Perturbation lemmas and C^1-generic dynamics

Christian Bonatti and Sylvain Crovisier*

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First Lecture: Description of the generic dynamics, perturbation lemmas

1 Topic of the lectures

In these three lectures, we will describe some aspects of the dynamics of the generic diffeomorphisms on a compact manifold.

We denote by M a compact manifold and by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms endowed with the C^1-topology. This is a complete space and thus a Baire space. A property is *generic* if it is satisfied on a set of diffeomorphisms which is a dense G_δ of $\mathrm{Diff}^1(M)$, i.e. (by Baire theorem) a countable intersection of dense open sets of $\mathrm{Diff}^1(M)$.

For instance, in order to prove that "the fixed points of a generic diffeomorphism are all hyperbolic", one can prove first that this property is open (this is the easy part) and then that it holds on a dense set. This is done by showing that any diffeomorphism can be perturbed in order to make all the fixed points hyperbolic. We need for that a perturbation lemma.

From this idea, which uses Thom's transversality theorem, the first genericity result is:

Theorem 1.1 (Kupka-Smale, 1963). *The Kupka-Smale diffeomorphisms (the periodic points are hyperbolic, their invariant manifolds are transverse) are generic in* $\mathrm{Diff}^1(M)$.

In fact, this result is true in any C^r-topology with $r \geq 1$. The other results we will present now are only proven in the C^1-topology.

The goal of these lectures is to present some main perturbations lemmas available in C^1-dynamics. The plan for these lectures is the following:

1. Description of the generic dynamics, perturbation lemmas.
2. Local perturbations, perturbation boxes.
3. Global perturbations.

*The authors thank Flavio Abdenur for a carefull reading of a preliminary version of this text.

2 Global dynamics: filtrations / recurrence

A good understanding of the dynamics involves a global description. Historically, two main directions were proposed for that:

1. the filtrations,
2. the notions of recurrence.

In the first approach, one tries to split the dynamics into different pieces. These pieces are ordered by the dynamics: when a given orbit leaves one piece, it never comes back later. In other words this organizes the wandering dynamics. This was the idea behind the notions of Lyapunov functions, Brouwer's lines on the plane, and more generally filtrations. (Of course this idea has no meaning in conservative dynamics.)

On the contrary, in the second approach, one wants to describe the other orbits, those one which are non-wandering. Different notions of recurrence where proposed for that. But the stronger recurrent behavior is realized by the periodic orbits. This is an important question, also in other dynamical settings: *Is it possible to approach the recurrent dynamics by periodic orbits?* It seems rather natural that by perturbation a recurrent orbit can be made periodic. To prove this one needs a perturbation lemma.

These two directions are complementary since the second one analyzes the dynamics in the pieces of the first one.

3 Filtrations: Conley's fundamental theorem of the dynamics

Before the description of the perturbation lemmas, we will spend some time on the problem of existence of filtrations: this problem was well understood after Conley's work [Co] in the 70's.

The filtrations are related to the existence of a Lyapunov function: a *Lyapunov function* for a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is a continuous map $h : M \to \mathbb{R}$ which decreases along the orbits:

$$\forall z \in M, \quad h(f(x)) \leq h(x).$$

Let us consider a value $c \in \mathbb{R}$ with the property that for any point $z \in M$,

$$h(z) = c \quad \Longrightarrow \quad h(f(z)) < h(z).$$

Then the open set $U = \lambda^{-1}(]-\infty, c[)$ is an *attractor*:

$$f(Cl(U)) \subset U.$$

An invariant compact set K is split by U if it intersects U and its complement. Two invariant sets K and K' are separated by U if for instance U contains K and $M \setminus U$ contains K'.

Taking different values $c_1 > c_2 > \cdots > c_n$, one gets a filtration of the manifold by a decreasing sequence of attractors which split the recurrent dynamics (see figure 1).

One also would like to get a filtration which is minimal in the sense that one can not get a finer sequence of attractors by changing the Lyapunov function. In other words, the set of points whose orbits stay between two values $c_k > c_{k+1}$ of the Lyapunov function should be indecomposable by attractors.

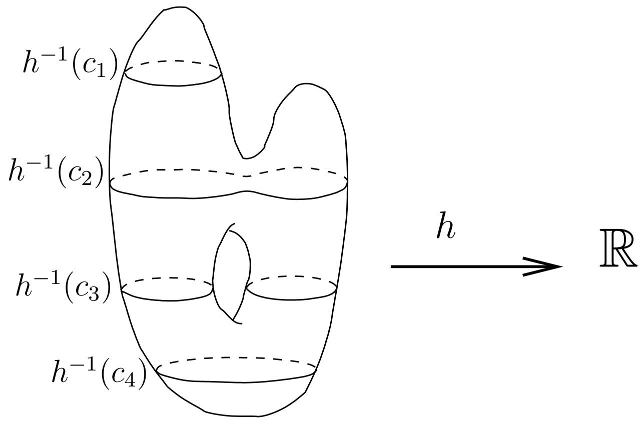

Figure 1: Filtration.

Clearly, an invariant set K can not be split by an attractor if for example there exists a dense orbit in K (one says that K is *transitive*). However, Conley remarked that a weaker property is sufficient: for every ε, a ε-pseudo-orbit is dense in K. Let us recall that a *ε-pseudo-orbit* is a sequence (z_n) in M such that for each n, $f(z_n)$ and z_{n+1} are at distance less than ε. If there exists a dense ε-pseudo-orbit in K, any attractor, such that $M \setminus U$ and $f(Cl(U))$ are at distance more than ε, can not split K into two parts.

This motivates the following definitions:

1. z' is a *chain-iterate* of z (and we write $z \dashv z'$) if for every $\varepsilon > 0$, there exists a ε-pseudo-orbit $z = z_0, z_1, \dots, z_n = z'$ with $n \geq 1$.

2. z and z' are *chain-equivalent* ($z \vdash\dashv z'$) if and only if $z \dashv z'$ and $z' \dashv z$.

3. The set of points that are chain-equivalent to themselves is closed and called the *chain recurrent set* $\mathcal{R}(f)$. The relation $\vdash\dashv$ is transitive and reflexive on $\mathcal{R}(f)$. This is an equivalence relation on $\mathcal{R}(f)$. The equivalence classes are called *chain recurrence classes*. These are the largest sets that can not be split by a filtration.

Conley's fundamental theorem is the following:

Theorem 3.1 (Conley, ∼1974). *There exists a Lyapunov function $h\colon M \to \mathbb{R}$ for f which has the following properties:*

- *h decreases along the orbits that are not in $\mathcal{R}(f)$:*
$$\forall z \in M \setminus \mathcal{R}(f), \quad h(f(z)) < h(z).$$

- *h is constant on each chain recurrent class and takes different values on different classes.*

- *The image of $\mathcal{R}(f)$ by h is a compact subset of $\mathbb{R}$ with empty interior.*

With these properties, two different classes are always separated by an attractor built using the function h.

The result is not only generic: it is always true. In fact, this theorem holds in a very general setting and uses only topological considerations. In order to understand the dynamics in the classes one should understand the recurrent dynamics.

4 Recurrence and perturbation lemmas

For the moment, we will forget Conley's description and consider the recurrent dynamics. Different definitions were proposed for the set of points which support the recurrent behavior:

1. the closure of the periodic points, $\mathcal{P}(f)$,
2. the non-wandering set, $\Omega(f)$,
3. the chain recurrent set, $\mathcal{R}(f)$.

We have the inclusions:

$$\mathcal{P}(f) \subset \Omega(f) \subset \mathcal{R}(f).$$

According to the Conley's theorem, the chain recurrent set is the most natural one. However, it is often simpler to work with the closure of the periodic orbits for two reasons:

- It seems easier to compute a periodic point than a non-wandering point.
- We have seen that generically, the periodic orbits are hyperbolic. Hence, each of them has a continuation after perturbation. This implies that on a generic set the closure of the periodic orbits varies continuously.

A first problem is to decide when these sets coincide. It is easy to build examples where they do not, the question is to decide if these are generic.

4.1 Pugh's closing lemma

The first result in this direction was due to C. Pugh [Pu].

Theorem 4.1 (Pugh, 1967). *For every generic diffeomorphism,*

$$\mathcal{P}(f) = \Omega(f).$$

This is a consequence of a local perturbation lemma, called the closing lemma (see figure 2).

Lemma 4.1 (Closing lemma, Pugh). *Let f be a C^1-diffeomorphism and z a non-wandering point. Then, there exists a C^1-small perturbation g of f such that z is a periodic point of g.*

This explains why the non-wandering set played so important a role in C^1-dynamics.

This result characterizes globally the recurrence but does not explain how to split it into pieces that could be studied individually. For that one should be able to say when two points are in the same piece.

One could propose the following relation on $\Omega(f)$: z and z' are in the same piece if and only if for any neighborhood U and U' of z and z' there exists a periodic orbit that intersects U and U'.

The problem is to prove the transitivity of this relation...

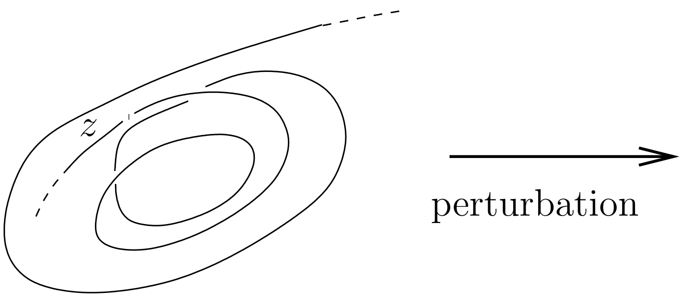

Figure 2: Closing an orbit to itself.

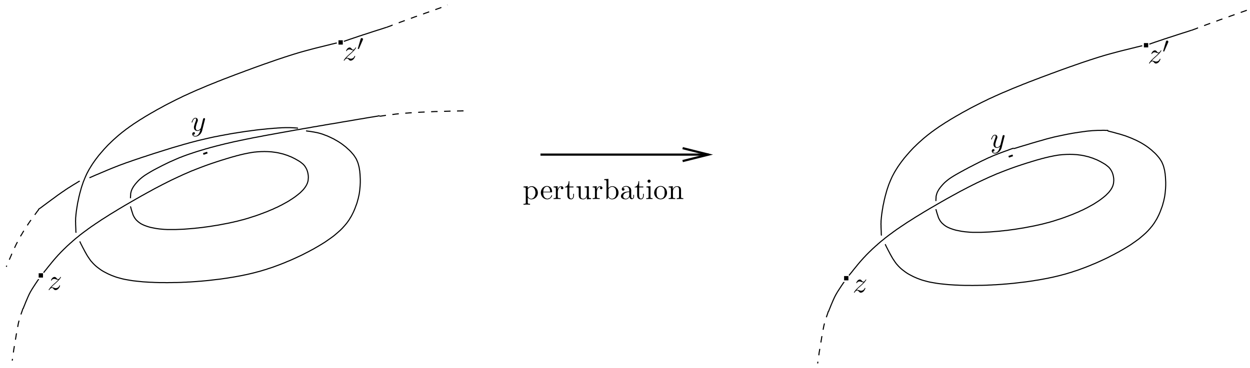

Figure 3: Connecting two orbits.

4.2 Hayashi's connecting lemma

This was only solved 30 years later thanks to another local perturbation lemma proved by S. Hayashi [Ha] (see figure 3).

Lemma 4.2 (Connecting lemma, Hayashi, 1997). *Let f be a diffeomorphism, z and z' two points in M such that:*

1. *$\omega(z)$ and $\alpha(z')$ accumulate on a same point y.*
2. *Either y is not periodic or y is a periodic hyperbolic point.*

Then, there exists a C^1-small perturbation g of f such that z' is on the forward orbit of z by g.

It has several consequences: some works by Carballo, Morales and Pacifico and by Arnaud use the connecting lemma to study the homoclinic classes. This will be discussed later.

The connecting lemma allows us to prove that the previous relation is transitive. One can restate it in a different way, due to M.-C. Arnaud and L. Wen:

1. z' is a *weak-iterate* of z (and we note $z \prec z'$) if for every neighborhoods U and U' of z and z', there exists an orbit $z_0, \dots, z_n$ with $n \geq 1$, $z_0 \in U$ and $z_n \in U'$.
2. z and z' are *weakly-equivalent* ($z \sim z'$) if $z \prec z'$ and $z' \prec z$.

3. The set of points that are weakly equivalent to themselves is the non-wandering set.

Now the connecting lemma implies:

Theorem 4.2 (Arnaud 2000, Wen). *For generic diffeomorphisms, the relation $\prec$ is transitive. The relation $\sim$ is an equivalence relation on $\Omega(f)$.*

The equivalence classes are closed.
Moreover, they are almost transitive:

Proposition 4.3. *Any class E of the relation $\sim$ is* weakly transitive*: for any open sets U and V intersecting E, there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$.*

One can compare this result with the definition of transitivity: a set $E \subset M$ is transitive if for any open sets U and V intersecting E, there exists $n \geq 1$ such that $f^n(U) \cap V \cap E \neq \emptyset$.
An important question to complete this theory remains:

Problem 1. *For a generic diffeomorphism, are the equivalent classes of $\sim$ transitive?*

With a positive answer, the equivalent classes would be the maximal transitive sets of f.

4.3 The connecting lemma for pseudo-orbits

The previous decomposition is interesting because of the weak transitivity property. In order to obtain the filtrations and to compare them to Conley's decomposition, we proved in [BC$_1$, BC$_2$] another perturbation lemma, which, this time, is global:

Lemma 4.4 (Connecting lemma for pseudo-orbits, [BC$_1$, BC$_2$]). *Let f be a diffeomorphism whose periodic orbits are hyperbolic. Then, for any $z, z' \in M$ such that $z \dashv z'$, there exists a C^1-small perturbation g of f such that z' is on the forward orbit of z by g.*

This implies the following result:

Theorem 4.3 ([BC$_1$, BC$_2$]). *For a generic diffeomorphism the relations $\dashv$ and $\prec$ are the same.*

In particular, we get the following:

Corollary 4.5. *For a generic diffeomorphism f we have,*

1. *$\mathcal{P}(f) = \Omega(f) = \mathcal{R}(f)$: the set containing the recurrent dynamics is well-defined.*
2. *Conley's decomposition and Arnaud-Wen's decomposition coincide.*

Remark 4.6. This perturbation lemma holds also under generic hypothesis among the set of conservative dynamics and by a result of Arnaud, Bonatti and Crovisier among the set of symplectic diffeomorphisms. This implies the following:

Theorem 4.4 (Arnaud-Bonatti-Crovisier). *On a compact connected manifold endowed with a volume or symplectic form, generic volume preserving diffeomorphisms and generic symplectic diffeomorphisms are transitive.*

5 Description of the classes

We now describe some of the results on generic dynamics that describe the chain recurrence classes (we sometimes adapted or generalized the original statement in terms of Conley's decomposition).

5.1 Homoclinic classes / aperiodic classes

Let us consider a periodic orbit Γ. One defines the homoclinic class of Γ as the closure of the transverse intersections between the stable and unstable manifolds of Γ:

$$H(\Gamma) = Cl(W^s(\Gamma) \pitchfork W^u(\Gamma)).$$

One sees easily that a homoclinic class is always contained in a recurrence class. In fact, Carballo, Morales, Pacifico [CMP] and Arnaud [A] have proven, using Hayashi's connecting lemma, that

Theorem 5.1 (Carballo-Morales-Pacifico, Arnaud, 2000). *For generic diffeomorphisms, the homoclinic classes are recurrence classes.*

These are the best understood classes. They are countably many homoclinic classes. However, C. Bonatti and L. Diaz [BD] have shown that other classes may exist: they do not contain any periodic orbit and are called *aperiodic classes.*

Theorem 5.2 (Bonatti-Diaz, 2000). *If $dim(M) \geq 3$, there exists a non-empty open set $U \subset \mathrm{Diff}^1(M)$ such that any generic diffeomorphism in U has uncountably many aperiodic classes.*

5.2 Isolated classes: tame dynamics

From the previous results, one can ask what is the cardinality of the classes: we know that it may be finite or uncountable, but does the intermediate situation occur?

Problem 2. *Is it possible for a generic diffeomorphism to have countably many recurrence classes?*

The case where there are only finitely many classes is interesting: the dynamics in this case is called *tame* since it looks like the dynamics of the hyperbolic diffeomorphisms (axiom A diffeomorphisms):

For Axiom A diffeomorphisms, Smale's spectral theorem decomposes the chain recurrent set $\mathcal{R}(f)$ to a finite union of disjoint basic sets which are hyperbolic homoclinic classes. There exists a filtration associated to this decomposition and this description is robust by perturbation.

In the more general case of generic tame diffeomorphisms, one obtains similar properties:

1. From the density of periodic orbits in $\mathcal{R}(f)$, we get that any isolated class in $\mathcal{R}(f)$ is a homoclinic class. Hence, if the set of chain recurrence classes is finite, all the classes are homoclinic classes.

2. Abdenur [Ab] proved that if E is an isolated chain recurrent class of a generic diffeomorphism f there exists a neighborhood U of E such that any diffeomorphism g close to f has exactly one chain recurrence class in U. In particular, the chain recurrence classes can be followed locally and their number remains the same after perturbation.

3. Moreover from results by Bonatti, Diaz and Pujals in [BDP], Abdenur has shown that each isolated chain recurrence class has a weak form of hyperbolicity called *volume partial hyperbolicity.*

5.3 Dynamics between the classes

We end by describing the dynamics between the chain recurrence classes. It would be interesting to describe the behavior of a generic point in M for any generic diffeomorphism. We would expect the following:

Problem 3. *For a generic diffeomorphism, is any generic point of M attracted by a chain recurrence class which is a topological attractor (i.e. the maximal invariant set in an open set which is an attractor)?*

Using previous results by Morales and Pacifico, we get the following partial result (first conjectured by Hurley):

Proposition 5.1. *For any generic diffeomorphism of a compact manifold M, any generic point of M accumulates on a chain recurrence class E which is a quasi-attractor: E is the countable intersection of topological (non-necessarily transitive) attractors.*

As we have seen, generically a chain recurrence class which is a topological attractor is a homoclinic class. Hence, a positive answer to problem 3 would also give a positive answer to the following open question:

Problem 4. *For a generic diffeomorphism, are the stable manifolds of periodic orbits dense in M?*

6 Conclusion

We have seen that the recurrent dynamics and its decomposition into elementary classes have been well identified. However little is known about the dynamics inside each class.

We finish this presentation with another old question: on surfaces, can some pathological dynamics occur like in higher dimension, or:

Problem 5. *For a surface S, are axiom A diffeomorphisms dense in* $\mathrm{Diff}^1(S)$*?*

Second Lecture: Local perturbations, perturbation boxes

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Third Lecture: Global perturbations

1 Introduction

In this last lecture, we prove the global perturbation lemma.

Theorem 1.1 (Connecting lemma for pseudo-orbits [$\mathbf{BC_1}$, $\mathbf{BC_2}$]). *Let f be a C^1-diffeomorphism of a compact manifold M and assume that the periodic orbits of f are hyperbolic. Let $\mathcal{U}$ be a C^1-neighborhood of f. Let x, y be two points in M.*

Then, if $x \dashv y$, there exists $g \in \mathcal{U}$ and $n \geq 1$ such that $g^n(x) = y$.

We won't give here the technical details of the proof which are written in [$\mathrm{BC_2}$]. The aim of these notes is to describe the main ideas.

Note that the perturbation of f cannot be realized with an arbitrarily small support. The perturbation should be global. However, this perturbation will be built as the composition of several local perturbations given by the perturbation boxes introduced in the last lecture.

In the proof, two ingredients will be used: the notion of perturbation domains, which is a generalization of the perturbation boxes (see section 2) and the existence of topological towers (see section 4). The presence of periodic orbits of low period is an obstruction for the construction of a topological tower. Hence, the proof of the Theorem is first given in the case where there is no periodic orbit of low period (section 5). In section 6 we explain how to proceed where there are such periodic orbits.

The global perturbation g will be build as a composition of local perturbations in $\mathcal{U}$. The number of local perturbations may be large, however g will remain in $\mathcal{U}$. We here explain the reason:

Any perturbation f' of f may be written as $f' = \varphi \circ f$ and the *support of the perturbation* is the (open) set of points x where $\varphi(x)$ and x are distinct. Let $f_1 = \varphi_1 \circ f$ and $f_2 = \varphi_2 \circ f$ be two perturbations of f in $\mathcal{U}$. If their supports are disjoint, the *composed perturbation* $g = \varphi_1 \circ \varphi_2 \circ f = \varphi_2 \circ \varphi_1 \circ f$ is well-defined.

One can now replace $\mathcal{U}$ by a smaller neighborhood of f which is a ball for the usual C^1 distance on $\mathrm{Diff}^1(M)$. With this choice of $\mathcal{U}$, for any pair of perturbations f_1 and f_2 of f with disjoint support, we have the announced property:

$$f_1, f_2 \in \mathcal{U} \quad \Longrightarrow \quad g \in \mathcal{U}.$$

2 First tool: the perturbation domains

We recall the definition of a *perturbation box* U around a point $x \in M$ for a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and for a neighborhood $\mathcal{U}$ of f, as was defined in the previous lecture.

In some local coordinates around x we have the following properties:

1. U is a tiled cube. Moreover the geometry of the tiling is bounded (the number of tiles is locally bounded and two adjacent tiles have comparable sizes, see figure 4).

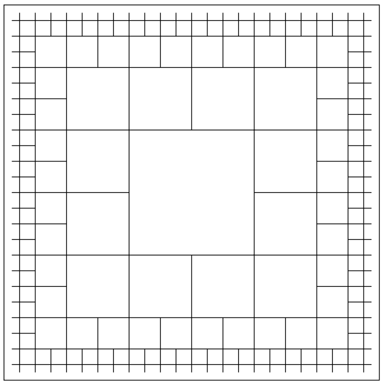

Figure 4: Tiled cube.

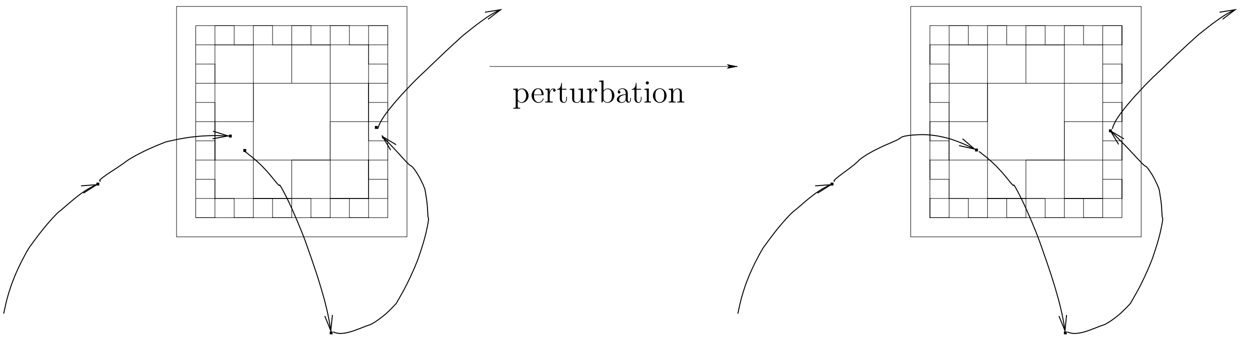

Figure 5: Perturbation in a perturbation box.

2. U is disjoint from its N first iterates. The integer N is the length of the box in time. The support of the box is $U \cup f(U) \cup \dots \cup f^N(U)$.

3. For any pseudo-orbit $(z_0, \dots, z_n)$ whose jumps are in the tiles of $\mathcal{U}$ (see figure 5) there exists a perturbation $g \in \mathcal{U}$ of f with support in $U \cup f(U) \cup \dots \cup f^{N-1}(U)$ and an integer $m \in \{1, \dots, n\}$ such that $g^m(z_0) = z_n$.

The fact that U is a cube is not so important: one could consider any small open set tiled by cubes (for the local coordinates around x) if the geometry of the tiling is bounded (see figure 6). This allows us to extend the definition of perturbation boxes and to consider *perturbation domains* of $(f,\mathcal{U})$ around x.

Hayashi's connecting lemma can be restated now as a theorem of existence of perturbation domains:

Theorem 2.1 (Hayashi's connecting lemma). *For any $(f,\mathcal{U})$, there exists an integer $N \geq 1$ and, around any point $x \in M$, there are local coordinates such that any tiled open set U close to x which is disjoint from its N first iterates is a perturbation domain of $(f,\mathcal{U})$.*

An important remark here is that N depends only on $(f,\mathcal{U})$ and not on the point x nor on the choice of the tiled open set U.

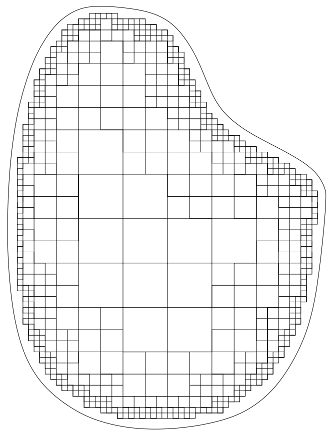

Figure 6: Tiled domain.

3 Strategy of the proof

Let us consider, as in the statement of theorem 1.1, a diffeomorphism f, a neighborhood $\mathcal{U}$ of f and a pair of points $x, y \in M$ such that $x \dashv y$.

The proof has two main steps:

- Choose some perturbation domains over the manifold M.
- Choose a good pseudo-orbit that connects x to y: the jumps should occur only in the tiles of the perturbation domains built previously.

If these two steps have been realized, one may perturb in any perturbation domain in order to remove the jumps inside (see figure 7).

Considering perturbations domain after perturbation domain, one can in this way eliminate all the jumps of the pseudo-orbit provided that the supports of the perturbation domains are pairwise disjoint. With our choice of $\mathcal{U}$, the composed perturbation will connect x to y by a "true" orbit and will belong to $\mathcal{U}$ as required.

4 Second tool: the topological towers

We explain how to perform the first step of the proof. The difficulty here is to introduce enough perturbation domains in order to cover a large part of the dynamics of f but also to guarantee that the perturbation domains have disjoint support.

This is allowed by the following result:

Theorem 4.1 (Existence of topological towers). *Let $N \geq 1$ be an integer and f be a diffeomorphism of M with no periodic orbit of period less that $Cte.N$.*

Then, there exists an open set U with the following properties:

- *U is disjoint from its N first iterates.*

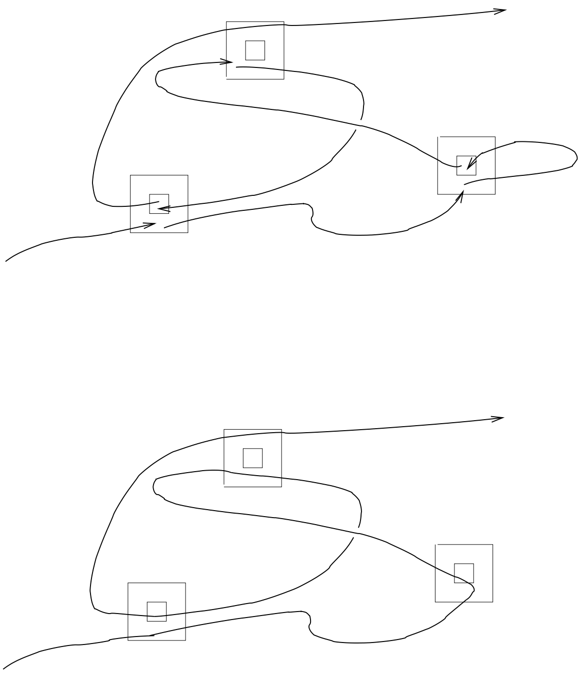

Figure 7: Global perturbation.

- *For any $x \in M$, there exists $n_x \geq 1$ such that $f^{n_x}(x) \in U$.*
- *The connected components of U are arbitrarily small.*

The assumption that f has no periodic orbits of low period is necessary: if f has a fixed point x, the orbit $\{x\}$ of x can not belong to an open set U disjoint from the $N \geq 1$ first iterates.

The constant in the assumption depends only on the dimension of the manifold M.

The name "topological towers" was given by analogy with the towers in ergodic theory. One may think here that a tower is a "section" of the dynamics of f.

The proof of this theorem reduces to a coloring problem. It won't be explained here.

In view of the assumption of this result it is much easier to prove the connecting lemma for pseudo-orbits when the diffeomorphism has no periodic orbits of low period (section 5). In this case, one can apply theorem 4.1 directly. The integer N is given by Hayashi's connecting lemma. Each connected component of the obtained open set U may then be tiled and becomes a perturbation domain. We get the following result:

Proposition 4.1. *If f has no periodic orbit of period less than $C(d).N$ (where $C(d)$ is a constant which depends only on the dimension d of the manifold M), then there exists some perturbation domains $B_1, \dots, B_s$ such that:*

- *Their support are pairwise disjoint.*
- *For any $x \in M$, there is $n_x \geq 1$ and a perturbation domain B_k s.t.*

$$f^{n_x}(x) \in B_k.$$

Working more and using the compactness of the manifold, one can improve this last result:

1. The return time n_x is uniformly bounded by some $m \geq 1$.
2. There is a finite number of tiles $C_1, \ldots, C_r$ taken from the domains $B_1, \ldots, B_s$ such that any point x has a return (maybe not the first return but some return among the m first iterates of x) inside one of these tiles.
3. There are some compact sets $D_1, \ldots, D_r$ contained in the interior of each tile $C_1, \ldots, C_r$ such that any point x has some return in one of these sets.
4. Not only the orbits of f have returns inside the compact sets D_i, but also the pseudo-orbits with small jumps.

5 An easy case: no periodic orbit. Construction of good pseudo-orbits and end of the proof of the theorem

We come now to the second step of the proof and assume that there are no periodic orbits of low period so that the assumption of theorem 4.1 is satisfied.

From the first step, one built:

- some perturbation domains $B_1, \ldots, B_s$ for $(f, \mathcal{U})$ with pairwise disjoint supports,
- some tiles $C_1, \ldots, C_r$ of these domains and some compact sets $D_1, \ldots, D_r$ contained in the interior of the tiles C_i,
- a return time $m \geq 1$.

The following property is satisfied:

any pseudo-orbit $z_0, \ldots, z_n$ of length at least m and with small jumps has an iterate inside one of the compact sets D_i.

We consider two points $x, y \in M$ such that $x \dashv y$: they are connected by pseudo-orbits with arbitrarily small jumps. However, one needs now that the jumps only occur in the tiles of the domains $B_1, \ldots, B_s$.

We will use the following trick: *any pseudo-orbit having small jumps may be modified into a pseudo-orbit whose jumps have been postponed* (see figure 8). For example, one can avoid the jump after one iteration, provided one realizes a larger jump after two iterations. The jumps of the new pseudo-orbit are quite larger but however controlled. One can also "push" in this way the jumps of a pseudo orbit a bounded number of time and keep a control on the size of the jumps.

We consider a first pseudo-orbit $z_0, \ldots, z_n$ with small jumps. As we see the time between the returns of the pseudo-orbit inside the compact sets D_i is bounded by a uniform constant m. One may now push the jumps of the pseudo-orbit to the time when the orbit returns to the compact sets D_i. If the size of the jumps of the initial pseudo-orbit is small enough, the new pseudo-orbit has jumps of size smaller that the distance between D_i and ∂C_i for each i. Consequently, the jumps of the new pseudo-orbit remain in the tiles C_i.

This ends the second step of the proof in the case where there are no periodic orbits of low period.

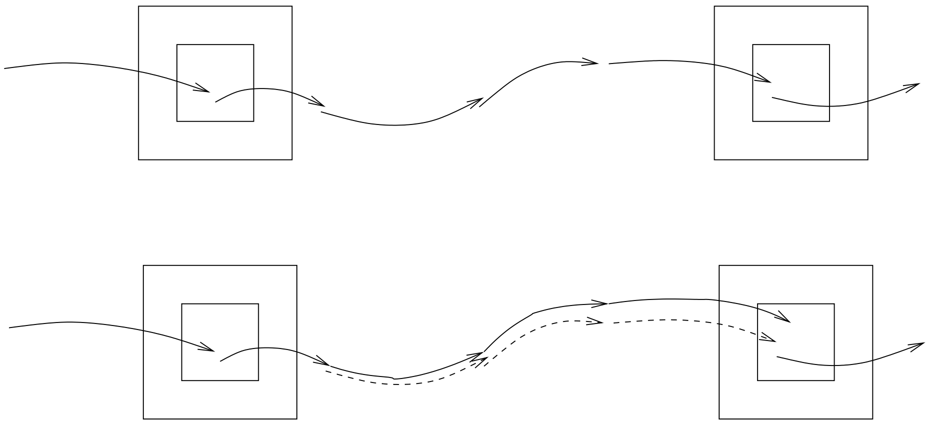

Figure 8: Modification of a pseudo-orbit.

6 How to deal with the periodic orbits?

We explain here the general case and how to modify the last section in order to allow periodic orbits of small period. At this point of the proof we use the fact that all the periodic orbits of f are hyperbolic.

One can adapt theorem 4.1 to the case where there are periodic orbits of small period: one chooses some very small neighborhoods V_i around the periodic points of period less than $C(d).N$. Then, it is possible to find an open set U disjoint from its N first iterates and disjoint from the neighborhoods V_i such that any segment of orbit of length larger some constant m, and disjoint from the V_i should intersect U. This allows to introduce some perturbation domains $B_1, \dots, B_s$ and tiles $D_1, \dots, D_r$ as before.

Let us consider a pair of points $x, y \in M$ such that $x \dashv y$. What happens if one tries to reproduce the proof of the previous section? Any pseudo-orbit that connects x to y will be divided according to its returns inside the sets D_i. *The problem here is that the time between two returns in U may not be bounded* because the orbit may spend a lot of time in the sets V_i, staying close to a periodic orbit.

The idea to bypass this difficulty is to again modify the pseudo-orbit when it crosses one of the open sets V_i and to replace it by a shorter pseudo-orbit: in each set V_i there is a unique hyperbolic periodic point z_i of low period. One can put close to z_i some additional perturbation boxes with disjoint supports which cover a fundamental domain for the stable and unstable manifolds of z_i.

If one considers a pair of perturbation boxes B^s and B^u intersecting the stable and the unstable manifolds of z_i respectively, there is a segment of orbit *chosen once for all* (and hence of bounded length) that connects B^s to B^u (see figure 9). These boxes will be called *entry and exit boxes* close to z_i.

Any pseudo-orbit that spends a lot of time close to the orbit of z_i should have iterates close to the stable and the unstable manifolds of z_i. Hence, it should cross such a pair of boxes B^s and B^u. The pseudo-orbit between these two times will be replaced by the chosen orbit that connects B^s to B^u.

The unbounded parts of the pseudo-orbit between the returns inside the perturbation do-

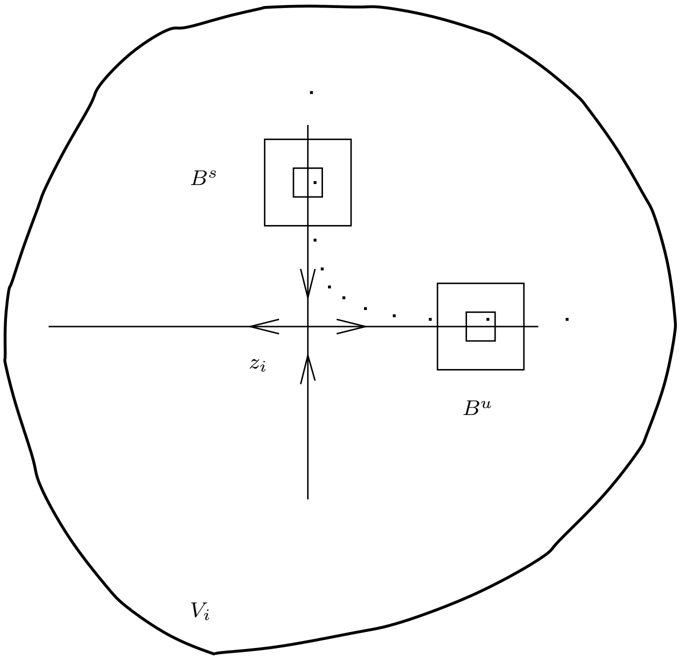

Figure 9: Entry and exit boxes close to a periodic point.

mains have been cut. The end of the proof is now similar to the previous case.

When one wants to prove the symplectic version of theorem 1.1, one should consider also points that are not hyperbolic: the derivative along the orbit may admit complex eigenvalue with modulus one and with an irrational argument. Some other arguments should be used in this case to cut the pseudo-orbits near these periodic points.

7 Summary of the proof

One sums up now the plan of the proof.

Let f be a diffeomorphism and $\mathcal{U}$ a neighborhood of f.

1. Hayashi's connecting lemma gives the size N for the time of the perturbation domains.
2. One builds entry and exit perturbation boxes near the periodic orbits of period less than $C(d).N$.
3. One introduces a topological tower far away from the periodic orbits of period less than $C(d).N$. This gives some other disjoint perturbation domains.

Let be $x, y \in M$ such that $x \dashv y$.

4. One builds a pseudo-orbit from x to y whose jumps are in the tiles of the perturbation domains previously introduced.
5. By perturbation of f, one removes the jumps of the pseudo-orbit.

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Christian Bonatti (bonatti@u-bourgogne.fr)

Sylvain Crovisier (sylvain.crovisier@u-bourgogne.fr)

CNRS - Institut de Mathématiques de Bourgogne, UMR 5584,
9, avenue A. Savary, B.P. 47 870,
F-21078 Dijon Cedex, France