mathbf{A}^1$. The following question has been investigated by several authors : Given a flat morphism $X \to \mathbf{A}^n$ with X affine smooth, if all fibres are isomorphic to $\mathbf{A}^d$, is X isomorphic to $\mathbf{A}^{n+d}$?

3.2. **Computing Brauer groups over a field.** Prove :

Proposition 3.2. *Let k be field of char. zero, let $\overline{k}$ be an algebraic closure and $G = Gal(\overline{k}/k)$. Let $P(x) \in k[x]$ be a separable, irreducible polynomial of degree d. Assume $K = k[x]/P(x)$ is a cyclic extension of k.*

Let X be the affine k-variety defined by the equation

$$yz = P(x).$$

Then $Br(X)/Br(k) \simeq \mathbf{Z}/d$. The cyclic algebra over $k(X)$ defined by $A = (K/k, \sigma, y)$ lies in $\mathrm{Br}\ (X)$ *and it generated* $\mathrm{Br}\ (X)/\mathrm{Br}\ (k)$.

Prove :

Proposition 3.3. *Let k be field of char. zero, let $\overline{k}$ be an algebraic closure and $G = Gal(\overline{k}/k)$. Let $a \in k^{\times}$. Let $P(x) \in k[x]$ be a separable polynomial and $P(x) = \prod_i P_i(x)$ a decomposition as a product of irreducible polynomials.*

Let X be the closed affine k-variety in $\mathbf{A}^3_k$ defined by

$$y^2 - az^2 = P(x).$$

Each quaternion class $A_i = (a, P_i(x)) \in \mathrm{Br}\ (k(X))$ lies in $\mathrm{Br}\ (X)$.

Question : is $\mathrm{Br}\ (X)/\mathrm{Br}\ (k)$ spanned by the A_i's ?

3.3. **Integral points.** Prove :

Proposition 3.4. *Let k be a number field. Let $Q(x) \in k[x]$ be a separable polynomial, $Q(0) \neq 0$. Strong approximation holds for the affine k-variety defined by*

$$yz = xQ(x).$$

Give a counterexample to strong approximation for an affine k-variety given by an equation

$$yz = P(x)$$

with $P(x)$ separable, irreducible, with associated field extension $K = k[x]/P(x)$ cyclic over k, of degree $d > 1$.

Prove :

Théorème 3.5. *Let $P(x) \in \mathbf{Z}[x]$ be separable as a polynomial in $\mathbf{Q}[x]$. Assume either that the degree of P is odd or that the leading coefficient of P is positive. Let $\mathcal{X}$ be the affine $\mathbf{Z}$-scheme defined by the equation*

$$y^2 + z^2 = P(x).$$

Let $X = \mathcal{X} \times_{\mathbf{Z}} \mathbf{Q}$. Assume

$$[\prod_p \mathcal{X}(\mathbf{Z}_p)]^{\mathrm{Br}\ (X)} \neq \emptyset.$$

If we assume Schinzel's hypothesis, then $\mathcal{X}(\mathbf{Z}) \neq \emptyset$, and given an element $\{M_p\} \in [\prod_p \mathcal{X}(\mathbf{Z}_p)]^{\mathrm{Br}\ (X)}$ one may find an element of $\mathcal{X}(\mathbf{Z})$ which is as close to $\{M_p\}$ in the variable x as we wish.

Discuss whether a similar result could be obtained with $y^2 - az^2$ (with $a \in \mathbf{Z}$, $a \neq 0$) in place of $y^2 + z^2$.