Observability of Baouendi-Grushin-type equations through resolvent estimates

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These slides present only part of our joint work.

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Introduction

We are interested in **sub-Laplacians** Δ, which are generalizations of the usual Euclidean (or Riemannian) Laplacian. We look at the associated PDEs; the present talk focuses on a family of Schrödinger-type equations. We want to understand **how energy of the solutions propagates**.

The concept of observability gives a way to understand this propagation. Consider a linear evolution PDE on a manifold M:

$$\partial_t U = PU, \quad U_{|t=0} = U_0 \in \mathcal{H}$$

(a) Conservative: if P is anti self-adjoint. e.g.: wave, Schrödinger;
(b) Dissipative: if P is self-adjoint and $P \leqslant 0$. e.g.: Heat.
The solution is given by the semi-group

$$U(t) = e^{tP} U_0, \quad t \geqslant 0.$$

Observability: Observability holds in time $T > 0$ on a smaller region $\emptyset \neq \omega \subset M$ if $\exists C > 0$ such that

$$\|e^{TP} U_0\|^2_{L^2(M)} \leqslant C(T, \omega) \int_0^T \|\mathbf{1}_\omega e^{tP} U_0\|^2_{L^2(M)} dt.$$

Introduction

Here we consider the Baouendi-Grushin operator

$$\Delta_\gamma = \partial_x^2 + x^{2\gamma}\partial_y^2, \quad \gamma > 0$$

with domain

$$D(\Delta_\gamma) = \{f \in L^2(M) : \partial_x^2 f, |x|^{2\gamma}\partial_y^2 f \in L^2(M),\ f_{|\partial M} = 0\},$$

where

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \qquad M = (-1,1)_x \times \mathbb{T}_y.$$

$$\Delta_\gamma = X_1^2 + X_2^2, \quad X_1 = \partial_x,\ X_2 = |x|^\gamma \partial_y.$$

- If $\gamma \in \mathbb{N}$: γ brackets to recover $T_{(0,y)}M$

$$\mathrm{Span}(X_1, [X_1, X_2], \ldots, \mathrm{Ad}_{X_1}^\gamma X_2) = \mathbb{R}^2 = T_{(0,y)}M$$

- Loss of regularity of order $\frac{\gamma}{\gamma+1}$:

$$\|f\|_{H^{\frac{1}{\gamma+1}}(M)} \lesssim \|\nabla_\gamma f\|_{L^2(M)}$$

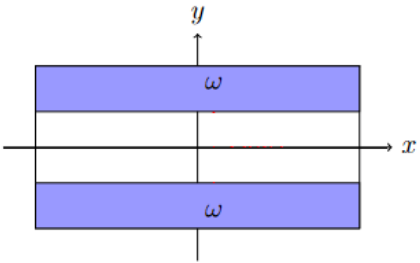
where the associated gradient is $\nabla_\gamma = (\partial_x, |x|^\gamma \partial_y)$.

Observability for subelliptic Schrödinger equations

Consider, for $\gamma \geqslant 1$ and $s \in \mathbb{N}$, the Schrödinger-type equation

$$i\partial_t u - (-\Delta_\gamma)^s u = 0 \qquad \text{with} \quad \Delta_\gamma = \partial_x^2 + x^{2\gamma}\partial_y^2 \tag{1}$$

in $(-1,1)_x \times \mathbb{T}_y$, and ω is an horizontal strip:



Recall that (1) is observable in ω at T if

$$\|u(0,\cdot)\|^2_{L^2(M)} \leqslant C_{T,\omega} \int_0^T \|u(t,\cdot)\|^2_{L^2(\omega)} dt$$

and $T_{\min}$ = minimal time of observability.

Main result

Recall that (1) is observable in ω at T if

$$\|u(0,\cdot)\|^2_{L^2(M)} \leqslant C_{T,\omega} \int_0^T \|u(t,\cdot)\|^2_{L^2(\omega)} dt$$

and $T_{\min}$ = minimal time of observability.

Theorem (C. L., C. Sun - 2020)

1. *If $s < \frac{\gamma+1}{2}$, then $T_{min} = +\infty$ (never observability);*
2. *If $s > \frac{\gamma+1}{2}$, then $T_{min} = 0$ (always observability);*
3. *If $s = \frac{\gamma+1}{2}$, then $0 < T_{min} < +\infty$.*

Balance between subelliptic effects (measured by γ) and the propagation strength (measured by s).

Remark: The Riemannian case corresponds to $\gamma = 0$: observability holds in finite time if $s = 1/2$ (half-wave equation) and in arbitrarily small time if $s = 1$ (Schrödinger equation), at least under the geometric control condition (GCC).

Idea of the proof: a resolvent estimate

To disprove observability, we use Gaussian beams arguments.
To prove observability (notably Point 2), we use a **resolvent estimate**, i.e., a time-independent inequality on the maximal possible concentration of quasimodes outside ω.

Theorem (C. L., C. Sun - 2020)

There exist $C, h_0 > 0$ *such that for any* $u \in D(\Delta_\gamma)$ *and any* $0 < h \leqslant h_0$, *there holds*

$$\|u\|_{L^2(M)} \leqslant C(\|u\|_{L^2(\omega)} + h^{-(\gamma+1)}\|(h^2\Delta_\gamma + 1)u\|_{L^2(M)}).$$

$\Rightarrow$ observability follows by abstract arguments, Burq-Zworski 2004.

Sketch of proof of Resolvent estimate: different regimes

For $(h^2\Delta_\gamma + 1)u = f$, we want to prove that

$$\|u\|_{L^2(M)} \leqslant C\|u\|_{L^2(\omega)} + Ch^{-(\gamma+1)}\|f\|_{L^2(M)}.$$

By contradiction, assume that $(h_n^2\Delta_\gamma + 1)u_n = f_n$,

$$\|u_n\|_{L^2(M)} \sim 1,\ \|u_n\|_{L^2(\omega)} = o(1),\ \|f_n\|_{L^2(M)} = o(h_n^{\gamma+1}),\ h_n \to 0.$$

- w.l.o.g., we may assume that $|h^2\Delta_\gamma| \sim 1$ since u_n is a quasimode. Hence, $u_n = \psi(-h_n^2\Delta_\gamma)u_n$, $\psi \equiv 1$ near 1.
- Subelliptic estimate: $\||D_y|^{\frac{2}{\gamma+1}}u\|_{L^2(M)} \leqslant C_1\|\Delta_\gamma u\|_{L^2(M)}$, i.e., $|D_y| \lesssim h^{-(\gamma+1)}$.

Different regimes

a. Horizontal propagation regime: $|D_y| \ll h^{-1} \to$ second microlocalization ideas.
b. GCC regime: $|D_y| \sim h^{-1} \to$ microlocal defect measures.
c. Degenerate regime: $h^{-1} \ll |D_y| \lesssim h^{-(\gamma+1)}$ (leads to $|x| \ll 1$).

One example: the degenerate regime

The key observation is that

$$[\Delta_\gamma, x\partial_x + (\gamma+1)y\partial_y] = 2\Delta_\gamma.$$

We insert a suitable truncation $\chi(y)$ to make the vector field $x\partial_x + (\gamma+1)y\partial_y$ well-defined. Choose χ such that

$$\mathrm{supp}(\chi'(y)) \subset \omega.$$

Computing

$$h_n^2([\Delta_\gamma, \chi_1(x)\chi(y)(x\partial_x + (\gamma+1)y\partial_y)]u_n, u_n)_{L^2(M)}$$

we obtain the estimate

$$\begin{aligned}\|h_n\nabla_\gamma u_n\|^2_{L^2(M)} \leqslant O(h_n)\|h_n\nabla_\gamma u_n\|^2_{L^2(M)} + O(1)\|h_n\nabla_\gamma u_n\|^2_{L^2(\mathrm{supp}(\chi'))} \\ + O(1)\|f_n\|_{L^2(M)}(\|\partial_x u_n\|_{L^2(M)} + \|\partial_y u_n\|_{L^2(M)}) + O(h_n).\end{aligned}$$

We have $\|\partial_y u_n\|_{L^2(M)} \lesssim h^{-(\gamma+1)}$, $\|\partial_x u_n\|_{L^2(M)} \lesssim h^{-1}$ and $\|f_n\|_{L^2(M)} = o(h^{\gamma+1})$, hence $\|h_n\nabla_\gamma u_n\|_{L^2(M)} = o(1)$. Contradiction with $h_n^2\Delta_\gamma \sim -1$ and $\|u_n\|_{L^2(M)} = 1$.

Other points of view and conclusion

With Clotilde Fermanian Kammerer, we used pseudodifferential calculus adapted to the group structure to prove observability results for subelliptic Schrödinger equations. This requires tools from noncommutative harmonic analysis (Fourier transform).

$$\mathrm{Op}_\varepsilon(\sigma)f(x) = \int_{G\times\widehat{G}} \mathrm{Tr}(\pi^\lambda_{y^{-1}x}\sigma(x,\varepsilon^2\lambda))f(y)|\lambda|^d d\lambda dy.$$

Conclusion: Energy propagates more slowly for subelliptic Schrödinger equations in the directions needing brackets to be generated. This has to be compared with our anterior result that subelliptic wave equations are never observable ("Subelliptic wave equations propagate at null speed").