Quantum limits of sub-Laplacians

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**Setting:** Riemannian manifold \((M, g)\) without boundary. There is a canonical **volume** \(d\text{vol}\) on the manifold.

We consider the **Laplacian** \(\Delta\) (Laplace-Beltrami operator). Assume that \(M\) is compact so that \(\Delta\) has a compact resolvent.

We consider an **orthonormal basis of eigenfunctions**

\[-\Delta \psi_k = \lambda_k \psi_k\]

with \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\) and \(\lambda_k \to +\infty\).

First kind of results: count **eigenvalues**, asymptotics of

\[N(\lambda) = \#\{\lambda_k \leq \lambda\}\]

(Weyl law, well-known).

**Our goal:** get information on general sequences of **eigenfunctions**, in the high-frequency limit \((k\text{ large})\).
Given a sequence of normalized eigenfunctions with eigenvalues $\rightarrow +\infty$, we are interested in

$$|\psi_k|^2 d\text{vol} \in \mathcal{P}(M)$$

where $\mathcal{P}(M)$ is the set of Radon probability measures on $M$. [Linked to probability of presence of quantum particles.]

There is at least one weak-* accumulation point $\nu \in \mathcal{P}(M)$, which verifies

$$\int_M \phi(q)|\psi_{k_n}(q)|^2 d\text{vol}(q) \xrightarrow{n \rightarrow +\infty} \int_M \phi(q)d\nu(q).$$

for some sequence $(k_n)_{n \in \mathbb{N}}$ tending to $+\infty$ and for any $\phi \in C^0(M)$.

**Interpretation:** $\nu$ captures the concentration of $|\psi_{k_n}|^2$ on the manifold, in the high-frequency limit.

Such a $\nu$ is called a **Quantum Limit** (QL).

**Question:** determine **the set of all QLs**.

**Example:** In $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$, the only QL is $\frac{1}{2\pi} dx$. 
**Sphere** $S^2$ with canonical metric.

Example of sequence of eigenfunctions: Spherical harmonics.

For instance $\psi_k(x) = c_k(x_1 + ix_2)^k$ concentrates on $x_3 = 0$ since

$$|\psi_k(x)|^2 d\text{vol} = c_k^2(x_1^2 + x_2^2)^k d\text{vol} = c_k^2(1 - x_3^2)^k d\text{vol} \xrightarrow[k \to +\infty]{} \delta_{x_3=0}.$$ 

QLs = All measures constant along big circles.

Proof: Linear (and orthogonal !) combinations of the above example.
**2D Torus** $\mathbb{T}^2$. An example of sequence of normalized eigenfunctions: $e^{i\tau \cdot x}$ for $\tau \in \mathbb{Z}^2$, with associated eigenvalue $|\tau|^2$.

The **spectrum is highly degenerate** (same eigenvalue for many different $\tau$-s). Therefore, we can imagine many other sequences of high-frequency eigenfunctions (taking linear combinations of elements of the above sequence), and hence the **set of all QLs can be very complicated**.

Jakobson characterized all Quantum Limits. In particular, he proved that

*Any QL is absolutely continuous w.r.t. the Lebesgue measure.*

⇒ The set of QLs strongly depends on the sequence of (high-frequency) eigenfunctions that one considers.
If the system is **completely integrable** (sphere, torus, etc), one may expect to characterize quite precisely the set of all QLs, as above, thanks to explicit formulas. This set may be large and complicated if the spectrum is very degenerate.

If the geodesic flow is **ergodic** (example: negative curvature) then “almost all” QLs are proportional to the volume measure. This is Shnirelman’s theorem. Removing the “almost all” in negative curvature is the Quantum Unique Ergodicity (QUE) conjecture of Sarnack and Rudnick.

In this talk, we address the same problem of finding QLs, but for **sub-Laplacians**.
Let $M$ be a smooth connected compact manifold of dimension $n$ and $\mu$ be a smooth volume on $M$. Let $X_1, \ldots, X_K$ be smooth vector fields on $M$, and $\mathcal{D} = \text{Span}(X_1, \ldots, X_K)$ (the “distribution”). We assume

$$\text{Lie}(\mathcal{D}) = TM.$$ 

We define the **sub-Laplacian**

$$\Delta = - \sum_{i=1}^{K} X_i^* X_i = \sum_{i=1}^{K} X_i^2 + \text{div}_\mu(X_i)X_i,$$

where

- $\text{Star} = \text{transpose in } L^2(M, \mu)$;
- $\text{div}_\mu X$ is defined by $L_X \mu = (\text{div}_\mu X)\mu$.

Sub-Laplacians are hypoelliptic, meaning that $\Delta u \in C^\infty \Rightarrow u \in C^\infty$. 
Examples of sub-Laplacians

[These examples are on non-compact manifolds but can be “made compact”.

- **Heisenberg:** $X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = \partial_y + x\partial_z$ in $\mathbb{R}^3$. Then $[X_1, X_2] = \partial_z$.

- **Engel:** $X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = \partial_y + x\partial_z + x^2\partial_w$ in $\mathbb{R}^4$. Here, $[X_1, X_2] = \partial_z + 2x\partial_w$ and $[X_1, [X_1, X_2]] = 2\partial_w$. This is a step 3 sub-Laplacian.

Sometimes, there are complicated relations between the vector fields and their brackets: $[X_1, X_3] = X_1$, etc.
Motivations:

- Understand subelliptic operators (Hörmander);
- In particular the role of brackets...
- ... and of geodesics (normal+abnormal)!
- Related problems of control/observability of subelliptic PDEs: for example, subelliptic wave equations are never observable.

See also heat equation, Schrödinger equation, ... Intrinsically hypoelliptic: Kolmogorov equation (probabilities).

Note: Sub-Riemannian geometry is used for describing motion under constraint.
Example: the Heisenberg (sub)-Laplacian

Natural sub-Laplacian: \( \Delta = \partial_x^2 + (\partial_y - x\partial_z)^2 \) on \( \mathbb{R}^3 \).

Endow \( \mathbb{R}^3 \) with the product law

\[
(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - xy').
\]

With this law, \( \tilde{H} = (\mathbb{R}^3, \star) \) is a Lie group, which is isomorphic to the group of matrices

\[
\begin{cases}
\begin{pmatrix} 1 & x & -z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, & x, y, z \in \mathbb{R}
\end{cases}
\]

endowed with the standard product law on matrices.

Natural vector fields on \( \mathbb{R}^3 \), i.e., left-invariant for \( \star \):

\[ X = \partial_x \] and \( Y = \partial_y - x\partial_z \).
We take a quotient of $\tilde{H} = (\mathbb{R}^3, \star)$: we note that
$\Gamma = \sqrt{2\pi} \mathbb{Z} \times \sqrt{2\pi} \mathbb{Z} \times 2\pi \mathbb{Z}$ is a discrete subgroup of $\tilde{H}$ and we consider
the left-quotient
$$H = \Gamma \backslash \tilde{H},$$
which is compact.

We still consider $\Delta_H = \partial_x^2 + (\partial_y - x \partial_z)^2$ on $H$.

This is the simplest example of a sub-Laplacian in a compact geometry.
Taking Fourier in $z$, the spectrum of $-\Delta_H = -(\partial_x^2 + (\partial_y - x\partial_z)^2)$ becomes explicit:

$$\text{sp}(-\Delta_H) = \{ \lambda_{\ell,\alpha} = (2\ell + 1)|\alpha| \mid \ell \in \mathbb{N}, \alpha \in \mathbb{Z} \setminus \{0\} \} \cup \{ \mu_{k_1,k_2} = 2\pi(k_1^2 + k_2^2) \mid (k_1, k_2) \in \mathbb{Z}^2 \}.$$ 

The multiplicity of $\lambda_{\ell,\alpha}$ is equal to $|\alpha|$ multiplied by the number of decompositions of $\lambda_{\ell,\alpha}$ into the form $(2\ell' + 1)|\alpha'|$.

$\Rightarrow$ The spectrum is very degenerate.

We recognize a factorization of $-\Delta_H$ as

$$-\Delta_H = R\Omega$$

with $R = |\partial_z|$ and $\Omega$ “a harmonic oscillator” (both are roughly in $\Psi^1(M)$).

The operators $R$ and $\Omega$ commute.
Quantum Limits of $\Delta_H$

Proposition

Any probability measure on $H$ which is invariant by the flow of $\partial_z$ is a QL of some normalized sequence of eigenfunctions of $-\Delta_H$.

In particular, a Dirac mass along a vertical line is a QL ("vertical" here means ‘z axis’): use ground state of the harmonic oscillator.

Therefore, there exist QLs which are not absolutely continuous with respect to the smooth volume $\mu$ (or the Lebesgue measure).

Goal of the next slide: detection of oscillations, in addition to concentration.
For $N \in \mathbb{Z}$, we write $\mathcal{S}^N$ for the homogeneous functions of order $N$ defined on $T^* M \setminus 0$ (space of “symbols”).

To $a \in \mathcal{S}^N$, we associate an operator $\text{Op}(a)$ (its (Weyl) quantization):

$$a \mapsto \text{Op}(a).$$

The application $a \mapsto \text{Op}(a)$ is linear and the operators $\text{Op}(a)$ belong to the set $\Psi^N$ of pseudodifferential operators of order $N$.

The map $\sigma_P : \Psi^N \to \mathcal{S}^N$ allows to recover the (principal) symbol from the operator: $\sigma_P(\text{Op}(a)) = a$.

Typical (principal) symbols: $1, \quad b(q), \quad p_j, \quad |p|^2$

Typical operators : $\text{Id}, \quad \times b(q), \quad i^{-1} \partial_{q_j}, \quad \Delta$.

$S^* M$: cosphere bundle $= S(T^* M) = (T^* M \setminus \{0\})/(0, +\infty)$. 

**Phase-space lift of Quantum Limits**

**Definition:** Let \((u_k)_{k \in \mathbb{N}^*}\) be a bounded sequence of \(L^2(M)\) which converges weakly to 0. We say that a measure \(\nu\) on \(S^*M\) is a **microlocal defect measure** of this sequence if, up to extraction of a subsequence, for any symbol \(a\) of order 0, there holds

\[
(\text{Op}(a)u_k, u_k)_{L^2(M)} \xrightarrow{k \to +\infty} \int_{S^*M} a \, d\nu.
\]

For example, if we consider the **wave-packet**

\[
u_k = k^{n/4} \rho(\sqrt{k}(q - q_0)) \, e^{ikp_0 \cdot q},
\]

it has only one microlocal defect measure, which is \(\delta_{q_0} \otimes \delta_{p_0}\).

**Definition:** Similarly, for two sequences \((u_k)_{k \in \mathbb{N}^*}\) and \((v_k)_{k \in \mathbb{N}^*}\), one can consider joint microlocal defect measures defined by

\[
(\text{Op}(a)u_k, v_k)_{L^2(M)} \xrightarrow{k \to +\infty} \int_{S^*M} a \, d\nu_{\text{joint}}.
\]

In the case where \(u_k = \psi_k\) is our sequence of (normalized) eigenfunctions, microlocal defect measures are called **microlocal Quantum Limits**.
Invariance property of microlocal QLs

Microlocal defect measures have no special properties in general, but microlocal QLs, since they are limits of eigenfunctions of the Laplacian are invariant by the geodesic flow \( \exp(t\vec{H}) \).

[Proof: To see it, we note that
\[
([\text{Op}(a), \sqrt{-\Delta_g}]\psi_k, \psi_k)
= (\text{Op}(a)\sqrt{-\Delta_g}\psi_k, \psi_k) - (\text{Op}(a)\psi_k, \sqrt{-\Delta_g}\psi_k) = 0
\]
for any \( a \in \mathcal{S}^0(M) \). The principal symbol of \([\text{Op}(a), \sqrt{-\Delta_g}]\) is \( \{a, H\}/i \), therefore, taking the limit \( k \to +\infty \), we get
\[
\int_{S^*M} \{a, H\} d\nu = 0.
\]

"Therefore", \( \int_{S^*M} a\{H, d\nu\} = 0 \) holds for any \( a \in \mathcal{S}^0(M) \), thus \( \{H, d\nu\} = 0 \), i.e., \( e^{t\vec{H}}\nu = 0 \).]
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$$= (\text{Op}(a)\sqrt{-\Delta_g}\psi_k, \psi_k) - (\text{Op}(a)\psi_k, \sqrt{-\Delta_g}\psi_k) = 0$$

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"Therefore", $\int_{S^*M} a\{H, d\nu\} = 0$ holds for any $a \in \mathcal{S}^0(M)$, thus $\{H, d\nu\} = 0$, i.e., $e^{t\vec{H}}\nu = 0$.]

This fact is not true in sub-Riemannian geometry: the above proof does not work anymore since $\sqrt{-\Delta}$ is not a pseudodifferential operator for a general sub-Laplacian $\Delta$.

$\Rightarrow$ Invariance properties of microlocal QLs in sR geometry ?
The cotangent space $T^*M$ in sR geometry

In sub-Riemannian geometry, the Hamiltonian $H(q, p) = \sigma_p(-\Delta)$ may vanish even when $p \neq 0$. We decompose $T^*M$ into two sets: the set $\Sigma$ where $H$ vanishes ("characteristic cone"), and the complementary set.

As we will see, $\Sigma$ plays a very important role in our problem: most microlocal QLs are supported in $\Sigma$ (‘they oscillate only in the directions of $\Sigma$’).

Example: for $H$, we have $H(x, y, z; p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + (p_y - xp_z)^2)$.

$$H(x, y, z; p_x, p_y, p_z) = 0 \iff p_x = p_y - xp_z = 0.$$ 

Above any point $q \in H$, $\Sigma$ is a line.

The sphere bundle $S\Sigma = \Sigma/\mathbb{R}^+$ is a twofold covering of $M=H$ (there are two points above each point of $M$).

We make the identification

$$S^*M = S\Sigma \cup U^*M,$$

where $U^*M = \{H = 1/2\}$.

**NB:** Differentiating $\sqrt{H}$, we get very large denominators near $\Sigma$. 
Let \((\psi_k)_{k \in \mathbb{N}}\) be an orthonormal basis of eigenfunctions of \(\Delta_H\).

Then there exists a \textbf{density-one subsequence} \((k_n)_{n \in \mathbb{N}}\) such that any microlocal QL associated to \((\psi_{k_n})_{n \in \mathbb{N}}\) is supported in \(S\Sigma\) (it gives no mass to \(U^*M\)).

Moreover, any microlocal QL \(\nu\) of \((\psi_k)_{k \in \mathbb{N}}\) may be decomposed as \(\nu = \nu_0 + \nu_\infty\) where

- \(\nu_\infty\) is supported in \(S\Sigma\) and is invariant by the lift of the Reeb flow.
- \(\nu_0(S\Sigma) = 0\) and \(\nu_0\) is invariant by the sub-Riemannian geodesic flow.

This result holds with far more generality, see Colin de Verdière, Hillairet, Trélat, \textit{Duke Math. Journal}, 2018.
In all the sequel, we work under the following assumption:

**Assumption (A).** There exist $Z_1, \ldots, Z_m$ smooth global vector fields on $M$ such that:

1. The vector fields $Z_1(x), \ldots, Z_m(x)$ complete $\mathcal{D}_x$ into a basis of $T_x M$ at any point $x \in M$ where $\mathcal{D}_x \neq T_x M$ (in particular, they are independent);

2. For any $1 \leq i, j \leq m$, there holds $[\Delta, Z_i] = [Z_i, Z_j] = 0$. 

It is satisfied in the following cases:

- Heisenberg
- Engel
- Quasi-contact
- Heisenberg-type sub-Laplacians (recent works by Fermanian-Fischer and Fermanian-L. using non-commutative Fourier analysis);

For manifolds obtained as products of the previous examples (and associated sub-Laplacians obtained by sum).
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It is satisfied in the following cases:

- Heisenberg, $\partial_x^2 + (\partial_y + x\partial_z)^2$, Engel $\partial_x^2 + (\partial_y + x\partial_z + x^2\partial_w)^2$
- Quasi-contact $\partial_x^2 + (\partial_y + x\partial_z)^2 + \partial_w^2$ on $\mathbb{R}^4$ (with abnormalities);
- Heisenberg-type sub-Laplacians (recent works by Fermanian-Fischer and Fermanian-L. using non-commutative Fourier analysis);
- For manifolds obtained as products of the previous examples (and associated sub-Laplacians obtained by sum).
Splitting of the cotangent bundle

Spectral theory ⇔ Geometry
Splitting of QLs ⇔ Splitting of $\Sigma$.

$\mathcal{P} = \text{set of all subsets of } \{1, \ldots, m\}$.
$\Sigma$ can be decomposed as a disjoint union

$$\Sigma = \bigcup_{\mathcal{J} \in \mathcal{P}} \Sigma_{\mathcal{J}}$$

where $\Sigma_{\mathcal{J}}$ is the set of points $(q, p) \in \Sigma$ with

$$(j \in \mathcal{J}) \iff (\sigma_{\mathcal{P}}(Z_j)(q, p) \neq 0).$$

We analyze microlocal QLs **separately** in each $\Sigma_{\mathcal{J}}$ thanks to the next result.
Let $\Delta$ satisfy Assumption (A) and $(\varphi_k)_{k \in \mathbb{N}^*}$ be a sequence of eigenfunctions of $-\Delta$ with eigenvalues $\lambda_k \to +\infty$. One can write

$$\varphi_k = \varphi_\emptyset^k + \sum_{\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}} \varphi_\mathcal{J}^k,$$

(up to extraction of a subsequence) with the following properties:

- The sequence $(\varphi_k)_{k \in \mathbb{N}^*}$ has a unique Quantum Limit $\nu$;
- $\varphi_\mathcal{J}^k$ is an eigenfunction of $-\Delta$ with eigenvalue $\lambda_k$;
- The sequence $(\varphi_\emptyset^k)_{k \in \mathbb{N}^*}$ admits a unique microlocal defect measure $\nu_\emptyset$, where $\nu_\emptyset \in \mathcal{M}_+(S^*M)$ and $\nu_\emptyset(S\Sigma) = 0$;
- $\forall \mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, $(\varphi_\mathcal{J}^k)_{k \in \mathbb{N}^*}$ admits a unique microlocal defect measure $\nu_\mathcal{J}$, having all its mass contained in $S\Sigma_\mathcal{J}$;
- For any $\mathcal{J} \neq \mathcal{J}' \in \mathcal{P}$, the joint microlocal defect measure of the sequences $(\varphi_\mathcal{J}^k)_{k \in \mathbb{N}^*}$ and $(\varphi_\mathcal{J}'^k)_{k \in \mathbb{N}^*}$ vanishes. As a consequence,

$$\nu = \sum_{\mathcal{J} \in \mathcal{P}} \nu_\mathcal{J}.$$
Idea: the operator

\[ E = -\Delta + \sum_{j=1}^{m} Z_j^* Z_j \in \Psi^2(M), \]

is elliptic. We do the joint spectral calculus of \( E, \Delta \) and \( Z_j^* Z_j \): we split

\[ L^2(M) = \bigoplus \mathcal{H}_{\alpha,\beta_1,\ldots,\beta_m}. \]

where, on \( \mathcal{H}_{\alpha,\beta_1,\ldots,\beta_m} \),

- \(-\Delta\) acts as \( \alpha \)
- \( Z_j \) acts as \( \beta_j \).

and hence \( E \) acts as \( \alpha + \sum |\beta_j|^2 \). Given an eigenfunction \( \varphi_k \), we select (asymptotically as \( k \to +\infty \)) to be in \( \varphi_k^{\mathcal{J}} \) (for \( \mathcal{J} \neq \emptyset \)) the parts of \( \varphi_k \) in the eigenspaces on which:

1. \(-\Delta \ll E \) (i.e, \( \alpha \ll \alpha + \sum |\beta_j|^2 \));
2. if \( i \notin \mathcal{J} \), then \( Z_j^* Z_j \ll E \);
3. if \( j \in \mathcal{J} \), then \( Z_j^* Z_j \gtrsim E \).

[More details later.]
Our next problem: We consider the manifold $H^m$ for some $m \geq 2$, and the associated sub-Laplacian, which is

$$\Delta = \partial_{x_1}^2 + (\partial_{y_1} - x_1 \partial_{z_1})^2 + \ldots + \partial_{x_m}^2 + (\partial_{y_m} - x_m \partial_{z_m})^2.$$  

Note that $Z_j = \partial_{z_j}$ yields an elliptic operator $E = -\Delta + \sum Z_j^* Z_j$, with the good commutation properties.

We know that most eigenfunctions concentrate in $S\Sigma$; can we say more?

In the sequel, we focus on microlocal QLs supported in $S\Sigma$. We will characterize all these microlocal QLs.

In the sequel, $m = 2$.

The underlying manifold $H^2$ is of dimension 6, and $\Sigma \subset T^* M$ is the set of points $(q, p)$ where

$$p_{x_i} = p_{y_i} - x_i p_{z_i} = 0, \quad i = 1, 2.$$  

Therefore, $\Sigma$ is of dimension 2 above any point of $H^2$, and the sphere bundle $S\Sigma$ is of dimension 1 above any point of $H^2$. 
We write

\[-\Delta = R_1 \Omega_1 + R_2 \Omega_2\]

where \( R_j = |\partial z_j| \) and \( \Omega_j = -\Delta_j / R_j \) is an harmonic oscillator. Here \( \Delta_j = \partial_{x_j}^2 + (\partial_{y_j} - x_j \partial_{z_j})^2 \).

Therefore, \( R_1, R_2, \Omega_1 \) and \( \Omega_2 \) commute. Again, we perform a careful joint spectral analysis:

\[ L^2(M) = \bigoplus G_{r_1, r_2, \omega_1, \omega_2} \]

where \( r_j \) runs over \( \mathbb{N}^* \) and \( \omega_j \) runs over \( 2\mathbb{N} + 1 \).

On \( G_{r_1, r_2, \omega_1, \omega_2} \), \( R_j \) acts as \( r_j \) and \( \Omega_j \) as \( \omega_j \), therefore \( -\Delta \) acts as \( r_1 \omega_1 + r_2 \omega_2 \).
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Therefore, \(R_1, R_2, \Omega_1\) and \(\Omega_2\) commute. Again, we perform a careful joint spectral analysis:
\[L^2(M) = \bigoplus \mathcal{G}_{r_1, r_2, \omega_1, \omega_2}\]
where \(r_j\) runs over \(\mathbb{N}^*\) and \(\omega_j\) runs over \(2\mathbb{N} + 1\).

On \(\mathcal{G}_{r_1, r_2, \omega_1, \omega_2}\), \(R_j\) acts as \(r_j\) and \(\Omega_j\) as \(\omega_j\), therefore \(-\Delta\) acts as \(r_1 \omega_1 + r_2 \omega_2\).

To know which subspaces \(\mathcal{G}_{r_1, r_2, \omega_1, \omega_2}\) correspond to microlocalization along \(\Sigma_J\) (when the eigenvalue \(r_1 \omega_1 + r_2 \omega_2\) tends to \(+\infty\)), we compare the respective sizes of \(R_1^2, R_2^2\) and \(E = -\Delta + R_1^2 + R_2^2\). We consider operators like
\[P_n^J = \chi_n \left( \frac{-\Delta}{E} \right) \prod_{i \notin J} \chi_n \left( \frac{R_i^* R_i}{E} \right) \prod_{j \in J} \left( 1 - \chi_n \right) \left( \frac{R_j^* R_j}{E} \right)\]
where \(\chi_n\) is a bump function around 0. We consider \(P_n^J \varphi_k\), then take limit in \(k\) and then in \(n\). We obtain the part of the QL in \(\Sigma_J\).
We are led to the decomposition

$$\varphi_k = \varphi_k^\emptyset + \sum_{\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}} \varphi_k^\mathcal{J},$$

of the first main result, with $\varphi_k^\mathcal{J}$ having a unique QL, supported in $S\Sigma J$.

We fix $\mathcal{J} \subset \{1, 2\}$, $\mathcal{J} \neq \emptyset$, and consider $\varphi_k^\mathcal{J}$. Recall that in $S\Sigma \mathcal{J}$, the action of $-\Delta$ is dominated by that of $R_j$ for $j \in \mathcal{J}$.

We are now looking for INVARIANCE properties of the corresponding QL $\nu^\mathcal{J}$. 
In the decomposition of $L^2(M)$, we also use the information coming from the $\Omega_j$’s for $j \in J$: on a joint eigenspace of the $\Omega_j$’s, $-\Delta$ roughly acts as

$$\sum_{j \in J} \omega_j R_j.$$ 

We are able to prove that the real important thing is not the sequence $\omega_j$, $j \in J$ but the ratios between the $\omega_j$’s. In other words, what matters is the element $s = (s_j)_{j \in J}$ in the simplex

$$S_J = \left\{ \sum_{j \in J} s_j = 1, \ s_j \geq 0 \ \forall j \right\}.$$ 

which is proportional to $(\omega_j)_{j \in J}$. If we set

$$R^J_s = \sum_{j \in J} s_j |\partial z_j|$$

and $(s_j)_{j \in J}$ is nearly colinear to $(\omega_j)_{j \in J}$, then

$$([\text{Op}(a), R^J_s] \varphi_k, (\omega_j)_{j \in J}, \varphi_k, (\omega_j)_{j \in J}) \xrightarrow{k \to +\infty} 0.$$
From

\[
([\text{Op}(a), R^J_s] \varphi_{k,(\omega_j)_{j \in J}}, \varphi_{k,(\omega_j)_{j \in J}}) \xrightarrow{k \to +\infty} 0,
\]

taking principal symbols (as done when computing \(([\text{Op}(a), \sqrt{-\Delta}] \varphi, \varphi))\)
we obtain a part of the QL which is invariant under the Hamiltonian
vector field corresponding to the principal symbol

\[
\rho^J_s = \sum_{j \in J} s_j |\sigma_P(\partial z_j)|
\]

of \(R^J_s = \sum_{j \in J} s_j |\partial z_j|\).

And all parts of the eigenfunction \(\varphi_k\) are orthogonal, i.e., joint microlocal
defect measures vanish!
A microlocal QL $\nu$ which is supported in $S\Sigma$ is not invariant by a single flow: to understand its structure and get invariance properties, it is necessary to **disintegrate** this measure.

The above analysis tells us that

$$\nu = \sum_{J \in \mathcal{P}} \int_{s_J} \nu_J^s dQ_J^s(s)$$

where $\nu_J^s$ is invariant by a flow $e^{t\tilde{\rho}_J^s}$ defined on $S\Sigma$ (invariance !).

**The invariance properties of** $\nu_J^s$ **are governed by a family of Hamiltonians, denoted by** $\rho_J^s$. 
Let $\nu$ be a microlocal Quantum Limit such that $\text{supp} \, \nu \subset S\Sigma$. Then, for any $\mathcal{J} \in \mathcal{P}$, there exist

- a non-negative Radon measure $dQ^\mathcal{J}$ on the simplex $S_\mathcal{J}$
- probability measures $\nu^\mathcal{J}_s$ indexed by $s \in S_\mathcal{J}$ and “supported on $S\Sigma_\mathcal{J}$”

such that for $Q^\mathcal{J}$-almost any $s \in S_\mathcal{J}$, $\nu^\mathcal{J}_s$ is invariant by the flow $e^{t\tilde{\rho}^\mathcal{J}_s}$, and

$$
\nu = \sum_{\mathcal{J} \in \mathcal{P}} \int_{S_\mathcal{J}} \nu^\mathcal{J}_s \, dQ^\mathcal{J}(s). \tag{1}
$$

Converse is true: any probability measure $\nu$ on $S\Sigma$ which writes (1) (with the invariance property) is a microlocal Quantum Limit.

($\sim$ Riemannian sphere, harmonic oscillators).
Conclusion

Perspectives:

- Other properties of eigenvalues/eigenfunctions: trace formulae, zero sets of eigenfunctions, etc.

- The first result can be generalized and used in other contexts: analysis of subelliptic Schrödinger equations under Assumption (A), i.e., include time in the analysis.

- Properties of subelliptic wave equations: propagation of singularities, etc.

Thank you for your attention!
We go backwards:

- **Step 1:** Given $\nu_s^J$ invariant by $e^{t\vec{\rho}_s^J}$, we construct a sequence of eigenfunctions having $\nu_s^J$ as unique microlocal QL.

- **Step 2:** We make $s$ run over $S_J$, i.e., we take linear/orthogonal combinations of eigenfunctions as in Step 1 to obtain any $\nu^J = \int_{S_J} \nu_s^J d\nu_s^J(s)$ as a microlocal QL.

- **Step 3:** We make $J$ run over $P$, i.e, we take linear/orthogonal combinations of eigenfunctions as in Step 2 to obtain any $\nu = \sum_{J \in P} \int_{S_J} \nu_s^J d\nu_s^J(s)$ as a microlocal QL.

At each step, we only add eigenfunctions which share the same eigenvalue: it is possible thanks to the algebraic properties of the set of eigenvalues.

For Step 1, we use ground states of harmonic oscillators.