Essential hyperbolicity versus homoclinic bifurcations

Partial Hyperbolicity at IM-UFRJ, Rio 2011

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joint work with E. Pujals, ArXiv:1011.3836

Caracterize non-hyperbolicity

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- Generate large sets of non-hyperbolic systems.

More generally: split the dynamics through dichotomies phenomenon/mechanisms.

Hyperbolic diffeomorphisms: definition

M: compact boundaryless manifold.

Definition

$f \in \mathrm{Diff}(M)$ is *hyperbolic* if there exists $K_0, \dots, K_d \subset M$ s.t.:

- each K_i is a hyperbolic invariant compact set

$$T_K M = E^s \oplus E^u,$$

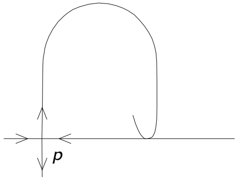
- for any $x \in M \setminus (\bigcup_i K_i)$, there exists $U \subset M$ open such that

$$f(\overline{U}) \subset U \text{ and } x \in U \setminus f(\overline{U}).$$

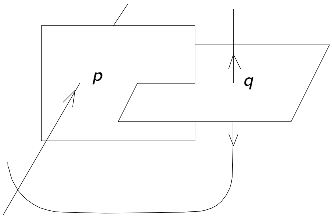
(Equivalent to "Axiom A $+$ no cycle condition".)

Obstructions to hyperbolicity

Homoclinic tangency associated to a hyperbolic periodic point p.



Heterodimensional cycle associated to two hyperbolic periodic points p, q such that $\dim(E^s(p)) \neq \dim(E^s(q))$.



Hyperbolicity conjecture

Conjecture (Palis)

Any $f \in \mathrm{Diff}(M)$ *can be approximated by a hyperbolic diffeomorphism or by a diffeomorphism exhibiting a homoclinic bifurcation (tangency or cycle).*

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In higher dimensions, we consider the C^1-topology.

Theorem (Pujals-Sambarino)

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Remark. The conjecture also holds in the conservative setting.

Essential hyperbolicity far from homoclinic bifurcations

Theorem (Pujals, C-)

Any generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is essentially hyperbolic.*

Essential hyperbolicity far from homoclinic bifurcations

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Definition of essential hyperbolicity. There exist hyperbolic attractors $A_1, \dots, A_k$ and repellors $R_1 \dots, R_\ell$ s.t.:

- the union of the basins of the A_i is (open and) dense in M,
- the union of the basins of the R_i is (open and) dense in M,

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Remarks.

- The set of these diffeomorphisms is not open apriori.
- In the setting of the theorem, the dynamics outside the basins is partially hyperbolic.

Program of the lectures

Goal. *Any generic* $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is essentially hyperbolic.*

- *Lecture 1.* Overview of the proof.
  Finiteness of attractors.
- *Lecture 2.* Classes of the dynamics.
  Chain-hyperbolicity, strong laminations.
- *Lecture 3.* Non-hyperbolic attractors.
  Perturbation and creation of strong connections.

Decomposition of the dynamics / quasi-attractors

The *chain-recurrent set* $\mathcal{R}(f)$: the set of $x \in M$ s.t. for any $\varepsilon > 0$, there exists a ε-pseudo-orbit $x = x_0, x_1, \dots, x_n = x$, $n \geq 1$.

The *chain-recurrence classes*: the equivalence classes of the relation "for any $\varepsilon > 0$, there is a periodic ε-pseudo-orbit containing x, y".

- This gives a partition of $\mathcal{R}(f)$ into compact invariant subsets.

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A *quasi-attractor* is a chain-recurrence class having a basis of neighborhoods U which satisfy $f(\overline{U}) \subset U$.

- There always exist quasi-attractors.

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- There always exist quasi-attractors.

For $f \in \mathrm{Diff}^1(M)$ generic:
1) Any chain-recurrence class which contains a periodic orbit O coincides with the *homoclinic class* $H(O) := \overline{W^s(O) \pitchfork W^u(O)}$.
(The other chain-recurrence classes are called *aperiodic classes*.)

2) The union of the basins of the quasi-attractors is dense in M.

Partial hyperbolicity far from homoclinic tangencies...

Theorem (C-, Sambarino, D. Yang)

Any generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$ *is partially hyperbolic:*

- *For aperiodic classes,* $TM = E^s \oplus E^c \oplus E^u$ *with* $\dim(E^c) = 1$ *and* $\dim(E^s), \dim(E^u) \geq 1$*.*
- *For homoclinic classes,* $TM = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_\ell \oplus E^u$*.*
  For each i *one has* $\dim(E^c_i) = 1$ *and the class has periodic points with Lyapunov exponent along* E^c_i *arbitrarily close to* 0*.*

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- *For homoclinic classes,* $TM = E^s \oplus E^c_1 \oplus \dots \oplus E^c_\ell \oplus E^u$. *For each* i *one has* $\dim(E^c_i) = 1$ *and the class has periodic points with Lyapunov exponent along* E^c_i *arbitrarily close to* 0.

... and far from heterodimensional cycles

If moreover $f \notin \overline{\mathrm{Cycle}}$, *then for each homoclinic class,*

- *each central bundle* E^c_i *is thin-trapped by* f *or* f^{-1},
- *there are at most two central bundles.*

The class is *chain-hyperbolic*. *(Cf. the second lecture.)*

Strong connexions

Let $H(O)$ be a homoclinic class with a splitting $TM = E^{cs} \oplus E^{cu} = (E^{ss} \oplus E^{c}) \oplus E^{cu}$, where $\dim(E^{c}) = 1$.

How does the strong stable lamination intersect the class?

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Theorem (Bonatti, C-)

If $W^{ss}(x) \cap H(O) = \{x\}$ for any $x \in H(O)$, then $H(O)$ is contained in a (loc. invariant) submanifold tangent to $E^{c} \oplus E^{cu}$.

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Definition. If $W^{ss}(x) \cap H(O) \neq \{x\}$ for some $x \in H(O)$ that is periodic and belongs to a transitive hyperbolic set containing O, one says that $H(O)$ has a strong connection.

- if one can choose K with a weak stable exponent, then there exists a C^1-perturbation of f with a heterodimensional cycle.

Topology inside center-stable leaves

Consider $H(O)$ a homoclinic class satisfying:

- there is a splitting $TM = (E^{ss} \oplus E^c) \oplus E^{cu}$, $\dim(E^c) = 1$,
- $E^{cs} = E^{ss} \oplus E^c$ and E^{cu} are thin trapped by f and f^{-1} resp.

Theorem (C-, Pujals)

If $H(O)$ has no strong connection and is not contained in a (loc. invariant) submanifold tangent to $E^c \oplus E^{cu}$, then the intersection of $H(O)$ with the center-stable plaques is totally disconnected.

Remark. This applies to hyperbolic sets.

Extremal bundles

Consider $H(O)$ with a splitting $TM = E^{cs} \oplus E^{cu}$ s.t.

- E^{cs}, E^{cu} are thin trapped by f and f^{-1} respectively,
- $\dim(E^{cu}) = 1$.

Theorem (C-, Pujals, Sambarino)

If f is C^1-generic and one of the following cases holds:

- $\dim(E^{cs}) = 1$,
- *E^{cs} is uniformly contracted,*
- *inside the center-stables plaques, $H(O)$ is totally disconnected,*

then E^{cu} is uniformly expanded.

Extremal bundles : corollaries

Corollary. For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$:

- Any homoclinic class $H(O)$ is either a sink/source or a part. hyperbolic set with non-degenenerated bundles E^s, E^u.
- If $H(O)$ has a non-uniform bundle $E^{cs} = E^s \oplus E^c$, then there exists $x \neq y$ in $H(O)$ such that $W^{ss}(x) = W^{ss}(y)$.
- The number of sinks/sources is finite.

Finiteness of attractors

Proposition

For a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$, *the set of non-trivial quasi-attractors is finite.*

Conclusion

We have seen that generically in $\text{Diff}^1(M) \setminus \overline{\text{Tangency} \cup \text{Cycle}}$,

- the union of the basins of the quasi-attractors is dense in M,
- quasi-attractors are finite.

It remain to prove that quasi-attractors are hyperbolic.

- *Lecture 2.* Quasi attractors (in fact all classes) are chain-hyperbolic and have nice properties.
- *Lecture 3.* One can perturb non-hyperbolic quasi-attractors to create a heterodimensional cycle.

Essential hyperbolicity versus homoclinic bifurcations (2)

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Partial hyperbolicity far from homoclinic bifurcations

Recall. Generically the dynamics splits into homoclinic and aperiodic classes.

Theorem 1

Any generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is part. hyperbolic:*

- *For aperiodic classes,* $TM = E^s \oplus E^c \oplus E^u$ *with* $\dim(E^c) = 1$ *and* $\dim(E^s), \dim(E^u) \geq 1$.
- *For homoclinic classes,* $TM = E^{cs} \oplus E^{cu}$*, where* E^{cs} *and* E^{cu} *are thin trapped by* f *and* f^{-1} *respectively.*

  If E^{cs} *is not uniformly contracted, then* $E^{cs} = E^s \oplus E^c$ *s.t.*
  - $\dim(E^c) = 1$ *and* E^s *is uniformly contracted,*
  - *the class has periodic points with Lyapunov exponent along* E^c *arbitrarily close to* 0.

Topological dynamics along invariant bundle

K an inv. compact set with a dom. splitting $TM = E^{cs} \oplus E^{cu}$.

Definition. A trapped plaque families tangent to E^{cs} is a continuous family of embedded plaques $\mathcal{D}_x$, $x \in K$, such that:

- $\mathcal{D}_x$ contains x and is tangent to to E^{cs}_x,
- The closure of $f(\mathcal{D}_x)$ is contained in $\mathcal{D}_{f(x)}$.

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Definition. The bundle E^{cs} is thin-trapped if there exists trapped plaque families tangent to E^{cs} with arbitrarily small diameters.

Example. If E^{cs} is uniformly contracted, it is thin-trapped.

Thm 1: How to use "far from homoclinic tangencies"?

Theorem (Wen)

Consider $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$ *and a sequence of hyperbolic periodic orbits* (O_n) *with the same stable dimension* d_s*.*
Then $\Lambda = \overline{\cup_n O_n}$ *has a dom. splitting* $T_\Lambda M = E \oplus F$ *with* $\dim(E) = d_s$*.*

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- This allows to build dominated splittings.

Corollary (Wen)

If μ *is an ergodic invariant probability, the support hat a dom. splitting* $TM = E \oplus E^c \oplus F$ *with* $\dim(E^c) \leq 1$*. The Lyapunov exponents of* μ *are* 0 *along* E^c *and non-zero along* E *and* F*.*

Thm 1: Decomposition of non-uniform bundles

Consider a generic $f \in \text{Diff}^1(M) \setminus \overline{\text{Tangency}}$ and an invariant compact set Λ with a splitting $T_\Lambda M = E \oplus_< F$.

Proposition (Decomposition principle)

If E is not uniformly contracted then one of the following holds:

- $\Lambda \subset H(p)$ *for some periodic p with* $\dim(E^s(p)) < \dim(E)$.
- $\Lambda \subset H(p)$ *for some periodic p with* $\dim(E^s(p)) = \dim(E)$.
  $H(p)$ contains periodic orbits with a weak stable exponent.
- *Λ contains K partially hyperbolic:* $T_K M = E^s \oplus_< E^c \oplus_< E^u$,
  with $\dim(E^c) = 1$, $\dim(E^s) < \dim(E)$.
  Any measure on K has a zero Lyapunov exponent along E^c.

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- In the two first cases, the bundle E splits $E = E' \oplus_< E^c$.
- In the third, one finds a periodic orbit in $H(p)$ which spends most of its time close to K. (Analyze the topol. central dyn.)

Chain-hyperbolic homoclinic classes: definition

Definition. A homoclinic class $H(O)$ is chain-hyperbolic if:

- there is a dominated splitting $TM = E^{cs} \oplus E^{cu}$,
- there are some plaque families $\mathcal{D}^{cs}, \mathcal{D}^{cu}$ tangent to E^{cs}, E^{cu} that are trapped by f and f^{-1} resp.
- $\mathcal{D}^{cs}_O \subset W^s(O)$ and $\mathcal{D}^{cu}_O \subset W^u(O)$.

Examples.

- The homoclinic classes of generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$.
- Some non-hyperbolic robustly transitive diffeomorphisms (Shub, Mañé, Bonatti-Viana,...).

Chain-hyperbolic homoclinic classes: properties

Let $H(O)$ be a chain-hyperbolic homoclinic class.

Proposition (Robustness)

(If $H(O)$ is a chain-recurrence class,)
$H(O_g)$ is chain-hyperbolic for any $g \in \mathrm{Diff}^1(M)$ close to f.

Proposition (Local product structure)

The plaques $\mathcal{D}^{cs}, \mathcal{D}^{cu}$ are resp. contained in the chain-stable and the chain-unstable sets of $H(O)$.
For x, y close, $\mathcal{D}^{cs}_x \cap \mathcal{D}^{cu}_y$ belongs to $H(O)$.

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For x, y close, $\mathcal{D}^{cs}_x \cap \mathcal{D}^{cu}_y$ belongs to $H(O)$.

This justifies the name "chain-hyperbolicity" however $H(O)$ can robustly contain periodic points of different stable dimension!

Chain-hyp. homoclinic classes: pointwise continuation

Start with f and a chain-hyperbolic class $H(O)$ s.t. $TM = E^{cs} \oplus E^{cu} = (E^s \oplus E^c) \oplus E^u$ and $\dim(E^c) = 1$.

By perturbation, any points has one or two continuations:

Proposition

If $f \in \mathrm{Diff}^r \setminus \overline{\text{strong connexions}}$, *there exists a lift dynamics* $(\tilde{H}, \tilde{f})$ *such that for each* g C^r*-close to* f *there is a semi-conjugacy* $\pi_g \colon \tilde{H} \to H(O_g)$ *satisfying:*

- *for each* $\tilde{x} \in \tilde{H}$ *the points* $\pi_f(\tilde{x}), \pi_g(\tilde{x})$ *are close,*
- *for each* $x \in H(O_g)$ *one has* $\#\pi_g^{-1}(x) \leq 2$.

Quasi-attractors

For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$, if they exist,
non-hyperbolic quasi-attractors are:

- *homoclinic classes,*
- *chain-hyperbolic with a splitting* $E^{cs} \oplus E^u = (E^s \oplus E^c) \oplus E^u$,
- *saturated by unstable leaves,*
- *not contained in a submanifold: they contain two different points* x, y *with a same strong-stable leaf.*

Goal. By perturbation, find p, q periodic in $H(O)$ such that $W^{ss}(p)$ and $W^u(q)$ intersect.
(This will give a heterodimensional cycle, hence a contradiction.)

Quasi-attractors: geometry of the unstable lamination

$H(O)$: quasi-attractor for a generic $f \notin \overline{\text{Tangency} \cup \text{Cycle}}$.
One looks at pairs (x, y) where $x \neq y$ in $H(O)$ have a same strong stable leaf.

- One can compare $W^u_{loc}(x)$ with the projection $\Pi^{ss}(W^u_{loc}(y))$ through strong stable holonomy.

Possible cases:

- transversal: for *some* pair (x, y),
  $W^u_{loc}(x)$ and $\Pi^{ss}(W^u_{loc}(y))$ cross,
- jointly integrable: for *some* pair (x, y),
  $W^u_{loc}(x)$ and $\Pi^{ss}(W^u_{loc}(y))$ coincide,
- stricly non-transversal: for *any* pair (x, y),
  $W^u_{loc}(x)$ and $\Pi^{ss}(W^u_{loc}(y))$ do not cross and do not coincide.

Boundary points

Definition. A *stable boundary point* $x \in H(O)$ is a point which is accumulated by $H(O)$ in only one component of $\mathcal{D}^{cs}_x \setminus W^{ss}(x)$.

Theorem. If the transversal case does not holds, then any stable boundary point belongs to the unstable manifold of a periodic point of $H(O)$.

Conclusion

Any non-hyperbolic quasi-attractor satisfies one of the following case robustly:

– Unstable case. There exists p_x, p_y periodic in $H(O)$ and $x \in W^u(p_x)$, $y \in W^u(p_y)$ distinct which share a same strong stable leaf.

– Stable case. There exists q periodic in $H(O)$ and $x, y \in W^s(q)$ distinct which share a same strong stable leaf.

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Tomorow, one will perturb to create a strong connexion.
$\Rightarrow$ all quasi-attractors are hyperbolic.

Essential hyperbolicity versus homoclinic bifurcations (3)

Program of the lectures

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Non-hyperbolic quasi-attractor

Take a quasi-attractor $H(O)$ which is a homoclinic class s.t.

- $TM = E^{cs} \oplus E^u$ and $E^{cs} = E^s \oplus E^c$, $\dim(E^c) = 1$.
- E^{cs} is thin-trapped.

Theorem. There exists g close to f such that

- either a submanifold tangent to $E^c \oplus E^u$ contains $H(O_g)$,
- or $H(O_g)$ has a strong connexion: it contains periodic points p, q such that $W^{ss}(p)$ and $W^u(q)$ intersect.

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- E^{cs} is thin-trapped.

Theorem. There exists g close to f such that

- either a submanifold tangent to $E^c \oplus E^u$ contains $H(O_g)$,
- or $H(O_g)$ has a strong connexion: it contains periodic points p, q such that $W^{ss}(p)$ and $W^u(q)$ intersect.

Remark.

- If f is C^1-generic and E^c is not uniform, then this gives heterodimensional cycles.
- The result also applies to hyperbolic sets with a one-codimensional strong stable bundle.
- If f is C^r, $r > 1$, then g is $C^{1+\alpha}$-close for some $\alpha > 0$.

The goal

One of the following cases holds robustly:

– Unstable case. There exists p_x, p_y periodic in $H(O)$ and $x \in W^u(p_x)$, $y \in W^u(p_y)$ distinct which share a same strong stable leaf.

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In the unstable case,

- either one builds g and a periodic point q such that $W^{ss}(q)$ meets $W^u(p_y)$,
- or one finds g such that $x_g \notin W^{ss}(y_g)$.

In the stable case, one breaks the joint integrability close to (x, y).

Unstable case: return time dichotomy

Consider closest returns $f^n(x)$ of x (or y) to x:

- the return comes along $E^c_{p_x}$.
- If N is the time spent close to p_x before visiting x,

$$d(f^n(x), x) \simeq \lambda_c^N, \qquad \text{for } \lambda_c = \text{central eigenvalue of } p_x.$$

Fix $K > 1$ large. Two cases occur:

- Fast returns. there are n large such that $n \leq K.N$.
- Slow returns. there are n large such that $n \geq K.N$.

The fast return case

There are large closest return $f^n(x)$ such that $n \leq K.N$.
One set $a \simeq K^{-1}|\log \lambda_c|$ and $b \simeq a(1 - K^{-1})$.

Lemma

There exist returns at time n large such that $f^n(x) \in B(x, e^{-an})$ and $f^m(x) \notin B(x, e^{-bn})$ for any $0 < m < n$.

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The perturbation is $C^{1+\alpha}$-small, where $1 + \alpha = a/b = \frac{K}{K-1}$.

The slow return case

There are large closest returns $f^n(x)$ *such that* $n \geq K.N$.
One perturbs in $B(f^{-1}(x), \lambda_c^N)$, moving x and "keeping" $\Pi^{ss}(y)$.

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- $d(y, y_g) \leq \lambda_u^{-n}$, where λ_u bounds E^u from below.
- $d(\Pi^{ss}(y), \Pi^{ss}(y_g)) \leq \lambda_u^{-\beta n}$, where Π^{ss} is β-Hölder.
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For K large, one has

$$\sigma^{-n} + \lambda_u^{-\beta n} < \lambda_c^{(1+\alpha)N}.$$

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For $m \geq 1$ large, one compares the intersections x', y' of $f^{-m}(D)$ with $W^u(x)$ and $W^u(y)$.

- y' crosses $W^{ss}(x')$ during the perturbation.

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What about the other chain-recurrence classes?