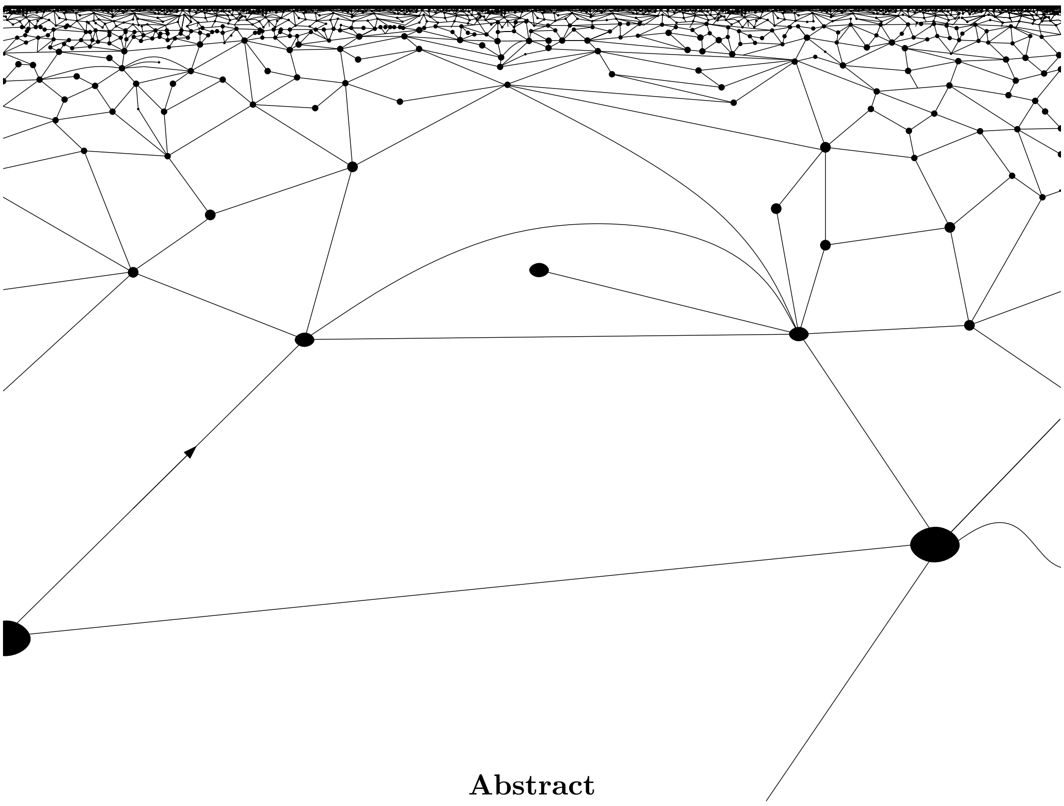
Local limits of random trees and maps



Abstract

These are the lecture notes of a mini-course given at the IX*th* Young European Probabilists conference that took place in Eindhoven (March 2012). They contain some material extracted from [2, 8, 9]. We survey some aspects of local limits of random trees and maps and deal in particular with the so-called geometric Galton-Watson tree conditioned to survive and the Uniform Infinite Planar Quadrangulation.

Acknowledgments: Many thanks go to Igor Kortchemski for a detailed reading of a first version of these lecture notes. I am also grateful to the organizers and the participants of the IX*th* YEP meeting.

1 Maps

1.1 What is a map?

We begin with one possible definition of a planar map.

Definition 1. *A planar map is a proper embedding of a finite connected graph in the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms of the sphere.*

There are other definitions of a map such as the surface obtained from the glueing of a certain number of polygons along their edges: If the resulting surface is a sphere then the map is planar. It might not be clear using Definition 1 to see that there is only a finite number of maps with n edges. This is however the case: A planar map can indeed be defined in purely combinatorial terms e.g. by saying that it is a planar graph together with coherent orientations around each vertex. We will not go into these formal definitions and will stick to the rather obvious intuition provided by Definition 1. All the maps considered in these pages are planar and we will then drop the adjective planar to simplify notation.

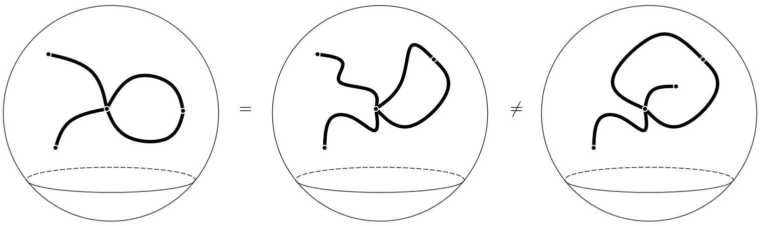


Figure 1: The same graphs but not the same maps.

A planar map m is thus composed of vertices, edges and faces (the connected components of the sphere minus the embedding). We denote the set of vertices, edges and faces of m by $\mathrm{V}(m), \mathrm{E}(m)$ and $\mathrm{F}(m)$ respectively. The cardinalities of these sets satisfy Euler's relation

$$\#\mathrm{V}(m) + \#\mathrm{F}(m) - \#\mathrm{E}(m) \quad = \quad 2. \tag{1}$$

Exercise 1. Prove it!

(Planar) maps are more rigid than planar graphs since they are given with an embedding (equivalently a planar orientation) whereas planar graphs only *possess* such an embedding. This rigidity enables us to enumerate planar maps more easily than planar graphs and this is mainly why we will consider maps instead of graphs. For a complete rigidity we will only consider *rooted* maps, that are, maps given with one distinguished oriented edge called the *root edge*. Once rooted, maps loose any non-trivial symmetry. From now on, all the maps considered are rooted.

1.2 Quadrangulations

The degree of a face $f \in \mathrm{F}(m)$ is the number of oriented edges adjacent to f. Hence, if an edge lies entirely into a face it is counted twice in the degree of the face. If a (rooted) map m has all its faces of degree 4, we say that m is a *quadrangulation*. Henceforth we will focus only on quadrangulations because they behave in many respects very nicely and form one of the simplest family of maps to enumerate (easier than triangulations for example). We denote the set of all quadrangulations with n faces by $\mathbf{Q}_n$.

Proposition 2. *There is a bijection between, on the one hand, the set of all quadrangulations with n faces, and on the other hand, the set of all planar maps with n edges.*

Proof. The one-to-one correspondence is given as follows: If m is a planar map with n edges, then in each face of m we put an extra point that we link to all (corners of) the vertices adjacent to this face, see Fig. 2. We then erase all the edges of m and are left with a quadrangulation q with n faces. The root edge is transferred from m to q as depicted on Fig. 2. □

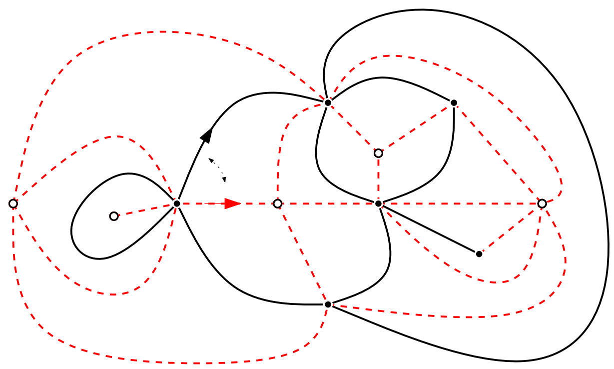


Figure 2: Illustration of the duality between maps and quadrangulations.

Remark. Notice that quadrangulations are bipartite, which means that we can color their vertices into black and white such that any adjacent vertices have different colors. See Fig.2.

As a consequence of the last proposition, the number of planar maps with n edges is the same as the cardinality of $\mathbf{Q}_n$ which turns out to be relatively simple:

Theorem 3 (Tutte)**.** *We have*

$$\#\mathbf{Q}_n \quad = \quad 3^n \frac{2}{n+2} \frac{1}{n+1} \binom{2n}{n}.$$

We will see a proof of the last theorem in Section 2 based on bijective techniques. Historically, the first tools used (by Tutte [18]) to enumerate planar maps were generating functions. A link was later established between enumeration of planar maps and matrix integrals [5].

1.3 What is it useful for?

The theory of planar maps has been triggered in the 60's motivated by the four colors problem. Since then, maps occurred in various fields such as algebra (Grothendieck's "dessins d'enfants"), physics (Feynman diagrams), matrix integrals or computer science (computational geometry) as data structures that encode various spatial objects (Google-Earth, surfaces, complex graphs...). More importantly they have been considered by physicists as a discretization of a fluctuating random surface (in the so-called "2D quantum gravity" theory). This triggered the probabilistic theory of random planar maps in the early 2000. Let us be a bit more precise. In the rest of this text we denote by Q_n a random variable uniformly distributed over $\mathbf{Q}_n$. The basic question is

What is the geometry of Q_n?

Very recently, Le Gall [12] and Miermont [15] proved that (in a certain sense) large random quadrangulations converge, once renormalized by their sizes to the power $1/4$, towards a continuum random surface called the *Brownian map*. Hence, in the same way discrete paths are

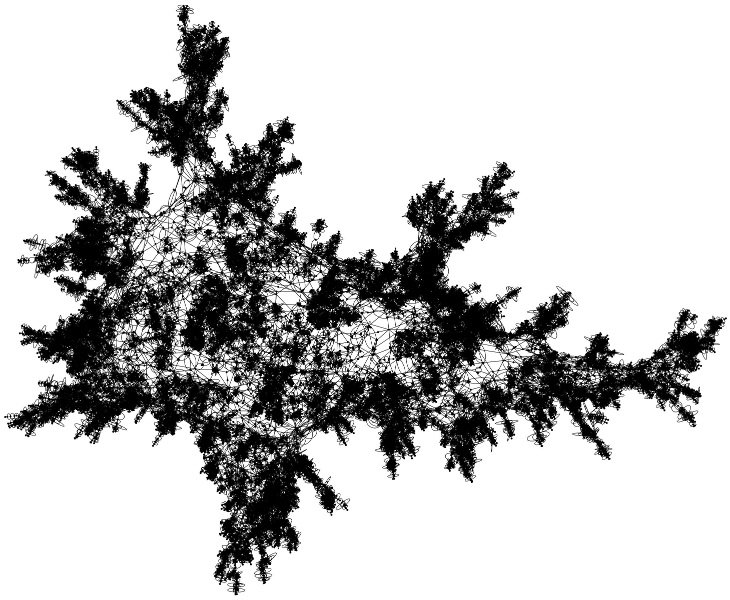


Figure 3: A random quadrangulation with $50\,000$ faces. Simulation made by Jean-François Marckert.

a discretization of the Brownian motion, discrete quadrangulations are a discretization of the Brownian map.

The point of view we are going to adopt here is different. We will rather focus on local properties of large random quadrangulations and prove that Q_n converge, without rescaling, towards a random infinite planar quadrangulation that is called the *Uniform Infinite Planar Quadrangulation* (Section 4). We will then study this random infinite network and show some of its amazing properties (Section 5).

1.4 Local limits of maps

If m is a map and $r \in \{0, 1, 2, 3, \dots\}$ we denote by $B_r(m)$ the map formed by all the faces of m whose vertices are all at graph distance less than r from the origin of the root edge in m. This map is called the ball of radius r in m. We now put a topology on the set $\mathbf{Q}_f$ of all finite quadrangulations: We say that a sequence $(q_n)_{n\geqslant 0} \in \mathbf{Q}_f$ converges *locally* if for all $r \geqslant 0$ the sequence $(B_r(q_n))_{n\geqslant 0}$ eventually stabilizes. This topology is induced by the following metric

$$\mathrm{d}_{\mathrm{map}}(q, q') \quad = \quad \Big(1 + \sup\{r \geqslant 0 : B_r(q) = B_r(q')\}\Big)^{-1}. \tag{2}$$

Note that the space $(\mathbf{Q}_f, \mathrm{d}_{\mathrm{map}})$ is not complete and we denote $\mathbf{Q}$ its completion. The elements in $\mathbf{Q}\backslash\mathbf{Q}_f$ are called infinite quadrangulations. See [9] for more details. The main theorem we are going to prove is the following (recall that Q_n is a random quadrangulation uniformly distributed over $\mathbf{Q}_n$):

Theorem 4 ([11])**.** *There exists a random infinite quadrangulation Q_∞ called the uniform infinite planar quadrangulation (UIPQ) such that we have the following convergence in distribution for the local topology*

$$Q_n \quad \xrightarrow[n\to\infty]{(d)} \quad Q_\infty.$$

This theorem is due to Krikun [11]. In a pioneer work, Angel & Schramm [1] defined a similar object (the UIPT) in the triangulation case. The proofs of Angel & Schramm and Krikun are both based on precise enumerative formulæ for the number of quadrangulations and triangulations with a boundary. The proof we are going to present is adapted from [9] and relies on a construction of quadrangulations from certain labeled trees.

2 The Cori-Vauquelin-Schaeffer bijection

One of the main tools for studying random quadrangulations is a bijection initially due to Cori & Vauquelin [7], and that was much developed by Schaeffer [17]. It establishes a one-to-one correspondence between rooted and pointed quadrangulations with n faces, and pairs consisting of a labeled tree with n edges and an element of $\{0, 1\}$. Let us describe this construction.

2.1 Labeled trees

Throughout this work we will use the standard formalism for plane trees as found in [16]. Let

$$\mathcal{U} \;=\; \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

where $\mathbb{N}^* = \{1, 2, \dots\}$ and $(\mathbb{N}^*)^0 = \{\varnothing\}$ by convention. An element u of $\mathcal{U}$ is thus a finite sequence of positive integers. We let $|u|$ be the length of the word u. If $u, v \in \mathcal{U}$, uv denotes the concatenation of u and v. If v is of the form uj with $j \in \mathbb{N}$, we say that u is the *parent* of v or that v is a *child* of u. More generally, if v is of the form uw, for $u, w \in \mathcal{U}$, we say that u is an *ancestor* of v or that v is a *descendant* of u. A *plane tree* τ is a (finite or infinite) subset of $\mathcal{U}$ such that

1. $\varnothing \in \tau$ ($\varnothing$ is called the *root* of τ),
2. if $v \in \tau$ and $v \neq \varnothing$, the parent of v belongs to τ
3. for every $u \in \mathcal{U}$ there exists $k_u(\tau) \geqslant 0$ such that $uj \in \tau$ if and only if $j \leqslant k_u(\tau)$.

A plane tree can be seen as a graph, in which an edge links two vertices u, v such that u is the parent of v or vice-versa. This graph is of course a tree in the graph-theoretic sense, and has a natural embedding in the plane, in which the edges from a vertex u to its children $u1, \dots, uk_u(\tau)$ are drawn from left to right. All the trees considered in these pages are plane trees. The integer $|\tau|$ denotes the number of edges of τ and is called the size of τ. For any vertex $u \in \tau$, we denote the shifted tree at u by $\sigma_u(\tau) := \{v \in \tau : uv \in \tau\}$. If a and b are two vertices of τ, we denote the set of vertices along the unique geodesic path going from a to b in τ by $[\![a, b]\!]$.

We denote by $\mathbf{T}$ the set of all plane trees and by $\mathbf{T}_n = \{\tau \in \mathbf{T} : |\tau| = n\}$ the set of all plane trees with n edges. We recall the classical enumeration result $\#\mathbf{T}_n = \mathrm{Cat}(n)$, where $\mathrm{Cat}(n)$ is the nth Catalan number:

$$\mathrm{Cat}(n) \;=\; \frac{1}{n+1}\binom{2n}{n}.$$

Exercise 2. Prove it!

A *labeled tree* is a pair $\theta = (\tau, (\ell(u))_{u\in\tau})$ that consists of a plane tree τ and a collection of integer labels assigned to the vertices of τ, such that $\ell(\varnothing) = 0$ and if $u, v \in \tau$ and v is a child of u, then $|\ell(u) - \ell(v)| \leqslant 1$. If $\theta = (\tau, \ell)$ is a labeled tree, $|\theta| := |\tau|$ is the size of θ. We denote by $\mathbf{L}_n$ the set of all labeled trees with n edges and by $\mathbf{L}_f = \cup_n \mathbf{L}_n$ the set of all finite labeled trees.

Exercise 3. Let τ be a finite plane tree. Prove that τ has $3^{|\tau|}$ different labelings.

A rooted and *pointed* quadrangulation is a pair $\mathbf{q} = (q, \rho)$ where q is a rooted quadrangulation and ρ is a distinguished vertex of q. We write $\mathbf{Q}_n^\bullet$ for the set of all rooted and pointed quadrangulations with n faces.

Theorem 5 (Cori-Vauquelin-Schaeffer)**.** *There exists a very nice bijection* $\Phi : \mathbf{L}_n \times \{0,1\} \longrightarrow \mathbf{Q}_n^\bullet$ *that has a lot of wonderful properties.*

The application Φ will be constructed in the following sections. Let us give an application of this result:

Proof of Theorem 3. By the last exercise we have $\#\mathbf{L}_n = 3^n \#\mathbf{T}_n = 3^n \mathrm{Cat}(n)$ which is also half of the cardinality of $\mathbf{Q}_n^\bullet$ by the above theorem. However, any quadrangulation with n faces has $n+2$ vertices by Euler's formula. We deduce that $\#\mathbf{Q}_n = 2 \cdot 3^n \mathrm{Cat}(n)/(n+2)$ as desired. □

2.2 From quadrangulations to trees

We first describe the inverse mapping $\Phi^{-1} : \mathbf{Q}_n^\bullet \longrightarrow \mathbf{L}_n \times \{0,1\}$. Details can be found in [6]. Let (q, ρ) be a finite quadrangulation rooted at $\vec{e}$ and given with a distinguished vertex $\rho \in V(q)$. We define a labeling ℓ of the vertices of the quadrangulation by setting

$$\ell(v) \quad = \quad \mathrm{d}_{\mathrm{gr}}^q(v, \rho), \quad v \in V(q),$$

where $\mathrm{d}_{\mathrm{gr}}^q$ denotes the graph distance in q. Since the map q is bipartite, if u, v are neighbors in q then $|\ell(u) - \ell(v)| = 1$. Thus the faces of q can be decomposed into two subsets: The faces such that the labels of the vertices listed in clockwise order are $(i, i+1, i, i+1)$ for some $i \geqslant 0$ or those for which these labels are $(i, i+1, i+2, i+1)$ for some $i \geqslant 0$. We then draw on top of the quadrangulation a "red" edge in each face according to the rules given by the figure below (which should display red edges inside the faces if the printer allows it).

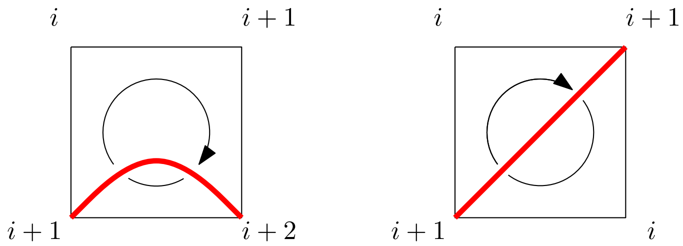


Figure 4: Rules for the reverse Schaeffer construction.

Proposition 6 ([6, Proposition 1])**.** *The graph* τ *formed by the edges red in the faces of* q *is a spanning tree of* $V(q) \backslash \{\rho\}$.

Proof. Suppose that q has n faces. Thus the graph formed by the red edges has cardinality n. Furthermore, it is easy to see that if u is a vertex of $V(q)\backslash\{\rho\}$ then u will be one of the extremities of at least one red edge. Indeed, consider a geodesic path going from u to ρ. Then since $i = \ell(u) > 0$ the first edge of this path is an edge linking a vertex of label i to a vertex of label $i-1$. According to the rules presented in Fig. 4, the face on the right of this edge oriented from $i \to (i-1)$ thus has a red edge adjacent to u.

By Euler's formula q has $n+2$ vertices. The graph formed by the red edges is thus a graph with n edges living on a set of $n+1$ points. It is then standard that this graph is a tree if and only if it has no cycle. Let us suppose by contradiction that the red edges form a cycle $\mathcal{C}$ and choose $u \in \mathcal{C}$ with minimal distance i to ρ.

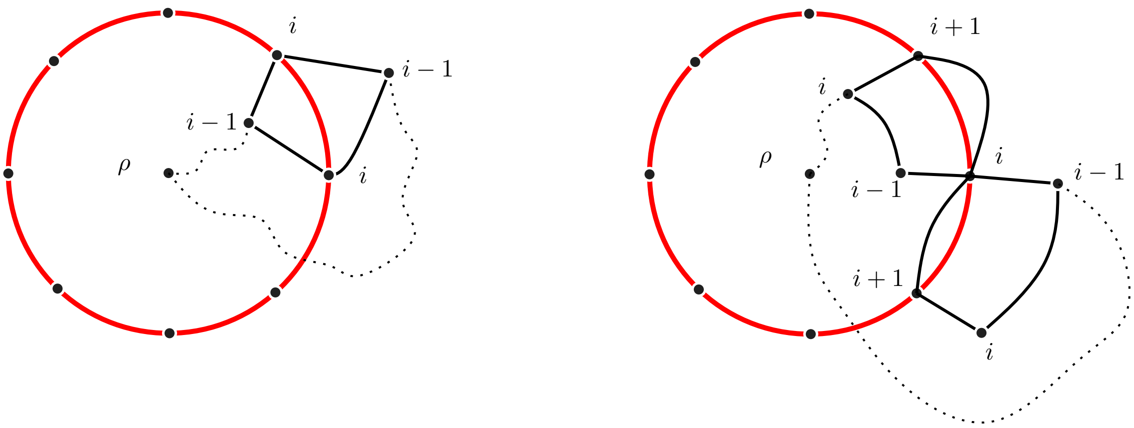


Figure 5: Proof of Proposition 6.

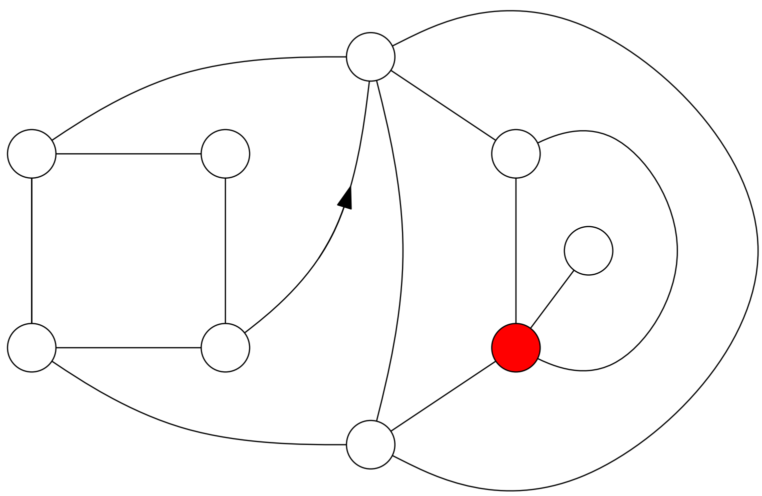
By analyzing the local structure around u, one is always able to find two vertices on both sides of $\mathcal{C}$ with labels $i-1$, see Fig. 5. This leads to a contradiction since (by the discrete Jordan's lemma) one of the geodesic paths from these vertices towards ρ has to cross $\mathcal{C}$ thus giving a vertex with label smaller than i, see Fig. 5. $\square$

This tree comes with a natural embedding in the plane, and we root τ on the corner incident to the vertex of minimal label of the root edge. Finally, we shift the labeling of τ inherited from the labeling on $V(q)\backslash\{\rho\}$ by the label of the root of τ,

$$\tilde{\ell}(u) \quad = \quad \ell(u) - \ell(\varnothing), \quad u \in \tau,$$

and we declare $\Phi^{-1}\big((q,\rho)\big) = \big((\tau,\tilde{\ell}), \mathbf{1}_{\ell(\vec{e}_+)>\ell(\vec{e}_-)}\big)$.

Exercise 4. Try the construction on this rooted pointed map:



2.3 From trees to quadrangulations

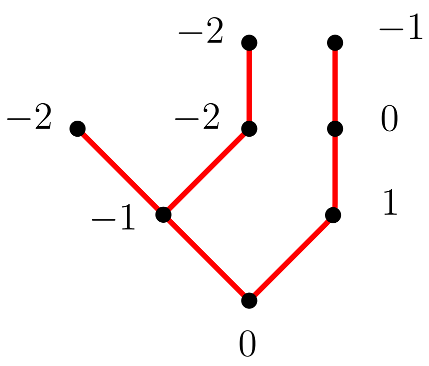
We now describe the mapping Φ from labeled trees to quadrangulations.

Let $\theta = (\tau,\ell)$ be an element of $\mathbf{L}_f$. We view τ as embedded in the plane. A *corner* of a vertex in τ is an angular sector formed by two consecutive edges in clockwise order around this vertex. Note that a vertex of degree k in τ has exactly k corners. If c is a corner of τ, $\mathcal{V}(c)$ denotes the vertex incident to c. By extension, the label $\ell(c)$ of a corner c is the label of $\mathcal{V}(c)$.

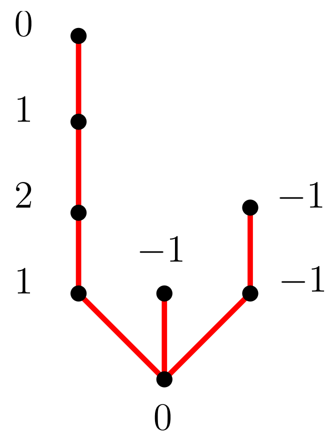
The corners are ordered clockwise cyclically around the tree in the so-called *contour order*. We index the corners by letting $(c_0, c_1, c_2, \ldots, c_{2n-1})$ be the sequence of corners visited during the contour process of τ, starting from the corner c_0 incident to $\varnothing$ that is located to the left of the oriented edge going from $\varnothing$ to 1 in τ. We extend this sequence of corners into a sequence $(c_i, i \geqslant 0)$ by periodicity, letting $c_{i+2n} = c_i$. For $i \in \mathbb{Z}_+$, the *successor* $\mathcal{S}(c_i)$ of c_i is the first corner c_j in the list $c_{i+1}, c_{i+2}, c_{i+3}, \ldots$ of label $\ell(c_j) = \ell(c_i) - 1$, if such a corner exists. In the opposite case, the successor of c_i is an extra element ∂, not in $\{c_i, i \geqslant 0\}$.

Finally, we construct a new graph as follows. Add an extra vertex ρ in the plane, that does not belong to (the embedding of) τ. For every corner c, draw an arc between c and its successor if this successor is not ∂, or draw an arc between c and ρ if the successor of c is ∂. The construction can be made in such a way that the arcs do not cross. After the interior of the edges of τ has been removed, the resulting embedded graph, with vertex set $\tau \cup \{\rho\}$ and edges given by the newly drawn arcs, is a quadrangulation q. In order to root this quadrangulation, we consider some extra parameter $\eta \in \{0, 1\}$. If $\eta = 0$, the root of q is the arc from c_0 to its successor, oriented in this direction. If $\eta = 1$ then the root of q is the same edge, but with the opposite orientation. We then let $q = \Phi(\theta, \eta) \in \mathbf{Q}_n^\bullet$ (q comes naturally with the distinguished vertex ρ).

Exercise 5. Try the construction on these two labeled trees...



$\eta = 0$



$\eta = 1$

... and check that the resulting quadrangulations are the same up to re-rooting and also correspond to the one of the last exercise!

Exercise 6. Prove that $q = \Phi(\theta, \eta)$ is indeed a quadrangulation and that for every vertex v of q not equal to ρ, one has

$$\mathrm{d}_{\mathrm{gr}}^{q}(v, \rho) \quad = \quad \ell(v) - \min_{u \in \tau} \ell(u) + 1 \,, \tag{3}$$

where we recall that every vertex of q not equal to ρ is identified with a vertex of τ. Prove that $\Phi(\Phi^{-1}(.)) = \mathrm{Id}$, you may consult [6] for help.

3 The uniform infinite labeled tree

3.1 Local limits of trees

In the same spirit as what we did for maps, we can define a distance on the set $\mathbf{T}$ of all plane trees. If $\tau \in \mathbf{T}$ and if $k \geqslant 0$ is an integer we denote by $[\tau]_k = \{u \in \tau : |u| \leqslant k\}$ the subtree of τ made of its k first generations and we set

$$\mathrm{d}_{\mathrm{tree}}(\tau, \tau') \quad = \quad \Big(1 + \sup\{r \geqslant 0 : [\tau]_r = [\tau']_r\}\Big)^{-1}. \tag{4}$$

Exercise 7. Prove that $\mathrm{d}_{\mathrm{tree}}$ is a metric on $\mathbf{T}$ such that $(\mathbf{T}, \mathrm{d}_{\mathrm{tree}})$ is Polish. Prove furthermore that $\tau_n \to \tau$ for $\mathrm{d}_{\mathrm{tree}}$ if and only if for every $k \in \{0, 1, 2, ...\}$ we eventually have $[\tau_n]_k = [\tau]_k$.

Theorem 7. *Let T_n be uniformly distributed over $\mathbf{T}_n$. Then we have the following convergence in distribution for* $\mathrm{d}_{\mathrm{tree}}$

$$T_n \quad \xrightarrow[n\to\infty]{(d)} \quad T_\infty,$$

where T_∞ is an infinite random plane tree known as the critical geometric Galton-Watson tree conditioned to survive.

This result is due to Kesten [10] (maybe not with this particular conditioning, but almost). Before proving this theorem, let us describe the distribution of the infinite random tree T_∞.

3.2 Description of T_∞

Let $\tau \in \mathbf{T}$ be an infinite tree. A *spine* in τ is an infinite sequence $u_0, u_1, u_2, \ldots$ in τ such that $u_0 = \varnothing$ and u_i is the parent of u_{i+1} for every $i \geqslant 0$. We let $\mathscr{S}$ be the set of all (infinite) trees with only one spine that we denote by $\varnothing = \mathrm{S}_\tau(0), \mathrm{S}_\tau(1), \mathrm{S}_\tau(2), \ldots$. If $\tau \in \mathscr{S}$, the spine then splits τ in two parts and every vertex $\mathrm{S}_\tau(n)$ of the spine determines a (plane) subtree of τ to its left and one to its right. These trees are denoted by $G_n(\tau), D_n(\tau)$. We can of course reconstruct the tree τ from the sequence $(G_n(\tau), D_n(\tau))_{n\geqslant 0}$.

We briefly recall the standard definition of Galton-Watson trees. Let ρ be a probability measure on $\mathbb{N}$ such that $\rho(1) < 1$. The law of a ρ-Galton-Watson tree τ is characterized by the following two properties:

(i) the distribution of $k_\varnothing$ is ρ,

(ii) conditionally on $\{k_\varnothing = j\}$, the j subtrees $\sigma_1(\tau), ..., \sigma_j(\tau)$ are i.i.d. ρ-Galton-Watson trees.

By standard facts on Galton-Watson trees, ρ has mean less than or equal to 1 if and only if τ is almost surely finite. In the sequel, $\rho_{1/2}$ is the geometric distribution of parameter $1/2$ that is for $k \in \{0, 1, 2, ...\}$ we have

$$\rho_{1/2}(k) \quad = \quad 2^{-k-1}.$$

Exercise 8. Let T be a Galton-Watson tree with offspring distribution $\rho_{1/2}$. We consider the contour function C of T defined by the picture below. Let $S_n = X_1 + ... + X_n$ be a simple random walk with i.i.d. increments $P(X_i = \pm 1) = 1/2$ and $\tau = \inf\{i \geqslant 1 : S_n = -1\}$. Show that C has the same distribution as (S_n) stopped at $\tau - 1$. In particular, deduce that

$$P(\mathrm{Height}(T) \geqslant n) \quad = \quad \frac{1}{n+1}. \tag{5}$$

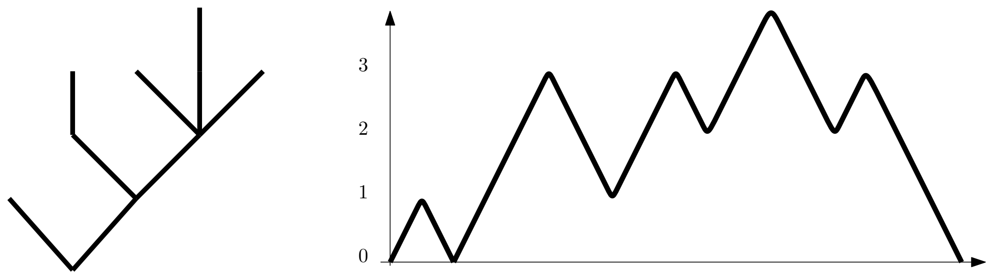


Figure 6: The contour function associated with a plane tree.

Definition 8. *The law of T_∞ (the critical geometric Galton-Watson tree conditioned to survive) is described by the following two properties*

(i) $T_\infty \in \mathscr{S}$ *almost surely,*

(ii) $(G_i(T_\infty))_{i\geqslant 0}$ *and* $(D_i(T_\infty))_{i\geqslant 0}$ *are independent sequences of i.i.d.* $\rho_{1/2}$*-Galton-Watson trees.*

Remark. The definition we gave of T_∞ is specific to the case of the geometric distribution. In general, if a distribution ξ has mean 1, the law of a ξ-Galton-Watson tree $\mathcal{T}_\infty$ conditioned to survive is described as follows [10, 14]. We let $\overline{\xi}$ be the size-biased distribution of ξ defined by $\overline{\xi}(k) = k\xi(k)$ for $k \geqslant 0$. Let $(D_i)_{i\geqslant 0}$ be a sequence of i.i.d. random variables distributed according to $\overline{\xi}$. Let also $(U_i)_{i\geqslant 1}$ be a sequence of random variables, such that, conditionally on $(D_i)_{i\geqslant 0}$, the $(U_i)_{i\geqslant 1}$ are independent and U_{k+1} is uniformly distributed over $\{1, 2, ..., D_k\}$ for every $k \geqslant 0$. The tree $\mathcal{T}_\infty$ has a unique spine, that is a unique infinite path $(\varnothing, U_1, U_1U_2, U_1U_2U_3, ...) \in \mathbb{N}^{*\mathbb{N}^*}$ and the degree of $U_1U_2...U_k$ is D_k. Finally, conditionally on $(U_i)_{i\geqslant 1}$ and $(D_i)_{i\geqslant 0}$ all the remaining subtrees are independent ξ-Galton-Waton trees, in particular $\mathcal{T}_\infty \in \mathscr{S}$. See Fig. 7.

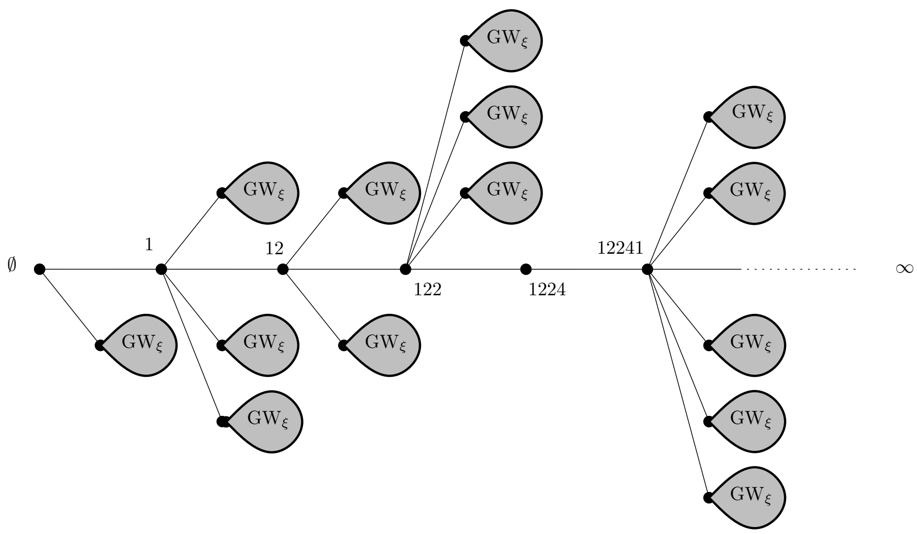


Figure 7: Construction of a general critical Galton-Watson tree conditioned to survive.

3.3 Proof of Theorem 7

In the following, T is a Galton-Watson tree with geometric offspring distribution of parameter $1/2$, T_n is uniformly distributed over $\mathbf{T}_n$ and T_∞ is the critical geometric Galton-Watson tree conditioned to survive.

First of all, if $\tau \in \mathbf{T}$ and $k \in \{0, 1, 2, ...\}$ we denote by $L_\tau(k) = \{u \in \tau : |u| = k\}$ the set of all individuals at generation k in τ. If $k = \infty$ then $L_k(\tau) = \varnothing$ by convention. The following exercise will be useful in the proof of Theorem 7.

Exercise 9. Show that if T is a Galton-Watson tree with geometric offspring distribution with parameter 1/2 (that is, $P(k_\varnothing = k) = 2^{-k-1}$), then for any plane tree τ and for any $k \in \{0, 1, 2, ...\} \cup \{\infty\}$ we have

$$P([T]_k = \tau) \quad = \quad 4^{-|\tau|} 2^{\# L_\tau(k) - 1}. \tag{6}$$

Using the above exercise, we deduce in particular that if $\tau \in \mathbf{T}_n$ then $P(T = \tau) = 4^{-n}/2$ is independent of the shape of τ, hence the distribution of T conditioned on $|T| = n$ is uniform over $\mathbf{T}_n$. Since $\#\mathbf{T}_n = \mathrm{Cat}(n)$ we deduce that

$$\sum_{k \geqslant 0} 4^{-k} \mathrm{Cat}(k) \quad = \quad 2. \tag{7}$$

By Stirling's formula, we also have the asymptotic

$$\mathrm{Cat}(n) \quad \sim \quad \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}, \quad \text{as } n \to \infty. \tag{8}$$

Exercise 10. Find another proof of (7).

Proof of Theorem 7. Let $k \in \{0, 1, 2, 3, ...\}$ and fix a finite plane tree τ_0 whose height does not exceed k. It is sufficient to show that $P([T_n]_k = \tau_0) \to P([T_\infty]_k = \tau_0)$ as $n \to \infty$, see Theorem 2.2 of [3].

We first compute $P([T_n]_k = \tau_0)$ (for $n \geqslant |\tau_0|$). On the event $\{[T_n]_k = \tau_0\}$, the tree T_n is obtained from τ_0 by grafting plane trees on top of the elements of $L_{\tau_0}(k)$ such that the sum of the sizes of these trees is equal to $n - |\tau_0|$. We set $i = \# L_{\tau_0}(k)$ and $n' = n - |\tau_0|$ to simplify notation. A simple counting argument shows that

$$P([T_n]_k = \tau_0) \quad = \quad \frac{1}{\mathrm{Cat}(n)} \sum_{k_1 + ... + k_i = n'} \prod_{j=1}^{i} \mathrm{Cat}(k_j), \tag{9}$$

We also set $p_k = 4^{-k} \mathrm{Cat}(k)$ and recall that $\sum p_k = 2$. Equation (9) thus becomes

$$P([T_n]_k = \tau_0) \quad = \quad \frac{4^{-|\tau_0|}}{p_n} \sum_{k_1 + ... + k_i = n'} \prod_{j=1}^{i} p_{k_j}.$$

We will show that when n is large, the main contribution in the previous sum is obtained when the indices $k_1, ..., k_i$ are such that only one of them is of order n' and the others are small in comparison. Let $A \geqslant 1$. Firstly, notice that at least one of the indices $k_1, ..., k_i$ is larger than n'/i. Secondly, let us evaluate the contribution of the sum when $k_1 \geqslant n'/i$ and k_2 is larger than A. Using the asymptotic behavior of the Catalan numbers (8) we have

$$\sum_{\substack{k_1 + \cdots + k_i = n' \\ k_1 \geqslant n'/i \\ k_2 \geqslant A}} \prod_{j=1}^{i} p_{k_j} \quad \leqslant \quad \sup_{k_1 \geqslant n'/i} p_{k_1} \cdot \left(\sum_{k_2 \geqslant A} p_{k_2} \right) \cdot \prod_{j=3}^{i} \left(\sum_{k_j = 1}^{\infty} p_{k_j} \right) \quad \leqslant \quad C n^{-3/2} A^{-1/2},$$

for some constant $C > 0$ which is independent of n. We thus deduce that the main contribution in (9) is made by those $k_1, ..., k_i$ for which one is big and the other are $O(1)$ thus

$$P([T_n]_k = \tau_0) \quad = \quad i\frac{4^{-|\tau_0|}}{p_n} \sum_{0 \leqslant k_2, \ldots, k_i \leqslant A} p_{n - \sum_{j \geqslant 2} k_j} \prod_{j=2}^{i} p_{k_j} + O(A^{-1/2}),$$

where the $O(A^{-1/2})$ is uniform in n. Thus letting $n \to \infty$ followed by $A \to \infty$ we deduce using (7) that the desired probability is equal to $i4^{-|\tau_0|}2^{\# L_{\tau_0}(k)-1}$.

Let us now compute $P([T_\infty]_k = \tau_0)$. To do this we split the event $\{[T_\infty]_k = \tau_0\}$ into $\# L_{\tau_0}(k)$ events according to the position of the kth vertex of the spine of T_∞ in the set $L_{\tau_0}(k)$. On each of these events, the critical subtrees grafted to the left and right-hand side of the jth vertex spine have a definite $[.]_{k-i}$ structure. But recall that for any $r \geqslant 0$ we know that $P([T]_r = |\tau|) = 4^{-|\tau|}2^{\# L_\tau(r)-1}$. From this, it is easy to see that $P([T_\infty]_k = \tau_0) = i4^{-|\tau_0|}2^{\# L_{\tau_0}(k)-1}$ thus completing the proof of the theorem. □

Remark. During the proof we encountered the formula $P([T_\infty]_k = \tau_0) = \# L_{\tau_0}(k) P([T]_k = \tau_0)$. This relation still holds in the general case of critical Galton-Watson trees and can serve as a characterization of the critical Galton-Watson conditioned to survive, see [13].

4 The UIPQ

Let $\mathbf{L}_\infty$ be the set of all labeled trees $\theta = (\tau, \ell)$ where $\tau \in \mathscr{S}$ and such that $\inf_i \ell(S_\tau(i)) = -\infty$. We now aim at extending the construction of Φ to elements of $\mathbf{L}_\infty$.

4.1 Extension of Φ

Let $\theta = (\tau, \ell) \in \mathbf{L}_\infty$. Again, we consider an embedding of τ in the plane, with isolated vertices. This is always possible (since τ is locally finite). The notion of a corner is unchanged in this setting, and there is still a notion of clockwise contour order for the corners of τ, this order being now a total order, isomorphic to $(\mathbb{Z}, \leqslant)$, rather than a cyclic order. We consider the sequence $(c_0^{(L)}, c_1^{(L)}, c_2^{(L)}, \ldots)$ of corners visited by the contour process of the left side of the tree in clockwise order. Similarly, we denote the sequence of corners visited on the right side by $(c_0^{(R)}, c_1^{(R)}, c_2^{(R)}, \ldots)$, in counterclockwise order. Notice that $c_0^{(L)} = c_0^{(R)}$ denotes the corner where the tree has been rooted. We now concatenate these two sequences into a unique sequence indexed by $\mathbb{Z}$, by letting, for $i \in \mathbb{Z}$,

$$c_i \quad = \quad \begin{cases} c_i^{(L)} & \text{if } i \geqslant 0 \\ c_{-i}^{(R)} & \text{if } i < 0\,. \end{cases}$$

In the sequel, we will write $c_i \leqslant c_j$ if $i \leqslant j$. For any $i \in \mathbb{Z}$, the *successor* $\mathcal{S}(c_i)$ of c_i is the first corner $c_j \geqslant c_{i+1}$ such that the label $\ell(c_j)$ is equal to $\ell(c_i) - 1$. From the assumption that $\inf_{i \geqslant 0} \ell(\mathrm{S}_\tau(i)) = -\infty$, and since all the vertices of the spine appear in the sequence $(c_i^{(L)})_{i \geqslant 0}$, it holds that each corner has one successor. We can associate with (τ, ℓ) an embedded graph q by drawing an arc between every corner and its successor. See Fig. 8. Note that, in contrast with the above description of the Schaeffer bijection on $\mathbf{L}_n \times \{0, 1\}$, we do not have to add an extra distinguished vertex ρ in this context. Hence, the vertex set of q exactly corresponds to the vertices of τ.

In a similar way as before, the embedded graph q is rooted at the edge emerging from the distinguished corner c_0 of $\varnothing$, that is, the edge between c_0 and its successor $\mathcal{S}(c_0)$. The direction

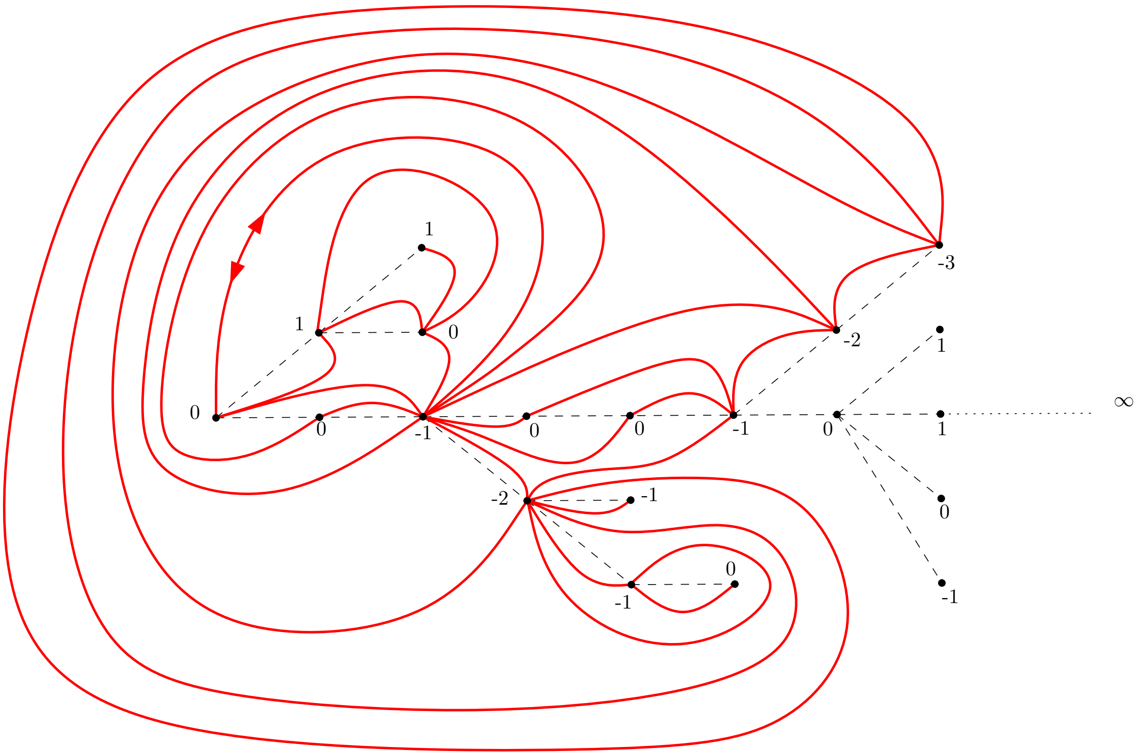


Figure 8: Illustration of the Schaeffer correspondence. The tree is represented in dotted lines and the quadrangulation in solid lines.

of the edge is given by an extra parameter $\eta \in \{0,1\}$, see Fig. 8. We leave the reader checking that $\Phi(\theta,\eta)$ is an infinite rooted quadrangulation with one end.

4.2 Bounds on distances

Let $q = \Phi(\theta,\eta)$ be constructed from a labeled tree $\theta = (\tau,\ell)$ in $\mathbf{L}_\infty$. We now present some useful bounds on the distances in q. Let us start with a trivial one: Since every pair $\{u,v\}$ of neighboring vertices in q satisfies $|\ell(u)-\ell(v)| = 1$ and thus for every $a,b \in q$ linked by a geodesic $a = a_0, a_1, \ldots, a_{\mathrm{d}^q_{\mathrm{gr}}(a,b)} = b$ we have the crude bound

$$\mathrm{d}^q_{\mathrm{gr}}(a,b) = \sum_{i=1}^{\mathrm{d}^q_{\mathrm{gr}}(a,b)} |\ell(a_i)-\ell(a_{i-1})| \geqslant \left| \sum_{i=1}^{\mathrm{d}^q_{\mathrm{gr}}(a,b)} \ell(a_i)-\ell(a_{i-1}) \right| = |\ell(a)-\ell(b)|. \tag{10}$$

A better lower bound is given by the so-called *cactus bound*

$$\mathrm{d}^q_{\mathrm{gr}}(a,b) \quad \geqslant \quad \ell(a)+\ell(b) - 2 \min_{v \in [\![a,b]\!]} \ell(v), \tag{11}$$

where we recall that $[\![a,b]\!]$ represents the geodesic line in τ between a and b. The idea goes as follows: let w be of minimal label on $[\![a,b]\!]$, and assume $w \notin \{a,b\}$ to avoid trivialities. Removing w breaks the tree τ into two connected parts, containing respectively a and b. Now a path from a to b has to "pass over" w using an arc between a corner (in the first component) to its successor (in the other component), and this can only happen by visiting a vertex with label less than $\ell(w)$. Using (10) we deduce that this path at length at least $\ell(a)-\ell(w)+\ell(b)-\ell(w)$, as wanted. We also have the upper bound

$$\mathrm{d_{gr}}(u,v) \quad \leqslant \quad 2+\ell(u)+\ell(v)-2\max\left\{\min_{c_1\leqslant c\leqslant c_2}\ell(\mathcal{V}(c)) : \{\mathcal{V}(c_1),\mathcal{V}(c_2)\}=\{u,v\}\right\}, \qquad (12)$$

where $\leqslant$ is the contour order on the corners of τ. Indeed, consider a corner c_i of u and a corner c_j of v and suppose that $i\leqslant j$. We construct the path starting from c_i and c_j following iteratively their successors. These two paths merge at the first corner after c_j with label $\min_{[c_i,c_j]}\ell-1$ and the concatenation of these two paths up to the merging point gives the bound $\mathrm{d_{gr}}(u,v)\leqslant 2+\ell(u)+\ell(v)-2\min\{\ell(c):c\in[c_i,c_j]\}$. The other cases are similar.

The topology induced by $\mathrm{d_{tree}}$ on the set of plane trees can obviously be extended to the set of labeled plane trees by considering the labeling as well. We will use this topology is the following proposition.

Proposition 9. *The extended mapping* $\Phi:\left(\mathbf{L}_\infty\cup\mathbf{L}_f\right)\times\{0,1\}\to\mathbf{Q}$ *is continuous.*

Proof. To prove the continuity of Φ, let $\theta_n=(\tau_n,\ell_n)$ be a sequence in $\mathbf{L}_\infty\cup\mathbf{L}_f$ converging to $\theta=(\tau,\ell)\in\mathbf{L}_\infty\cup\mathbf{L}_f$. Fix also $\eta\in\{0,1\}$. To simplify notation we let $\Phi(\theta):=\Phi(\theta,\eta)$ and $\Phi(\theta_n):=\Phi(\theta_n,\eta)$ in the following. If $\theta\in\mathbf{L}_f$ then $\theta_n=\theta$ for every n large enough, so the fact that $\Phi(\theta_n)\to\Phi(\theta)$ is obvious. So let us assume that $\theta\in\mathbf{L}_\infty$, with spine vertices $\mathrm{S}_\tau(0),\mathrm{S}_\tau(1),\ldots$.

Fix $r\geqslant 0$, we have to prove that $B_r(\Phi(\theta))=B_r(\Phi(\theta_n))$ for n large enough. We let $i=i(r)$ be the first $i\geqslant 1$ such that $\ell(\mathrm{S}_\tau(i))=-r-2$ (i exists since $\inf\ell(\mathrm{S}_\tau(.))=-\infty$). And let H be the height of the subtree obtained from τ by pruning off at $\mathrm{S}_\tau(i)$. Now, for every n large enough, it holds that $[\theta_n]_H=[\theta]_H$. We claim that this implies $B_r(\Phi(\theta))=B_r(\Phi(\theta_n))$. Indeed, by the cactus bound, any vertex $u\notin[\theta_n]_H$ satisfies $\mathrm{d_{gr}}(\varnothing,u)\geqslant r+2$ and thus does not belong to $B_r(\theta_n)$. Thus all the edges belonging to $B_r(\Phi(\theta_n))$ are arcs drawn between two vertices of $[\theta_n]_H$, it is then easy to check that all these edges have correspondents in $[\theta_i]$ and vice-versa. $\square$

4.3 The UIPQ

If τ is random (finite or infinite) plane tree, we can assign labels to its vertices in a uniform way: Conditionally on τ, we consider a sequence of independent random variables uniformly distributed over $\{-1,0,+1\}$ carried by each edge of τ. For any vertex $u\in\tau$, the label $\ell(u)$ of u is then defined as the sum of the variables carried by the edges along the geodesic path from the root $\varnothing$ to u, in particular $\ell(\varnothing)=0$. The random labeled tree $\theta=(\tau,\ell)$ hence obtained is called the uniform labeling of τ. In particular, when $\tau=T_n$ is uniformly distributed over the set of all plane trees with n edges, the labeled tree $\Theta_n=(T_n,\ell_n)$ obtained by this procedure is uniformly distributed over $\mathbf{L}_n$. When $\tau=T_\infty$ is the critical geometric Galton-Watson tree conditioned to survive, the labeled tree $\Theta_\infty=(T_\infty,\ell)$ obtained is called the uniform infinite labeled tree and almost surely belongs to $\mathbf{L}_\infty$.

Let η be uniformly distributed over $\{0,1\}$ independent of the Θ's. An easy consequence of Theorem 7 then entails that

$$(\Theta_n,\eta) \quad \xrightarrow[n\to\infty]{(d)} \quad (\Theta_\infty,\eta), \qquad (13)$$

for the local topology. By Theorem 5, the rooted and pointed quadrangulation $\Phi(\Theta_n,\eta)$ is uniformly distributed over $\mathbf{Q}_n^\bullet$. Hence, the quadrangulation obtained from it after forgetting the pointed vertex is uniformly distributed over $\mathbf{Q}_n$. Using (13) and Proposition 9 we deduce that Q_n converges towards some random infinite rooted quadrangulation Q_∞ that satisfies $Q_\infty=\Phi(\Theta_\infty,\eta)$ in distribution. Of course, this object is identified with the UIPQ introduced in Theorem 4.

Exercise 11. Using the construction $Q_\infty = \Phi(\Theta_\infty, \eta)$ compute the probability that the origin of the root edge in Q_∞ is of degree 1.

5 Applications

5.1 Reconstruction

Since $Q_\infty = \Phi((T_\infty, \ell), \eta)$ in distribution, a natural question is: Are the tree T_∞, the labeling ℓ (and the variable η) measurable functions of Q_∞? The answer is yes. In fact, it is sufficient to recover the labeling ℓ from Q_∞ and then the tree T_∞ can be recovered by an inverse Schaeffer construction as in Section 2.2.

Theorem 10 ([9])**.** *If $Q_\infty = \Phi(\Theta_\infty, \eta)$, then, with the usual identification of the vertices of Q_∞ with those of T_∞, one has, almost surely,*

$$\ell(u) - \ell(v) \quad = \quad \lim_{z \to \infty} \left(\mathrm{d}_{\mathrm{gr}}^{Q_\infty}(u, z) - \mathrm{d}_{\mathrm{gr}}^{Q_\infty}(v, z) \right) , \qquad \forall u, v \in Q_\infty . \tag{14}$$

We are not going to prove this theorem. It relies on fine properties of the discrete geodesics in the UIPQ. Note however that (14) can be interpreted as a generalization of (3) in the infinite setting. See [9] for more details.

5.2 Volume growth

If $(Y_n)_{n \geqslant 0}$ is a random process indexed by $\mathbb{N}$ with values in $\mathbb{R}_+$, we write $Y_n \succeq n^\alpha$ resp. $Y_n \preceq n^\alpha$ for $\alpha > 0$ if there exists a constant $\kappa > 0$ such that we almost surely have

$$\liminf_{n \to \infty} \frac{Y_n}{n^\alpha \log^{-\kappa}(n)} = \infty \quad \text{resp.} \quad \limsup_{n \to \infty} \frac{Y_n}{n^\alpha \log^{\kappa}(n)} = 0.$$

If we have both $Y_n \preceq n^\alpha$ and $n^\alpha \preceq Y_n$ we write $Y_n \approx n^\alpha$. In words $Y_n \approx n^\alpha$ means that almost surely Y_n grows like n^α up to polylogarithmic fluctuations.

The goal of this section is to show:

Theorem 11. *We have $|B_r(Q_\infty)| \approx r^4$.*

Although planar, the geometry of the UIPQ is thus weird since the volume growth of Q_∞ is very different from that of $\mathbb{Z}^2$ for example. The UIPQ has many folds, bottlenecks, mushrooms and bubbles at every scale. This nested "baby universe" structure (name used by some physicists) is responsible of the large volume growth of the UIPQ.

In the following, (T_∞, ℓ) always denotes the tree constructed above that we call the uniform infinite labeled tree. For $n \geqslant 0$ we denote by $\mathcal{T}_n$ the (labeled) subtree obtained from (T_∞, ℓ) after pruning at the n^{th} vertex of the spine s_n, that is, we remove all the offspring of s_n (but we keep s_n). Recall that $|\mathcal{T}_n|$ is the number of vertices of $\mathcal{T}_n$. We also denote by $\varnothing(\mathcal{T}_n)$ the diameter $\max\{\mathrm{dist}(u, v) : u, v \in \mathcal{T}_n\}$ where $\mathrm{dist}(.,.)$ is the graph distance in $\mathcal{T}_n$.

Proposition 12. *We have*

$$\begin{aligned} \varnothing(\mathcal{T}_n) &\approx n, \\ |\mathcal{T}_n| &\approx n^2, \qquad (15) \\ \Delta(\mathcal{T}_n) &\approx n^{1/2}. \qquad (16) \end{aligned}$$

Proof. Let $n \geqslant 0$. The tree $\mathcal{T}_n$ is composed of the first $n+1$ vertices on the spine together with $2n$ independent critical geometric Galton-Watson trees grafted to the right-hand side and to the left-hand side of $s_0, s_1, \ldots, s_{n-1}$ (when there is no tree on one side of a vertex of spine we consider that we grafted the tree with a single vertex). Thus we have

$$|\mathcal{T}_n| = 1 - n + \sum_{i=1}^{2n} X_i \quad \text{and} \quad n \leqslant \varnothing(\mathcal{T}_n) \leqslant 2\big(n + \max_{1 \leqslant i \leqslant 2n} H_i\big)$$

where $X_1, H_1, X_2, H_{2,} \ldots$ are respectively the size and the height of the $2n$ critical geometric Galton-Watson trees grafted on the n first vertices of the spine. Recall from that $P(H_1 \geqslant n) \sim n^{-1}$. From this we easily deduce using Borel-Cantelli lemma that eventually $H_i \leqslant i \log^2(i)$ and thus $\varnothing(\mathcal{T}_n) \approx n$.

Concerning the size $|\mathcal{T}_n|$, recall (8).The analogue of the law of the iterated logarithm in the case of infinite variance (see [4, Section 3.9]) directly show that $S_n = \sum_{i=1}^{2n} X_i \approx n^2$ which implies $|\mathcal{T}_n| \approx n^2$.

Let us now turn to (16). Recall that conditionally on the tree structure of $\mathcal{T}_n$ the labels evolve along the branches of $\mathcal{T}_n$ as a random walk $(Z_k)_{k \geqslant 0}$ whose increments are uniform in $\{-1, 0, +1\}$. Looking at the labels of $s_0, \ldots, s_{n-1}$ we deduce that $\Delta(\mathcal{T}_n) \geqslant \max_{0 \leqslant i \leqslant n-1} |\ell(s_i)|$ which gives the lower bound $\Delta(\mathcal{T}_n) \succeq n^{1/2}$. For the upper bound, we have

$$P\Big(\Delta(\mathcal{T}_n) > \log^2(n) n^{1/2} \,\Big|\, \text{Structure of } \mathcal{T}_n\Big) \quad \leqslant \quad |\mathcal{T}_n| P\Big(\sup_{0 \leqslant k \leqslant \varnothing(\mathcal{T}_n)} Z_k \geqslant \log^2(n) n^{1/2}\Big).$$

On the event $A_n := \{|\mathcal{T}_n| \leqslant n^3 \text{ and } \varnothing(\mathcal{T}_n) \leqslant n \log^2(n)\}$ the right-hand side of the last display is $O(n^{-2})$. But the previous estimates imply that A_n eventually occur and thus an application of Borel-Cantelli proves $\Delta(\mathcal{T}_n) \preceq n^{1/2}$. □

Proof of Theorem 11. Here also we consider that $Q_\infty = \Phi(T_\infty, \ell)$. We begin with the upper bound $|\text{Ball}(Q_\infty, r)| \preceq r^4$. Let $r \geqslant 1$. We introduce the tree $\mathcal{T}_{H_r}$ consisting of T_∞ pruned at the first vertex s_{H_r} of the spine reaching label $-r$. Thanks to (11) the vertices at distance less than $r+2$ from the origin of Q_∞ must be vertices of $\mathcal{T}_{H_{r+3}}$, hence

$$|\text{Ball}(Q_\infty, r)| \quad \leqslant \quad |\mathcal{T}_{H_{r+3}}|. \tag{17}$$

Since H_r is the hitting time of $-r$ by a random walk with steps distribution uniform in $\{-1, 0, +1\}$, we have $H_r = H_1^{(1)} + \ldots + H_1^{(r)}$ where de $H_1^{(i)}$ are i.i.d. and distributed as H_1. Standard calculations show that $P(H_1 \geqslant n) \sim C n^{-1/2}$ for some $C > 0$. Hence similar arguments as those presented in the proof of Proposition 12 show that $H_r \approx r^2$. Thus we can combine this fact together with (15) and (17) to complete the upper bound.

We now turn to the lower bound. For $r \geqslant 1$, we put

$$L_r \quad = \quad \sup\{i \geqslant 0 : \Delta(\mathcal{T}_i) < r\}.$$

Consistently with the preceding notation we write $\mathcal{T}_{L_r}$ for the tree T_∞ pruned at s_{L_r}. Using the bound (12), one sees that all the vertices in $\mathcal{T}_{L_r}$ are at a graph distance at most $3r+2$ from $\varnothing$ in Q_∞, which implies

$$\mathcal{T}_{L_r} \quad \leqslant \quad \text{Ball}(Q_\infty, 3r+4), \tag{18}$$

in terms of vertex sets. Using (16) we deduce that $L_r \approx r^2$. Henceforth by (15) we have $|\mathcal{T}_{L_r}| \approx r^4$ which together with (18) completes the proof of the proposition. □

5.3 The uniform infinite planar map

We can use the local convergence of random quadrangulations towards the UIPQ to deduce the local convergence of uniform planar maps towards an infinite random map that we call the *uniform infinite planar map*. Specifically, we let $\mathbf{M}_n$ be the set of all (rooted) planar maps with n edges and $\mathbf{M}_f$ the set of all finite planar maps. Similarly as in Section 1.4 we define a topology on $\mathbf{M}_f$ by saying that $m_n \to m$ if for all $r \geqslant 0$ we have

$$B_r(m_n) \quad \xrightarrow[n\to\infty]{} \quad B_r(m).$$

This topology is metrizable by $\mathrm{d_{map}}$. Here also, $\mathbf{M}_f$ is not complete for this topology and we have to work in its completion $\mathbf{M}$.

Theorem 13. *Let M_n be uniformly distributed over $\mathbf{M}_n$. Then there exists a random infinite map M_∞ called the uniform infinite planar map such that we have the following convergence in distribution for* $\mathrm{d_{map}}$

$$M_n \quad \xrightarrow[n\to\infty]{(d)} \quad M_\infty.$$

Proof. Obviously, this theorem will follow from Theorem 4 using the bijection between planar maps and quadrangulations: Recall from Proposition 2 the bijection $\mathcal{G} : \mathbf{Q}_f \to \mathbf{M}_f$ that sends a rooted quadrangulation with n faces on a rooted planar map with n edges. In particular, if Q_n is uniformly distributed over $\mathbf{Q}_n$ then $\mathcal{G}(Q_n)$ is uniformly distributed over $\mathbf{M}_n$. The mapping $\mathcal{G}$ can be extended continuously $\mathcal{G} : \mathbf{Q} \to \mathbf{M}$. Indeed suppose that $q_n \to q$ and suppose $q \notin \mathbf{Q}$ to avoid trivialities. Since q_n is bipartite we can partition the vertices of q_n into black and white vertices $q_n^\circ \cup q_n^\bullet$. We suppose that the origin of q_n is in q_n° such that the vertices of $\mathcal{G}(q_n)$ are in correspondence with q_n°. The key observation is that if $u, v \in q_n^\circ$ then we have

$$\frac{1}{2}\mathrm{d}_{\mathrm{gr}}^{q_n}(u,v) \quad \leqslant \quad \mathrm{d}_{\mathrm{gr}}^{\mathcal{G}(q_n)}(u,v). \tag{19}$$

Hence the ball $B_r(\mathcal{G}(q_n))$ of radius r in $\mathcal{G}(q_n)$ is a deterministic function of $B_{2r}(q_n)$. Since $B_{2r}(q_n)$ eventually stabilizes the same holds for $B_r(\mathcal{G}(q_n))$. This shows that $\mathcal{G} : \mathbf{Q}_f \to \mathbf{M}_f$ can be extended continuously to $\mathbf{Q} \to \mathbf{M}$.

It then flows from Theorem 4 that

$$M_n \overset{(d)}{=} \mathcal{G}(Q_n) \quad \xrightarrow[n\to\infty]{(d)} \quad M_\infty \overset{(d)}{=} \mathcal{G}(Q_\infty),$$

in distribution for $\mathrm{d_{map}}$, thus completing the proof of the theorem. □

The uniform infinite planar map (UIPM) can thus be seen as the infinite map $\mathcal{G}(Q_\infty)$ constructed by duality from the UIPQ. But there is another notion of duality: If m is a planar map we define the dual map $\mathcal{D}(m)$ by placing in each face of m a vertex of $\mathcal{D}(m)$ and for each edge e in m we introduce a new edge in $\mathcal{D}(m)$ connecting the two vertices in $\mathcal{D}(m)$ corresponding to the two faces in m that share the edge e.

Proposition 14. *The UIPM is self-dual, that is $M_\infty = \mathcal{D}(M_\infty)$ in distribution.*

Proof. We can suppose $M_\infty = \mathcal{G}(Q_\infty)$ and $Q_\infty = \Phi(\Theta_\infty, \eta)$. Then notice that $\mathcal{D}(M_\infty) = \mathcal{G}(\overleftarrow{Q_\infty})$ where $\overleftarrow{q}$ is the quadrangulation obtained from q by reversing the orientation of the oriented edge of q (draw a picture). We can conclude since

$$M_\infty = \mathcal{G}(Q_\infty) = \mathcal{G}\Big(\Phi(\Theta_\infty, \eta)\Big) \quad \overset{(d)}{=} \quad \mathcal{G}\Big(\Phi(\Theta_\infty, 1-\eta)\Big) = \mathcal{G}(\overleftarrow{Q_\infty}) = \mathcal{D}(M_\infty).$$

□

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