Subelliptic wave equations are never observable

Cyril Letrouit

Ecole Normale Supérieure (DMA)
Sorbonne Université (Laboratoire Jacques-Louis Lions)

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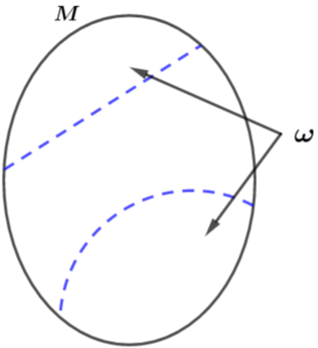
In a few words

We consider the **wave equation**

$$\partial_{tt}^2 - \Delta u = 0, \quad (u_{t=0}, \partial_t u_{t=0}) = (u_0, u_1)$$

in a manifold M equipped with a volume μ. Here, Δ is a **sub-Riemannian** (or subelliptic) Laplacian.
We fix $\omega \subset M$ (measurable).



In a few words

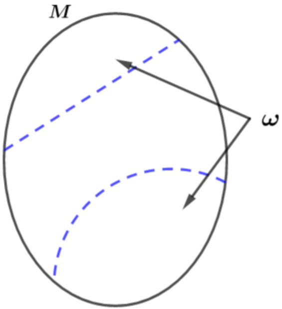
We say that the wave equation is **observable** in time $T_0 > 0$ in ω if $\exists C > 0$ such that for any initial data $(u_0, u_1) \in \mathcal{H} \times L^2$,

$$\int_0^{T_0} \int_\omega |\partial_t u(t,x)|^2 d\mu(x) dt \geqslant C \|(u_0, u_1)\|^2_{\mathcal{H} \times L^2}.$$

Main result: If $M \backslash \omega$ has non-empty interior, then the wave equation is **never** observable (i.e., observable for no time $T_0 < +\infty$).

Remark: Observability $\Leftrightarrow$ Controllability.

Related goal: Understand speed of propagation of information/singularities for subelliptic evolution equations.



Plan of the talk

I - Introduction and main result

Sub-Laplacians

Let M be a smooth connected compact manifold of dimension n and μ be a smooth volume on M. Let $X_1, \ldots, X_m$ be smooth vector fields on M (not necessarily linearly independent). We assume

$$\mathrm{Lie}(X_1, \ldots, X_m) = TM.$$

We define the **sub-Laplacian**

$$\Delta = -\sum_{i=1}^{m} X_i^* X_i = \sum_{i=1}^{m} X_i^2 + \mathrm{div}_\mu(X_i) X_i,$$

where

- Star = transpose in $L^2(M, \mu)$;
- $\mathrm{div}_\mu X$ is defined by $L_X \mu = (\mathrm{div}_\mu X)\mu$.

Sub-Laplacians are hypoelliptic, i.e., $\Delta u \in C^\infty(V) \Rightarrow u \in C^\infty(V)$. They satisfy subelliptic estimates:

$$\|u\|_{H^{2/k}} \leqslant C(\|u\|_{L^2} + \|\Delta u\|_{L^2}).$$

Here k = step = degree of subellipticity !

Examples of sub-Laplacians

- Heisenberg: $\Delta = X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = \partial_y - x\partial_z$ in $\mathbb{R}^3$. Then $[X_1, X_2] = -\partial_z$.
- Grushin: $\Delta = X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = x\partial_y$ in $\mathbb{R}^2$.
- Martinet: $X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = \partial_y + x^2\partial_z$ in $\mathbb{R}^3$. Here, $[X_1, X_2] = 2x\partial_z$ and $[X_1, [X_1, X_2]] = 2\partial_z$. This is a **step 3** sub-Laplacian.

Sometimes, there are complicated relations between the vector fields and their brackets: $[X_1, X_3] = X_1$, etc.

Sub-Riemannian distance

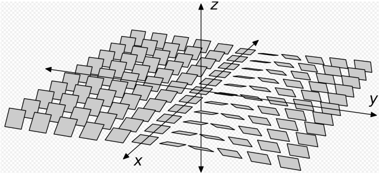
We set $\mathcal{D} = \text{Span}(X_1, \ldots, X_m)$ (the "distribution"). There is a metric g associated to the X_j, namely

$$g_q(v) = \inf \left\{ \sum_{j=1}^{m} u_j^2, \quad v = \sum_{j=1}^{m} u_j X_j(q) \right\},$$

and an associated distance

$$d_{\text{sR}}(q, q') = \inf_{\substack{\gamma(0)=q, \gamma(1)=q' \\ \dot\gamma(t) \in \mathcal{D}, \text{ a.e. } t}} \int_0^1 \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))} dt.$$

According to Chow-Rashevsky, $d_{\text{sR}}(q, q') < +\infty$ for any $q, q' \in M$.



Sub-Riemannian wave equation

We consider a bounded subset $M \subset \mathbb{R}^n$ with $\partial M \neq \emptyset$, a time $T > 0$, and the **free wave equation**

$$\begin{cases} \partial_{tt}^2 u - \Delta u = 0 \quad \text{in } (0,T) \times M \\ u = 0 \quad \text{on } (0,T) \times \partial M, \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1). \end{cases}$$

The natural **energy** of a solution is

$$E(u(t,\cdot)) = \frac{1}{2} \int_M \left(|\partial_t u(t,x)|^2 + |\nabla^{\mathrm{sR}} u(t,x)|^2 \right) d\mu(x).$$

where

$$\nabla^{\mathrm{sR}} \phi = \sum_{j=1}^m (X_j \phi) X_j.$$

Then $\frac{d}{dt} E(u(t,\cdot)) = 0$. **Initial data:**

$$\|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2 = \|u_0\|_{\mathcal{H}}^2 + \|u_1\|_{L^2(M,\mu)}^2$$

with

$$\|v\|_{\mathcal{H}} = \left(\int_M |\nabla^{\mathrm{sR}} v(x)|^2 d\mu(x) \right)^{\frac{1}{2}}.$$

Main result

Definition

Let $T_0 > 0$ and $\omega \subset M$ be a μ-measurable subset. The subelliptic wave equation is **exactly observable** on ω in time T_0 if there exists a constant $C_{T_0}(\omega) > 0$ such that, for any $(u_0, u_1) \in \mathcal{H}(M) \times L^2(M)$, the solution u of the wave equation satisfies

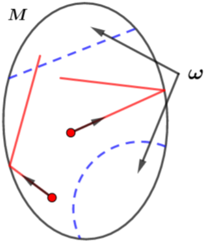
$$\int_0^{T_0} \int_\omega |\partial_t u(t,x)|^2 d\mu(x) dt \geqslant C_{T_0}(\omega) \|(u_0, u_1)\|^2_{\mathcal{H} \times L^2}.$$

Theorem (C.L.-2020)

Let $\omega \subset M$ be a measurable subset. We assume that $M \backslash \omega$ contains in its interior a point x such that $[X_i, X_j](x) \notin \mathrm{Span}(X_1(x), \ldots, X_m(x)) = \mathcal{D}_x$ for some i, j. Then the subelliptic wave equation is **not exactly observable** *on ω in time T_0, for any $T_0 > 0$.*

If Δ is a **Riemannian** Laplacian,

$$(\text{Observability in time } T_0 \text{ in } \omega) \Leftrightarrow$$
$$(\omega \text{ satisfies GCC in time } T_0)$$



where (GCC):

any ray of geometrical optics (=geodesic)
travelled at speed 1 meets ω within time T_0.

(Bardos-Lebeau-Rauch, 1992).
Proof: $\Rightarrow$ Solutions with energy localized along a ray.
[Rmk: Gaussian beams $\approx$ Coherent states $\approx$ WKB]
If GCC not satisfied, take geodesic not entering ω in time T_0. Observability contradicted by associated Gaussian beam.
Reminder: A geodesic is a local minimizer of the sR distance

$$d_{\mathrm{sR}}(q,q') = \inf_{\substack{\gamma(0)=q,\gamma(1)=q' \\ \dot\gamma(t)\in\mathcal{D}, \text{ a.e. } t}} \int_0^1 \sqrt{g(\dot\gamma(t),\dot\gamma(t))}dt.$$

Structure of the proof

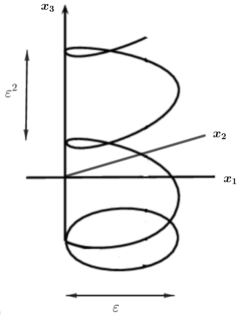
Two ingredients:

- Find a *sub-Riemannian* geodesic which does not enter ω within time T_0: in other words, (GCC) never holds because there exist *spiraling geodesics* which stay very long in $M\backslash\omega$.
- Construct a Gaussian beam along this geodesic: all the energy concentrates near this geodesic, hence outside ω.
  Therefore observability does not hold.
  More generally, along any (normal) sub-Riemannian geodesic, one may construct Gaussian beams.

Remark: Second point is not surprising (although not explicitly in the literature), first point is new.

Example: $M_H = (-1,1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, with $\mathbb{T} = \mathbb{R}/\mathbb{Z} \approx (-1,1)$.
Vector fields $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1\partial_{x_3}$.
Laplacian $\Delta = X_1^2 + X_2^2$. [Measure $\mu =$Lebesgue.]
Distribution $\mathcal{D}_H = \mathrm{Span}(X_1, X_2)$.
Metric g_H: (X_1, X_2) is a g_H-orthonormal frame of $\mathcal{D}_H$.
Then, $(M_H, \mathcal{D}_H, g_H)$= "Heisenberg manifold with boundary".
We note that $[X_1, X_2] = -\partial_{x_3}$ ($\Rightarrow \Delta$ subelliptic).
Spiraling geodesics:

$$\begin{aligned} x_1(t) &= \varepsilon \sin(t/\varepsilon) \\ x_2(t) &= \varepsilon \cos(t/\varepsilon) - \varepsilon \\ x_3(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4). \end{aligned}$$



They spiral around the x_3 axis $x_1 = x_2 = 0$.

Remark: This geodesic is travelled at speed 1;
The x_3 coordinate remains small for $0 \leqslant t \leqslant T_0$.

Sub-Riemannian geodesics

Two types of **geodesics** (i.e., local minimizers of distance) in sR geometry:

- normal geodesics (projections of bicharacteristics);
- abnormal geodesics (discovered by Montgomery).

We focus on **normal geodesics** (sufficient for our proof).
Definitions/Notations. The **Hamiltonian** is

$$g^*(x,\xi) = \sigma_p(-\Delta) = \sum_{i=1}^{m} h_{X_i}^2.$$

where, for X smooth vector field, $h_X : T^*M \to \mathbb{R}$ denotes the momentum map

$$h_X(x,\xi) = \xi(X(x)).$$

Example: For Heisenberg, $g^* = \xi_1^2 + (\xi_2 - x_1\xi_3)^2$.
The **wave operator** $P = \partial_{tt}^2 - \Delta$ has principal symbol

$$p_2(t,\tau,x,\xi) = -\tau^2 + g^*(x,\xi).$$

Sub-Riemannian geodesics, II

Null-bicharacteristics = maximal solutions of

$$\begin{cases} \dot{t}(s) = -2\tau(s)\,, \\ \dot{x}(s) = \nabla_\xi g^*(x(s),\xi(s))\,, \\ \dot{\tau}(s) = 0\,, \\ \dot{\xi}(s) = -\nabla_x g^*(x(s),\xi(s)) \end{cases}$$

together with condition $p_2 = 0$, i.e., $\tau^2 = g^*$ (preserved by the flow). They are integral curves of $\vec{p}_2 = (\partial_\tau p_2, \partial_\xi p_2, -\partial_t p_2, -\partial_x p_2)$ lying in the characteristic manifold $\{p_2 = 0\}$.

By **homogeneity**, fix $\tau = -1/2$, which gives $g^*(x(s),\xi(s)) = 1/4$. In other words, we use t as a time variable for the null-bicharacteristic $(x(t),\xi(t))$, and it is **travelled at speed** 1.

The **normal geodesics** are the **projections** on M of the null-bicharacteristics: we keep $x(t)$ but forget $\xi(t)$. They **locally minimize** the sub-Riemannian distance.

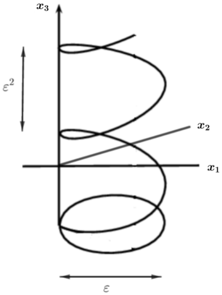
Example on Heisenberg geodesics

$\Delta = \partial_{x_1}^2 + (\partial_{x_2} - x_1\partial_{x_3})^2$; Hamiltonian $g^* = \xi_1^2 + (\xi_2 - x_1\xi_3)^2$.
The bicharacteristic equations are

$$\begin{aligned}
\dot{x}_1(t) &= 2\xi_1, & \dot{\xi}_1(t) &= 2\xi_3(\xi_2 - x_1\xi_3),\\
\dot{x}_2(t) &= 2(\xi_2 - x_1\xi_3), & \dot{\xi}_2(t) &= 0,\\
\dot{x}_3(t) &= -2x_1(\xi_2 - x_1\xi_3), & \dot{\xi}_3(t) &= 0.
\end{aligned}$$

Take $\xi_3 = \varepsilon^{-1}$. Since the geodesic is travelled at speed 1, i.e., $\xi_1^2 + (\xi_2 - x_1\xi_3)^2 = 1/4$, we take for example $\xi_1 = \cos(2t/\varepsilon)/2$ and $\xi_2 = 0$. Then

$$\begin{aligned}
x_1(t) &= \frac{\varepsilon}{2}\sin\left(\frac{2t}{\varepsilon}\right)\\
x_2(t) &= \frac{\varepsilon}{2}\cos\left(\frac{2t}{\varepsilon}\right) - \frac{\varepsilon}{2}\\
x_3(t) &= \frac{\varepsilon}{4}\left(t - \frac{\varepsilon}{4}\sin\left(\frac{4t}{\varepsilon}\right)\right).
\end{aligned}$$



Geodesics do not go far from their initial point !

Observability for small $|\xi_3|$: "repairing" the main result

Idea: Apart from these geodesics with large $|\xi_3|$, all other geodesics meet ω in bounded time.
Again, $M_H = (-1,1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$ with $\Delta = \partial_{x_1}^2 + (\partial_{x_2} - x_1\partial_{x_3})^2$. Let ω be an horizontal strip $(-1,1)_{x_1} \times \mathbb{T}_{x_2} \times I_{x_3}$.

Theorem (C.L.-2020)

Let a be a non-negative symbol of order 0 such that $Supp(a)$ contains $T^*\omega$ and

$$V_\varepsilon = \left\{ (x,\xi) \in T^*M_H : |\xi_3| > \frac{1}{\varepsilon}(g_x^*(\xi))^{1/2} \right\}.$$

For T large enough (depending on ε), there exists $C > 0$ such that

$$C\|(u(0), \partial_t u(0))\|^2_{\mathcal{H}_0 \times L^2_0} \leqslant \int_0^T |(Op(a)\partial_t u, \partial_t u)_{L^2}|\, dt + \|(u(0), \partial_t u(0))\|^2_{L^2_0 \times \mathcal{H}_0'}$$

for any solution u of the subelliptic wave equation.

II - Ideas of proof for Theorem 1

Proof: Gaussian beams along normal sR geodesics

Two ingredients:

- Find a *sub-Riemannian* geodesic which does not enter ω within time T_0: in other words, (GCC) never holds because there exist *spiraling geodesics* which stay very long in $M \backslash \omega$.
- Construct a Gaussian beam along this geodesic: all the energy concentrates near this geodesic, hence outside ω.

Two steps for constructing Gaussian beams (GBs):
- **Approximate solutions:** $\partial_{tt}^2 v_k - \Delta v_k \sim 0$ with energy concentrated along the geodesic;
- **Exact solutions:** $\partial_{tt}^2 u_k - \Delta u_k = 0$ and concentrated energy.

We focus on **approximate solutions** and fix a null-bicharacteristic $(x(t), \xi(t))_{t \in [0,T]}$ which not hitting ∂M in the time-interval $(0, T)$.

Important: The construction is the same as for the Riemannian wave equation since normal geodesics stay in the elliptic part of the symbol.

Proof: Approximate solutions

We look for **approximate solutions** of the wave equation under the form

$$v_k(t,x) = k^{\frac{n}{4}-1} a_0(t,x) e^{ik\psi(t,x)}.$$

[n is the dimension of M, k is a large parameter ($k \sim 1/h$).]

Important: ψ is complex-valued (conjugate points).

Plug into the wave equation:

$$\partial_{tt}^2 v_k - \Delta v_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi}$$

with

$$\begin{aligned}
A_1(t,x) &= [-(\partial_t \psi(t,x))^2 + g^*(x, \nabla\psi(t,x))] a_0(t,x) \\
A_2(t,x) &= L a_0(t,x), \\
A_3(t,x) &= \partial_{tt}^2 a_0(t,x) - \Delta a_0(t,x).
\end{aligned}$$

L is a linear first-order transport operator.

Proof: Approximate solutions, II

There exist $a_0, \psi \in C^2((0,T)\times M)$ such that

$$v_k(t,x) = k^{\frac{n}{4}-1} a_0(t,x) e^{ik\psi(t,x)}$$

- is an approximate solution of the wave equation:

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1((0,T);L^2(M))} \leqslant Ck^{-\frac{1}{2}}.$$

- has energy bounded below (uniformly w.r.t k and $t \in [0,T]$):

$$\exists A > 0, \forall t \in [0,T], \quad \liminf_{k\to+\infty} E(v_k(t,\cdot)) \geqslant A.$$

- has energy small off $x(t)$: for any small δ,

$$\sup_{t\in[0,T]} \int_{M\setminus B_g(x(t),\delta)} \left(|\partial_t v_k(t,x)|^2 + |\nabla^{sR} v_k(t,x)|^2\right) d\mu(x) \underset{k\to+\infty}{\to} 0.$$

Proof: Approximate solutions, III

For simplicity, $M \subset \mathbb{R}^n$.
Recall that with $v_k(t,x) = k^{\frac{n}{4}-1} a_0(t,x) e^{ik\psi(t,x)}$,

$$\partial_{tt}^2 v_k - \Delta v_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi}.$$

We take

$$\psi(t,x) = \xi(t) \cdot (x - x(t)) + \frac{1}{2}(x - x(t))^T \cdot M(t) \cdot (x - x(t)).$$

with a well-chosen $n \times n$ matrix $M(t)$.
Consequence: $A_1(t,x), A_2(t,x), A_3(t,x)$ vanish at high order along $\Gamma = \{(x(t), \xi(t))\}$ (i.e. where $e^{ik\psi}$ is not negligible).
The **new unknown** is the $n \times n$ complex-valued matrix $M(t)$:

- It is **complex-valued** and $\mathrm{Im}(M(t)) > 0$;
- It is chosen so that the second derivatives of A_1 vanish along Γ (resolution of a **Riccati equation**).

Proof: Approximate solutions, IV

To sum up, with these choices of a_0 and ψ, we have all desired properties for v_k:

- $E(v_k(t,\cdot)) \geqslant A$ (independently of k and t).
- The energy of v_k "looks like" $k^{\frac{n}{2}} e^{-ckd^2(x,x(t))}$, hence is ~ 0 outside a small ball centered at $x(t)$.
- $\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^2} \leqslant Ck^{-\frac{1}{2}}$ thanks to the choices of ψ and a_0.

Remark: Interpretation as propagation of complex Lagrangian spaces, or propagation of coherent states.

Proof: Existence of spiraling geodesics

Forget about the wave equation ! The rest is pure **geometry**.

Two ingredients:

- Find a *sub-Riemannian* geodesic which does not enter ω within time T_0: in other words, (GCC) never holds because there exist *spiraling geodesics* which stay very long in $M \backslash \omega$.
- Construct a Gaussian beam along this geodesic: all the energy concentrates near this geodesic, hence outside ω.

Proposition

For any $T_0 > 0$, any $q \in M$ and any open neighborhood V of q in M, there exists a geodesic $t \mapsto x(t)$ of $(M, \mathcal{D}, g)$ travelled at speed 1 and such that $x(t) \in V$ for any $t \in (0, T_0)$.

Remark: These geodesics lose quickly their optimality.
Main idea: Isolate a "Heisenberg structure".
Two steps: Nilpotent case and then general case.

Idea: Given a sR structure $(M, \mathcal{D}, g)$ and $q \in M$, it is possible to "approximate" it around q by a (simpler) **nilpotent** structure.
Example: $X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$ and $X_2 = \partial_\theta$ on $\mathbb{R}^2_{x,y} \times \mathbb{T}_\theta$. At $q = 0$, (x, θ) have "**weight** 1" and y has "weight 2". Also, ∂_x and ∂_θ have weight -1, and ∂_y has weight -2.
Taking **Taylor expansions** we write

$$X_1 = \partial_x - \frac{\theta^2}{2}\partial_x + \ldots + \theta\partial_y - \frac{\theta^3}{6}\partial_y + \ldots, \quad \text{hence}$$

$$X_1 = X_1^{(-1)} + X_1^{(0)} + X_1^{(1)} + \ldots, \quad \text{with}$$

$$X_1^{(-1)} = \partial_x + \theta\partial_y, \qquad X_1^{(0)} = 0, \qquad X_1^{(1)} = -\frac{\theta^2}{2}\partial_x - \frac{\theta^3}{6}\partial_y \ldots$$

Similarly $X_2^{(-1)} = X_2 = \partial_\theta$.

We set $\widehat{X}_1 = X_1^{(-1)} = \partial_x + \theta\partial_y$ and $\widehat{X}_2 = X_2^{(-1)} = \partial_\theta$.
Then $\widehat{X}_1$ and $\widehat{X}_2$ generate a **nilpotent** sR structure $(\widehat{M}, \widehat{\mathcal{D}}, \widehat{g})$. The **geodesics** of $(\widehat{M}, \widehat{\mathcal{D}}, \widehat{g})$ approximate those of $(M, \mathcal{D}, g)$ (near 0).

Proof. Existence of spiraling geodesics: General case

The geodesics are the **integral curves** of $\vec{g}^*$ with

$$g^* = \sum_{i=1}^{m} h_{X_i}^2$$

$[h_{X_i}(x,\xi) = \xi(X_i(x)).]$ Since $X_i \approx \widehat{X}_i$, it is not difficult to prove that they remain close to the integral curves of $\vec{\widehat{g}}^*$ with

$$\widehat{g}^* = \sum_{i=1}^{m} h_{\widehat{X}_i^q}^2.$$

Therefore, it is **sufficient** to prove the result in **nilpotent** sR structures.

Proof. Nilpotentization

Question: In which coordinates do we write the vector fields?

Sub-Riemannian **flag** of $(M, \mathcal{D}, g)$: $\mathcal{D}^0 = \{0\}$, $\mathcal{D}^1 = \mathcal{D}$, and

$$\forall j \geqslant 1, \qquad \mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}, \mathcal{D}^j].$$

Fix $q \in M$. We have a flag

$$\{0\} = \mathcal{D}_q^0 \subset \mathcal{D}_q^1 \subset \ldots \subset \mathcal{D}_q^{r-1} \subsetneq \mathcal{D}_q^{r(q)} = T_qM.$$

Weights: $w_i(q) = 1+$ number of brackets needed to generate the i^{th} direction at q (for $1 \leqslant i \leqslant n$).
A family $(Z_1, \ldots, Z_n)$ of n vector fields is **adapted** to the flag at q if it is a frame of T_qM at q and if $Z_i(q) \in \mathcal{D}_q^{w_i(q)}$ for $1 \leqslant i \leqslant n$.
The inverse of the local diffeomorphism

$$(x_1, \ldots, x_n) \mapsto \exp(x_1 Z_1) \circ \cdots \circ \exp(x_n Z_n)(q)$$

defines **exponential coordinates** of the 2^{nd} kind at q. We now work in these coordinates. They are **privileged coordinates**: for any $1 \leqslant j \leqslant n$,

$$\mathrm{ord}_q(x_j) := \sup\{s \in \mathbb{R} : f(p) = O(d(q,p)^s)\} = w_j.$$

Proof. Nilpotentization, II

Setting: SR structure $(M, \mathcal{D}, g)$ with orthonormal frame $(X_1, ..., X_m)$. Every vector field X_i has a Taylor expansion

$$X_i(x) \sim \sum_{\alpha, j} a_{\alpha,j} x^\alpha \partial_{x_j}.$$

As above, we **group** terms together

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \dots$$

and we set $\widehat{X}_i = X_i^{(-1)}$.
We take $\widehat{M} \simeq \mathbb{R}^n$, $\widehat{\mathcal{D}}^q = \text{Span}\,(\widehat{X}_1^q, \dots, \widehat{X}_k^q)$ and $\widehat{g}^q(\widehat{X}_i^q, \widehat{X}_j^q) = g_q(X_i, X_j)$.
It defines a **nilpotent sR structure** $(\widehat{M}^q, \widehat{\mathcal{D}}^q, \widehat{g}^q)$. Very good approximation of $(M, \mathcal{D}, g)$ only around q.

Proof. Existence of spiraling geodesics: Nilpotent case

Second reduction: it is possible to reduce to the case where all brackets of length $\geqslant 3$ between $X_1, \ldots, X_m$ vanish. We work under this assumption called (A) in the sequel. Also we set

$$n_2 = \dim(\mathrm{Span}(X_1, \ldots, X_m, [X_1, X_2], \ldots, [X_j, X_k], \ldots))$$

The normal geodesics satisfy

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)), \tag{1}$$

where $u_i(t) = h_{X_i}(x(t), \xi(t))$. Thanks to (A), we rewrite (1) as

$$\dot{x}(t) = F(x(t))u(t), \tag{2}$$

where $F = (a_{ij})$ has size $n_2 \times m$, and $u = {}^t(u_1, \ldots, u_m)$. Differentiating the equation for u_i, we have

$$\dot{u}(t) = G(x(t), \xi(t))u(t)$$

where G is the Goh matrix

$$G = (\{h_{X_i}, h_{X_j}\})_{1 \leqslant i,j \leqslant m}.$$

Proof. Existence of spiraling geodesics: Nilpotent case

Due to (A), $G(t) \equiv G$ is constant in t. We know that $G \neq 0$ and that G is antisymmetric. The whole control space $\mathbb{R}^m$ is the direct sum of the image of G and the kernel of G, and G is nondegenerate on its image.

We take u_0 in an invariant plane of G; in other words its projection on the kernel of G vanishes. Then $u(t) = e^{tG}u_0$ and since e^{tG} is an orthogonal matrix, we have $\|e^{tG}u_0\| = \|u_0\|$. We have by integration by parts

$$
\begin{aligned}
x(t) &= \int_0^t F(x(s))e^{sG}u_0\, ds \\
&= F(x(t))G^{-1}(e^{tG} - I)u_0 - \int_0^t \frac{d}{ds}(F(x(s))G^{-1}(e^{sG} - I)u_0\, ds.
\end{aligned}
$$

We take an initial covector ξ^ε so that $G(0, \xi^\varepsilon(t)) = \frac{1}{\varepsilon}G_0$ for some matrix $G_0 \neq 0$. In the above equation, this brings a factor ε, and we can conclude thanks to Gronwall's lemma.

III - Further comments

Bibliographical comments

Two important equations:

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{(Heat equation)} \\ i\partial_t u - \Delta u &= 0 \quad \text{(Schrödinger equation)} \end{aligned}$$

with same (subelliptic) Laplacian.

Heat equation: Minimal time of observability for Grushin and Heisenberg heat equations.
[Beauchard-Cannarsa-Gugliemi 2014], [Beauchard-Cannarsa 2017].

Proofs : Explicit computations with harmonic oscillators, Fourier series, etc.

Ideas for subelliptic Schrödinger equation

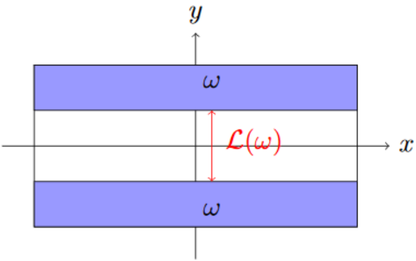
Only one paper: Burq-Sun (2019) for the Grushin Schrödinger

$$i\partial_t u - (\partial_x^2 + x^2\partial_y^2)u = 0 \qquad \text{on } \mathbb{R}_t \times (-1,1)_x \times \mathbb{T}_y.$$

Observation set of the form $\omega = (-1,1)_x \times \omega_y$ (union of strips).

Result:
Existence of a minimal time of control $\mathcal{L}(\omega)$ related to the maximal height of the strips of $M_G \backslash \omega$.

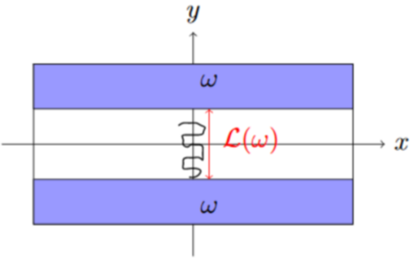
Proof: Semiclassical analysis and construction of vertical Gaussian beams (along degenerated direction).



Ideas for subelliptic Schrödinger equation

With spiraling geodesics: We find again their result heuristically.

Different frequencies travel at different speed (**dispersion**). If ξ_y is large, the spiraling geodesic is more "folded", makes very small meanders, but it is travelled more quickly. All in all, geodesics starting from 0 but with different ξ_y reach ω at the same time ($\neq$ waves).



Ongoing work with C. Fermanian-Kammerer: Study of Schrödinger equation in H-type groups (with semiclassical measures with "two scales" and representation theory).

Ongoing work with C. Sun: Study of Schrödinger equations which are both fractional and subelliptic.

Conclusion

Open questions:
- How fast does information (singularities, defect measures, etc) propagate along directions transverse to the distribution?
- Can these spiraling geodesics be used for other purposes (spectral analysis, etc)?

Thank you very much for your attention !