STABLY RATIONAL SURFACES OVER A QUASIFINITE FIELD

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Abstract. Let k be a field and X a smooth, projective, stably k-rational surface. If X is split by a cyclic extension, for instance if the field k is finite or more generally quasifinite, then the surface X is k-rational.

Soient k un corps et X une k-surface projective, lisse, stablement k-rationnelle. Si X est déployée par une extension cyclique, par exemple si le corps k est fini ou plus généralement quasi-fini, alors la surface X est k-rationnelle.

1. Introduction

Most of the following result is well known (cf. [5, §1, §2], [6, §2.A]).

Theorem 1.1. *Let k be a field and $\overline{k}$ a separable closure of k. Let X be a smooth, projective, geometrically integral k-variety. Assume that $\overline{X} = X \times_k \overline{k}$ is $\overline{k}$-rational, i.e $\overline{k}$-birational to a projective space. Let us consider the following statements:*

(i) The k-variety X is k-rational, i.e. k-birational to a projective space $\mathbf{P}^d_k$.

(ii) The k-variety X is stably k-rational, i.e. there exists integers n, m such that $X \times_k \mathbf{P}^n_k$ is k-birational to $\mathbf{P}^m_k$.

(iii) The k-variety X is a direct factor of a k-rational variety, i.e. there exists a smooth, projective, geometrically connected k-variety Y such that $X \times_k Y$ is k-birational to a projective space.

(iv) The Galois lattice $\mathrm{Pic}(\overline{X})$ is stably permutation, i.e. there exist finitely generated permutation Galois lattices P_1 et P_2 and an isomorphism of Galois modules $\mathrm{Pic}(\overline{X}) \oplus P_1 \simeq P_2$.

(v) The Galois module $\mathrm{Pic}(\overline{X})$ is a direct factor of a permutation module, i.e. there exists a Galois module M, a finitely generated permutation Galois lattice P and an isomorphism of Galois modules $\mathrm{Pic}(\overline{X}) \oplus M \simeq P$.

(vi) For any finite separable extension of fields k'/k, one has $H^1(k', \mathrm{Pic}(\overline{X})) = 0$.

(vii) For any finite separable extension k'/k, the natural map of Brauer groups $\mathrm{Br}(k') \to \mathrm{Br}(X_{k'})$ is surjective.

Then: (i) implies (ii), which implies (iii); (ii) implies (iv); (iii) implies (v); (iv) implies (v); (v) implies (vi); (vi) implies (vii).

Proof. For (i) implies (iv), see [6, Prop. 2A1, p. 461]. That (ii) implies (iv) and (iii) implies (v) is a consequence of the computation of the Picard group of a product [5, Lemme 11]. For any Galois permutation module P and any finite separable extension k'/k, we have $H^1(k', P) = 0$. Thus (v) implies (vi). If X is a smooth, projective, geometrically integral k-variety and k'/k is a finite separable extension, one has an

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exact sequence (cf. [6, (1.5.0), p. 386]):

$$\mathrm{Br}(k') \to \mathrm{Ker}[\mathrm{Br}(X_{k'}) \to \mathrm{Br}(\overline{X})] \to H^1(k', \mathrm{Pic}(\overline{X})).$$

If moreover $\overline{X}$ is $\overline{k}$-rational, then $\mathrm{Br}(\overline{X}) = 0$. This is well known if $\mathrm{char}(k) = 0$ and this still holds for the ℓ-torsion of $\mathrm{Br}(\overline{X})$ for ℓ a prime number not equal to the characteristic exponent of k. That $\mathrm{Br}(\overline{X}) = 0$ holds for any separably closed field $\overline{k}$ if $\overline{X}$ is $\overline{k}$-rational may be seen by combining [11, Cor. 5.8] and [3, Prop. 2.1.9]. Thus (vi) implies (vii). □

Let $K \subset \overline{k}$ be a Galois subextension. By definition, a smooth, projective, geometrically rational k-surface is split by K if $X(K) \neq \emptyset$ and the natural inclusion of lattices $\mathrm{Pic}(X_K) \to \mathrm{Pic}(\overline{X})$ is an isomorphism. Under the extra hypothesis $X(k) \neq \emptyset$, this is equivalent to the assumption that $\mathrm{Gal}(\overline{k}/K)$ acts trivially on the lattice $\mathrm{Pic}(\overline{X})$.

The main result of this paper is the following theorem.

Theorem 1.2. *Let k be a field and X a smooth, projective, geometrically rational k-surface. Assume $X(k) \neq \emptyset$ and X is split by a cyclic extension of k. If X is not k-rational, then there exists a finite separable field extension k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$, and the k-variety X is not stably k-rational.*

The proof will be given in §4 (Theorem 4.1), where the results of §2 (conic bundle surfaces) and §3 (del Pezzo surfaces) are gathered. It is a case by case proof which builds in an essential manner on tables giving the Galois actions on the Picard group of del Pezzo surfaces of degree 3, 2, 1. These tables are the outcome of the work of various authors.

When looking at Theorem 1.2 one should keep in mind the following fact. Let k be a field, $\mathrm{char}(k) \neq 2$, and let $P(x) \in k[x]$ be a degree 3 separable, irreducible polynomial whose discriminant $a \in k^*$ is not a square. In [2], we prove that the k-surface given by the affine equation $y^2 - az^2 = P(x)$ is stably k-rational but is not k-rational. There thus exist such surfaces over any field k ($\mathrm{char}(k) \neq 2$) which admits a Galois field extension with Galois group the symmetric group on three letters $\mathfrak{S}_3$, for instance the rational field $\mathbb{Q}$ or the field $F = \mathbb{C}(t)$ of rational functions in one variable over the complex field.

By definition, a quasifinite field is a perfect field whose Galois closure is the procyclic group $\hat{\mathbb{Z}}$ [19, Chap. XIII, §2]. There are two classical examples of such fields: finite fields and the field of formal power series in one variable over an algebraically closed field of characteristic zero.

If k is a field of either type, then any smooth, projective, geometrically rational surface over k has a k-point (Proposition 1.6), any finite field extension K/k is cyclic, and any smooth conic over k is k-isomorphic to $\mathbf{P}^1_k$.

Theorem 1.2 and Theorem 1.1 then give the following result. For a finite field k, this answers a question raised by B. Hassett and mentioned by A. Pirutka in [18].

Theorem 1.3. *Let k be a field and X a smooth, projective, geometrically rational k-surface. Under any of the following hypotheses:*

(i) $X(k) \neq \emptyset$ and k is a quasifinite field,

(ii) k is a finite field,

(iii) $k = \mathbb{C}((t))$ is the field of formal power series in one variable over an algebraically closed field $\mathbb{C}$ of characteristic zero,

the following holds:

(a) If for any finite field extension k'/k one has $H^1(k', \mathrm{Pic}(\overline{X})) = 0$, then X is k-rational.

(b) For the surface X, conditions (i) to (vii) of Theorem 1.1 are equivalent. In particular, over a quasifinite field k, any stably k-rational k-surface is k-rational.

Corollary 1.4. *Let k be a field and X a smooth, projective, geometrically rational k-surface. In any of the following three cases:*

(i) $X(k) \neq \emptyset$ and k is a quasifinite field,

(ii) k is a finite field,

(iii) $k = \mathbb{C}((t))$ is the field of formal power series in one variable over an algebraically closed field $\mathbb{C}$ of characteristic zero,

if one of the following three hypotheses is fulfilled:

(a) the gcd of degrees of finite field extension K/k such that X_K is stably K-rational is equal to 1,

(b) X is k-unirational and the gcd of degrees of dominant generically separable rational maps from $\mathbf{P}^2_k$ to X is equal to 1,

(c) the Chow group of zero-cycles on X is universally trivial,

then the surface X is k-rational.

Proof. Let $CH_0(X)$ denote the Chow group of classes of zero–cycles on X, and let $A_0(X)$ denote the subgroup of classes of degree zero. These groups are k-birational invariants of smooth, projective, geometrically integral k-varieties. Under some restrictions, e.g. $\mathrm{char}(k) = 0$, this was proved in [4, Prop. 6.3]. Fulton's proof [9, Example 16.1.11] holds over any field.

Hypothesis (c) means: for *any* field extension F/k, the degree map

$$deg_F : CH_0(X_F) \to \mathbb{Z}$$

is an isomorphism. This holds if X is smooth, projective, geometrically integral and stably k-rational.

If $\mathrm{char}(k) = 0$, according to [4, Prop. 6.4], either of the hypotheses (a) or (b) for X implies (c). Via correspondances [9, Chap. 16] this holds over any field.

Under hypothesis (c), the Galois module $\mathrm{Pic}(\overline{X})$ is a direct factor of a permutation lattice ([10, Appendix A]). For any field extension E/k, one thus has $H^1(E, \mathrm{Pic}(\overline{X})) = 0$. Theorem 1.3 then gives the result. □

Let us recall the k-birational classification of smooth, projective, geometrically rational k-surfaces. This is due to Enriques, Manin, Iskovskikh [14], and Mori. See [15, Thm. III.2.3].

Theorem 1.5. *Let k be a field and X a smooth, projective, geometrically rational k-surface. Then there exists a k-birational morphism $X \to Y$ where Y is a smooth, projective, k-minimal surface of one of the following types :*

(i) Relatively minimal conic bundle over a smooth conic;

(ii) del Pezzo surface of degree d with $1 \leq d \leq 9$.

As noticed by Manin and the author (cf. [15, Thm. IV.6.8]), this gives a case by case proof for the following result.

Proposition 1.6. *Let k be a field and X a smooth, projective, geometrically rational k-surface. If k is a C_1-field, then $X(k) \neq \emptyset$. This holds in particular if k is a finite field or if $k = \mathbb{C}((t))$.*

For k a finite field, there is a uniform proof (A. Weil, cf. [17, Thm. 27.1, Cor. 27.1.1]).

2. Conic bundles over the projective line

Let k be a field, $\overline{k}$ a separable closure of k and $g = \mathrm{Gal}(\overline{k}/k)$. If X is a smooth, projective, geometrically connected k-surface equipped with a relatively minimal k-morphism $f : X \to \mathbf{P}^1_k$ whose generic fibre is a smooth, genus zero curve, then the closed points M with nonsmooth fibre $X_M/k(M)$ have their residue field $k(M)$ separable over k, and over a separable quadratic field extension $L(M)/k(M)$ the singular fibre $X_M/k(M)$ decomposes as the union of two lines $\mathbf{P}^1_{L(M)}$ which transversally intersect at a $k(M)$-point. Such closed points $M \in \mathbf{P}^1_k$ are referred to as the bad reduction points of the fibration $f : X \to \mathbf{P}^1_k$. For the proof of these statements, see [14]. As a consequence, over a separable closure $\overline{k}$ de k, there exists a blow-down map $\overline{X} \to Y$ over $\mathbf{P}^1_{\overline{k}}$ such that the fibres of $Y \to \mathbf{P}^1_{\overline{k}}$ are all isomorphic to $\mathbf{P}^1$. Thus the element in $\mathrm{Br}(\overline{k}(\mathbf{P}^1))$ associated to the generic fibre of $X \times_k \overline{k} \to \mathbf{P}^1_{\overline{k}}$ belongs to $\mathrm{Br}(\mathbf{P}^1_{\overline{k}}) = 0$. We have $\mathrm{Br}(\mathbf{P}^1_{\overline{k}}) \subset \mathrm{Br}(\overline{k}(\mathbf{P}^1))$, and Tsen's theorem implies that the second group is p-primary for p equal to the characteristic exponent of k. One actually has $\mathrm{Br}(\mathbf{P}^1_{\overline{k}}) = 0$ for any separably closed field $\overline{k}$ (Grothendieck [11, Cor. 5.8]).

Thus the generic fibre of $\overline{f} : \overline{X} \to \mathbf{P}^1_{\overline{k}}$ has a rational point. As $\mathbf{P}^1_{\overline{k}}$ is regular of dimension 1 and $\overline{f}$ is a proper morphism, any such point extends to a section of $\overline{f} : \overline{X} \to \mathbf{P}^1_{\overline{k}}$. The k-variety X is split over $\overline{k}$. The generic fibre of $f : X \to \mathbf{P}^1_k$ is thus associated to a class $\beta \in H^2(g, \overline{k}(\mathbf{P}^1)^*) \subset \mathrm{Br}(k(\mathbf{P}^1))$. The generic fibre of $X \to \mathbf{P}^1_k$ is a smooth curve of genus zero over $k(\mathbf{P}^1)$, hence it admits a point over a separable, degree 2 extension of $k(\mathbf{P}^1)$. The class β is thus killed by 2. At a closed, bad reduction point $M \in \mathbf{P}^1_k$, the divisor map defines a g-equivariant homomorphism

$$\overline{k}(\mathbf{P}^1)^* \to \oplus_{k(M)\subset \overline{k}}\, \mathbb{Z},$$

where $k(M) \subset \overline{k}$ runs through the k-embeddings of the separable extension $k(M)/k$ into $\overline{k}$. This homomorphism induces a residue map

$$\partial_M : H^2(g, \overline{k}(\mathbf{P}^1)^*) \to H^2(g, \oplus_{k(M)\subset \overline{k}}\, \mathbb{Z}) = H^2(k(M), \mathbb{Z}) = H^1(k(M), \mathbb{Q}/\mathbb{Z}).$$

The image of β under this map describes the quadratic extension of $k(M)$ corresponding to the bad fibre. For $k \subset L \subset \overline{k}$, with L/k finite, we have the following commutative diagram:

$$\begin{array}{ccc} H^2(g_k, \overline{k}(\mathbf{P}^1)^*) & \to & H^1(k(M), \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^2(g_L, \overline{k}(\mathbf{P}^1)^*) & \to & \oplus_{N\to M} H^1(k(N), \mathbb{Q}/\mathbb{Z}), \end{array}$$

where N runs through the closed points of $\mathbf{P}^1_L$ with image M under the projection map $\mathbf{P}^1_L \to \mathbf{P}^1_k$, the horizontal maps are the above defined residue maps, and the vertical maps are restriction maps.

Lemma 2.1. *Let $f : X \to \mathbf{P}^1_k$ be a relatively minimal conic bundle over a field k, with smooth generic fibre and smooth total space X/k. Let $M \in \mathbf{P}^1_k$ be a closed point with residue field $k(M)$ where the fibration f has bad reduction. Let $K = k(M)$ denote the residue field at M. Let N be a K-point of $\mathbf{P}^1_K$ over $M \in \mathbf{P}^1_k$. The fibration $f_K : X_K \to \mathbf{P}^1_K$ has bad reduction at N.*

Proof. Let $\beta \in H^2(g_k, \overline{k}(\mathbf{P}^1)^*)$ be associated to the generic fibre of $f : X \to \mathbf{P}^1_k$. The fibration f has bad reduction at M if and only if the residue

$$\gamma := \partial_M(\beta) \in H^1(k(M), \mathbb{Q}/\mathbb{Z}) = H^1(K, \mathbb{Q}/\mathbb{Z})$$

is nontrivial. By the above diagram,

$$\partial_N(\beta_K) = \gamma \in H^1(K(N), \mathbb{Q}/\mathbb{Z}) = H^1(K, \mathbb{Q}/\mathbb{Z}),$$

hence the result. □

Iskovskikh proved the following result ([12, Thm. 2], [14, Thm. 4, Thm. 5]).

Proposition 2.2. *Let k be a field and X/k a smooth, projective, geometrically connected surface over k, equipped with a relatively minimal conic bundle structure $X \to \mathbf{P}^1_k$. If the number of geometric degenerate fibres is at most 3, and if X has a k-rational point, then X is a k-rational surface.*

Proposition 2.3. *Let k be a field and X/k a smooth, projective, geometrically connected surface over k, equipped with a relatively minimal conic bundle structure $X \to \mathbf{P}^1_k$. Assume that X is split over a cyclic field extension K/k. If the number of singular geometric fibres of the fibration is at least 4, then there exists a finite separable extension k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.*

Proof. Let us recall a few facts on the Galois lattices defined by the geometric Picard group of a surface with a conic bundle structure over $\mathbf{P}^1_k$. For more details, see [7, §2].

Let as above $\overline{k}$ be a separable closure of k and let $\overline{X} = X \times_k \overline{k}$. There is an exact sequence of Galois modules

$$0 \to P \to \mathbb{Z}.f \oplus Q \to \mathrm{Pic}(\overline{X}) \to \mathbb{Z} \to 0,$$

where P is the permutation modules on $\overline{k}$-points of $\mathbf{P}^1$ with singular fibre, Q is the permutation module on components of the singular fibres over $\overline{k}$, and $\mathbb{Z}.f$ is spanned by the fibre over a k-point of $\mathbf{P}^1_k$. The map $\mathrm{Pic}(\overline{X}) \to \mathbb{Z}$ is induced by restriction to the generic fibre. [1]

Let M denote the kernel of this restriction map. There are short exact sequences of Galois modules

$$0 \to P \to \mathbb{Z} \oplus Q \to M \to 0$$

and

$$0 \to M \to \mathrm{Pic}(\overline{X}) \to \mathbb{Z} \to 0.$$

Galois cohomology then yields the exact sequences

$$0 \to \mathbb{Z}/2 \to H^1(k, M) \to H^1(k, \mathrm{Pic}(\overline{X})) \to 0$$

and

$$0 \to H^1(k, M) \to H^2(k, P) \to H^2(k, \mathbb{Z} \oplus Q).$$

This last sequence gives rise to an exact sequence

$$0 \to H^1(k, M) \to \oplus_{i=1}^r \mathbb{Z}/2 \to H^1(k, \mathbb{Z}/2),$$

[1](added in the English translation) By hypothesis, X is split by a cyclic extension K/k. This implies that for each bad reduction point $P \in \mathbf{P}^1_k$, the residue field $k(P)$ is a (cyclic) subfield of K and that the above $\mathrm{Gal}(\overline{k}/k)$-modules are $\mathrm{Gal}(K/k)$-modules.

where i runs through the $r \geq 1$ closed points P_i of $\mathbf{P}^1_k$ with singular fibre, split by a separable quadratic field extension, of class $a_i \in H^1(k(P_i), \mathbb{Z}/2)$, and the map $\theta_i : \mathbb{Z}/2 \to H^1(k, \mathbb{Z}/2)$ sends 1 to the norm (from $k(P_i)$ to k) of $a_i \in H^1(k(P_i), \mathbb{Z}/2)$.

Furthermore, there is a reciprocity relation [7, §2, Remark] which implies that the image of $(1, \dots, 1) \in \oplus_i \mathbb{Z}/2$ is the trivial class in $H^1(k, \mathbb{Z}/2)$.

We want to show: if the number of degenerate geometric fibres of $X \to \mathbf{P}^1_k$ is at least equal to 4, then there exists a separable finite field extension k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

If E/k is a separable finite field extension of odd degree, then over E the family $X_E \to \mathbf{P}^1_E$ is still relatively minimal: no residue vanishes. We may thus assume that the degree of any closed point of bad reduction is either 1 or is of even degree.

Assume there exists a closed point P of bad reduction of even degree at least 4. Then by Lemma 2.1, after going over from k to the Galois subextension $L = k(P)$ of the cyclic extension K/k, we reduce to the situation where we have at least 4 rational points P_1, P_2, P_3, P_4 with singular fibre. Since the surface is split by a cyclic extension, the nontrivial classes $a_i \in H^1(L(P_i), \mathbb{Z}/2) = H^1(L, \mathbb{Z}/2)$ all coincide with a unique nontrivial class $a \in H^1(L, \mathbb{Z}/2)$.

The map $\oplus_{i=1}^r \mathbb{Z}/2 \to H^1(L, \mathbb{Z}/2)$ induces a map $(\mathbb{Z}/2)^4 \to H^1(L, \mathbb{Z}/2)$ which thus factorizes through $(\mathbb{Z}/2)^4 \to \mathbb{Z}/2$. The group $H^1(L, M) = \mathrm{Ker}[\oplus_{i=1}^r \mathbb{Z}/2 \to H^1(L, \mathbb{Z}/2)]$ thus contains the kernel of a map $(\mathbb{Z}/2)^4 \to \mathbb{Z}/2$, it is or order at least 8, thus $H^1(L, \mathrm{Pic}(\overline{X}))$ has order at least 4.

Let us now assume that all bad reduction closed points P_i have degree 2 or 1 over k. If there are at least 4 closed points of degree 1 with bad reduction, the above argument gives $H^1(k, \mathrm{Pic}(\overline{X})) \neq 0$.

Suppose there are at least two bad reduction closed points P_1, P_2 of degree 2. Since X is split by a cyclic extension K/k which contains all $k(P_i)$, the field $k(P_1)$ and $k(P_2)$ coincide with the unique quadratic subfield extension L/k of K/k. Applying Lemma 2.1, one sees that $X_L \to \mathbf{P}^1_L$ has singular fibres over at least 4 L-rational points, and we conclude $H^1(L, \mathrm{Pic}(\overline{X})) \neq 0$.

To prove the proposition, we are reduced to considering the cases where the set of degrees of closed points with bad reduction is either $(1, 1, 1, 2)$ or $(1, 1, 2)$.

Let us consider the case $(1, 1, 1, 2)$. Since X is split by a cyclic extension, the quadratic extensions associated to $a_i \in H^1(k(P_i), \mathbb{Z}/2)$ for k-rational points P_i are all equal a nontrivial class $a \in H^1(k, \mathbb{Z}/2)$. Let R be the degree 2 closed point, and let $b \in H^1(k(R), \mathbb{Z}/2)$ be the residue at that point. The map $\oplus_{i=1}^3 \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to H^1(k, \mathbb{Z}/2)$ send (x, y, z, t) to $(x + y + z).a + t.\mathrm{Norm}_{k(R)/k}(b) \in H^1(k, \mathbb{Z}/2)$. By reciprocity, the class $(1, 1, 1, 1)$ has trivial image. Thus $3a + \mathrm{Norm}_{k(R)/k}(b)$ is trivial in $H^1(k, \mathbb{Z}/2)$. This implies $a = \mathrm{Norm}_{k(R)/k}(b)$, and this element is nontrivial in $H^1(k, \mathbb{Z}/2)$. The map $\oplus_{i=1}^3 \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to H^1(k, \mathbb{Z}/2)$ thus sends (x, y, z, t) to $(x+y+z+t).a$ in $H^1(k, \mathbb{Z}/2)$. Is is thus the sum $(\mathbb{Z}/2)^4 \to \mathbb{Z}/2$. Its kernel is $(\mathbb{Z}/2)^3$, hence $H^1(k, \mathrm{Pic}(\overline{X})) = (\mathbb{Z}/2)^2$.

Let us show that $(1, 1, 2)$ does not occur. Just as above the quadratic extensions associated to $a_i \in H^1(k(P_i), \mathbb{Z}/2)$ for the k-rational points P_i are all equal to a unique nontrivial class $a \in H^1(k, \mathbb{Z}/2)$. Let Q be the closed degree 2 point and let $b \in H^1(k(Q), \mathbb{Z}/2)$ be the residue at that point. This corresponds to a separable quadratic field extension $L/k(Q)$. Under our hypotheses, the field extension L/k is cyclic with Galois group $\mathbb{Z}/4$. Under this hypothesis, one checks that the norm map $H^1(k(Q), \mathbb{Z}/2) \to H^1(k, \mathbb{Z}/2)$ sends the class b to the class a. On the other hand, by reciprocity, the class $(1,1,1)$ has trivial image under the map $\mathbb{Z}/2 \oplus \oplus_{i=1}^{2} \mathbb{Z}/2 \to H^1(k, \mathbb{Z}/2)$. Thus $3a = a \in H^1(k, \mathbb{Z}/2)$ vanishes, and this is a contradiction. □

3. del Pezzo surfaces

The following result is well known (Châtelet, Manin, Iskovskikh, see [23, Thm. 2.1]):

Proposition 3.1. *Let X be a del Pezzo surface of degree $d \geq 5$ over a field k. If X has a k-rational point, then X is k-rational.*

Proposition 3.2. *Let k be a field and $X \subset \mathbf{P}^4_k$ be a del Pezzo surface of degree 4 over k. Assume that it is k-minimal and that it is split by a cyclic exension K/k. Then:*

(i) There exists a field extension E/k such that $H^1(E, \mathrm{Pic}(\overline{X})) \neq 0$.[2]

(ii) The Galois module $\mathrm{Pic}(\overline{X})$ is not a direct factor of a permutation lattice.

(iii) If X has a k-rational point, then there exists a finite separable field extension k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Proof. Since the del Pezzo surface X of degree 4 is k-minimal, it is not k-rational [13, Thm. 2].

If X has a k-point P which is not located on any of the 16 (geometric) exceptional lines, by blowing up P one produces a smooth cubic surface Y over k equipped with a conic bundle structure $Y \to \mathbf{P}^1_k$. This fibration has 5 geometric degenerate fibres. Under the hypothesis that X is split over a cyclic extension of k, the components of the bad fibres are all defined over the cyclic extension K/k. If this fibration is not relatively minimal, let $Z \to \mathbf{P}^1_k$ be a relatively minimal model. Let s be the number of geometric degenerate fibres. If we had $s \leq 3$, then according to Proposition 2.2, Y and hence X would be k-rational. Thus $s \geq 4$. Proposition 2.3 then gives the existence of a finite separable field extension k'/k such that $H^1(k', \mathrm{Pic}(\overline{Z})) \neq 0$. One goes over from X to Z by blow-ups at separable closed points. The Galois modules $\mathrm{Pic}(\overline{X})$ and $\mathrm{Pic}(\overline{Z})$ are thus isomorphic up to addition of permutation lattices. Thus $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

If X has a k-point and the field k has at least 23 elements, then there exists a k-point on X outside the 16 lines [17, Chap. 4, §8, Teor. 8.1= Thm. 30.1]. Assume that k is finite and X is split over the cyclic extension K/k. There exists a finite extension L/k which is linearly dispoint from K over which X has an L-rational points outside the 16 lines.

The degree 4 del Pezzo $X \times_k L$ over L is L-minimal and is split by the cyclic extension $K.L/L$. The above argument then produces a finite field extension k' of L such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

This completes the proof of (iii).

To prove (i), we use the trick of going over to the generic point (cf. [6, Thm. 2.B.1]). Let $F = k(X)$ be the field of rational functions of X. Since k is algebraically closed in $F = k(X)$, and X is k-minimal, X_F is F-minimal. The Galois module $\mathrm{Pic}(\overline{X})$ does not change by going over from k to F, it is split by the extension K/k and by the extension F'/F, where $F' := F.K$. The generic point of X gives an F-point of X_F which is not located on the (geometric) lines of X_F. The F-minimal surface X_F is split by the cyclic extension F'/F. By (iii), there exists a finite separable extension E/F such that $H^1(E, \mathrm{Pic}(\overline{X})) \neq 0$, which gives (i).

[2] In this statement, E/k need not be a finite field extension.

This implies that the Galois lattice $\mathrm{Pic}(\overline{X})$ is not a direct factor of a permutation lattice (Theorem 1.1), which gives (ii). $\square$

We now turn to del Pezzo surfaces of degree 3, 2 et 1, over a field k, which are split by a finite, cyclic field extension K/k.

We let $Frob$ denote a generator of the cyclic group $\mathrm{Gal}(K/k)$. The possible actions of the finite cyclic group $\mathrm{Gal}(K/k)$ on the group $\mathrm{Pic}(X_K) = \mathrm{Pic}(\overline{X})$ were listed by Frame [8], then Swinnerton-Dyer [21]. They were amended by Manin [17, Chapitre IV], and further amended and completed by Urabe [22] and more recently by Banwait, Fité and Loughran [1].

In [17, Chap. IV, Table I, Column 5] and in [1, Table 7.1, Column 5], a surface is attributed the symbol $\prod_m m^{n_m}$, with all $n_m \geq 0$, if, for given $m \geq 1$, the Galois invariant set of primitive m-th roots of unity among the eigenvalues of $Frob$ has n_m elements. In other words, one decomposes the characteristic polynomial of $Frob$ acting on $\mathrm{Pic}(\overline{X}) \otimes_{\mathbb{Z}} \mathbf{C}$ by grouping together the orbits of roots under the action of the Galois group. This symbol is called the characteristic symbol of $Frob$ (for its action on $\mathrm{Pic}(\overline{X}) \otimes_{\mathbb{Z}} \mathbf{C}$).

Urabe [22, Supplement] uses the Frame symbol [8]. The Frame symbol $\prod_m m^{n_m}$, with $n_m \in \mathbb{Z}$, corresponds to a rewriting of the characteristic polynomial of $Frob$ for its action on $\mathrm{Pic}(\overline{X}) \otimes_{\mathbb{Z}} \mathbf{C}$. Namely, one writes it as a product $\prod_m (t^m - 1)^{n_m}$, with $n_m \in \mathbb{Z}$. There is a unique way to write the characteristic polynomial as such a product, with distinct integers $m > 0$, and with $n_m \neq 0$. Let $r > 1$. To compute the Frame symbol of $Frob^r$, in the product $\prod_m (t^m - 1)^{n_m}$ attached to the Frame symbol $\prod_m m^{n_m}$ of $Frob$, for each integer m one writes $r = uv$ and $m = uw$ (u, v, w positive integers) with $(v, w) = 1$, and one replaces $(t^m - 1)$ by $(t^w - 1)^u$, then one gathers the terms together.

In the tables of [22] et [1], the symbols (of either type) associated to different surfaces may coincide but this seldom happens.

K. Shramov showed me the next proposition.

Proposition 3.3. *(A. Trepalin) Let X be a smooth cubic surface over a field k. Assume that is is k-minimal and is split by a cyclic extension K/k. Then there exists a finite separable extension of fields k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.*

Proof. We use Table 7.1 in the paper [1]. The actions corresponding to k-minimal surfaces, i.e. surfaces of index $i(X) = 0$, are those numbered 1, 2, 3, 4, 5 in Table 7.1 of [1]. For numbers 3 et 5, we have $H^1(k, \mathrm{Pic}(\overline{X})) \neq 0$. For the other ones, one has $H^1(k, \mathrm{Pic}(\overline{X})) = 0$.

For number 1, the eigenvalues of $Frob$ are $1, 3^2, 12^4$, that is 1, the two primitive cubic root of 1 and the 4 primitive 12-th roots of 1. If one replaces $Frob$ by $Frob^4$, i.e. if one goes over to the subfield extension k'/k of K/k of degree 4, the eigenvalues of $Frob^4$ are $1, 3^6$. In the table, only number 3 has these eigenvalues, and at this level the table gives $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

For number 2, the eigenvalues of $Frob$ are $1, 3^2, 6^4$. If one replaces $Frob$ by $Frob^2$, i.e. if one goes over to the subfield extension k'/k of K/k of degree 2, the eigenvalues of $Frob^2$ are $1, 3^6$. In the table, only number 3 has these eigenvalues, and the table gives $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

For number 4, the eigenvalues of $Frob$ are $1, 9^6$. If one replaces $Frob$ by $Frob^3$, i.e. if one goes over to the subfield extension k'/k of K/k of degree 3, the eigenvalues of $Frob^3$ are $1, 3^6$. In the table, only number 3 has these eigenvalues, and the table gives $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$. □

Proposition 3.4. *Let X be a del Pezzo of degree 2 over a field k. Assume it is k-minimal and split by a cyclic extension K/k. Then there exists a finite separable extension of fields k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.*

Proof. Here we use Table 1 in Urabe's paper [22], and we use Frame symbols.

Surfaces which are k-minimal have the numbers 1 to 19. As Daniel Loughran pointed out to me, number 1 of Table 1 contains a mistake. Its index is at least 2, the surface is not k-minimal. We thus only discuss cases numbered 2 to 19.

For surfaces with $H^1(k, \mathrm{Pic}(\overline{X})) \neq 0$ there is nothing to do.

In each of the following cases, we consider a power $Frob^r$ of $Frob$ and we denote k' the fixed field of $Frob^r$.

Case 5. Taking $Frob^5$, one finds $1^{-4}.2^6$ as new Frame symbol. The only possibility is case 2. If k' is the fixed field of $Frob^5$, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 6. Taking $Frob^2$, one finds 4^2. The only possibility is case 3, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 7. Taking $Frob^3$, one finds $1^{-4}.2^6$. The only possibility is case 2, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 15. Taking $Frob^9$, one finds $1^{-4}.2^6$. The only possibility is case 2, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 16. Taking $Frob^7$, one finds $1^{-6}.2^7$. The only possibility is case 8, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 17. Taking $Frob^3$, one finds $1^{-2}.2^1.4^2$. The only possibility is case 9, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 18. Taking $Frob^3$, one finds $1^{-1}.2^2.5^{-1}.10^1$. The only possibility is case 13, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 19. Taking $Frob^3$, one finds $1^{-6}.2^7$. The only possibility is case 8, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$. □

Remark 3.5. In [16], the author points out three types of degree 2 del Pezzo surface in Urabe's Table 1 which would have all their $H^1(k', \mathrm{Pic}(\overline{X}))$ for varying k' trivial. These are the types 1, 5 and 16. In case 1, we have seen that the surface is not k-minimal. For the two other types, presumably there is a computational mistake in [16].

Proposition 3.6. *Let X be a del Pezzo of degree 1 over a field k. Assume it is k-minimal and split by a cyclic extension K/k. Then there exists a finite separable extension of fields k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.*

Proof. Here we use Urabe's Table 2 in [22]. We only consider surfaces of index 0. If $H^1(k, \mathrm{Pic}(\overline{X})) \neq 0$, then the conclusion is clear. If not, then we are in one of the following cases, for which we use Frame symbols. We consider a power $Frob^r$ of $Frob$ and we denote k' the fixed field of $Frob^r$.

Case 5. Taking $Frob^3$, one finds $1^1.2^{-2}.4^3$ as new Frame symbol. The only possibility is case 3, on a $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 6. Taking $Frob^5$, one finds $1^{-3}.2^4.4^1$. The only possibility is case 1, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 7.Taking $Frob^3$, one finds $1^{-1}.2^3.4^{-1}.8^1$. The only possibility is case 4, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 29.Taking $Frob^{10}$, one finds $1^{-3}.3^4$. The only possibility is case 9, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 30.Taking $Frob^3$, one finds $1^1.4^{-2}.8^2$. The only possibility is case 23, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 31.Taking $Frob^5$, one finds $1^1.2^{-4}.4^4$. The only possibility is case 17, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 32.Taking $Frob^3$, one finds $1^1.2^{-4}.4^4$. The only possibility is case 17, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 33.Taking $Frob^2$, one finds 9^1. The only possibility is case 14, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 34.Taking $Frob^3$, one finds $1^{-1}.5^2$. The only possibility is case 11, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 35.Taking $Frob^2$, one finds $1^{-1}.5^2$. The only possibility is case 11, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 36.Taking $Frob^3$, one finds $1^{-3}.2^2.4^2$. The only possibility is case 10, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

Case 37.Taking $Frob^2$, one finds $1^{-3}.3^4$. The only possibility is case 9, one has $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$. □

Remark 3.7. As a referee points out, in Propositions 3.4 and 3.6 a number of cases may be reduced to the case of conic bundle surfaces, already handled in Proposition 2.3.

Remark 3.8. The statement of Theorem 5.4.3 of [16] is identical to the above statement. One needs however to correct the proof given in [16] for case 6, since the choice there made of $Frob^4$ leads, as the author writes, to surface number 102, but this surface has $H^1(k, \mathrm{Pic}(\overline{X})) = 0$.

4. Conclusion

Let us gather our results together.

Theorem 4.1. *Let k be a field and X a smooth, projective, geometrically rational k-surface. Assume that X has a k-point and is split over a cyclic extension of k. If X is not k-rational, then:*

(i) There exists a finite separable field extension k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

(ii) The Galois module $\mathrm{Pic}(\overline{X})$ is not a direct factor of a permutation lattice.

(iii) The k-variety X is not stably k-rational.

Proof. One may assume that X is k-minimal. Indeed, if $f : Y \to X$ is a k-birational *k-morphism* between geometrically rational, smooth, projective k-surfaces, if Y is split by a cyclic extension of k, so is X. According to Theorem 1.5, one may further assume that X either is a k-minimal del Pezzo surface of degree d with $1 \leq d \leq 9$ or admits a relatively minimal conic bundle fibration $X \to \mathbf{P}^1_k$.

If X is a del Pezzo of degree d with $5 \leq d \leq 9$, then X is k-rational according to Propositions 1.6 and 3.1.

If X is a k-minimal del Pezzo surface of degree $d = 4$, resp. $d = 3$, resp. $d = 2$, resp. $d = 1$, then according to Proposition 3.2, resp. 3.3, resp. 3.4, resp. 3.6, there exists a finite separable field extension k'/k with $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

If X is a relatively minimal conic bundle $X \to \mathbf{P}^1_k$, then Propositions 2.2 and 2.3 ensure that X either is k-rational, or there exists a finite separable field extension k'/k such that $H^1(k', \mathrm{Pic}(\overline{X})) \neq 0$.

This gives (i). The other statements follow (Theorem 1.1). □

Remark 4.2. One would wish for a proof of Theorem 4.1 which avoided the case by case analysis used in this paper, in particular which avoided the use of tables for the Galois action on geometric Picard groups of del Pezzo surfaces of degree 1,2,3. In this remark, I freely use notions from [6]. Let k be a quasifinite field and let X be a smooth, projective, geometrically rational k-surface with a k-point. As the absolute Galois group of k is a procyclic group, under the hypothesis $H^1(k', \mathrm{Pic}(\overline{X})) = 0$ for any finite extension k'/k, a theorem of Endo and Miyata [5, Prop. 2, p. 184] implies that the Galois lattice $\mathrm{Pic}(\overline{X})$ is a direct factor of a permutation lattice. Let S be the k-torus whose character group is the Galois lattice $\mathrm{Pic}(\overline{X})$. Is is then a direct factor of a quasitrivial torus. Since the cohomological dimension of k is 1, up to isomorphism there exists a unique universal torsor $\mathcal{T} \to X$ over X. It is a torsor under the k-torus S. By Hilbert's theorem 90, any torsor under a direct factor of a quasitrivial k-torus is generically split. Thus the k-variety $\mathcal{T}$ is k-birational to the product $X \times_k S$. Over any field k, it is an open question whether the underlying k-variety of a universal torsor over any smooth, projective, geometrically rational k-surface X, with a k-point, is a (stably) k-rational variety. If that were the case, then for k quasifinite the above argument would show that under the hypothesis $X(k) \neq \emptyset$ and $H^1(k', \mathrm{Pic}(\overline{X})) = 0$ for any finite extension k'/k, the k-variety X is a direct factor of a k-rational variety.

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